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# Applications of the full Kostant-Toda lattice and hyper-functions to unitary representations of the Heisenberg groups

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We consider a new orbit method for unitary representations which determines the explicit values of the multiplicities of the irreducible components of unitary representations of the connected Lie groups. We provide the polarized symplectic affine space on which the Lie group acts. This polarization is obtained by the Hamiltonian flows of the full Kostant-Toda lattice. The Hamiltonian flows of the ordinary Toda lattice are not sufficient for constructing this polarization. In this paper we do an experiment on the case of the unitary representations of the Heisenberg groups. The irreducible representations of the Heisenberg group are obtained and classified by $\mathbb{R}$ by the Stone-von Nuemann theorem. The multiplicities are obtained by using spontaneous symmetry breaking and Sato hyper-functions.

## 1. Introduction

The aims of this paper are to explore the tower structure of the flag manifold by using an integrable system called the full Kostant-Toda lattice and its application to unitary representations of the Heisenberg groups. We know that there exist two generalizations of the Toda lattice. One is the Toda flow on a generic orbit studied by P.Deift, L.Li, T.Nanda and C.Tomei[4]. Another is the full Kostant-Toda lattice studied by N.Ercolani, H.Flaschka and S.Singer[5]. The Lax operators of [4] are full symmetric matrices. On the other hand the Lax operators of [5] are of the form $\Lambda + \bar{\mathfrak{b}}$, where $\Lambda$ is the shift matrix and $\bar{\mathfrak{b}}$ is the lower triangular Borel subalgebra. They have a lot in common but have not been proven to be equivalent yet as far as we know. The full Kostant-Toda lattice includes an ordinary Toda lattice of a tri-diagonal Lax operator, which is reduced from the full Kostant-Toda lattice. To study the geometry of the flag manifold, we must consider the full Kostant-Toda lattice. The full Kostant-Toda lattice is the integrable system of lower triangular Lax-type matrices. This system has ordinary well-known Hamiltonian flows of the Toda lattice. It also has Hamiltonian flows associated with chop integlals[4,5,8]. Ordinary Hamiltonian flows and Hamiltonian flows of chop integrals commute each other. Kostant classifies integrable generalized Toda lattice by Dynkin diagram and solves them explicitly[13]. The key method which links integrability of the generalized Toda lattice with representation theory is the coadjoint orbit method. Put $G = GL_n(\mathbb{R})$. Let $B \subset G$ be the upper triangular Borel subgroup and $N \subset B$ be the nilpotent subgroup of strictly upper triangular matrices. Let $\bar{B}$ be ${}^tB$ and $\bar{N}$ be ${}^tN$. Put $\bar{\mathfrak{b}} = \mathrm{Lie}\bar{B}$. Let $\Lambda = \sum_{i=1}^{n-1} E_{i,i+1}$ be the shift matrix, where $E_{i,j}$ is the $i, j$ matrix unit. We define the affine space of Lax operators $Lax$ by $Lax = \Lambda + \bar{\mathfrak{b}}$. Fix an element $L_0 \in Lax$. Thus the adjoint orbit through $L_0$ is defined by

$$\mathscr{O}(L_0) = \{aL_0a^{-1} | a \in \bar{N}\} \subset Lax$$

The orbit $\mathscr{O}(L_0)$ corresponds to the coadjoint orbit $Z$ studied in [13]. However there exist two noticeable differences. At first $Z$ consists of tri-diagonal Jacobi elements, and on the other hand $\mathscr{O}(L_0)$ consists of lower triangular Lax-type matrices. Secondly $Z$ is a $2n-2$ dimensional symplectic manifold. On the other hand $\mathscr{O}(L_0)$ is an $\frac{n(n-1)}{2}$ dimensional Poisson manifold. To obtain explicit solutions, we consider the Gauss decomposition of the potential function $\Phi(\mathbf{t}) = \exp(\sum_{j=1}^{n-1} t_j L_0^j)$

$$w(\mathbf{t})^{-1} b(\mathbf{t}) = \Phi(\mathbf{t}), \tag{1.1}$$

where $w(\mathbf{t}) \in \bar{N}$ and $b(\mathbf{t}) \in B$. Thus the orbit $L(\mathbf{t}) = w(\mathbf{t})L_0 w(\mathbf{t})^{-1}$ satisfies the ordinary Lax equations

$$\frac{\partial}{\partial t_j} L(\mathbf{t}) = [(L(\mathbf{t})^j)_+, L(\mathbf{t})],\ j = 1, \ldots, n-1, \tag{1.2}$$

where $(\cdot)_+$ is the projection to the upper triangular Borel part. We rewrite (1.1) by

$$w(\mathbf{t})^{-1} = \Phi(\mathbf{t}) b(\mathbf{t})^{-1}. \tag{1.3}$$

Then we have

$$w(\mathbf{t})^{-1} B = \Phi(\mathbf{t}) B. \tag{1.4}$$

The equation (1.4) means that the orbit of the Lax equation in *Lax* is translated into the orbit in the flag manifold $G/B$. This translation is called the companion embedding[5](This translation is also called a dressing operator[16]). In this paper we consider the tower structure of the flag manifold $G/B$. Put the $\ell \times \ell$ identity matrix into $E_\ell$. Let $P \supset B$ be the parabolic subgroup whose Levi part is $GL_1(\mathbb{R}) \times GL_{n-2}(\mathbb{R}) \times GL_1(\mathbb{R})$ and $U \subset \bar{N}$ be the $2n-3$ dimensional Heisenberg group[2]

$$U = \{ \begin{pmatrix} 1 & {}^t\mathbf{0} & 0 \\ \mathbf{a} & E_{n-2} & \mathbf{0} \\ c & {}^t\mathbf{b} & 1 \end{pmatrix} | \mathbf{a}, \mathbf{b} \in \mathbb{R}^{n-2}, c \in \mathbb{R} \}.$$

We can identify the open dense cell of $G/P$ with $U$ as topological space. The flag manifold $G/B$ has an iterated fiber bundle structure where the total base space is $G/P$. We call this structure of an iterated fiber bundle the tower structure of $G/B$. The points in $G/B$ can not move along the fibers of this tower structure by ordinary Hamiltonian flows of (1.2). The Hamiltonian flows along these fibers are given by chop integrals[4,5,8]. In section 2, we use the Hamiltonian flows of the chop integrals. Put $\mathfrak{u} = \mathrm{Lie}U$. Let $R$ be the center of $U$, i.e. $R = \{E_n + rE_{n,1} | r \in \mathbb{R}\}$ and $\mathfrak{r}$ be $\mathrm{Lie}R$. In section 2, we demonstrate that $U = E_n + \mathfrak{u}$ and $U$ is the affine space with vector space $\mathfrak{u}$. Similarly, we demonstrate that $R = E_n + \mathfrak{r}$ and $R$ is the affine space with vector space $\mathfrak{r}$. Therefore we observe that $X = U/R$ is the affine space $X = E_n + \mathfrak{u}/\mathfrak{r}$ with vector space $\mathfrak{u}/\mathfrak{r}$. We can identify $\mathfrak{u}/\mathfrak{r}$ with the vector space

$$\mathfrak{u}_0 = \{ \begin{pmatrix} 0 & {}^t\mathbf{0} & 0 \\ \mathbf{p} & O & \mathbf{0} \\ 0 & {}^t\mathbf{q} & 0 \end{pmatrix} | \mathbf{p}, \mathbf{q} \in \mathbb{R}^{n-2} \}.$$

The precise definition of the identification $X$ with $\mathfrak{u}/\mathfrak{r}$ is addressed in section 2. Since $Lax$ has the form $\Lambda + \bar{\mathfrak{b}}$, we can define the projection $proj : Lax \to \mathfrak{u}$. Let $res$ be the canonical projection from $\mathfrak{u}$ to $\mathfrak{u}/\mathfrak{r}$. Thus $res \circ proj$ defines the projection from $Lax$ to $X$. The precise definition of $proj$ and $res$ will be given in section 3. The affine space of standard form of Lax operators s-$Lax_0$ is defined by

$$\text{s}-Lax_0 = \{ \Lambda + \begin{pmatrix} 0 & {}^t\mathbf{0} & 0 \\ \mathbf{0} & O & \mathbf{0} \\ 0 & {}^t\mathbf{q} & 0 \end{pmatrix} | \mathbf{q} \in \mathbb{R}^{n-2} \}.$$

To obtain a polarized symplectic structure, it is essential to consider a foliation structure[9,19]. By using the Kostant's orbit method, we see that $\mathrm{Ad}\bar{N}$ orbit of $\Lambda + x,\ x \in \mathfrak{u}_0$ meets $s\text{-}Lax_0$ at only one point such as $a(x)(\Lambda + x)a(x)^{-1} = L(\mathbf{q}) \in s\text{-}Lax_0,\ a(x) \in \bar{N}$. Put $\Phi(\mathbf{t}, \mathbf{q}) = \exp(t_1 L(\mathbf{q}) + \cdots + t_{n-2} L(\mathbf{q})^{n-2})$. Thus we see that the point $\Lambda + x$ exists on the Toda lattice orbit

$$\mathscr{O}(\mathbf{q}, x) := \{ L(\mathbf{t}, \mathbf{q}) = w(\mathbf{t}, \mathbf{q}) L(\mathbf{q}) w(\mathbf{t}, \mathbf{q})^{-1} |$$

$$w(\mathbf{t}, \mathbf{q}) \text{ satisfies } w(\mathbf{t}, \mathbf{q})^{-1} b(\mathbf{t}, \mathbf{q}) = a(x)\Phi(\mathbf{t}, \mathbf{q}) \}.$$

But we only know $res \circ proj(L(\mathbf{0}, \mathbf{q})) = x$. Thus we can not parameterize $X$ by $\mathbf{t}$ and $\mathbf{q}$ as it is. In §3, we show that the points $\Lambda + x,\ x \in \mathfrak{u}_0$ can be transferred to the points of minimal Lax orbits which correspond to base space $G/P$ of the tower structure along the curve $res \circ proj = x$. To move along $res \circ proj = x$, we need variables $\mathbf{t}$ and $\mathbb{S}$, where $\mathbf{t}$ is the time variables of ordinary Toda lattice and $\mathbb{S}$ is the time variables corresponding to chop integrals of the full Kostant-Toda lattice. Let $\Sigma$ be the $2n-4$-dimensional surface made by minimal Lax orbits. Since any point $\Lambda + x,\ x \in \mathfrak{u}_0$ is transferred to a point of a minimal Lax orbit along $res \circ proj = x$, we see that $res \circ proj(\Sigma) = X$. This allow us to parameterize $X$ by $\mathbf{t}$ and $\mathbf{q}$. The restricted Poisson structure of $Lax$ to $\Sigma$ brings the symplectic structure of $X$ and the parameterization of $X$ defines the polarization of $X$($\mathbf{t}$-direction and $\mathbf{q}$-direction). In §4, we introduce canonical commutation relations and the Stone-von Neumann theorem which determines all irreducible unitary representations of the Heisenberg group parameterized by $\mathbb{R}$. We discuss the problem of irreducible decomposition of the unitary representation of Heisenberg group in formal way. In §5, we consider the unitary representation of the Heisenberg group $U$ on $L^2$-sections of the line bundle over $X$ which is caused by the symplectic action of $U$ on $X$ and study its irreducible decomposition. At the beginning of the section, we discuss the multiplicities of irreducible components from point of the view of the orbit method. But by this orbit method, we can hardly grasp the multiplicity as explicit natural number. To obtain multiplicities with finer texture, we consider the spontaneous symmetry breaking(SSB) method as follows. Let $\rho$ be a unitary representation of $U$ on the line bundle over $X$. In §3, we consider $X$ as polalized symplectic affine space $X = \mathfrak{u}_0$. In this case, the irreducible components are of the form $(\rho_h, L^2(\mathbb{R}^{n-2})),\ h \in \mathbb{R}$, where $\rho_h$ are unitary dual of $U$ of the Stone-von Nuemann theorem. But $X$ is also quotient space $X = U/R$. It is natural to think that the irreducible components are $(\rho_h, \hat{R} \otimes L^2(\mathbb{R}^{n-2})), h \in \mathbb{R}$, where $\hat{R} \otimes L^2(\mathbb{R}^{n-2})$ is the space of $\hat{R}$-valued $L^2$ functions on $\mathbb{R}^{n-2}$ rather than $L^2(\mathbb{R}^{n-2})$ if we consider $X$ as the quotient space $U/R$. We can regard that irreducible components $(\rho_h, \hat{R} \otimes L^2(\mathbb{R}^{n-2}))$ have $R$-symmetries. If we use just $(\rho_h, L^2(\mathbb{R}^{n-2}))$ as the irreducible component, we must break $R$-symmetry. This is an analogy of the spontaneous symmetry breaking(SSB) which appears in particle physics and condensed matter physics[21]. The $\hat{R}$-valued $L^2$ function on $\mathbb{R}^{n-2}$ has form $\chi_\rho^h \otimes f(\mathbf{x})$, where $\chi_\rho^h$ is a character of $R$ determined by $\rho$ and $h \in \mathbb{R}$. However, irreducible components are $L^2(\mathbb{R}^{n-2})$. We do not designate the representation space $\hat{R} \otimes L^2(\mathbb{R}^{n-2})$ but $L^2(\mathbb{R}^{n-2})$. Take $g \in R$ arbitrary; the designation is described as

$$\chi_\rho^h \otimes f(\mathbf{x}) \mapsto \chi_\rho^h(g) f(\mathbf{x}). \tag{1.5}$$

