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Unique toric structure on a Fano Bott manifold

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We prove that if there exists a c_1-preserving graded ring isomorphism between integral cohomology rings of two Fano Bott manifolds, then they are isomorphic as toric varieties. As a consequence, we give an affirmative answer to McDuff's question on the uniqueness of a toric structure on a Fano Bott manifold.

1. Introduction

To each symplectic manifold (M, ω), one can associate the Hamiltonian diffeomorphism group $\mathrm{Ham}(M, \omega)$. It is a normal subgroup of the symplectomorphism group $\mathrm{Symp}(M, \omega)$ and governs all possible Hamiltonian Lie group actions on (M, ω). The group $\mathrm{Ham}(M, \omega)$ is infinite dimensional and non-compact in general, and it might possess more than one maximal torus with distinct conjugacy classes. It was proved by Karshon–Kessler–Pinsonnault [KKP07] and Pinsonnault [Pin08] that the number of conjugacy classes of maximal tori in $\mathrm{Ham}(M, \omega)$, allowing conjugations by elements of $\mathrm{Symp}(M, \omega)$, is finite in dimension four. Also, McDuff [McD11, Proposition 3.1] proved that the number of conjugacy classes of maximal tori of dimension $\frac{\dim M}{2}$ (called *toric actions*) is finite in any dimension.

Recall that a symplectic form ω is called *monotone* if $c_1(M) := c_1(TM, J) = \lambda \cdot [\omega]$ for some $\lambda > 0$ and an ω-compatible almost complex structure J on M. Throughout this paper, we always assume that $\lambda = 1$ unless stated otherwise. In this paper, we consider the following question posed by McDuff [McD11].

Question 1.1 (McDuff). *[McD11, Question 1.11] Let (M, ω) be a $2n$-dimensional closed monotone symplectic manifold. If T_1 and T_2 are n-tori in* $\mathrm{Ham}(M, \omega)$*, are (M, ω, T_1) and (M, ω, T_2) equivariantly symplectomorphic? Equivalently, is the number of symplectic conjugacy classes of n-tori in* $\mathrm{Ham}(M, \omega)$ *precisely one?*

Note that Question 1.1 is the negation of the original question [McD11, Question 1.11] as we believe the answer to Question 1.1 would be true. (See Conjecture 1.4.) We also note that one can replace $\mathrm{Ham}(M, \omega)$ by $\mathrm{Symp}(M, \omega)$ in Question 1.1 since M is a closed symplectic toric manifold, which in particular implies that any symplectic torus action is Hamiltonian so that any connected subgroup of $\mathrm{Symp}(M, \omega)$ is indeed in $\mathrm{Ham}(M, \omega)$.

Due to Delzant's theorem [Del88], any closed symplectic toric manifold is equivariantly symplectomorphic to a smooth projective toric variety equipped with a torus invariant Kähler form. When a symplectic form is monotone, it is equivariantly symplectomorphic to a smooth Fano toric variety by Kleiman's ampleness criterion [Kle66, Theorem 1 in Section III-1] and its moment polytope becomes a *reflexive*[1] polytope.

In the algebro-geometric aspects, Question 1.1 asks whether a smooth Fano toric variety has a unique toric structure up to isomorphism. McDuff [McD11] gave an affirmative answer to Question 1.1 when $M = \mathbb{C}P^k \times \mathbb{C}P^m$, and Fanoe [Fan14] generalized McDuff's result to the case of $\mathbb{C}P^k$-bundle over $\mathbb{C}P^m$. To the best of the authors' knowledge, Question 1.1 is still open.

This paper addresses Question 1.1 in case that M is a Bott manifold. A *Bott tower*, first introduced by Grossberg and Karshon [GK94], is an iterated $\mathbb{C}P^1$-bundle starting with a point

$$\mathcal{B}_n \xrightarrow{\pi_n} \mathcal{B}_{n-1} \xrightarrow{\pi_{n-1}} \cdots \longrightarrow \mathcal{B}_1 = \mathbb{C}P^1 \xrightarrow{\pi_1} \mathcal{B}_0 = \{\text{a point}\}$$

where each $\mathcal{B}_i$ is obtained by projectivizing the direct sum of the trivial line bundle $\underline{\mathbb{C}}$ and a complex line bundle ξ_i over $\mathcal{B}_{i-1}$, i.e., $\mathcal{B}_i = P(\underline{\mathbb{C}} \oplus \xi_i)$. The total space $\mathcal{B}_n$ is called a *Bott manifold*.

We may equip a Bott manifold with a natural complex structure by taking each ξ_i as a holomorphic line bundle so that $\mathcal{B}_n$ becomes a complex manifold with a natural $(\mathbb{C}^*)^n$-action constructed in an iterative way using a toric structure of a base space and a $\mathbb{C}^*$-action on a fiber at each stage, see [Oda78, Section I-7.6′]. Indeed, $\mathcal{B}_n$ is a smooth projective toric variety.

Any Bott manifold $\mathcal{B}_n$ can be characterized by an $n \times n$ lower triangular integer matrix called a *Bott matrix*. Roughly speaking, a Bott manifold $\mathcal{B}_n$ is a smooth projective toric variety, where the corresponding fan is combinatorially equivalent to the normal fan of the n-cube. After fixing one reference maximal cone of the fan and making it into the "first quadrant" via some basis change by multiplying a suitable element of $\mathrm{GL}(n, \mathbb{Z})$, we obtain residual n (column) vectors which form a Bott matrix. Note that a Bott matrix

[1] A polytope is *reflexive* if it is integral and has a unique interior lattice point such that the affine distance from the point to each facet is equal to one.

presentation of a Bott manifold is not unique and it depends on the choice of a reference cone and an element of $\mathrm{GL}(n,\mathbb{Z})$. In Section 3, we illustrate some operations (temporarily denoted by **Op.1** and **Op.2** in this paper) on the set of Bott matrices and describe cohomology ring isomorphisms induced by the operations that will be crucially used in the proof of our main theorem.

Not all Bott manifolds are Fano. For instance, there are only two Fano Bott manifolds in dimension four: $\mathbb{C}P^1 \times \mathbb{C}P^1$ and a Hirzebruch surface $P(\underline{\mathbb{C}} \oplus \mathcal{O}(1))$. Recently, Suyama [Suy19] classified all Fano Bott manifolds in terms of Bott matrices, see Section 2 for details. Now we state our main theorem which says that every Fano Bott manifold is characterized (as a toric variety) by its integral cohomology ring and the first Chern class.

Theorem 1.2 (Theorem 4.2). *Let X and Y be Fano Bott manifolds. If there exists a c_1-preserving graded ring isomorphism*

$$\varphi\colon H^*(X;\mathbb{Z}) \to H^*(Y;\mathbb{Z}),$$

then X and Y are isomorphic[2] *as toric varieties, i.e., the fans associated with X and Y are unimodularly equivalent.*

We call a monotone symplectic manifold (M,ω) a *monotone Bott manifold* if M is diffeomorphic to a Bott manifold. Using Theorem 1.2, we obtain a positive answer to Question 1.1.

