On GIT quotients of the symplectic group, stability and bifurcations of periodic orbits (with a view towards practical applications)

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In this article, we will introduce a collection of tools aimed at studying periodic orbits of Hamiltonian systems, their (linear) stability, and their bifurcations. We will provide topological obstructions to the existence of orbit cylinders of symmetric orbits, for mechanical systems preserved by anti-symplectic involutions (e.g. the circular restricted three-body problem). Such cylinders induce continuous paths which do not cross the *bifurcation locus* of suitable GIT quotients of the symplectic group, which are branched manifolds whose topology provide the desired obstructions. Namely, the complement of the corresponding loci consist of several connected components which we enumerate and explicitly describe; by construction these cannot be joined by a path induced by an orbit cylinder. We also provide preferred normal forms for each regular and singular component. We further introduce a notion of signature for symmetric orbits, which extends the notion from Krein theory (which only applies for elliptic orbits), to allow also for the case of symmetric orbits which are hyperbolic. This signature helps predict at which points of a symmetric orbit a bifurcation arises. This gives a general theoretical framework for the study of stability and bifurcations of symmetric orbits, with a view towards practical and numerical implementations within the context of space mission design. This is the subject of the follow-up paper [7], where this framework is supported by numerical work.

1. Introduction

In this article, we will introduce a collection of tools aimed at studying periodic orbits of Hamiltonian systems, their stability, and their bifurcations[1]. In the case where the system admits symmetries in the form of anti-symplectic involutions, we will obtain richer refinements for the case of symmetric orbits. The prototypical system of interest is the circular restricted three-body problem, although these tools apply in full generality. The motivation stems from the study of families of orbits, which are ubiquitous in the space mission design industry, where closed trajectories are used to place satellites in orbit around a celestial body[2]. This is the subject of the follow-up article [7], where Euler characteristic of suitable local Floer homology groups are considered, which help predict the existence of orbits before and after a bifurcation. Moreover, the mathematical framework here provided is used also in [7] in the study of families of orbits in the Jupiter-Europa and Saturn-Enceladus systems. These systems are of tremendous current interest, as the icy moons are believed to be candidates for harboring conditions for extraterrestrial life. The list of orbits that have been found, for the three-body problem alone, is certainly long (see e.g. [21, Chapter 9] for numerical work, [10] for a quantitative analysis of bifurcations, [5] for symmetric planar orbits, [13] and references therein for a very recent numerical investigation on Hill's lunar problem); this poses the necessity of keeping track on how they relate to each other.

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[1]Recall that a bifurcation of a periodic orbit occurs whenever one considers 1-parameter families of orbits $t \mapsto \gamma_t$, and at some singular time t_0 the orbit γ_{t_0} becomes degenerate, i.e. the monodromy matrix has 1 as an eigenvalue; see Abraham–Marsden's book [3, p. 599] for a complete generic classification in dimension 4.

[2]For instructive videos on the role of the Halo orbits in the NASA Gateway space mission, see e.g. `https://www.youtube.com/watch?v=kpgj6Im4Ol4`.

This motivates our general direction, encapsulated in the following guiding questions:

> **Guiding Question 1.** *(Classification) Can we tell when two orbits are* **qualitatively** *different[a]?*
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> **Guiding Question 2.** *(Catalogue/Data Science) Can we resource-efficiently refine "data bases" of known orbits or known bifurcations between them, so that one may use techniques from data science to study them (e.g. machine learning)?*
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> **Guiding Question 3.** *(Practical tests) Can we use Floertheoretical invariants to test the accuracy of the algorithms, and to guide/organise the numerical work?*
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> [a]We say that two orbits are *qualitatively* different if there is no regular family joining them.

In this article we will address the first two questions. The third one is addressed in [7], with supporting numerical work. The concrete aims here pursued, which provide the underlying theoretical framework, are to:

1) **(Global topological methods)** Introduce global topological methods for the study of periodic orbits of Hamiltonian systems and their bifurcations, for which we give an algebraic interpretation, in the form of the sequence of maps and spaces

$$(1) \quad \mathrm{Sp}^{\mathcal{I}}(2n)//\mathrm{GL}_n(\mathbb{R}) \to \mathrm{Sp}(2n)//\mathrm{Sp}(2n) \to \mathrm{M}_{n\times n}(\mathbb{R})//\mathrm{GL}_n(\mathbb{R}) \cong \mathbb{R}^{2n},$$

which we call the *GIT sequence*[3]. It consists of GIT quotients of spaces of matrices with respect to suitable actions. The action on the second and third space is by conjugation, and the first one we will explain shortly.