This designation breaks $R$-symmetry. We call it as SSB. Note that the representation of $U$, $(\rho_h, \hat{R} \otimes L^2(\mathbb{R}^{n-2}))$ induces the outer tensor product representation of $R \times U$, $(\chi_\rho^h \boxtimes \rho_h, \mathbb{C}_h \otimes L^2(\mathbb{R}^{n-2}))$ and vice-versa, where $\mathbb{C}_h$ is the one-dimensional Hilbert space on which $R$ acts. In this paper we define the weight $M_\rho(h) = \frac{1}{2\pi}\int_0^{2\pi} e^{im_\rho(h)s} ds$ and the map $\hat{R} \otimes \hat{U} \to \hat{U}\ \chi_\rho^h \boxtimes \rho_h \mapsto M_\rho(h)\rho_h$ to recover the $R$-symmetry from SSB. Therefore we obtain the weight decomposition of $\rho$ such that

$$\rho \simeq \int_{h \in \mathbb{R}} M_\rho(h) \rho_h dh. \tag{1.6}$$

In §5, we give a precise definition of the integral of (1.6), as well as an explanation to explain why $[|M_\rho(h)|]$ is the multiplicity of $\rho$ on the irreducible component, where [ ] is the Gauss symbol. If $m_\rho(h) = 0$, then we see that $\chi_\rho^h \boxtimes \rho_h$ is equivalent to $\rho_h$. Thus irreducible components of (1.6) satisfying $m_\rho(h) = 0$ are proper irreducible components. Recall that the irreducible components are $(\rho_h, L^2(\mathbb{R}^{n-2}))$, $h \in \mathbb{R}$, when we consider $X$ as the polarized symplectic affine space $\mathfrak{u}_0$ rather than as the quotient space $X = U/R$. Thus the proper components are irreducible components which appear in the irreducible decomposition of $\rho$ when we consider $X = \mathfrak{u}_0$. To pick up proper irreducible components from the weight decomposition of (1.6), we use the amplifier $1/m_\rho(h)$. The amplifier is a function of $h$. We multiply the integrand of (1.6) by the amplifier to magnify certain weights $M_\rho(h)$. The function $1/m_\rho(h)$ is not well defined because the amplifier since $m_\rho(h)$ has zero points. To justify $1/m_\rho(h)$ as the amplifier, we use the hyperfunction[14].

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## 2. Tower structure of the flag manifold and the full Kostant-Toda lattice

Let us fix the notations. Let $G$ be $GL_n(\mathbb{R})$. Let $B \subset G$ be the upper triangular Borel subgroup and $N \subset B$ be the unipotent radical given by

$$N = \{ \begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix} \}.$$

The opposites are defined by $\bar{B} = {}^tB$ and $\bar{N} = {}^tN$. The corresponding German letters denote corresponding Lie algebras, that is, $\mathfrak{g} = \mathrm{Lie}G$, $\mathfrak{b} = \mathrm{Lie}B, \mathfrak{n} = \mathrm{Lie}N$, $\bar{\mathfrak{b}} = \mathrm{Lie}\bar{B}$, $\bar{\mathfrak{n}} = \mathrm{Lie}\bar{N}$. Let $E_k$ be the $k \times k$ identity matrix.

The $2n-3$-dimensional Heisenberg group $U \subset \bar{N}$ is defined by

$$U = \{ \begin{pmatrix} 1 & {}^t\mathbf{0} & 0 \\ \mathbf{a} & E_{n-2} & \mathbf{0} \\ c & {}^t\mathbf{b} & 1 \end{pmatrix} | \mathbf{a}, \mathbf{b} \in \mathbb{R}^{n-2}, c \in \mathbb{R}\}.$$

The center of $U$ is

$$R := \{ \begin{pmatrix} 1 & {}^t\mathbf{0} & 0 \\ \mathbf{0} & E_{n-2} & \mathbf{0} \\ c & {}^t\mathbf{0} & 1 \end{pmatrix} | c \in \mathbb{R}\}.$$

The elements of $U$ are expressed as follows $e^X,\ X \in \mathfrak{u}$. Because $X^2 \in \mathfrak{r}$ and then $X^k\ k = 1, 2, 3, \ldots$ are included in $\mathfrak{u}$. Therefore, we observe that $U \subset E_n + \mathfrak{u}$; certainly $E_n + \mathfrak{u} \subset U$. Then, we observe that the Heisenberg group $U$ satisfies $U = E_n + \mathfrak{u}$ and $U$ is the affine space with vector space $\mathfrak{u}$. Similarly $R$ coincides with the affine space $E_n + \mathfrak{r}$ with vector space $\mathfrak{r}$. Hence, we observe that $U/R$ is the affine space $E_n + \mathfrak{u}/\mathfrak{r}$ with vector space $\mathfrak{u}/\mathfrak{r}$. We coordinatize $X = U/R$ by adopting the dual basis of $\mathfrak{u}/\mathfrak{r}$. The vector space $\mathfrak{u}/\mathfrak{r}$ is identified with

$$\mathfrak{u}_0 = \{ \begin{pmatrix} 0 & {}^t\mathbf{0} & 0 \\ \mathbf{p} & O & \mathbf{0} \\ 0 & {}^t\mathbf{q} & 0 \end{pmatrix} | \mathbf{p}, \mathbf{q} \in \mathbb{R}^{n-2}\}.$$

Put $\mathbf{p} = {}^t(p_1, \ldots, p_{n-2})\, \mathbf{q} = {}^t(q_1, \ldots, q_{n-2})$. Then, $(p_1, \ldots, p_{n-2}, q_1, \ldots, q_{n-2})$ are the coordinates on $X$. Let us consider the sequence of the parabolic subgroups of $G = GL_n(\mathbb{R})$ as follows.

$$G \supset P_1 \supset P_2 \supset \cdots \supset P_{[n/2]} = B,$$

where $P_k$ is the parabolic subgroup whose Levi factor is

$$\underbrace{GL_1(\mathbb{R}) \times \cdots \times GL_1(\mathbb{R})}_{k} \times GL_{n-2k}(\mathbb{R}) \times \underbrace{GL_1(\mathbb{R}) \times \cdots \times GL_1(\mathbb{R})}_{k}.$$

We often write $P_1$ by $P$. We see that $G/P_2$ is the fiber bundle over $G/P_1$ whose fiber is $G/P_2 \Big/ G/P_1 \simeq P_1/P_2 \simeq U_{2n-7}$ as a topological space, where $U_{2n-7}$ is the $2n-7$ dimensional Heisenberg group$(U = U_{2n-3})$. We write this by

$$
\begin{tikzcd}
G/P_2 & U_{2n-7} \arrow[l] . \\
G/P_1 \arrow[from=1-1, to=2-1]
\end{tikzcd}
\tag{2.1}
$$

Iterating this process we obtain the following diagram for odd(even) $n$

$$
\begin{tikzcd}
G/B = G/P_{[n/2]} \arrow[d] & P_{[n/2]-1}/P_{[n/2]} \simeq U_3(U_1) \arrow[l] \\
G/P_{[n/2]-1} \arrow[d] & P_{[n/2]-2}/P_{[n/2]-1} \simeq U_7(U_5) \arrow[l] \\
\vdots & \vdots \\
G/P_2 \arrow[d] & P_2/P_1 \simeq U_{2n-7} \arrow[l] \\
G/P_1 \simeq U
\end{tikzcd}
\tag{2.2}
$$

We call (2.2) the tower structure of the flag manifold $G/B$ over the base space $G/P = G/P_1$. Note that the open dense cell of $G/P_k$ is homeomorphic to $\bar{N}_k,\ k = 1, \dots, [n/2]$ which are the subgroups of a unipotent radical $\bar{N}$. The subgroup $\bar{N}_k$ has the following form $E_n + \sum_{1 \le j < i \le n} a_{i,j} E_{i,j},\ a_{i,j} = 0$ for $k+1 \le i < j \le n-k$, where $E_{i,j}$ is the $i,j$ matrix unit. Then we have the sequence of subgroups of $\bar{N}$ as follows

$$
U = \bar{N}_1 \subset \bar{N}_2 \subset \cdots \subset \bar{N}_{[n/2]} = \bar{N} \tag{2.3}
$$

Let $\Lambda = \sum_{i=1}^{n-1} E_{i,i+1}$ be the shift matrix. We denote the affine space of Lax operators by $Lax = \{L \in \Lambda + \bar{\mathfrak{b}} | \mathrm{tr} L = 0\}$. Let $L_{i,j},\ 1 \le j \le i \le n$ be the dual basisi of $\bar{\mathfrak{b}}$. We impose on them the relation $L_{1,1} + \cdots + L_{n,n} = 0$. The dual basis satisfying the relation $L_{1,1} + \cdots + L_{n,n} = 0$ are generators of the algebra of coordinate functions of $Lax$. Note that there exists a Poisson structure on $Lax$ defined by

$$
\{L_{i,j}, L_{k,\ell}\} = \delta_{j,k} L_{i,\ell} - \delta_{\ell,i} L_{k,j}. \tag{2.4}
$$

Once we fix $L \in Lax$, each $G/P_k$ of (2.2) has the Lax operator form

$$uLu^{-1}, u \in \bar{N}_k, \tag{2.5}$$

called the companion embedding of $G/P_k$ into $Lax$. The full Kostant-Toda Lattice is the Gelfand-Cetlin system [6] defined as follows. Put $\mathfrak{Q} = \{Q(r, \mathbf{q}) = \begin{pmatrix} 0 & {}^t\mathbf{0} & 0 \\ \mathbf{0} & O & \mathbf{0} \\ r & {}^t\mathbf{q} & 0 \end{pmatrix} | r \in \mathbb{R}, \mathbf{q} \in \mathbb{R}^{n-2}\}$. The set of the standard Lax operators $s$-$Lax$ is defined by $s$-$Lax = \{L = \Lambda + Q(r, \mathbf{q}) | Q(r, \mathbf{q}) \in \mathfrak{Q}\}$. For $L_0 \in s$-$Lax$, we define the potential function by

$$\Phi(\mathbf{t}) = \exp(t_1 L_0 + \cdots + t_{n-2} L_0^{n-2}),$$

where $\mathbf{t} = (t_1, \ldots, t_{n-2})$. Suppose that $\Phi(\mathbf{t})$ has the following $U$-$P$ decomposition

$$u_0(\mathbf{t})^{-1} p_0(\mathbf{t}) = \Phi(\mathbf{t}), \tag{2.6}$$

where $u_0(\mathbf{t}) \in U$ and $p_0(\mathbf{t}) \in P$. We can suppose that the Levi factor of $p_0(\mathbf{t})$ has the form

$$q_1(\mathbf{t}) \times A_1(\mathbf{t}) \Phi_1(\mathbf{s}_1) \times q_2(\mathbf{t}),$$

where $q_1(\mathbf{t}), q_2(\mathbf{t}) \in GL_1(\mathbb{R}),\ A(\mathbf{t}) \in GL_{n-2}(\mathbb{R})$ and

$$\Phi_1(\mathbf{s}_1) = \exp(s_1^1 \nabla I_{1,1} + \cdots + s_{n-2}^1 \nabla I_{n-2,1}). \tag{2.7}$$