Corollary 1.3. *Any monotone Bott manifold has a unique toric structure.*

Proof. Suppose that (M,ω) is a $2n$-dimensional monotone Bott manifold and T_1, T_2 are n-tori in $\mathrm{Ham}(M,\omega)$. From Delzant's theorem [Del88], each T_i-action makes M into a toric Fano variety (which we denote by X_i) with T_i-invariant complex structure J_i on M. Note that each J_i can be chosen to be ω-compatible so that $c_1(X_1) = c_1(X_2) = [\omega] \in H^2(M;\mathbb{Z})$.

On the other hand, it follows from [MP08, Corollary 3.5 and Theorem 5.5] that any smooth projective toric variety whose integral cohomology ring is isomorphic to that of a Bott manifold is in fact isomorphic to a Bott manifold as a toric variety. (See Remark 2.3 for details.) Thus we may

[2]We say that two toric varieties X and Y are *isomorphic as toric varieties* if there exists a toric isomorphism ϕ from X to Y, i.e., $\phi(t\cdot x) = \phi(t)\cdot\phi(x)$ for every element t in the torus $T_X \subset X$ and $x \in X$. In fact, it was proved in [Ber03] that there exists a toric isomorphism if X and Y are isomorphic as abstract varieties.

assume that X_1 and X_2 are Fano Bott manifolds. Then the identity map $H^*(M;\mathbb{Z}) \to H^*(M;\mathbb{Z})$ induces a graded ring isomorphism

$$H^*(X_1;\mathbb{Z}) \to H^*(X_2;\mathbb{Z}), \qquad c_1(X_1) = [\omega] \mapsto [\omega] = c_1(X_2).$$

Thus the result follows from Theorem 1.2. □

It is worth mentioning a relation between Theorem 1.2 and a problem posed by the third author and Suh [MS08, Problems 1 and 4] which asks whether two smooth complete toric varieties having isomorphic cohomology rings (as graded rings) are diffeomorphic or not. This problem is now called the *cohomological rigidity* for toric varieties. There are many partial affirmative answers to the problem. For instance, two smooth complete toric varieties with Picard number 2 are diffeomorphic if and only if their integral cohomology rings are isomorphic as graded rings, see [CMS10]. We also refer the reader to [CMS11, BEM$^+$17] and references therein for recent accounts of this problem.

Inspired by Theorem 1.2, we pose the following conjecture.

Conjecture 1.4. *Suppose that X and Y are smooth toric Fano varieties. If there exists a c_1-preserving graded ring isomorphism between their integral cohomology rings, then X and Y are isomorphic as toric varieties.*

Conjecture 1.4 was verified for some other classes of smooth toric Fano varieties. Indeed, the authors confirmed Conjecture 1.4 for smooth toric Fano varieties with Picard number 2, whose proof will be provided in an upcoming manuscript [CLMP23]. Also the third author together with Higashitani and Kurimoto [HKM22] proved Conjecture 1.4 for smooth toric Fano varieties with small dimension ($\dim X_{\mathbb{C}} \leq 4$) or with large Picard number.)

Note that if Conjecture 1.4 is true, then the answer to Question 1.1 is positive. More precisely, if (M, ω, T_1) and (M, ω, T_2) are two toric structures over the same monotone symplectic manifold (M, ω), then the identity map on $H^*(M;\mathbb{Z})$ satisfies the hypothesis in Conjecture 1.4. Therefore, Conjecture 1.4 can be thought of as a stronger version of Question 1.1.

We also note that we can count the number of isomorphism classes (as verieties) of Fano Bott manifolds, that agrees with the number of rooted triangular cacti. The number goes to infinity as the dimension approaches infinity. On the other hand, the number of isomorphism classes of smooth toric Fano varieties is explicitly counted up to some dimension by Obro and we may give a table of these numbers to compare them up to some dimension. See [CLMP21].

As a final remark, we would like to mention a recent work of Pabiniak and Tolman. In [PT20], they considered the following question which they called *symplectic cohomological rigidity.*

Question 1.5. *[PT20, p.3] Let (M_1, ω_1) and (M_2, ω_2) be symplectic toric manifolds. If there exists a graded ring isomorphism between their integral cohomology rings sending $[\omega_1]$ to $[\omega_2]$, are (M_1, ω_1) and (M_2, ω_2) symplectomorphic?*

They also gave a positive answer to Question 1.5 under the assumptions that ω_1 and ω_2 are rational symplectic forms and that $H^*(M_1; \mathbb{Q}) \cong H^*(M_2; \mathbb{Q}) \cong H^*(\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1; \mathbb{Q})$.

This paper is organized as follows. In Section 2, we explain the notion of Bott manifolds and also discuss their cohomological properties. In Section 3, we introduce two operations **Op.1** and **Op.2** on Bott matrices and prove that any two Bott matrices which represent isomorphic Bott manifolds are obtained by applying those operations repeatedly. In Section 4, we give the proof of Theorem 1.2.

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2. Bott manifolds

We begin by recalling the definition of Bott towers and Bott manifolds.

Definition 2.1. [GK94, §2.1] A *Bott tower* $\mathcal{B}_\bullet$ is an iterated $\mathbb{C}P^1$-bundle starting with a point:

$$\begin{array}{ccccccccc} \mathcal{B}_n & \xrightarrow{\pi_n} & \mathcal{B}_{n-1} & \xrightarrow{\pi_{n-1}} & \cdots & \xrightarrow{\pi_2} & \mathcal{B}_1 & \xrightarrow{\pi_1} & \mathcal{B}_0, \\ \| & & & & & & \| & & \| \\ P(\underline{\mathbb{C}} \oplus \xi_n) & & & & & & \mathbb{C}P^1 & & \{\text{a point}\} \end{array} \tag{2.1}$$

where each $\mathcal{B}_i$ is the complex projectivization of the Whitney sum of a holomorphic line bundle ξ_i and the trivial bundle $\underline{\mathbb{C}}$ over $\mathcal{B}_{i-1}$. The total space $\mathcal{B}_n$ is called a *Bott manifold*.

Let γ_j be the tautological line bundle over $\mathcal{B}_j$ and $\gamma_{i,j}$ the pullback of γ_j by the projection $\pi_i \circ \cdots \circ \pi_{j+1} \colon \mathcal{B}_i \to \mathcal{B}_j$ for $i > j$. We also define $\gamma_{j,j} \coloneqq \gamma_j$ for convenience. The Picard group of $\mathcal{B}_{i-1}$ is isomorphic to the free abelian group of rank $i-1$, and is generated by the line bundles $\gamma_{i-1,j}$ for $1 \leq j < i$ by [Har77, Exercise II.7.9]. Therefore, for each $i = 2, \ldots, n$, there exist $a_{i,j} \in \mathbb{Z}$ for $1 \leq j < i$ such that

$$\xi_i = \bigotimes_{1 \leq j < i} \gamma_{i-1,j}^{\otimes a_{i,j}}. \tag{2.2}$$

Thus the set of integers $\{a_{i,j}\}_{1 \leq j < i \leq n}$ determines a Bott manifold.

Each projection $\pi_i \colon \mathcal{B}_i \to \mathcal{B}_{i-1}$ admits a section induced from the zero section of $\underline{\mathbb{C}} \oplus \xi_i$. This implies that

$$\pi_i^* \colon H^*(\mathcal{B}_{i-1}; \mathbb{Z}) \to H^*(\mathcal{B}_i; \mathbb{Z})$$

is an injective ring homomorphism. By abuse of notation, we continue to write $x_j \in H^2(\mathcal{B}_{i-1}; \mathbb{Z})$ for the first Chern class of the dual of $\gamma_{i-1,j}$ for each $i > j$. From (2.2), we obtain

$$c_1(\xi_i) = -\sum_{j=1}^{i-1} a_{i,j} x_j \in H^2(\mathcal{B}_{i-1}).$$

On the other hand, a Bott manifold $\mathcal{B}_n$ is a smooth projective toric variety by the construction (cf. [GK94] and [Oda78, Section I-7.6′]). If $\mathcal{B}_n$ is obtained from $\{a_{i,j}\}_{1 \leq j < i \leq n}$, then it is known from [Civ05, §3] that its fan

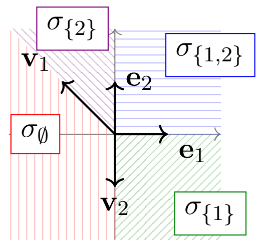

Figure 1: The fan of $\mathcal{H}_1$.

has $2n$ rays and their generators are column vectors of the following matrix

$$(2.3)\qquad (E \mid A) := \begin{bmatrix} 1 & & & & & -1 & & & & \\ & 1 & & & & a_{2,1} & -1 & & & \\ & & 1 & & & a_{3,1} & a_{3,2} & -1 & & \\ & & & \ddots & & \vdots & \vdots & \ddots & \ddots & \\ & & & & 1 & a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & -1 \end{bmatrix},$$

where E is the $n \times n$ identity matrix. We call an integer matrix of the form A in (2.3) a *Bott matrix*. For a given Bott matrix A, we denote by

$$\mathbf{e}_j := j\text{-th standard basis vector}, \qquad \mathbf{v}_j := j\text{-th column vector of } A.$$

The fan Σ_A of the Bott manifold $\mathcal{B}_n$ has 2^n maximal cones $\Sigma(n) = \{\sigma_I \mid I \subset [n]\}$, where $[n] := \{1, \ldots, n\}$ and

$$(2.4)\qquad \sigma_I = \operatorname{Cone}\left(\{\mathbf{e}_i \mid i \in I\} \cup \{\mathbf{v}_j \mid j \in I^c\}\right).$$

Example 2.2. Let $n = 2$. Then the Bott manifold determined by $\{a_{2,1}\}$ is a Hirzebruch surface $\mathcal{H}_{a_{2,1}} := P(\underline{\mathbb{C}} \oplus \mathcal{O}(-a_{2,1}))$. There are four maximal cones:

$$\begin{aligned} \sigma_\emptyset &= \operatorname{Cone}\{\mathbf{v}_1, \mathbf{v}_2\}, & \sigma_{\{1\}} &= \operatorname{Cone}\{\mathbf{e}_1, \mathbf{v}_2\}, \\ \sigma_{\{2\}} &= \operatorname{Cone}\{\mathbf{e}_2, \mathbf{v}_1\}, & \sigma_{\{1,2\}} &= \operatorname{Cone}\{\mathbf{e}_1, \mathbf{e}_2\}. \end{aligned}$$

We present the fan of $\mathcal{H}_1$ in Figure 1.

Let $\mathcal{M}_n$ be the set of all Bott matrices of size $n \times n$, i.e., the set of all $n \times n$ lower triangular integer matrices with -1's on the main diagonal as in (2.3). Since a Bott matrix A determines the fan Σ_A of a Bott manifold, we denote the corresponding Bott manifold by $\mathcal{B}(A)$. Note that it happens that $\mathcal{B}(A)$ and $\mathcal{B}(A')$ are isomorphic as toric varieties even if A and A' are different.

2.1. Cohomology rings

By the Borel–Hirzebruch formula [BH58], the integral cohomology ring of a Bott manifold $\mathcal{B}(A)$ for $A \in \mathcal{M}_n$ is described by

$$\begin{aligned} H^*(\mathcal{B}(A);\mathbb{Z}) &\cong \mathbb{Z}[x_1,\dots,x_n]/\langle x_i^2 + c_1(\xi_i)x_i \mid i=1,\dots,n\rangle \\ &\cong \mathbb{Z}[x_1,\dots,x_n]/\langle x_i^2 - (a_{i,1}x_1 + \cdots + a_{i,i-1}x_{i-1})x_i \mid i=1,\dots,n\rangle \\ &\cong \mathbb{Z}[x_1,\dots,x_n]/\langle x_i^2 - \alpha_i x_i \mid i=1,2,\dots,n\rangle. \end{aligned} \tag{2.5}$$

Here, x_i is the first Chern class of the dual of $\gamma_{n,i}$, and we set

$$\alpha_i \coloneqq a_{i,1}x_1 + \cdots + a_{i,i-1}x_{i-1} \in H^2(\mathcal{B}(A);\mathbb{Z}). \tag{2.6}$$

Note that $x_1,\dots,x_n$ are of degree two and they generate the cohomology ring $H^*(\mathcal{B}(A);\mathbb{Z})$.

Remark 2.3. For a given smooth projective toric variety M, if its integral cohomology ring is isomorphic to that of a certain Bott manifold $\mathcal{B}$ as graded rings, then the fan of M is combinatorially equivalent to that of $\mathcal{B}$, i.e., it is combinatorially equivalent to the normal fan of the n-cube (see [MP08, Theorem 5.5]). Moreover, such a toric variety M is again a Bott manifold by [MP08, Corollary 3.5]. Accordingly, any smooth projective toric variety whose integral cohomology ring is isomorphic to that of a Bott manifold is isomorphic to a Bott manifold.

When working with rational coefficients, we often use the following notation

$$y_i \coloneqq x_i - \frac{1}{2}\alpha_i \in H^2(\mathcal{B}(A);\mathbb{Q}). \tag{2.7}$$

Note that the y_i's may not be integral classes but they generate $H^*(\mathcal{B}(A);\mathbb{Q})$ as a ring.

Recall from [Oda88, Theorem 3.12] that the total Chern class of a Bott manifold $\mathcal{B}(A)$ is given by

$$c(\mathcal{B}(A)) = \prod_{i=1}^n (1+x_i)(1+x_i-\alpha_i) = \prod_{i=1}^n (1+2x_i-\alpha_i).$$

Substituting (2.7) in the above, we get

$$c(\mathcal{B}(A)) = \prod_{i=1}^{n}(1+2y_i) \tag{2.8}$$

which implies that

$$c_1(\mathcal{B}(A)) = 2\sum_{i=1}^{n} y_i \quad \text{and} \quad c_n(\mathcal{B}(A)) = 2^n \prod_{i=1}^{n} y_i. \tag{2.9}$$

Remark 2.4. Note that the cohomology ring description in (2.5) can be obtained from the Danilov–Jurkiewicz theorem, see [BP15, Theorem 5.3.1]. It follows that the even Betti number $b_{2i}(\mathcal{B}(A))$ is $\binom{n}{i}$ for $1 \le i \le n$. Since $H^*(\mathcal{B}(A);\mathbb{Z})$ is generated by degree two elements $\{x_i \mid 1 \le i \le n\}$, we obtain the following.