We have to explain $I_{j,1}, j = 1, \ldots, n-2$. In general $I_{j,1}, j = 1, \ldots, n-2$ are meromorphic functions on the coordinate ring of $Lax$, where they are invariant under the Adjoint action of $P$. Thus the meromorphic functions $I_{j,1},\ j = 1, \ldots, n-2$ pairwise-commute with respect to the Poisson bracket defined by (2.4). Let $I_{j,0},\ j = 1, \ldots n-1$ be the coefficients of $\det(\lambda - L)$. Of course $I_{j,0},\ j = 1, \ldots, n-1$ are Ad$P$ invariant. Then $I_{j,0},\ j = 1, \ldots, n-1$ Poisson commute with $I_{j,1}\ j = 1, \ldots, n-2$. Since $I_{j,1}\ j = 1, \ldots, n-2$ are also Ad$GL_{n-2}(\mathbb{R})$-invariant, their gradients are realized in $\mathfrak{gl}_{n-2}(\mathbb{R})$ as pairwise-commuting matrices. We call this a torus realization. The torus realization of $I_{j,0},\ j = 1, \ldots, n-1$ are $L_0, \ldots, L_0^{n-1}$. But we omit $L_0^{n-1}$ in $\Phi(\mathbf{t})$ since we use the freedom corresponding to $L_0^{n-1}$ for the central extension of $\mathfrak{u}$. Although $I_{i,0}, I_{j,1},\ i = 1, \ldots, n-1,\ j = 1, \ldots, n-2$ Poisson commute, $I_{j,1},\ j = 1, \ldots, n-2$ are not Ad$G$ invariant. Then there is not a simultaneous torus realization where their gradients are commutative. In general we embedded $g \in GL_{n-2k}(\mathbb{R})$ into $GL_n(\mathbb{R})$ by

$$\underbrace{1 \times \cdots \times 1}_{k} \times g \times \underbrace{1 \times \cdots \times 1}_{k}. \tag{2.8}$$

For simplicity we write $g$ as it is instead of (2.8) if reader can distinguish from context. One can assume $p_0(\mathbf{t})$ has parameter $\mathbf{s}_1$ such that

$$p_0(\mathbf{t}, \mathbf{s}_1) = (1 \times A_1(\mathbf{t})\Phi_1(\mathbf{s}_1) \times 1) \cdot p_0(\mathbf{t}, \mathbf{s}_1)'.$$

Let $\mathring{P}_k$ be the parabolic subgroup of $GL_{n-2k}(\mathbb{R})$ which has Levi factor $GL_1(\mathbb{R}) \times GL_{n-2k-2}(\mathbb{R}) \times GL_1(\mathbb{R})$. Let $U_{2n-4k-3}$ be the $2n-4k-3$ dimensional Heisenberg subgroup of $GL_{n-2k}(\mathbb{R})$. For $g \in GL_{n-2k}(\mathbb{R})$, we also call the decomposition $g = up$, where $u \in U_{2n-4k-3}$, $p \in \mathring{P}_k$, the $U$-$P$ decomposition in brief. We can assume $U$-$P$ decomposition in $GL_{n-2}(\mathbb{R})$ defined by

$$u_1(\mathbf{t}, \mathbf{s}_1)^{-1} p_1(\mathbf{t}, \mathbf{s}_1) = A_1(\mathbf{t})\Phi_1(\mathbf{s}_1).$$

Let $I_{1,2}, \dots, I_{n-4,2}$be the $\mathrm{Ad}P_2$ invariant meromorphic functions in $Lax$. As above $I_{j,2}, j = 1, \dots, n-4$ Poisson commute with each other and $I_{i,0}, i = 1, \dots, n-1$, $I_{j,1}$, $j = 1, \dots, n-2$, $I_{k,2}$, $k = 1, \dots, n-4$ Poisson commute with each other. $I_{k,2}$, $k = 1, \dots, n-4$ have a torus realization in $\mathfrak{gl}_{n-4}(\mathbb{R})$. Suppose that the Levi factor of $p_1(\mathbf{t}, \mathbf{s}_1)$ is decomposed into $q_1' \times A_2(\mathbf{t}, \mathbf{s}_1)\Phi_2(\mathbf{s}_2) \times q_2'$, where

$$\Phi_2(\mathbf{s}_2) = \exp(s_1^2 \nabla I_{1,2} + \cdots + s_{n-4}^2 \nabla I_{n-4,2}),$$

$q_1'$ , $q_2' \in GL_1(\mathbb{R})$, $\nabla I_{i,2}$ are torus realizations of $I_{i,2}$ and $A_2(\mathbf{t}, \mathbf{s}_1) \in GL_{n-4}(\mathbb{R})$. Suppose that $p_1(\mathbf{t}, \mathbf{s}_1)$ has a parameter $\mathbf{s}_2$ given by

$$p_1(\mathbf{t}, \mathbf{s}_1, \mathbf{s}_2) = (1 \times A_2(\mathbf{t}, \mathbf{s}_1)\Phi_2(\mathbf{s}_2) \times 1) p_1'(\mathbf{t}, \mathbf{s}_1)$$

in $GL_{n-2}(\mathbb{R})$. One can assume the $U$-$P$ decomposition holds in $GL_{n-4}(\mathbb{R})$ defined by

$$u_2(\mathbf{t}, \mathbf{s}_1, \mathbf{s}_2)^{-1} p_2(\mathbf{t}, \mathbf{s}_1, \mathbf{s}_2) = A_2(\mathbf{t}, \mathbf{s}_1)\Phi_2(\mathbf{s}_2).$$

Iterating the same manipulations, we obtain $u_k(\mathbf{t}, \mathbf{s}_1, \dots, \mathbf{s}_k) \in U_{2n-4k-3}$ and $p_k(\mathbf{t}, \mathbf{s}_1, \dots, \mathbf{s}_k)'$, $k = 0, \dots, [n/2]-1$. Put

$$w(\mathbf{t}, \mathbb{S}) = u_{[n/2]-1}(\mathbf{t}, \mathbf{s}_1, \dots, \mathbf{s}_{[n/2]-1}) \cdots u_1(\mathbf{t}, \mathbf{s}_1) u_0(\mathbf{t}) \in \bar{N}$$

and

$$b(\mathbf{t}, \mathbb{S}) = p'_{[n/2]-1}(\mathbf{t}, \mathbf{s}_1, \ldots, \mathbf{s}_{[n/2]-1}) \cdots p'_1(\mathbf{t}, \mathbf{s}_1) p'_0(\mathbf{t}) \in B,$$

where we put $\mathbf{s}_1, \ldots, \mathbf{s}_{[n/2]-1}$ all together $\mathbb{S}$. Here we use the same symbol $g \in GL_{n-2k}(\mathbb{R})$ as embedded image of (2.8). Note that the $U$-$P$ decompositions of $A_i(\mathbf{t}, \mathbf{s}_1, \ldots, \mathbf{s}_{i-1})\Phi_i(\mathbf{s}_i), i = 1, \ldots, [n/2]-1$ do not make influence on the original $U$-$P$ decomposition of (2.6). Then we have the Gauss decomposition

$$w(\mathbf{t}, \mathbb{S})^{-1} b(\mathbf{t}, \mathbb{S}) = \Phi(\mathbf{t}). \tag{2.9}$$

We regard $w(\mathbf{t}, \mathbb{S}) \in G/B$ as the Gelfand-Cetlin system on $G/B$ and call it as full Kostant-Toda Lattice. Put $L := L(\mathbf{t}, \mathbb{S}) = w(\mathbf{t}, \mathbb{S}) L_0 w(\mathbf{t}, \mathbb{S})^{-1}$. From (2.9), we see that $L$ satisfies ordinary Lax equations on $\mathbf{t}$

$$dL/dt_j = [(L^j)_+, L], \ j = 1, \ldots, n-2 \tag{2.10}$$

where $(*)_+$ is the projection onto $\mathfrak{b}$. By the flows of the full Kostant-Toda lattice on the variables $\mathbb{S}$, it moves along the fibers $P_k/P_{k+1}, \ k = 1, \ldots, [n/2]-1$ of the tower structure of (2.2). The examples of chop integrals are given in [4],[5] and [8] as follows. For $X \in Mat_{n \times n}(\mathbb{R})$, Let $X_{(k)} \in Mat_{(n-k)\times(n-k)}(\mathbb{R})$ be the matrix removing first $k$ rows and right $k$ colums from $X$. It is shown in general that

$$\det(\mathrm{Ad}(\begin{pmatrix} q_1 & *** & * \\ \mathbf{0} & Q & \vdots \\ 0 & 0 & q_2 \end{pmatrix})X)_{(k)} = (\det(q_1)/\det(q_2)) \det(X_{(k)}),$$

where $q_i \in GL_k(\mathbb{R}), \ i = 1, 2$ and $Q \in GL_{n-2k}(\mathbb{R})$. Let us consider the characteristic polynomial of $L \in Lax$ such as

$$\det(\lambda - L) = \lambda^n + I_{1,0}\lambda^{n-2} + \cdots + I_{n-1,0}.$$

Then the coefficients $I_{1,0}, \ldots, I_{n-1,0}$ are $\mathrm{Ad}G$-invariant and they pairwise commute on the Poisson bracket (2.4). The $P_k$ invariants are constructed as follows. Consider

$$\det(\lambda - L)_{(k)} = \tilde{I}_{0,k}\lambda^{n-2k} + \tilde{I}_{1,k}\lambda^{n-2k-1} + \cdots + \tilde{I}_{n-2k,k}.$$

And put $I_{i,k} = \tilde{I}_{i,k}/\tilde{I}_{0,k}, \ i = 1, \ldots, n-2k$. Then $I_{i,k}, \ i = 1, \ldots, n-2k$ also pairwise commute on (2.4) and have torus realizations in $\mathfrak{gl}_{n-2k}(\mathbb{R})$.