1) Since $x_1, \dots, x_n$ are linearly independent (over $\mathbb{Z}$), so are $y_1, \dots, y_n$ (over $\mathbb{Q}$).
2) The set $\{x_{i_1} \cdots x_{i_k} \mid 1 \le i_1 < \cdots < i_k \le n\}$ is a $\mathbb{Z}$-basis of $H^{2k}(\mathcal{B}(A);\mathbb{Z})$.
3) $\prod_{i=1}^{n} y_i$ is the orientation class of $\mathcal{B}(A)$. Indeed, since the Euler characteristic of $\mathcal{B}(A)$ is 2^n and it agrees with $c_n(\mathcal{B}(A))$ evaluated on the fundamental class of $\mathcal{B}(A)$, $\prod_{i=1}^{n} y_i$ evaluated on the fundamental class is 1.

In terms of y_i's, we can obtain a simple description of (any) isomorphisms of $H^*(\mathcal{B}(A);\mathbb{Q})$ as follows. In the following, we denote x_i, y_i and α_i in $H^*(\mathcal{B}(A))$ by x_i^A, y_i^A and α_i^A, respectively.

Proposition 2.5. *[CMM15, Proposition 4.1] For A and A' in $\mathcal{M}_n$, if we have a graded ring isomorphism*

$$\varphi\colon H^*(\mathcal{B}(A);\mathbb{Q}) \to H^*(\mathcal{B}(A');\mathbb{Q}),$$

then there are nonzero $q_1, \dots, q_n \in \mathbb{Q}$ and a permutation σ on $[n]$ such that

$$\varphi(y_i^A) = q_i y_{\sigma(i)}^{A'} \qquad \text{for } i = 1, \dots, n.$$

Theorem 2.6. *Suppose that there is a c_1-preserving graded cohomology ring isomorphism between two Bott manifolds. Then all q_i's in Proposition 2.5 are equal to 1. Moreover, it preserves their total Chern classes and hence all the Chern numbers of the two Bott manifolds are the same.*

Proof. Let $\mathcal{B}$ and $\mathcal{B}'$ be Bott manifolds determined by Bott matrices A and A', respectively. Let φ be a c_1-preserving graded cohomology ring isomorphism between $\mathcal{B}$ and $\mathcal{B}'$. By (2.9), we have

$$\varphi\left(2\sum_{i=1}^{n} y_i^A\right) = 2\sum_{i=1}^{n} y_i^{A'}.$$

On the other hand, it follows from Proposition 2.5 that

$$\varphi\left(2\sum_{i=1}^{n} y_i^A\right) = 2\sum_{i=1}^{n} q_i y_{\sigma(i)}^{A'}.$$

Comparing these two identities, we obtain

$$2\sum_{i=1}^{n} y_i^{A'} = 2\sum_{i=1}^{n} q_i y_{\sigma(i)}^{A'}.$$

Here, $y_1^{A'}, \dots, y_n^{A'}$ are linearly independent, so we conclude $q_i = 1$ for any i. This together with (2.8) shows that φ preserves their total Chern classes, proving the former part of the theorem.

The latter part of the theorem follows from the former part and the fact that φ preserves the orientation classes $\prod_{i=1}^{n} y_i^A$ and $\prod_{i=1}^{n} y_i^{A'}$ (as well as top Chern classes). □

Remark 2.7. We note the following.

1) Not every graded cohomology ring isomorphism between Bott manifolds is c_1-preserving. For instance, one can find such an isomorphism for Hirzebruch surfaces $\mathcal{H}_0$ and $\mathcal{H}_2$. See Example 2.8.

2) Recall that two Hirzebruch surfaces $\mathcal{H}_a$ and $\mathcal{H}_b$ are isomorphic if and only $|a| = |b|$. However, for any integers a and b with the same parity, there is a c_1-preserving graded cohomology ring isomorphism between Hirzebruch surfaces $\mathcal{H}_a$ and $\mathcal{H}_b$. Therefore, the existence of such a cohomology ring isomorphism does not imply that two varieties are isomorphic. We would need to restrict our attention to Fano Bott manifolds to conclude a variety isomorphism. See Example 2.8 and Remark 4.3.

Example 2.8. It follows from (2.5) that

$$H^*(\mathcal{H}_0;\mathbb{Z}) = \mathbb{Z}[x_1,x_2]/\langle x_1^2, x_2^2\rangle \quad \text{and}$$
$$H^*(\mathcal{H}_2;\mathbb{Z}) = \mathbb{Z}[x_1',x_2']/\langle (x_1')^2, x_2'(x_2'-2x_1')\rangle.$$

Note that $\mathcal{H}_0$ is Fano with $c_1(\mathcal{H}_0) = 2x_1 + 2x_2$ and $\mathcal{H}_2$ is not with $c_1(\mathcal{H}_2) = 2x_2'$. The map $\varphi\colon H^*(\mathcal{H}_0) \to H^*(\mathcal{H}_2)$ given by $\varphi(x_1) = x_1'$ and $\varphi(x_2) = x_1' - x_2'$ is a graded ring isomorphism which does not preserve the first Chern class. On the other hand, the map φ' defined by $\varphi'(x_1) := x_1'$ and $\varphi'(x_2) := x_2' - x_1'$ is a c_1-preserving isomorphism from $H^*(\mathcal{H}_0)$ to $H^*(\mathcal{H}_2)$. (Note that $\varphi'(c_1(\mathcal{H}_0)) = \varphi'(2x_1 + 2x_2) = 2x_2' = c_1(\mathcal{H}_2)$.)

2.2. Fano Bott manifolds

In this subsection, we recall a description of Fano Bott manifolds from [Suy19].

Theorem 2.9. *[Suy19, Theorem 8] A Bott manifold $\mathcal{B}(A)$ is Fano if and only if each column of $A + E$ has values in $\{-1, 0, 1\}$ and it satisfies one of the following:*

1) *all entries are zero,*
2) *there is at most one* 1 *and every other entry below the* 1 *vanishes (if there is* 1 *on the column),*
3) *if there is* -1 *at the i-th row, then the entries below the* -1 *coincide with the entries on the i-th column below the diagonal* $a_{ii} = -1$.

Example 2.10. Consider the following Bott matrices.

$$A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

One can easily check that only A_4 satisfies all the conditions in Theorem 2.9 and so $\mathcal{B}(A_4)$ is the only Fano Bott manifold among $\mathcal{B}(A_k)$'s for $1 \le k \le 4$. Indeed, A_4 is (10) on the list of Fano threefolds in the book of Oda [Oda88, Figure 2.6] and (12) on the list of 'Smooth toric Fano varieties' [Obr07] in the Graded Ring Database [BK].