## 3. Polarized symplectic structure of $X$ and symplectic action of $U$ on $X$

The following fact is shown by Kostant[12]. For any $L \in Lax$, there exists a unique $w \in \bar{N}$ satisfying

$$wLw^{-1} \in s\text{-}Lax. \tag{3.1}$$

Recall that $Lax = \{L \in \Lambda + \bar{\mathfrak{b}} | \mathrm{tr} L = 0\}$. We observe that $\bar{\mathfrak{b}}$ has the vector space decomposition $\bar{\mathfrak{b}} = \mathfrak{u} \oplus (\bar{\mathfrak{b}} \cap \mathfrak{p})$. Suppose $\Lambda + x \in Lax$. Let $proj'$ be the projection from $\bar{\mathfrak{b}}$ onto $\mathfrak{u}$, relative to the decomposition $\bar{\mathfrak{b}} = \mathfrak{u} \oplus (\bar{\mathfrak{b}} \cap \mathfrak{p})$. We define the projection $proj$ from $Lax$ onto $\mathfrak{u}$ by $proj(\Lambda + x) := proj'(x)$. In addition, we define the canonical projection $res$ from $\mathfrak{u}$ to $\mathfrak{u}_0$ by

$$res\left(\begin{pmatrix} 0 & {}^t\mathbf{0} & 0 \\ \mathbf{p} & O & \mathbf{0} \\ r & {}^t\mathbf{q} & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & {}^t\mathbf{0} & 0 \\ \mathbf{p} & O & \mathbf{0} \\ 0 & {}^t\mathbf{q} & 0 \end{pmatrix}.$$

We define the submanifold $s\text{-}Lax_0$ of $s\text{-}Lax$ by

$$s\text{-}Lax_0 = \{L(\mathbf{q}) = \Lambda + \begin{pmatrix} 0 & {}^t\mathbf{0} & 0 \\ \mathbf{0} & O & \mathbf{0} \\ 0 & {}^t\mathbf{q} & 0 \end{pmatrix} | \mathbf{q} \in \mathbb{R}^{n-2}\}.$$

Let $\Phi(\mathbf{t}, \mathbf{q})$ be the following matrix

$$\Phi(\mathbf{t}, \mathbf{q}) = \exp(t_1 L(\mathbf{q}) + \cdots + t_{n-2} L(\mathbf{q})^{n-2}), \tag{3.2}$$

where $L(\mathbf{q}) \in s\text{-}Lax_0$. We consider the Gauss decomposition with phase shift $A$,

$$w_A(\mathbf{t}, \mathbf{q})^{-1} b_A(\mathbf{t}, \mathbf{q}) = A\Phi(\mathbf{t}, \mathbf{q}), \tag{3.3}$$

where $w_A(\mathbf{t}, \mathbf{q}) \in \bar{N}$, $b_A(\mathbf{t}, \mathbf{q}) \in B$ and $A \in \bar{N}$ is constant. By using the solutions of (3.3), we define the orbits $\tilde{\mathscr{O}}(\mathbf{q}, A)$ for $\mathbf{q} \in \mathbb{R}^{n-2}$ by

$$\tilde{\mathscr{O}}(\mathbf{q}, A) := \{w_A(\mathbf{t}, \mathbf{q})(L(\mathbf{q})) w_A^{-1}(\mathbf{t}, \mathbf{q}) | \mathbf{t} \in \mathbb{R}^{n-2}\}.$$

We call $\tilde{\mathscr{O}}(\mathbf{q}, A)$ the orbit of the Toda lattice. Suppose that there exists point $w_A(\mathbf{t}_0, \mathbf{q}) L(\mathbf{q}) w_A(\mathbf{t}_0, \mathbf{q})^{-1} \in \tilde{\mathscr{O}}(\mathbf{q}, A)$ such that

$$res \circ proj(w_A(\mathbf{t}_0, \mathbf{q}) L(\mathbf{q}) w_A(\mathbf{t}_0, \mathbf{q})^{-1}) = x \in \mathfrak{u}_0.$$

But $(\mathbf{t}_0, A)$ is not unique for $x$ as we show later. To determine $(\mathbf{t}_0, A)$ uniquely for $x \in \mathfrak{u}_0$, we need minimal Lax orbits. Put $\tilde{\mathscr{O}}(\mathbf{q}) := \tilde{\mathscr{O}}(\mathbf{q}, E_n)$. By the companion embedding [5], the orbit $\mathscr{O}(\mathbf{q})$ in $Lax$ is translated into the orbit $\{\Phi(\mathbf{t}, \mathbf{q})B = w_{E_n}(\mathbf{t}, \mathbf{q})^{-1}B |\ \mathbf{t} \in \mathbb{R}^{n-2}\} \subset G/B$. Then it holds that $\tilde{\mathscr{O}}(\mathbf{q}) \cap \tilde{\mathscr{O}}(\mathbf{q}') = \emptyset$ if and only if $\mathbf{q} \neq \mathbf{q}'$. We also consider the $U$-$P$ decomposition

$$u(\mathbf{t}, \mathbf{q})^{-1} p(\mathbf{t}, \mathbf{q}) = \Phi(\mathbf{t}, \mathbf{q}), \tag{3.4}$$

where $u(\mathbf{t}, \mathbf{q}) \in U,\ p(\mathbf{t}, \mathbf{q}) \in P$. The orbits of this parabolic Toda lattice corresponding to the $U$-$P$ decomposition (3.4) is

$$\mathscr{O}(\mathbf{q}) := \{u(\mathbf{t}, \mathbf{q}) L(\mathbf{q}) u(\mathbf{t}, \mathbf{q})^{-1} | \mathbf{t} \in \mathbb{R}^{n-2}\}.$$

Because $\mathscr{O}(\mathbf{q})$ can be identified with the orbit $\Phi(\mathbf{t}, \mathbf{q})P$ on $G/P$, we also have $\mathscr{O}(\mathbf{q}) \cap \mathscr{O}(\mathbf{q}') = \emptyset$ if and only if $\mathbf{q} \neq \mathbf{q}'$. Put $\hat{\mathscr{O}}(\mathbf{q}) := res(proj\,\mathscr{O}(\mathbf{q}))$. We verify that $X = \mathfrak{u}_0$ has the foliation structure of the codimension $n-2$.

**Theorem 3.1** *It holds that* $X = \sqcup_{\mathbf{q} \in \mathbb{R}^{n-2}} \hat{\mathscr{O}}(\mathbf{q})$.

*proof.* For $x \in \mathfrak{u}_0$, put $L_0 := \Lambda + x$. There exists $w \in \bar{N}$, such that $w(\Lambda + x)w^{-1} \in s$-$Lax$. Hence, we have

$$w(\Lambda + x)w^{-1} = \Lambda + \begin{pmatrix} 0 & {}^t\mathbf{0} & 0 \\ \mathbf{0} & O & \mathbf{0} \\ r & {}^t\mathbf{q} & 0 \end{pmatrix}. \tag{3.5}$$

Because $\mathbb{R}E_{n,1}$ is also the center of $\bar{\mathfrak{n}}$, we have

$$w(\Lambda + x)w^{-1} = \Lambda + \begin{pmatrix} 0 & {}^t\mathbf{0} & 0 \\ \mathbf{0} & O & \mathbf{0} \\ 0 & {}^t\mathbf{q} & 0 \end{pmatrix} + rE_{n,1}$$

and

$$w(\Lambda + x - rE_{n,1})w^{-1} = \Lambda + \begin{pmatrix} 0 & {}^t\mathbf{0} & 0 \\ \mathbf{0} & O & \mathbf{0} \\ 0 & {}^t\mathbf{q} & 0 \end{pmatrix}.$$

Put $\tilde{x} := x - rE_{n,1}$. Accordingly, we have

$$res(proj(\Lambda + \tilde{x})) = x \quad \text{and} \quad w(\Lambda + \tilde{x})w^{-1} \in s\text{-}Lax_0.$$

Therefore, we can take $L(\mathbf{q}) \in s$-$Lax_0$ and $a \in \bar{N}$ satisfying

$$\Lambda + \tilde{x} = aL(\mathbf{q})a^{-1}. \tag{3.6}$$

Let us consider the Gauss decomposition with phase shift $a^{-1}$

$$w(\mathbf{t},\mathbf{q})^{-1}b(\mathbf{t},\mathbf{q}) = a^{-1}\Phi(\mathbf{t},\mathbf{q}), \tag{3.7}$$

where $w(\mathbf{t},\mathbf{q})^{-1} \in \bar{N}$ and $b(\mathbf{t},\mathbf{q}) \in B$. The decomposition (3.7) implies that $L(\mathbf{t},\mathbf{q}) := w(\mathbf{t},\mathbf{q})L(\mathbf{q})w(\mathbf{t},\mathbf{q})^{-1}$ satisfies the Lax equations and $L(\mathbf{0},\mathbf{q}) = \Lambda + \tilde{x}$. Recall that the time parameters $\mathbb{S}$ corresponding to chop integrals can be consistently imposed on $w(\mathbf{t},\mathbf{q})$ and $b(\mathbf{t},\mathbf{q})$ with (3.7). Then, we have

$$aw(\mathbf{t},\mathbb{S},\mathbf{q})^{-1}b(\mathbf{t},\mathbb{S},\mathbf{q}) = \Phi(\mathbf{t},\mathbf{q}). \tag{3.8}$$

Because $aw(\mathbf{t},\mathbb{S},\mathbf{q})^{-1} \in \bar{N}$, we can regard $aw(\mathbf{t},\mathbb{S},\mathbf{q})^{-1}$ as the point of $G/B$. We deform $aw(\mathbf{t},\mathbb{S},\mathbf{q})^{-1}$ into the point of the base space of $G/P$ of (2.2), along with fibers consistent with (3.8). The deformation along the fibers of (2.2) amount to the multiplication of $1 \times m(\mathbf{t},\mathbb{S}) \times 1$ from the right, where $m(\mathbf{t},\mathbb{S})$ is an $(n-2)\times(n-2)$ lower nilpotent matrix. For simplicity, we rewrite $1 \times m(\mathbf{t},\mathbb{S}) \times 1$ as $m(\mathbf{t},\mathbb{S})$ . Put $u(\mathbf{t},\mathbb{S},\mathbf{q})^{-1} := aw(\mathbf{t},\mathbb{S},\mathbf{q})^{-1}m(\mathbf{t},\mathbb{S}) \in U$. The consistency with (3.8) is expressed by

$$u(\mathbf{t},\mathbb{S},\mathbf{q})^{-1}(m(\mathbf{t},\mathbb{S})^{-1}b(\mathbf{t},\mathbb{S},\mathbf{q})) = \Phi(\mathbf{t},\mathbf{q}). \tag{3.9}$$

Note that $p(\mathbf{t},\mathbb{S},\mathbf{q}) := m(\mathbf{t},\mathbb{S})^{-1}b(\mathbf{t},\mathbb{S},\mathbf{q})$ is an element of $P$. Hence, (3.9) is the $U$-$P$ decomposition of $\Phi(\mathbf{t},\mathbf{q})$. In the decomposition (3.9), the point starts along the fiber from $w(\mathbf{t}_0,\mathbf{q})B = aB$ toward the point of the base space of $G/P$. Hence, we can write $\mathbb{S} = \mathbb{S}(\mathbf{t}_0)$. Then, we have the $U$-$P$ decomposition without phase shift

$$u(\mathbf{t},\mathbb{S}(\mathbf{t}_0),\mathbf{q})^{-1}p(\mathbf{t},\mathbb{S}(\mathbf{t}_0),\mathbf{q}) = \Phi(\mathbf{t},\mathbf{q})$$

and obtain the orbit of the parabolic Toda Lattice $\mathscr{O}(\mathbf{q})$ in $Lax$. The deformation $aw(\mathbf{t},\mathbf{q})^{-1}$ to $u(\mathbf{t},\mathbb{S}(\mathbf{t}_0),\mathbf{q})^{-1}$ along the fibers of (2.2) and adjusting time parameters $\mathbf{t}$ provides deformation along the curve $res \circ proj$ =constant in companion coordinate in $Lax$. The orbit $\tilde{\mathscr{O}}(\mathbf{q},a^{-1})$ goes though the initial point $\Lambda + \tilde{x}$ at $\mathbf{t} = \mathbf{0}$. Hamiltonian flows of chop integrals, together with ordinary Hamiltonian flows, provide the curve of $res \circ proj =$ constant in $Lax$, as aforementioned. Thus, there exists a unique $\tilde{\mathbf{t}}$, $\mathbb{S}(\mathbf{t}_0)$ such that

$$res \circ proj(u(\tilde{\mathbf{t}},\mathbb{S}(\mathbf{t}_0),\mathbf{q})(\Lambda + \tilde{x})u(\tilde{\mathbf{t}},\mathbb{S}(\mathbf{t}_0),\mathbf{q})^{-1}) = x,$$

where $u(\tilde{\mathbf{t}}, \mathbb{S}(\mathbf{t}_0), \mathbf{q})$ satisfies the $U$-$P$ decomposition

$$u(\tilde{\mathbf{t}}, \mathbb{S}(\mathbf{t}_0), \mathbf{q})^{-1} p(\tilde{\mathbf{t}}, \mathbb{S}(\mathbf{t}_0), \mathbf{q}) = \Phi(\tilde{\mathbf{t}}, \mathbf{q}).$$

Consider the $U$-$P$ decomposition

$$u(\tilde{\mathbf{t}}, \mathbf{q})^{-1} p(\tilde{\mathbf{t}}, \mathbf{q}) = \Phi(\tilde{\mathbf{t}}, \mathbf{q}).$$

Then, we have

$$u(\tilde{\mathbf{t}}, \mathbf{q}) = u(\tilde{\mathbf{t}}, \mathbb{S}(\mathbf{t}_0), \mathbf{q}),\ p(\tilde{\mathbf{t}}, \mathbf{q}) = p(\tilde{\mathbf{t}}, \mathbb{S}(\mathbf{t}_0), \mathbf{q}). \tag{3.10}$$

From (3.10), we can omit $\mathbb{S}(\mathbf{t}_0)$ hereafter. Because the Adjoint orbits of ordinary Hamiltonians of Toda lattice (orbits of the time parameters $\mathbf{t}$) and the fibers of (2.2) over the base space $G/P$(orbits of the time parameters of the chop integrals $\mathbb{S}$) give the foliation structure in $G/B$, therefore, $(\tilde{\mathbf{t}}, \mathbb{S}(\mathbf{t}_0), \mathbf{q})$ is unique for $x \in \mathfrak{u}_0$. If $(\tilde{\mathbf{t}}, \mathbb{S}(\mathbf{t}_0), \mathbf{q})$ and $(\tilde{\mathbf{t}}', \mathbb{S}'(\mathbf{t}_0), \mathbf{q})$ correspond with the same $x$, then we have $(\tilde{\mathbf{t}}, \mathbb{S}(\mathbf{t}_0), \mathbf{q}) = (\tilde{\mathbf{t}}', \mathbb{S}'(\mathbf{t}_0), \mathbf{q})$ by uniqueness. This implies $(\tilde{\mathbf{t}}, \mathbf{q}) = (\tilde{\mathbf{t}}', \mathbf{q})$. Thus there exists unique point in $\mathscr{O}(\mathbf{q})$, such that $res(proj(L(\mathbf{t}, \mathbf{q}))) = x$. Then we have $X = \sqcup_{\mathbf{q} \in \mathbb{R}^{n-2}} \hat{\mathscr{O}}(\mathbf{q})$. $\square$

Let $\Sigma$ be the $2n-4$ dimensional surface in $Lax$ defined by $\Sigma = \sqcup_{\mathbf{q} \in \mathbb{R}^{n-2}} \mathscr{O}(\mathbf{q})$. Based on Theorem 3.1, each point of $X$ is expressed by

$$P(\mathbf{t}, \mathbf{q}) = res(proj(L(\mathbf{t}, \mathbf{q})) = \begin{pmatrix} 0 & {}^t\mathbf{0} & 0 \\ \mathbf{L_1} & O & \mathbf{0} \\ 0 & {}^t\mathbf{L_n} & 0 \end{pmatrix},$$

where $\mathbf{L_1} = \begin{pmatrix} L_{2,1}(\mathbf{t}, \mathbf{q}) \\ \vdots \\ L_{n-1,1}(\mathbf{t}, \mathbf{q}) \end{pmatrix}$, $\mathbf{L_n} = \begin{pmatrix} L_{n,2}(\mathbf{t}, \mathbf{q}) \\ \vdots \\ L_{n,n-1}(\mathbf{t}, \mathbf{q}) \end{pmatrix}$ and

$$L(\mathbf{t}, \mathbf{q}) = \Lambda + (L_{i,j}(\mathbf{t}, \mathbf{q}))_{1 \le j \le i \le n} \in \mathscr{O}(\mathbf{q}).$$

Then, we have the polarization of $X$ with $\mathbf{q}$ and $\mathbf{t}$-directions. Let us consider the Poisson bracket in $Lax$ restricted to $\Sigma$ given by

$$\{L_{i,1}(\mathbf{t}, \mathbf{q}), L_{n,j}(\mathbf{t}, \mathbf{q})\} = -\delta_{i,j} L_{n,1}(\mathbf{t}, \mathbf{q}) \tag{3.11}$$

and other bracket relationships are zero. Put

$$q_i = q_i(\mathbf{t}, \mathbf{q}) = L_{i+1,n}(\mathbf{t}, \mathbf{q}),\ i = 1, \dots, n-2$$

$$p_j = p_j(\mathbf{t},\mathbf{q}) = L_{n,j+1}(\mathbf{t},\mathbf{q}),\ j = 1,\dots,n-2.$$

and we regard $L_{n,1}(\mathbf{t},\mathbf{q})$ as the structure constant of the Poisson structure. The Poisson bracket of (3.11) brings 2-form on $T_{P(\mathbf{t},\mathbf{q})}X$ defined by

$$\omega(\partial/\partial p_i, \partial/\partial q_j) = L_{n,1}(P(\mathbf{t},\mathbf{q}))\delta_{i,j}. \tag{3.12}$$

Recall that $\mathscr{O}(\mathbf{q})$ is homeomorphic to the orbit $u(\mathbf{t},\mathbf{q})^{-1}P$ in the flag manifold $G/P$ by companion embedding. Cleary, the orbit $u(\mathbf{t},\mathbf{q})^{-1}P$ is the orientable submanifold of $G/P$. Hence, we observe that $\mathscr{O}(\mathbf{q})$ is orientable. Because $L_{n,1}(P(\mathbf{t},\mathbf{q}))E_{n,1}$ is the vector that determines the orientation of $\mathscr{O}(\mathbf{q})$, $L_{n,1}(P(\mathbf{t},\mathbf{q}))$ never disappears for any $\mathbf{t}$ and $\mathbf{q}$. This implies that the 2-form of (3.12) defines the symplectic structure on $X$. This symplectic structure is determined by the surface $\Sigma$, and we write this symplectic structure by $\omega_\Sigma$. Let $\ell(\mathbf{q})$ be the $C^\infty$ function on $\mathbf{q}$. We consider the action $\tau$ of $U$ on $X$ by

$$\tau(\begin{pmatrix} 1 & {}^t\mathbf{0} & 0 \\ \mathbf{a} & E_{n-2} & \mathbf{0} \\ c & {}^t\mathbf{b} & 1 \end{pmatrix})P(\mathbf{t},\mathbf{q}) = P(\mathbf{t}+\ell(\mathbf{q})\mathbf{b},\mathbf{q}). \tag{3.13}$$

**Theorem 3.2** *The action $\tau$ of* (3.13) *is the symplectic action of $U$ on $X$.*

*proof.* Take an orbit leaf $\hat{\mathscr{O}}(\mathbf{q}) \subset X$. The symplectic structure along $\hat{\mathscr{O}}(\mathbf{q})$ is determined by the Poisson relation on the orbit $\mathscr{O}(\mathbf{q})$. Hence, we observe that $\tau(g)^*\omega_\Sigma = \omega_{\tau(g^{-1})^*\Sigma}$ for any $g \in U$. We consider that $\Sigma$, the surface of the fiber bundle over $X$ and $\tau(g^{-1})^*\Sigma$, is the pull back of $\Sigma$ by $\tau(g^{-1})$. Because $\omega_\Sigma$ and $\omega_{\tau(g^{-1})^*\Sigma}$ have the same conserved quantities $\mathbf{q}$ along the $\mathbf{t}$ orbit, $\omega_\Sigma$ and $\omega_{\tau(g^{-1})^*\Sigma}$ must be equivalent. Then, $\tau(g)$ are symplectic isomorphism for any $g \in U$, and we observe that $\tau$ is the symplectic action of $U$ on $X$. $\square$

## 4. Canonical commutation relations(CCR) and Stone-von Neumann theorem

For $\mathbf{a},\mathbf{b} \in \mathbb{R}^{n-2},\ c \in \mathbb{R}$, put $X_{\mathbf{a},\mathbf{b},c} = \begin{pmatrix} 0 & {}^t\mathbf{0} & 0 \\ \mathbf{a} & O & \mathbf{0} \\ c & {}^t\mathbf{b} & 0 \end{pmatrix} \in \mathfrak{u}$. The non vanishing Lie bracket relations of $\mathfrak{u}$ are expressed as:

$$[X_{\mathbf{0},\mathbf{b},0}, X_{\mathbf{a},\mathbf{0},0}] = X_{\mathbf{0},\mathbf{0},{}^t\mathbf{b}\mathbf{a}}. \tag{4.1}$$

In the Lie group level, the relation of (4.1) are lifted up so that they satisfy

$$e^{X_{\mathbf{0},\mathbf{b},0}}e^{X_{\mathbf{a},\mathbf{0},0}} = e^{X_{\mathbf{0},\mathbf{0},{}^t\mathbf{ba}}}e^{X_{\mathbf{a},\mathbf{0},0}}e^{X_{\mathbf{0},\mathbf{b},0}} \tag{4.2}$$

The relations (4.2) are called the Weyl form of CCR. Let $\rho$ be the unitary representation of $U$ on the Hilbert space $\mathscr{H}$. There exists a family of Borel measure on $\mathbb{R}^{n-2}$, $\{\mu_1, \mu_2, \ldots, \mu_\infty\}$, such that $\mathscr{H}$ is isomorphic to $\oplus_{m\in\mathbb{N}} mL^2(\mathbb{R}^{n-2}, \mu_m)$, where $mL^2(\mathbb{R}^{n-2}, \mu_m)$ represents $m$ copies of $L^2(\mathbb{R}^{n-2}, \mu_m)$ and $X_{\mathbf{a},\mathbf{0},0} \in \mathfrak{u}$ acts as ${}^t\mathbf{xa}f(\mathbf{x})$ for $f(\mathbf{x}) \in L^2(\mathbb{R}^{n-2}, \mu_m)$. Via the CCR of (4.2), we observe that all $\mu_m, m \in \mathbb{N}\cup\{\infty\}$ are equivalent to the Lebesgue measure $|d\mathbf{x}|$. Put $\mathscr{H}_m = mL^2(\mathbb{R}^{n-2}, d\mathbf{x}), m \in \mathbb{N}\cup\{\infty\}$. Suppose that $(\rho, \mathscr{H})$ is irreducible. Hence, we can assume $\mathscr{H} \subset \mathscr{H}_m$ for some $m \in \mathbb{N}\cup\{\infty\}$. Accordingly, we can write $\mathscr{H} = W \otimes L^2(\mathbb{R}^{n-2}, d\mathbf{x})$, where $W$ represents the $m$-dimensional vector space. The action $\rho(X_{\mathbf{a},\mathbf{0},0})$ on $\mathscr{H}$ is translated into $1 \otimes V(e^{X_{\mathbf{a},\mathbf{0},0}})$ on $\mathscr{H}$, where $V(e^{X_{\mathbf{a},\mathbf{0},0}})f(\mathbf{x}) = e^{i{}^t\mathbf{xa}}f(\mathbf{x})$. From CCR(4.2), we observe that $\rho(e^{X_{\mathbf{0},\mathbf{b},0}})$ is translated into $1 \otimes V(e^{X_{\mathbf{0},\mathbf{b},0}})$, where $V(e^{X_{\mathbf{0},\mathbf{b},0}})f(\mathbf{x}) = f(\mathbf{x}+\mathbf{b})$. Because $X_{\mathbf{0},\mathbf{0},c}$ is the center and $(\rho, \mathscr{H})$ is irreducible, $\rho(e^{X_{\mathbf{0},\mathbf{0},c}})$ is translated into $1 \otimes V(e^{X_{\mathbf{0},\mathbf{0},c}})$, where $V(e^{X_{\mathbf{0},\mathbf{0},c}})f(\mathbf{x}) = e^{i\lambda c}f(\mathbf{x})$, and $\lambda \in \mathbb{R}$ is a constant. Note that

$$\begin{pmatrix} 1 & {}^t\mathbf{0} & 0 \\ \mathbf{a} & E_{n-2} & \mathbf{0} \\ c & {}^t\mathbf{b} & 1 \end{pmatrix} = e^{X_{\mathbf{a},\mathbf{b},c-\frac{1}{2}{}^t\mathbf{ba}}}.$$