3. Operations on Bott matrices

For two Bott matrices A and A' in $\mathcal{M}_n$, we say that A and A' are *isomorphic* if $\mathcal{B}(A)$ and $\mathcal{B}(A')$ are isomorphic as toric varieties. Equivalently, A and A' are isomorphic if the corresponding fans Σ_A and $\Sigma_{A'}$ are unimodularly equivalent, i.e., there is a $\mathbb{Z}$-linear map in $\mathrm{GL}(n,\mathbb{Z})$ which sends a maximal cone in Σ_A to a maximal cone in $\Sigma_{A'}$. For a given Bott matrix A, there are two natural ways of producing (possibly new) isomorphic Bott matrices as follows.

For $A \in \mathcal{M}_n$ and $I \subset [n]$, we consider the $n \times n$ lower triangular matrix L_I whose j-th column $\mathbf{c}_j$ is defined by

$$\mathbf{c}_j := \begin{cases} \mathbf{e}_j & \text{if } j \in I, \\ \mathbf{v}_j & \text{if } j \in I^c, \end{cases}$$

where $\mathbf{v}_j$ is the j-th column of A.

Proposition 3.1. *For $A \in \mathcal{M}_n$ and $I \subset [n]$, the matrix*

$$A_I := L_I^{-1} \cdot L_{I^c}$$

is also a Bott matrix. Moreover, A and A_I are isomorphic. We denote the operation $A \mapsto A_I$ by **"Op.1"**.

Proof. Observe that A_I is a lower triangular integer matrix and

$$(A_I)_{ii} = \sum_{j=1}^{n} (L_I^{-1})_{ij} (L_{I^c})_{ji} = (L_I^{-1})_{ii} (L_{I^c})_{ii} = -1$$

since $(L_I^{-1})_{ij} = 0$ for $j > i$ and $(L_{I^c})_{ji} = 0$ for $i > j$. Thus the first claim easily follows.

For the latter statement, consider a $\mathbb{Z}$-linear map given by $L_I^{-1} \in \mathrm{GL}(n,\mathbb{Z})$. Then it induces a map

$$\Sigma_A \to \Sigma_{A_I}$$

which sends each maximal cone $\sigma_J \in \Sigma_A$ to $L_I^{-1}\sigma_J \in \Sigma_{A_I}$ for each $J \subset [n]$ (in particular σ_I to the first quadrant). Therefore, two fans Σ_A and Σ_{A_I} are unimodularly equivalent. $\square$

Remark 3.2. The column vectors of L_I are the ray generators of the maximal cone σ_I in (2.4). Therefore, the operation **Op.1** is nothing but a procedure of selecting a *reference cone* σ_I and sending it to the *first quadrant* by $L_I^{-1} \in \mathrm{GL}(n,\mathbb{Z})$. Accordingly, if we take a Bott matrix A which defines a *Fano* Bott manifold, all the matrices A_I obtained by the operation **Op.1** define *Fano* Bott manifolds because being *Fano* is an intrinsic property of a toric variety.

Depending on A, it could happen that a reordering (by some permutation $\pi \in S_n$) of the standard basis $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ changes A into another Bott matrix (denoted by A_π). Equivalently, $A_\pi = P_\pi A P_\pi^{-1}$, where P_π is the row permutation matrix of π, that is, P_π has 1 on $(i,\pi(i))$-entry for $i=1,\dots,n$ and all the other entries are zero. We call the operation $A \mapsto A_\pi$ **"Op.2"** when A_π is still a Bott matrix. It is straightforward that A and A_π are isomorphic since Σ_A and Σ_{A_π} are the same up to reordering coordinates.

Example 3.3. Consider $A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

1) **(Op.1)** For $I=\{1\} \subset [3]$, we have

$$L_I = L_I^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad L_{I^c} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_I = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

2) **(Op.2)** For $\pi=(2,3)$, we have

$$A_{(2,3)} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

The following proposition tells us that for a given Bott matrix A, all Bott matrices isomorphic to A are produced by applying **Op.1** and **Op.2** to A.

Proposition 3.4. *Two Bott matrices are isomorphic if and only if one is obtained from the other by applying the two operations* **Op.1** *and* **Op.2***.*

Proof. The "if" part is straightforward. Thus we only need to prove the "only if" part.

Suppose that A and B are isomorphic Bott matrices, i.e., there exists a $\mathbb{Z}$-linear map $C \in \mathrm{GL}(n,\mathbb{Z})$ sending maximal cones in Σ_A to maximal cones in Σ_B. The $\mathbb{Z}$-linear map C sends $\sigma^A_{[n]} \in \Sigma_A$ to $\sigma^B_I \in \Sigma_B$ for some $I \subset [n]$. (We denote by σ^A_I the maximal cone in Σ_A given by $I \subset [n]$.) Since B and $B_I := (L^B_I)^{-1} \cdot L^B_{I^c}$ are isomorphic, we may think of $(L^B_I)^{-1} \cdot C$ as a $\mathbb{Z}$-linear map which sends $\sigma^A_{[n]} \in \Sigma_A$ to $\sigma^{B_I}_{[n]} \in \Sigma_{B_I}$. In other words, $(L^B_I)^{-1} \cdot C$ is a reordering of the standard basis and hence it corresponds to **Op.2**. Therefore, B_I is obtained from A by **Op.2**. Accordingly, B is given by A applying **Op.2** and **Op.1** in order and this completes the proof. □

Note that the rational cohomology class y^A_i given in (2.7) depends on the Bott matrix A. In the rest of the section, we will show that if A and A' are isomorphic, then there is a graded ring isomorphism

$$\varphi\colon H^*(\mathcal{B}(A);\mathbb{Z}) \to H^*(\mathcal{B}(A');\mathbb{Z}) \tag{3.1}$$

sending the set $\{2y^A_i\}^n_{i=1}$ to $\{2y^{A'}_i\}^n_{i=1}$. This fact can be obtained as a byproduct of Propositions 3.4, 3.5, and 3.6.

Proposition 3.5. *Let $A \in \mathcal{M}_n$. For $I \subset [n]$, there exists a c_1-preserving graded ring isomophism*

$$\varphi_I\colon H^*(\mathcal{B}(A);\mathbb{Z}) \to H^*(\mathcal{B}(A_I);\mathbb{Z})$$

such that $\varphi_I(2y^A_i) = 2y^{A_I}_i$ for $i = 1, \dots, n$.

Proof. We first consider the case where $I = \{k\}^c = [n] \setminus \{k\}$ for some $k \in [n]$ so that $A_I = L^{-1}_{\{k\}^c} L_{\{k\}}$. Let $\mathbf{a}_i$ and $\mathbf{a}^I_i$ be the i-th row of $A + E$ and $A_I + E$, respectively. By direct computations, we obtain

1) $\mathbf{a}^I_i = \mathbf{a}_i$ for $i < k$;
2) $\mathbf{a}^I_i = -\mathbf{a}_k$ for $i = k$; and
3) $\mathbf{a}^I_i = \mathbf{a}_i + a_{i,k}\mathbf{a}_k$ for $i > k$.

Then the Bott tower $\mathcal{B}^I_\bullet$ corresponding to A_I is given by

$$\mathcal{B}^I_i = \begin{cases} \mathcal{B}_i & \text{for } i < k, \\ P(\underline{\mathbb{C}} \oplus \xi_k^{-1}) & \text{for } i = k, \\ P(\underline{\mathbb{C}} \oplus (\xi_i \otimes \xi_k^{a_{i,k}})) & \text{for } i > k. \end{cases}$$

Here, $\xi_1, \dots, \xi_n$ are the line bundles used to construct the Bott tower $\mathcal{B}_\bullet = \mathcal{B}(A)$.

We define a map φ_I by

$$\varphi_I(x_i^A) = \begin{cases} x_i^{A_I} & \text{for } i \neq k, \\ x_k^{A_I} + \sum_{j<k} a_{k,j} x_j^{A_I} & \text{for } i = k. \end{cases} \tag{3.2}$$

Then, by (2), we obtain that

$$\varphi_I(x_k^A) = x_k^{A_I} - \alpha_k^{A_I} \quad \text{and} \quad \varphi_I(\alpha_k^A) = -\alpha_k^{A_I}. \tag{3.3}$$

We claim that φ_I is well-defined and is indeed a graded ring isomorphism such that $\varphi_I(2y_i^A) = 2y_i^{A_I}$ for every $i = 1, \dots, n$. The well-definedness follows by showing that $\varphi_I(x_i^A(x_i^A - \alpha_i^A)) = 0$ for all i.

- For $i < k$, we get $\varphi_I(x_i^A(x_i^A - \alpha_i^A)) = x_i^{A_I}(x_i^{A_I} - \alpha_i^{A_I}) = 0$;
- for $i = k$, by (3.3), we have

$$\begin{aligned} \varphi_I(x_k^A(x_k^A - \alpha_k^A)) &= (x_k^{A_I} - \alpha_k^{A_I})(x_k^{A_I} - \alpha_k^{A_I} - \varphi_I(\alpha_k^A)) \\ &= x_k^{A_I}(x_k^{A_I} - \alpha_k^{A_I}) = 0; \end{aligned}$$

- for $i > k$, we have $\varphi_I(x_i^A(x_i^A - \alpha_i^A)) = x_i^{A_I}(x_i^{A_I} - \varphi_I(\alpha_i^A)) = x_i^{A_I}(x_i^{A_I} - \alpha_i^{A_I}) = 0$;

where the second last equality is obtained from (3.2) and (3):

$$\begin{aligned} \varphi_I(\alpha_i^A) &= \sum_{j<i} a_{i,j} \varphi_I(x_j^A) \\ &= \sum_{j \neq k, j<i} a_{i,j} x_j^{A_I} + a_{i,k} \left(x_k^{A_I} + \sum_{j<k} a_{k,j} x_j^{A_I} \right) \\ &= \alpha_i^{A_I} \qquad \text{for } i > k. \end{aligned} \tag{3.4}$$

To show $\varphi_I(2y_i^A) = 2y_i^{A_I}$, we only need to check the case when $i = k$ because $\varphi_I(x_i^A) = x_i^{A_I}$ and $\varphi_I(\alpha_i^A) = \alpha_i^{A_I}$ for $i \neq k$ by (3.2) and (3.4). Then, by (3.3), we obtain

$$\varphi_I(2y_k^A) = \varphi_I(2x_k^A - \alpha_k^A) = 2(x_k^{A_I} - \alpha_k^{A_I}) + \alpha_k^{A_I} = 2x_k^{A_I} - \alpha_k^{A_I} = 2y_k^{A_I}$$

and this completes the proof for the case of $I = \{k\}^c$.

For a general $I = \{i_1 < \cdots < i_m\} \subset [n]$, using the fact $L_I = L_{\{i_1\}^c} \cdot L_{\{i_2\}^c} \cdots L_{\{i_m\}^c}$, we obtain

$$\varphi_I = \varphi_{\{i_1\}^c} \circ \varphi_{\{i_2\}^c} \circ \cdots \circ \varphi_{\{i_m\}^c}.$$

Applying the previous procedure repeatedly, the result follows. One can immediately check that φ_I is c_1-preserving by (2.7). □

For the cohomology ring isomorphism between Bott manifolds induced from **Op.2**, we recall the result [CMM15].

Proposition 3.6 ([CMM15, Lemma 6.1]). *Let $A \in \mathcal{M}_n$. For a permutation π on $[n]$, if $A_\pi \in \mathcal{M}_n$, then there is a c_1-preserving graded ring isomorphism*

$$\varphi_\pi \colon H^*(\mathcal{B}(A);\mathbb{Z}) \to H^*(\mathcal{B}(A_\pi);\mathbb{Z})$$

such that $\varphi_\pi(x_i^A) = x_{\pi(i)}^{A_\pi}$ for $i = 1, \dots, n$. Indeed, we have $\varphi_\pi(2y_i^A) = 2y_{\pi(i)}^{A_\pi}$ for $i = 1, \dots, n$.

4. Main Theorem

In this section, we will prove Theorem 1.2. Throughout this section, we take coefficient in $\mathbb{Z}$ for cohomology unless stated otherwise.

Before to begin with, we explain some notations used in this section. We use letters $\mathcal{B}$, $\mathcal{B}'$ and $\mathcal{B}''$ to indicate Bott manifolds and we similarly denote by (x_i, y_i, α_i), (x_i', y_i', α_i'), and $(x_i'', y_i'', \alpha_i'')$ the elements x, y, and α defined in (2.5) and (2.7) for $\mathcal{B}, \mathcal{B}'$, and $\mathcal{B}''$, respectively.

Lemma 4.1. *Let $\mathcal{B}$ and $\mathcal{B}'$ be Fano Bott manifolds. If there is a c_1-preserving graded ring isomorphism $\varphi \colon H^*(\mathcal{B}) \to H^*(\mathcal{B}')$, then there exists a Fano Bott manifold $\mathcal{B}''$ together with a c_1-preserving graded ring isomorphism $\psi \colon H^*(\mathcal{B}) \to H^*(\mathcal{B}'')$ such that $\mathcal{B}'$ and $\mathcal{B}''$ are isomorphic and $\psi(x_1) = x_1''$. In particular, we have $\psi(2y_1) = 2y_1''$.*

Proof. Suppose that $\mathcal{B}$ and $\mathcal{B}'$ are Fano Bott manifolds determined by Bott matrices $A = (a_{ij})$ and $A' = (a_{ij}')$, respectively. By Theorem 2.6, there exists a permutation σ on $[n]$ such that $\varphi(2y_i) = 2y_{\sigma(i)}'$ for each $i = 1, \dots, n$. If $\sigma(1) \neq 1$, then we get $a_{\sigma(1),j}' = 0$ for every $j < \sigma(1)$. Indeed, since $2y_1 = 2x_1$

and $2y'_{\sigma(1)} = 2x'_{\sigma(1)} - \sum_{j<\sigma(1)} a'_{\sigma(1),j}x'_j$, we get

$$2(\varphi(x_1) - x'_{\sigma(1)}) = - \sum_{j<\sigma(1)} a'_{\sigma(1),j}x'_j$$

which is divisible by 2. Since $a'_{\sigma(1),j}$ belongs to $\{0, \pm 1\}$, we conclude that $a'_{\sigma(1),j} = 0$ for every $j < \sigma(1)$ and $\varphi(x_1) = x'_{\sigma(1)}$. This fact tells us that the $\sigma(1)$-th row of $A' + E$ is zero, and therefore we may apply **Op.2** to A' for the permutation $\pi = s_1 s_2 \dots s_{\sigma(1)-1}$ where s_i denotes the simple transposition $(i, i+1)$. Here, we note that $(\pi \circ \sigma)(1) = 1$.