Hence, we adopt the notation $E_{\mathbf{a},\mathbf{b},c} = e^{X_{\mathbf{a},\mathbf{b},c-\frac{1}{2}{}^t\mathbf{ba}}}$. By the actions of $V(E_{\mathbf{a},\mathbf{b},c})$ on $L^2(\mathbb{R}^{n-2})$(we omit $d\mathbf{x}$ hereafter), we define the unitary representations of the $2n-3$ dimensional Heisenberg group $U$ parameterized by $h \in \mathbb{R}$ such as

$$\rho_h(E_{\mathbf{a},\mathbf{b},c})f(\mathbf{x}) = e^{i(hc+{}^t\mathbf{xa})}f(\mathbf{x}+h\mathbf{b}). \tag{4.3}$$

The Theorem of Stone-von Nuemann states that

*For each non-zero real number $h$, there is an irreducible representation $\rho_h$ acting on Hilbert space $L^2(\mathbb{R}^{n-2})$ by* (4.3). *All these representations are unitary inequivalent and any irreducible representation that is non trivial on the center of the Heisenberg group is unitary equivalent to exactly one of these.*

**Remark.** In the case of $h = 0$, we can consider two types of irreducible representations as follows.

(i) $$\rho_0(E_{\mathbf{a},\mathbf{b},c})f(\mathbf{x}) = e^{i{}^t\mathbf{x}\mathbf{a}}f(\mathbf{x})$$

(ii) $$\rho_{\mu,\nu}(E_{\mathbf{a},\mathbf{b},c})f(\mathbf{x}) = e^{i({}^t\mu\mathbf{a}+{}^t\mathbf{b}\nu)}f(\mathbf{x}),\ \mu,\ \nu \in \mathbb{R}^{n-2}$$

Let $\tilde{\rho}_h,\ h \in \mathbb{R}$ be the unitary irreducible representations of $U$ on $\mathscr{H}$, which are equivalent to $1 \otimes \rho_h$ on $W \otimes L^2(\mathbb{R}^{n-2})$. We consider the irreducible decomposition of a unitary representation $\rho$ on the Hilbert space $\mathscr{H}$, :hence, the irreducible components are $\tilde{\rho}_h,\ h \in \mathbb{R}$. Hereafter, we assume that the 0 component of $\rho$, $\tilde{\rho}_0$, is solely equivalent to type (i) of the above Remark.

The unitary representation $\rho$ is decomposed into irreducible components $\{(\tilde{\rho}_h, V_h)\}_{h\in\mathbb{R}}$, where $\mathscr{H} = \int_{h\in\mathbb{R}}^{\oplus} V_h dh$. Each $V_h$ is isomorphic to $W_h \otimes L^2(\mathbb{R}^{n-2})$, where $W_h$ is an $m(h)$ dimensional vector space. In this case, the multiplicity of $\rho$ in $V_h$ is equal to $m(h)$. Hence, the irreducible decomposition of $\rho$ amounts to determine the map from $\mathbb{R}$ to $\{0\} \cup \mathbb{N} \cup \{\infty\}$( if $\rho$ does not have an irreducible component $V_h$, then $m(h) = 0$). Because the Heisenberg group is Type I, we are aware of the following well-known result. We define the appropriate Borel measure on $\hat{U}$, $d\mu$.

**Theorem.** *Any unitary representation has irreducible decomposition given by*

$$\rho \simeq \int_{\hat{U}}^{\oplus} m(\rho_h)\rho_h d\mu(\rho_h),$$

*where $\rho_h,\ h \in \mathbb{R}$ are the irreducible unitary representations parameterized by $h \in \mathbb{R}$ , and $m(\rho_h) \in \mathbb{N} \cup \{\infty\}$ is the multiplicity function.*

However, it appears hard to determine clues on how to construct the map $m(\rho_h) : \mathbb{R} \to \mathbb{N} \cup \{\infty\}$. In the following section, we will grasp the above Stone-von Neumann theorem as the orbit method of the Heisenberg group and apply the SSB method to obtain detailed information on the multiplicities of the irreducible components.

## 5. Irreducible decomposition of the unitary representations of Heisenberg groups and hyper-functions

In section 3, we give the polarized symplectic structure to $X$ and demonstrate the action $\tau$ of $U$ on $X$

$$\tau(E_{\mathbf{a},\mathbf{b},c})P(\mathbf{t},\mathbf{q}) = P(\mathbf{t} + \ell(\mathbf{q})\mathbf{b}, \mathbf{q})$$

is a symplectic action in Theorem 3.2. Let $\mathscr{L}$ be the line bundle over $X$. Let $\nabla$ be the connection of $\mathscr{L}$ whose curvature form is $\omega_\Sigma$(mentioned in Theorem 3.2). We assume that there exists $\nabla$-invariant Hermitian form on $\mathscr{L}$. Let $\mathscr{K}$ be the sub bundle of $\mathscr{L}$ defined by

$$\mathscr{K} = \{u \in \Gamma(X;\mathscr{L}) | \nabla_{\frac{\partial}{\partial q_i}} u = 0,\ i = 1, \dots, n-2\}.$$

Let $\mathscr{H}$ be the Hilbert space of the $L^2$-section of $\oplus_{\mathbf{q}\in\mathbb{R}^{n-2}}\Gamma(\hat{\mathscr{O}}(\mathbf{q});\mathscr{K})$. We consider the unitary representation of $U$, which are caused by the aforementioned symplectic action $\tau$. Let $\rho$ be the unitary representation of $U$ on $\mathscr{H}$ caused by $\tau$. From the orbit method[10], we observe that the $U$-orbit through $P(\mathbf{t},\mathbf{q})$

$$\mathfrak{O}(\mathbf{q}) := \tau(U)P(\mathbf{t},\mathbf{q}) = \{P(\mathbf{t} + \ell(\mathbf{q})\mathbf{b}, \mathbf{q}) | \mathbf{b} \in \mathbb{R}^{n-2}\} \tag{5.1}$$

corresponds with an irreducible component of $(\rho, \mathscr{H})$. We write $\ell(\mathbf{q}) = \ell_\rho(\mathbf{q})$. The irreducible unitary representation of $U$ is realized in $L^2(\mathbb{R}^{n-2})$ by the Stone-von Neumann theorem

$$\rho_h(E_{\mathbf{a}.\mathbf{b},c})f(\mathbf{x}) = e^{i(hc+{}^t\mathbf{x}\mathbf{a})}f(x + h\mathbf{b}),\ h \in \mathbb{R}. \tag{5.2}$$

If $\ell_\rho(\mathbf{q}) = \ell_\rho(\mathbf{q}') = h$, the orbits $\mathfrak{O}(\mathbf{q})$ and $\mathfrak{O}(\mathbf{q}')$ correspond to $\rho_h$, although $\mathbf{q} \neq \mathbf{q}'$. Put $\mathfrak{M}_\rho(h) = \{\mathbf{q} \in \mathbb{R}^{n-2} | \ell_\rho(\mathbf{q}) = h\}$. Considering naively, the cardinality $|\mathfrak{M}_\rho(h)|$ may be the multiplicity of $\rho$ on the irreducible component $(\rho_h, \mathscr{H}_h)$. However, almost all case $\mathfrak{M}_\rho(h)$ is a hyper surface in $\mathbb{R}^{n-2}$ and $|\mathfrak{M}_\rho(h)| = \aleph$ or $\mathfrak{M}_\rho = \phi$ almost all $h$. Hence, $|\mathfrak{M}_\rho(h)|$ is not suitable for explicit multiplicities. We will introduce the *density* of the multiplicity that indicates the multiplicities of non-negative integers. As we introduced in §4, the unitary representation $(\rho, \mathscr{H})$ is decomposed into $(\rho_h, L^2(\mathbb{R}^{n-2}))$ with multiplicity. Because $\mathscr{H}$ is the line bundle over $X = U/R$, the irreducible components should have forms $\mathbb{C}_h \otimes L^2(\mathbb{R}^{n-2}),\ h \in \mathbb{R}$, where $\mathbb{C}_h$ are the one-dimensional Hilbert spaces on which $R$ acts. The Hilbert space $\mathbb{C}_h \otimes L^2(\mathbb{R}^{n-2}),\ h \in \mathbb{R}$ are the representation spaces of $R \times U$ rather than $U$. The representation of $R \times U$ on $\mathbb{C}_h \otimes L^2(\mathbb{R}^{n-2})$ should be $\chi^h_\rho \boxtimes \rho_h$, where $\chi^h_\rho$ is the one-dimensional unitary representation of $R$ defined by

$$\chi^h_\rho(E_{\mathbf{0},\mathbf{0},s}) = e^{im_\rho(h)s}. \tag{5.3}$$

On the right hand side of (5.3), $m_\rho(h)$ is a real constant determined by the unitary representation $\rho$ and $h \in \mathbb{R}$. We call $m_\rho(h)$ as the density of the multiplicity. $\chi_\rho^h \boxtimes \rho_h$ acts on $\mathbb{C}_h \otimes L^2(\mathbb{R}^{n-2})$ as

$$\chi_\rho^h \boxtimes \rho_h(E_{\mathbf{0},\mathbf{0},s}, E_{\mathbf{a},\mathbf{b},c})\mathbf{1}_h \otimes f(\mathbf{x}) = e^{im_\rho(h)s}\mathbf{1}_h \otimes \rho_h(E_{\mathbf{a},\mathbf{b},c})f(\mathbf{x}), \tag{5.4}$$

where $\mathbf{1}_h$ is the base of $\mathbb{C}_h$, and we sometime write $e^{im_\rho(h)s} \cdot \mathbf{1}_h \otimes f(\mathbf{x})$ by $e^{im_\rho(h)s} \otimes f(\mathbf{x})$. We consider $X = \mathfrak{u}_0$ as $2n-4$-dimensional polarized symplectic manifold. The irreducible components of $\rho$ on $\mathscr{H}$ should have forms $(\rho_h, L^2(\mathbb{R}^{n-2}))$, $h \in \mathbb{R}$. Therefore we must reduce $\mathbb{C}_h \otimes L^2(\mathbb{R}^{n-2}) \to L^2(\mathbb{R}^{n-2})$, such that

$$\mathbf{1}_h \otimes f(\mathbf{x}) \mapsto \chi_\rho^h(E_{\mathbf{0},\mathbf{0},s})f(\mathbf{x}) = e^{im_\rho(h)s}f(\mathbf{x}), \tag{5.5}$$

where $s \in [0, 2\pi]$ is arbitrary. Certainly, $[0, 2\pi]$ is an artificial interval. If $m_\rho(h) \notin \mathbb{Z}$, $e^{im_\rho(h)s}$ does not have $2\pi$ as the period on $s$. However, the choice of the interval does not have an effect on the results. We regard the map (5.5) as spontaneous symmetry breaking(SSB) because the map of (5.5) abuses the symmetry of $R$. More precisely, the $R$-symmetry of $\mathbb{C}_h \otimes L^2(\mathbb{R}^{n-2})$ is described as follows. We observe that $\mathbb{C}_h$ is an $R$-module. For $E_{\mathbf{0},\mathbf{0},r} \in R$, we can regard

$$A(E_{\mathbf{0},\mathbf{0},r})(\mathbf{1}_h \otimes L^2(\mathbb{R}^{n-2})) := \chi_\rho^h(E_{\mathbf{0},\mathbf{0},r}) \otimes L^2(\mathbb{R}^{n-2})$$

as the base change of $\mathbb{C}_h$. Because $R$ is commutative, we have

$$\chi_\rho^h(g)A(E_{\mathbf{0},\mathbf{0},r}) = A(E_{\mathbf{0},\mathbf{0},r})\chi_\rho^h(g)$$

for any $g \in R$. The equivalency relative to $A$ is regarded as the symmetry of $R$. To adjust $\mathbb{C}_h \otimes L^2(\mathbb{R}^{n-2})$ to $L^2(\mathbb{R}^{n-2})$, we lose $R$ symmetry. However, we can recover $R$ symmetry as follows. We regard $\chi_\rho^h(E_{\mathbf{0},\mathbf{0},s}) = e^{im_\rho(h)s}$ as a function on $[0, 2\pi]$, and extend it to outside of the interval by $\chi_\rho^h(E_{\mathbf{0},\mathbf{0},s\pm 2\pi}) = \chi_\rho^h(E_{\mathbf{0},\mathbf{0},s})$. Consider the summation

$$\frac{1}{n}\sum_{k=1}^{n} \chi_\rho^h(E_{\mathbf{0},\mathbf{0},\frac{2\pi k}{n}})\rho_h. \tag{5.6}$$