Consider the Bott matrix $A'' := A'_\pi$ and let $\mathcal{B}''$ be the Bott manifold associated with A''. Suppose that $\mathcal{B}''$ is Fano. Then $\mathcal{B}'$ and $\mathcal{B}''$ are isomorphic and there is a graded ring isomorphism $\varphi' \colon H^*(\mathcal{B}') \to H^*(\mathcal{B}'')$ such that $\varphi'(x'_{\sigma(1)}) = x''_{\pi(\sigma(1))} = x''_1$ by Proposition 3.6. Then $\varphi' \circ \varphi$ is the desired isomorphism and it completes the proof. □

Theorem 4.2 (Theorem 1.2). *Let $\mathcal{B}$ and $\mathcal{B}'$ be Fano Bott manifolds. Assume that there is a c_1-preserving graded ring isomorphism $\varphi \colon H^*(\mathcal{B}) \to H^*(\mathcal{B}')$. Then $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic as toric varieties.*

Proof. We will use an induction on k, where k is the smallest positive integer such that $\varphi(x_k) \neq x'_k$ and modify $\mathcal{B}'$ into $\mathcal{B}''$ using either **Op.1** or **Op.2** so that the new isomorphism

$$\psi := \begin{cases} \phi_I \circ \phi & (\text{for } \mathbf{Op.1}) \\ \phi_\pi \circ \phi & (\text{for } \mathbf{Op.2}) \end{cases} \quad : H^*(\mathcal{B}) \to H^*(\mathcal{B}'')$$

satisfies $\psi(x_k) = x''_k$.

Suppose that the Fano Bott manifolds $\mathcal{B}$ and $\mathcal{B}'$ are determined by the Bott matrices $A = (a_{ij})$ and $A' = (a'_{ij})$, respectively. From Theorem 2.6, there is a permutation σ on $[n]$ such that $\varphi(2y_i) = 2y'_{\sigma(i)}$ for all $i = 1, \dots, n$. We may further assume that $\varphi(x_1) = x'_1$ (or equivalently $\varphi(2y_1) = 2y'_1$) by Lemma 4.1, i.e., $\sigma(1) = 1$.

Suppose there exists $2 \leq k \leq n$ such that $\varphi(x_i) = x'_i$ for every $i < k$ and $\varphi(x_k) \neq x'_k$, that is, $\varphi(2y_i) = 2y'_i$ for all $i < k$. Then there are two possibilities:

1) $\sigma(k) = k$; or

2) $\sigma(k) > k$.

We first consider the case where $\sigma(k)=k$. That is, $\varphi(2y_k)=2y'_k$ and $\varphi(x_k)\neq x'_k$. Since $\varphi(2y_k)=\varphi(2x_k-\alpha_k)=2\varphi(x_k)-\varphi(\alpha_k)$ and $\varphi(2y_k)=2y'_k=2x'_k-\alpha'_k$, we get

$$2(\varphi(x_k)-x'_k)=\varphi(\alpha_k)-\alpha'_k \tag{4.1}$$

and hence $\varphi(\alpha_k)-\alpha'_k$ is divisible by 2. Note that $\varphi(\alpha_k)-\alpha'_k=\sum_{j<k}(a_{k,j}-a'_{k,j})x'_j$ by the definition of α_i and α'_i in (2.5) and from the choice of k. Comparing this with Equation (4.1), we have $\varphi(x_k)=x'_k+\sum_{j<k}d_jx'_j$, where

$$2d_j=a_{k,j}-a'_{k,j} \qquad \text{for } j=1,\dots,k-1 \tag{4.2}$$

which implies that

$$d_j=a_{k,j}=-a'_{k,j} \quad \text{when } d_j\neq 0 \tag{4.3}$$

by the Fano condition in Theorem 2.9. Note that

(4.4) not every d_j is zero for $j<k$ by our assumption.

Now we claim that Equation (4.3) holds even when $d_j=0$. By the well-definedness of the isomorphism φ, we have that

$$\begin{aligned}
0=\varphi(x_k(x_k-\alpha_k)) &= \left(x'_k+\sum_{j<k}d_jx'_j\right)\left(x'_k+\sum_{j<k}d_jx'_j-\sum_{j<k}a_{k,j}x'_j\right)\\
&=(x'_k)^2+x'_k\left(2\sum_{j<k}d_jx'_j-\sum_{j<k}a_{k,j}x'_j\right)\\
&\quad+\left(\sum_{j<k}d_jx'_j\right)\left(\sum_{j<k}d_jx'_j-\sum_{j<k}a_{k,j}x'_j\right)\\
&=(x'_k)^2+x'_k\sum_{j<k}(2d_j-a_{k,j})x'_j\\
&\quad+\left(\sum_{j<k}d_jx'_j\right)\left(\sum_{j<k}(d_j-a_{k,j})x'_j\right)
\end{aligned}$$

in $H^*(\mathcal{B}')$. Since

- there is no term having x'_k in $\left(\sum_{j<k}d_jx'_j\right)\left(\sum_{j<k}(d_j-a_{k,j})x'_j\right)$, and

- $(x'_k)^2 + x'_k \sum_{j<k}(2d_j - a_{k,j})x'_j$ can be expressed as a linear combination of the linearly independent set $\{x'_k x'_j\}_{j<k}$,

both $(x'_k)^2 + x'_k \sum_{j<k}(2d_j - a_{k,j})x'_j$ and $(\sum_{j<k} d_j x'_j)(\sum_{j<k}(d_j - a_{k,j})x'_j)$ vanish in $H^*(\mathcal{B}')$. In particular, it follows from the vanishing of the latter term above that

$$\begin{aligned} 0 &= \left(\sum_{j<k} d_j x'_j\right)\left(\sum_{j<k}(d_j - a_{k,j})x'_j\right) \\ &= \left(\sum_{\substack{j<k \\ d_j\neq 0}} d_j x'_j\right)\left(-\sum_{\substack{\ell<k \\ d_\ell=0}} a_{k,\ell}x'_\ell\right) = -\sum_{\substack{j,\ell<k \\ d_j\neq 0 \text{ and } d_\ell=0}} d_j a_{k,\ell} x'_j x'_\ell \end{aligned}$$

in $H^*(\mathcal{B})$, where the second equality follows from (4.3). However, $\{x'_j x'_\ell \mid d_j \neq 0,\ d_\ell = 0,\ j,\ell < k\}$ is linearly independent by Remark 2.4 (2), so $a_{k,\ell} = 0$ if $d_\ell = 0$. Moreover, since $2d_j = a_{k,j} - a'_{k,j}$ by (4.2), we conclude that

- if $d_j = 0$, then $a_{k,j} = a'_{k,j} = 0$, and
- if $d_j \neq 0$, then $a_{k,j} = -a'_{k,j}$ by (4.3).