Taking limit $n \to \infty$ in (5.6), we have

$$\frac{1}{2\pi}\int_0^{2\pi} e^{im_\rho(h)s}ds\rho_h. \tag{5.7}$$

We write the integral of (5.7) by $M_\rho(h)$ and call it a weight of $\rho$ on $\mathbb{C}_h \otimes L^2(\mathbb{R}^{n-2})$. We regard $e^{im_\rho(h)s}$ as a function on $s \in [0, 2\pi]$. We extended

$e^{im_\rho(h)s}$ to outside of $[0, 2\pi]$ by periodicity. Let us consider two vector spaces over $\mathbb{C}$

$$V_0 = \mathbb{C}\chi_\rho^h \boxtimes \rho_h \text{ and } V_1 = \oplus_{s\in\mathbb{R}}\mathbb{C}e^{im_\rho(h)s}\rho_h.$$

The spontaneous symmetry breaking(SSB) is a linear map from $V_0$ to $V_1$, such that

$$\chi_\rho^h \boxtimes \rho_h \mapsto \chi_\rho^h(E_{\mathbf{0},\mathbf{0},s_0})\rho_h = e^{im_\rho(h)s_0}\rho_h,$$

where $s_0 \in \mathbb{R}$ is arbitrary. We consider the representation $T(E_{\mathbf{0},\mathbf{0},s_1})$ by

$$T(E_{\mathbf{0},\mathbf{0},s_1})e^{im_\rho(h)s_0}\rho_h = e^{im_\rho(h)(s_0+s_1)}\rho_h.$$

Let $V_2$ be the vector space $\mathbb{C}M_\rho(h)\rho_h$. We consider the map from $V_1$ to $V_2$ defined by

$$\Xi : e^{im_\rho(h)s_0}\rho_h \mapsto \int_0^{2\pi} e^{im_\rho(h)(s+s_0)}ds\rho_h \tag{5.8}$$

and extend it as a whole $V_1$ by linearity. By periodicity, we have

$$\int_0^{2\pi} e^{im_\rho(h)(s+s_0)}ds = M_\rho(h).$$

Therefore we have the following diagram.

$$\begin{tikzcd} V_0 \arrow[r, "SSB"] \arrow[dr, "SSB"'] & V_1 \arrow[r, "\Xi"] \arrow[d, "T(E_{\mathbf{0},\mathbf{0},s})"] & V_2 \arrow[d, "id"] \\ & V_1 \arrow[r, "\Xi"'] & V_2 \end{tikzcd} \tag{5.9}$$

The diagram (5.9) indicates that the map $\Xi$ recovers $R$-symmetry, which is taken away by SSB. From (5.9), we can also understand the equivalence of each component

$$\begin{aligned}(\chi_\rho^h \boxtimes \rho_h, \mathbb{C}_h \otimes L^2(\mathbb{R}^{n-2})) &\simeq (id \boxtimes \rho_h, M_\rho(h) \otimes L^2(\mathbb{R}^{n-2})) \\ &= (\rho_h, M_\rho(h)L^2(\mathbb{R}^{n-2})).\end{aligned}$$

However this is no longer a unitary equivalent. This implies non unitary equivalence

$$\rho \simeq \int_{h\in\mathbb{R}}^{\oplus} M_\rho(h)\rho_h dh,$$

where the right-hand side is the direct integral. Comprehensively, we show that above direct integral has an analytical meaning, namely, we will show

$$\rho \simeq \int_{h\in\mathbb{R}} M_\rho(h)\rho_h dh, \tag{5.10}$$

where the right-hand side of (5.10) is the Lebesugue integral of the map from $U$ to $B(L^2(\mathbb{R}^{n-2}), L^2(\mathbb{R}^{n-2}))$ (the space of bounded operators on $L^2(\mathbb{R}^{n-2})$) valued function. Via SSB, we replace the irreducible components $\mathbb{C}_h \otimes L^2(\mathbb{R}^{n-2})$ with $L^2(\mathbb{R}^{n-2})$, $h \in \mathbb{R}$. We consider the completion of $\oplus_{h\in\mathbb{R}} L^2(\mathbb{R}^{n-2})$. Let us consider the direct product $\Pi_{h\in\mathbb{R}}\mathscr{H}_h$, where $\mathscr{H}_h = L^2(\mathbb{R}^{n-2})$. For $f_h(\mathbf{x}), g_h(\mathbf{x}) \in \mathscr{H}_h = L^2(\mathbb{R}^{n-2})$, we define the inner product $f_h \cdot g_h$ as:

$$f_h \cdot g_h = \int_{\mathbb{R}^{n-2}} f_h(\mathbf{x})\bar{g}_h(\mathbf{x})d\mathbf{x}.$$

Let $\mathscr{V}$ be the Hilbert space defined by

$$\mathscr{V} = \{\{f_h\} \in \Pi_{h\in\mathbb{R}}\mathscr{H}_h | \int_{\mathbb{R}} f_h \cdot f_h dh < \infty\}.$$

We regard $\mathscr{V}$ as the completion of $\oplus_{h\in\mathbb{R}}\mathscr{H}_h$. We sometimes write $\{f_h\}_{h\in\mathbb{R}} \in \mathscr{V}$ as $\int_{h\in\mathbb{R}}^{\oplus} f_h dh$. Let $\mathscr{G}$ be a Hilbert space and $B(\mathscr{G}, \mathscr{G})$ be the space of bounded operators from $\mathscr{G}$ to $\mathscr{G}$. Let $\mathscr{M}(U, B(\mathscr{G}, \mathscr{G}))$ be the space of measurable maps from the Heisenberg group $U$ to $B(\mathscr{G}, \mathscr{G})$. For $n \in \mathbb{N}$, we define the step function $\Pi_n^\rho(h)$ as:

$$\Pi_n^\rho(h) = M_\rho(\frac{2k+1}{2n})\rho_{\frac{2k+1}{2n}}, \quad \frac{k}{n} \leq h < \frac{k+1}{n}, \ k \in \mathbb{Z}. \tag{5.11}$$

We define the integral of $\mathscr{M}(U, B(L^2(\mathbb{R}^{n-2}), L^2(\mathbb{R}^{n-2})))$-valued step function

$$\int_{h\in\mathbb{R}} \Pi_n^\rho(h)dh. \tag{5.12}$$

Because (5.12) is the integral of a step function, $g \in U$ acts on $\mathscr{V}$ as:

$$\int_{h\in\mathbb{R}} \Pi_n^\rho(h)dh(g)\int_{h\in\mathbb{R}}^{\oplus} f_h dh = \{\Pi_n^\rho(h)(g)f_h\}_{h\in\mathbb{R}} \in \Pi_{h\in\mathbb{R}}\mathscr{H}_h. \tag{5.13}$$

We impose a condition upon $m_\rho(h)$ that $M_\rho(h)$ becomes a bounded measurable function. Hence, we have

$$\int_{h\in\mathbb{R}} ||\Pi_n^\rho(g)f_h||^2 dh = \sum_{k\in\mathbb{Z}} \int_{k/n}^{(k+1)/n} |M_\rho(\frac{2k+1}{2n})|^2 ||\rho_{\frac{2k+1}{2n}}(g)f_h||^2 dh$$

Because $\rho_{\frac{2k+1}{2n}}(g)$ is unitary

$$\begin{aligned} &= \sum_{k\in\mathbb{Z}} |M_\rho(\frac{2k+1}{2n})|^2 \int_{k/n}^{(k+1)/n} ||f_h||^2 dh \\ &\leq |M|^2 \sum_{k\in\mathbb{Z}} \int_{k/n}^{(k+1)/n} ||f_h||^2 dh = |M|^2 \int_{h\in\mathbb{R}} f_h \cdot f_h dh < \infty, \end{aligned}$$

where $M = \sup_{h\in\mathbb{R}}|M_\rho(h)|$. Hence, we see that

$$\int_{h\in\mathbb{R}} \Pi_n^\rho(h)dh \in \mathscr{M}(U, B(\mathscr{V}, \mathscr{V})). \tag{5.14}$$

Certainly, $\lim_{n\to\infty} \Pi_n^\rho(h) = M_\rho(h)\rho_h$ (a.e. $h$). Let us consider the limit

$$\lim_{n\to\infty}(\int_{\mathbb{R}} \Pi_n^\rho(h)dh(g)\int_{\mathbb{R}}^{\oplus} f_h dh) = \lim_{n\to\infty}\{\Pi_n^\rho(h)(g)f_h\}_{h\in\mathbb{R}}.$$

Because we can take limit by each fiber, we have

$$\lim_{n\to\infty}\{\Pi_n^\rho(h)(g)f_h\}_{h\in\mathbb{R}} = \{\lim_{n\to\infty}\Pi_n^\rho(g)f_h\}_{h\in\mathbb{R}} = \{M_\rho(h)\rho_h(g)f_h\}_{h\in\mathbb{R}} \tag{5.15}$$

We define $\int_{h\in\mathbb{R}} M_\rho(h)\rho_h dh$ by

$$\int_{h\in\mathbb{R}} M_\rho(h)\rho_h dh(g)\{f_h\}_{h\in\mathbb{R}} = \{M_\rho(h)\rho_h(g)f_h\}_{h\in\mathbb{R}} \tag{5.16}$$

We can demonstrate that $\int_{h\in\mathbb{R}} M_\rho(h)\rho_h dh$ belongs to $\mathscr{M}(U, B(\mathscr{V}, \mathscr{V}))$, similar to showing (5.14). Moreover, $\int_{h\in\mathbb{R}} M_\rho(h)\rho_h dh$ is the limit of step functions $\int_{h\in\mathbb{R}} \Pi_n^\rho(h)dh$. Then, we obtain the conclusion that

$$\int_{h\in\mathbb{R}} M_\rho(h)\rho_h dh \in \mathscr{M}(U, B(\mathscr{V}, \mathscr{V}))$$

is the Lebesgue integral of $\mathscr{M}(U, B(L^2(\mathbb{R}^{n-2}), L^2(\mathbb{R}^{n-2})))$-valued function. Then, we can treat the integral $\int_{h\in\mathbb{R}} M_\rho(h)\rho_h dh$ as usual analysis. We regard the integral of (5.10) as the weight decomposition of $\rho$. We define the multiplicity of $\rho$ at $\mathscr{H}_h$ by $N_\rho(h) = [|M_\rho(h)|]$(maximal integer$\leq |M_\rho(h)|$). We must verify the validity to regard $N_\rho(h)$ as multiplicity. Suppose that $f(x)$ is a function of $L^1[0,2\pi]$, we extend $f(x)$ to $(2\pi,\infty)$ and $(-\infty,0)$ as periodic functions, with period $2\pi$. For any $n \in \mathbb{N}$, we have identity

$$\int_0^{2\pi} f(x)dx = \int_0^{2\pi} f(nx)dx. \tag{5.17}$$

The integral of the right hand side of (5.17) is split into the integrals on the sheets of the $n$ covering space of $[0,2\pi]$ as follows.