Consequently, we have $d_j = a_{k,j} = -a'_{k,j}$ for every $j < k$. Therefore, we obtain

$$\varphi(x_k) = x'_k + \sum_{j<k} d_j x'_j = x'_k - \sum_{j<k} a'_{k,j}x'_j = x'_k - \alpha'_k.$$

Now, we let $\mathcal{B}''$ be a Fano Bott manifold whose Bott matrix A'' is obtained from A' by applying **Op.1** with $I = \{k\}^c$. Then there is a c_1-preserving graded ring isomorphism $\varphi'\colon H^*(\mathcal{B}') \to H^*(\mathcal{B}'')$ such that $\varphi'(x'_j) = x''_j$ for every $j \neq k$ and $\varphi'(x'_k) = x''_k - \alpha''_k$ by (3.2) and (3.3), and also $\varphi'(2y'_i) = 2y''_i$ for all i by Proposition 3.5. Then, the composition $\psi := \varphi' \circ \varphi$ is a graded ring isomorphism $\psi\colon H^*(\mathcal{B}) \to H^*(\mathcal{B}'')$ such that $\psi(2y_i) = 2y''_i$ for every $i = 1, \dots, n$, $\mathcal{B}'$ and $\mathcal{B}''$ are isomorphic as toric varieties, and $\psi(x_j) = x''_j$ for every $j \le k$. Indeed, by (3.3), we have

$$\psi(x_k) = \varphi'(x'_k - \alpha'_k) = x''_k - \alpha''_k + \alpha''_k = x''_k.$$

Now, we consider the second case, $\sigma(k) > k$. Since $\varphi(2y_k) = 2\varphi(x_k) - \sum_{j<k} a_{k,j}x'_j$ and $\varphi(2y_k) = 2y'_{\sigma(k)}$, we get

$$2(\varphi(x_k) - x'_{\sigma(k)}) = \sum_{j<k} a_{k,j}x'_j - \sum_{\ell<\sigma(k)} a'_{\sigma(k),\ell}x'_\ell. \tag{4.5}$$

Because $a_{k,j}$ and $a_{\sigma(k),\ell}$ belong to $\{0, 1, -1\}$ and the left hand side of (4.5) is divisible by 2, we have

$$a'_{\sigma(k),\ell} = 0 \qquad \text{for } k \le \ell < \sigma(k).$$

Indeed, the $\sigma(k)$-th row of A' has consecutive zeros from $(\sigma(k), k)$ to $(\sigma(k), \sigma(k) - 1)$. Therefore by applying **Op.2** to A' for the permutation $\pi = s_k \cdots s_{\sigma(k)-1}$, we get a new Bott matrix A'' such that $\mathcal{B}(A'')$ is Fano and it is isomorphic to $\mathcal{B}'$ and there is a graded ring isomorphism $\varphi' \colon H^*(\mathcal{B}') \to H^*(\mathcal{B}'')$ satisfying

$$\varphi'(x'_i) = \begin{cases} x''_i & \text{for } i < k \text{ or } i > \sigma(k), \\ x''_{i+1} & \text{for } k \le i \le \sigma(k) - 1, \\ x''_k & \text{for } i = \sigma(k) \end{cases}$$

by Proposition 3.6. Since $\pi \circ \sigma(i) = i$ for every $i \le k$, the composition $\varphi' \circ \varphi$ is a c_1-preserving graded ring isomorphism $\varphi' \circ \varphi \colon H^*(\mathcal{B}) \to H^*(\mathcal{B}'')$ satisfying

$$\varphi' \circ \varphi(x_i) = x''_i \quad (i < k) \qquad \text{and} \qquad \varphi' \circ \varphi(2y_k) = \varphi'(2y'_{\sigma(k)}) = 2y_{\pi\circ\sigma(k)} = 2y''_k.$$

Hence this case reduces to the first case.

We may repeat the above argument as many times as necessary. Since the indices are bounded above by n, this process must stop and eventually we get an isomorphism

$$\widetilde{\psi} : H^*(\mathcal{B}) \to H^*(\widetilde{\mathcal{B}})$$

sending x_i to $\widetilde{x}_i$ for every $i = 1, \ldots, n$ where $\widetilde{\mathcal{B}}$ is the resulting Bott manifold obtained from $\mathcal{B}'$ by applying a sequence of **Op.1** and **Op.2** in an inductive way. Since $\widetilde{\psi}(x_i) = \widetilde{x}_i$ and $\widetilde{\psi}(y_i) = \widetilde{y}_i$, it follows from (2.7) that $\widetilde{\psi}(\alpha_i) = \widetilde{\alpha}_i$, which implies that the Bott matrices corresponding to $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ coincide by definition of α_i in (2.6). Therefore $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic. This finishes the proof. □

Remark 4.3. We cannot extend Theorem 1.2 to weak Fano Bott manifolds. Note that the Hirzebruch surfaces $\mathcal{H}_0$ and $\mathcal{H}_2$ are weak Fano[3] Bott manifolds. As we saw in Example 2.8, $\mathcal{H}_0$ and $\mathcal{H}_2$ are not isomorphic but there is a c_1-preserving graded cohomology ring isomorphism between them.

Remark 4.4. One may wonder whether we can extend Theorem 1.2 to Bott manifolds whose Bott matrices have entries $0, 1$, or -1. However, the set of such Bott matrices is not closed under the operation **Op.1**. For example, consider a matrix

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

Then, we have that

$$A_\emptyset = A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ -2 & -1 & -1 \end{bmatrix}$$

whose entry has $-2 \notin \{0, 1, -1\}$. Note that the set of Bott matrices obtained from Fano Bott manifolds is closed under the operation **Op.1** as we mentioned in Remark 3.2.

Remark 4.5. For $n = 3$, there are five Bott matrices associated with Fano Bott manifolds up to isomorphisms.

Oda's list Øbro's list	6 21	7 11	8 18	9 17	10 12
Bott matrix	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

Here, we refer the lists provided by Oda [Oda88, Figure 2.6] and Øbro [Obr07] in the Graded Ring Database [BK]. Note that the cohomology rings of the Bott manifolds of the second and the third matrices are isomorphic as graded rings and hence they are diffeomorphic by the smooth rigidity theorem of Bott manifolds in dimension six. See [CMS10, Theorem 7.1]. Thus they provide an example of diffeomorphic but non-isomorphic Bott manifolds. Indeed, one can check that the degrees of the two Bott manifolds are

[3] We call a variety X *weak Fano* if $\langle c_1(TX), [C]\rangle \geq 0$ for every algebraic curve $C \subset X$.

different, so there does not exist c_1-preserving cohomology ring isomorphism between them.

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