$$\int_0^{2\pi} f(nx)dx = \sum_{k=0}^{n-1}\int_{\frac{2\pi k}{n}}^{\frac{2\pi(k+1)}{n}} f(nx)dx = \sum_{k=0}^{n-1}\int_{2k\pi}^{2(k+1)\pi} f(x)\frac{dx}{n}$$

Because $f(x)$ is the periodic function with period $2\pi$, we have

$$= \sum_{k=0}^{n-1}\int_0^{2\pi} f(x)\frac{dx}{n} = n\int_0^{2\pi} f(x)\frac{dx}{n}. \tag{5.18}$$

It appears that the formula of (5.17) amounts to a trivial identity

$$\int_0^{2\pi} f(x)dx = n\int_0^{2\pi} f(x)\frac{dx}{n}. \tag{5.19}$$

However, we can regard $\int_0^{2\pi} f(x)\frac{dx}{n}$ as an integral on a sheet of the $n$ covering space. We abbreviate $N_\rho(h)$ by $N$ for simplicity. The integral $M_\rho(h)$ splits into integrals on the sheets of the $N$ covering space as follows:

$$M_\rho(h) = N\int_0^{2\pi} e^{im_\rho(h)s}\frac{ds}{N} \tag{5.20}$$

Because

$$|\int_0^{2\pi} e^{im_\rho(h)s}\frac{ds}{N}| = \frac{|\int_0^{2\pi} e^{im_\rho(h)s}ds|}{N},$$

we observe that the component of $M_\rho(h)\rho_h$ splits into the $N$ components $\frac{1}{N}M_\rho(h)\rho_h$ whose multiplicity is $[|\frac{1}{N}M_\rho(h)|]$. We also observe that

$$|\int_0^{2\pi} e^{im_\rho(h)s}ds| = N + \Theta_{\rho,h},$$

where $\Theta_{\rho,h} = |\int_0^{2\pi} e^{im_\rho(h)s}ds| - N$ so $0 \leq \Theta_{\rho,h} < 1$. Then we have $[|\int_0^{2\pi} e^{im_\rho(h)s}\frac{ds}{N}|] = 1$. This implies that $\rho$ splits into the $N$ representations that are multiplicity free on each sheet of the $N$ covering space of $\mathscr{H}_h$. If $N' > N$, we have

$$0 \leq |\int_0^{2\pi} e^{im_\rho(h)} \frac{ds}{N'}| = |\int_0^{2\pi} e^{im_\rho(h)s} ds|/N' \leq \frac{(N+\Theta_{\rho,h})}{N'}$$

$$< (N+1)/N' \leq 1 \tag{5.21}$$

The inequality of (5.21) implies $[|\int_0^{2\pi} e^{im_\rho(h)s}\frac{ds}{N'}|] = 0$. Hence, $N$ is the largest number that $\rho$ splits into multiplicity free components. We summarize as follows.

(i) $\rho$ has weight decomposition by a Lebesgue integral

$$\rho \simeq \int_{h\in\mathbb{R}} M_\rho(h)\rho_h dh$$

(ii) $[|M_\rho(h)|]$ is the multiplicity of $\rho$ of the irreducible component $\mathscr{H}_h$.

In the Lebesgue integral (5.10), we replace $\rho_h$ with $\phi(h)\rho_h$, where $\phi(h)$ is a measurable function. We refer to $\phi(h)$ as an amplifier. For example, the cut off function

$$\phi(h) = \begin{cases} 1 & 0 \leq h \leq 1 \\ 0 & h < 0,\ h > 1 \end{cases}$$

is an amplifier satisfying:

$$\int_{h\in\mathbb{R}} M_\rho(h)(\phi(h)\rho_h)dh = \int_0^1 M_\rho(h)\rho_h dh.$$

However, the modification $\rho_h \mapsto \phi(h)\rho_h$ spoils the unitarity of $\rho_h$. We can save the unitarity of $\rho_h$ by adopting the identity $\int_{h\in\mathbb{R}} M_\rho(h)(\phi(h)\rho(h))dh = \int_{h\in\mathbb{R}}(\phi(h)M_\rho(h))\rho(h)dh$. We interpret $\phi(h)$ as an amplifier of the weight function $M_\rho(h)$, rather than that of irreducible representation $\rho_h$. Hence, the multiplicity $[|M_\rho(h)|]$ is replaced by $[|\phi(h)M_\rho(h)|]$. Certainly, if we take different amplifiers for each unitary representation $\rho$, it is meaningless to compare weights $\phi(h)M_\rho(h)$ among unitary reresentations. However, if there exists canonical amplifier $\phi(h)$ of Lebesgue integrals (5.10) for every unitary representation, we can adopt $[|\phi(h)M_\rho(h)|]$ as the multiplicity of $\rho$ of the irreducible component $\mathscr{H}_h$.

Recall that the weight function $M_\rho(h)$ caused by SSB of $\rho$ is

$$M_\rho(h) = \frac{1}{2\pi}\int_0^{2\pi} e^{im_\rho(h)s}ds = \begin{cases} \frac{1}{2\pi i}\frac{e^{2\pi i m_\rho(h)}-1}{m_\rho(h)} & m_\rho(h) \neq 0 \\ 1 & m_\rho(h) = 0 \end{cases} .$$

Note that $M_\rho(h) = 0$, if and only if $m_\rho(h) \in \mathbb{Z} - \{0\}$ and $\phi(h)M_\rho(h) = 0$ for any amplifier $\phi(h)$ if $M_\rho(h) = 0$. Before SSB, every irreducible component of the unitary representation has the form $(\chi_\rho^h \boxtimes \rho_h, \mathbb{C}_h \otimes L^2(\mathbb{R}^{n-2}))$. If $m_\rho(h) = 0$(and then $M_\rho(h) = 1$), the irreducible component is $(1 \boxtimes \rho_h, \mathbb{C}_h \otimes L^2(\mathbb{R}^{n-2})) = (\rho_h, L^2(\mathbb{R}^{n-2}))$. We regard such components as the proper irreducible components of $\rho$. We have to consider the weight decomposition solely with proper irreducible components when we regard $X = \mathfrak{u}_0$ as the polarized symplectic affine space. Before SSB, the irreducible components have the form $\hat{R} \boxtimes \rho_h$. We refer to the $\hat{R}$ part as spin. We construct the weight decomposition with proper irreducible components without the spin from integral (5.10) by multiplying canonical amplifiers $\phi(h)$: in addition, we obtain multiplicities $[|\phi(h)M_\rho(h)|]$(see Fig.I. The black arrows denote representations of $\rho_h, h \in \mathbb{R}$ and the accompanying red arrows are representations of the spin. We extract irreducible representations without red arrows).

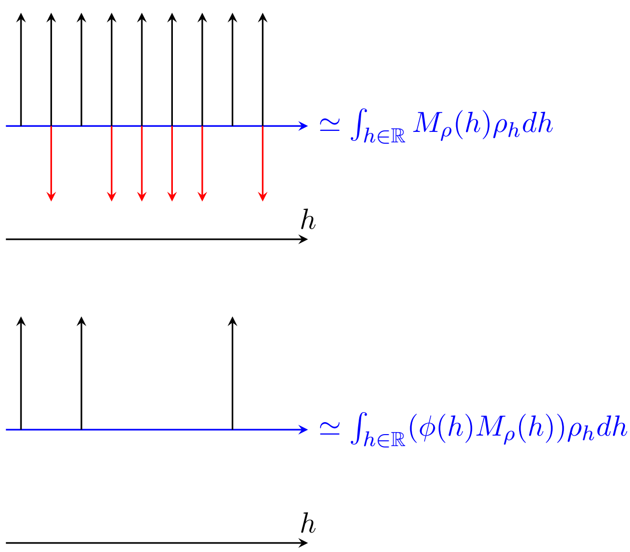

Fig.I

**Discrete decomposition.** First, we assume that $f(h) = m_\rho(h)$ has countable simple zero points $a_n,\ n \in \mathbb{Z}$ (Fig.II).

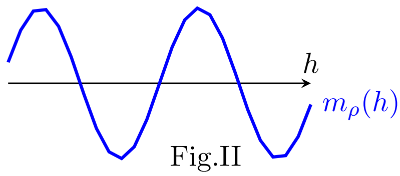


Fig.II

To justify $1/f(h)$ as an amplifier, we assume that $f(h)$ may be extended to $F(z)$ in the neighborhood of real axis $V_r := \{h+iy|h \in \mathbb{R},\ |y| < r\}$ as the holomorphic function and $F(z) \neq 0$ $z \in V_r - \{a_n\}_{n\in\mathbb{Z}}$. From the assumption on the form of $f(h)$, we have the following partial fraction decomposition

$$\frac{1}{F(z)} = \sum_{n\in\mathbb{Z}} \frac{A_n}{z-a_n} + G(z), \tag{5.22}$$

where $A_n,\ n \in \mathbb{Z}$ are constants and $G(z)$ is the holomorphic function on $V_r$. We take

$$\frac{1}{F(h+i\epsilon)} - \frac{1}{F(h-i\epsilon)}$$

as the amplifier. Then we have

$$\begin{aligned}
&\int_{h\in\mathbb{R}} \{\frac{1}{F(h+i\epsilon)} - \frac{1}{F(h-i\epsilon)}\} M_\rho(h)\rho_h dh \\
&= \int_{h\in\mathbb{R}} \sum_{n\in\mathbb{Z}} \{A_n(\frac{1}{h-a_n+i\epsilon} - \frac{1}{h-a_n-i\epsilon}) \\
&\qquad + G(h+i\epsilon) - G(h-i\epsilon)\} M_\rho(h)\rho_h dh.
\end{aligned}$$

Let us take the limit $\lim_{\epsilon\to 0+0}$. Because the $G(z)$ is holomorphic, as well as the delta function expression of the hyper-function [14], we have

$$\to \int_{h\in\mathbb{R}} \sum_{n\in\mathbb{Z}} A_n \delta(h-a_n) M_\rho(h)\rho_h dh = \sum_{n\in\mathbb{Z}} A_n M_\rho(a_n)\rho_{a_n}. \tag{5.23}$$

Moreover $M_\rho(a_n) = 1$ because $m_\rho(a_n) = 0$. Then, we have the weight decomposition

$$\rho \simeq \sum_{n\in\mathbb{Z}} A_n \rho_{a_n}. \tag{5.24}$$

Hence, we have the proper multiplicity of $\rho$ on $\mathscr{H}_{a_n} = L^2(\mathbb{R}^{n-2})$, $[|A_n|]$, $n \in \mathbb{Z}$. The decomposition (5.24) is the weight decomposition of $\rho$ on $\Gamma(X = \mathfrak{u}_0; \mathscr{H})$.

**Continuous decomposition** Suppose that $f(h) = m_\rho(h) = 0$ in the cer-

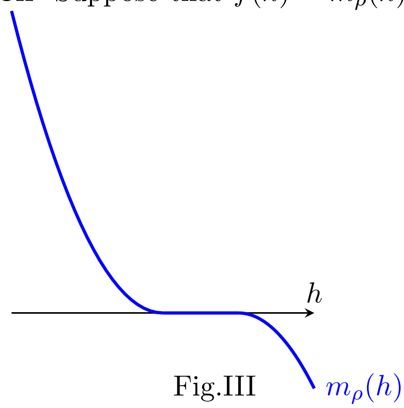


Fig.III

tain interval $[\alpha, \beta]$ (Fig.III).

We can regard the integral

$$\rho = \int_\alpha^\beta M_\rho(h)\rho_h dh = \int_\beta^\alpha \rho_h dh. \tag{5.25}$$

as the weight decomposition of $\rho$. We refer to (5.25) as the continuous decomposition of $\rho$.

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