2) **(*B*-signature)** Introduce the notion of *B-signature*, a sign attached to each elliptic and hyperbolic eigenvalue of the monodromy matrix of a *symmetric* periodic orbit, and which generalizes the notion of

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[3]Recall that the GIT quotient $X//G$ of the action of a group G on a space X is defined via the relation $x \sim y$ if the closures of their G-orbits intersect. It is Hausdorff, and needed for topological reasons, as X/G may not be.

the classical Moser–Krein [14–17, 20] signature to also apply to the hyperbolic case. This sign is relevant for bifurcations of symmetric orbits, as it can help predict at which symmetric point a bifurcation occurs (see Figure 1 and also [7]).

3) **(Branching structure)** Describe the "singular regions" of $M_{n \times n}(\mathbb{R})//GL_n(\mathbb{R}) \cong \mathbb{R}^{2n}$ the *base* of the GIT sequence for the cases $n = 1, 2$, enumerate the connected components lying above the "regular components" of each of the GIT quotients $\mathrm{Sp}^{\mathcal{I}}(2n)//\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{Sp}(2n)//Sp(2n)$, and determine the covering degree for each component of the maps in the sequence. See Figure 3.

4) **(Labelling)** Identify each component of $\mathrm{Sp}^{\mathcal{I}}(2n)//GL_n(\mathbb{R})$ and $\mathrm{Sp}(2n)//Sp(2n)$ with the corresponding B-signs (which agree with the Krein signs in the second case). See Figure 5.

5) **(Normal forms)** Describe a choice of a favourite representative for each lift of the regular components, as well as the singular ones.

The case $n = 1$ of the GIT sequence is elementary and appears e.g. in [23, Appendix B], where it plays an important role when trying to define a real version of Embedded Contact Homology (ECH). Moreover, as we learned after finishing a first version of this article, for $n = 2$, the GIT sequence is a refinement and reformulation of the "Broucke stability diagram" [4], which is already used in the space mission design industry (see e.g. [18]). In this context, stability of an orbit is of central importance not only theoretically but also for space mission design, as this minimizes the fuel corrections of a spacecraft needed to stay in orbit, and this notion is already captured by the diagram (see the end of Appendix B). The cases $n = 1, 2$ apply to e.g. to the (circular) *planar* and *spatial* restricted three-body problem, respectively. The cases $n = 3, 4$ were also studied by Howard–Mackay [11].

Symmetric orbits and monodromy matrices. After fixing the energy and projecting out the direction of the flow of a Hamiltonian system with $n + 1$ degrees of freedom, the linearization of the dynamics at any given point of an orbit gives a $2n \times 2n$-symplectic matrix (the *reduced monodromy matrix*). A different choice of point gives another matrix is the same symplectic conjugacy class. Therefore an orbit induces a point in the space $\mathrm{Sp}(2n)//\mathrm{Sp}(2n)$, where $\mathrm{Sp}(2n)$ acts by conjugation on itself. This is the "middle" space in the GIT sequence, designed to study *arbitrary* closed orbits.

Now, the first space in the GIT sequence is designed to study *symmetric* orbits, see Figure 1. This is the case for several families which appear in numerical work, and therefore entail practical interest. A symmetric orbit can be thought of as a chord or open string, i.e. an orbit segment with its endpoints lying in the fixed-point set of the antisymplectic involution. If we pick one of the symmetric points of the symmetric orbit (an endpoint of the chord), the corresponding monodromy matrix at this point satisfies special symmetries. Concretely, such a matrix has the form

$$(2) \qquad M = M_{A,B,C} = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix} \in M_{2n \times 2n}(\mathbb{R}),$$

where A, B, C are $n \times n$-matrices that satisfy the equations

$$(3) \quad B = B^T, \quad C = C^T, \quad AB = BA^T, \quad A^T C = CA, \quad A^2 - BC = I,$$

which ensure that M is symplectic. We will denote the space of such symplectic matrices by $\mathrm{Sp}^{\mathcal{I}}(2n)$ (where the $\mathcal{I}$ stands for *involution*). A general algebraic fact due to Wonenburger [22] is that *every* symplectic matrix is symplectically conjugated to a matrix in $\mathrm{Sp}^{\mathcal{I}}(2n)$[4]. We will therefore call a matrix of the form $M_{A,B,C}$, a *Wonenburger* matrix.

The expression for $M_{A,B,C}$ implies the choice of a basis for the tangent space to the fixed-point locus along the symmetric point. A different choice of basis amounts to acting with an invertible matrix $R \in GL_n(\mathbb{R})$, via

$$(4) \qquad R_*(A, B, C) = \left(RAR^{-1}, RBR^T, (R^T)^{-1}CR^{-1} \right),$$

i.e. $M_{A,B,C}$ is replaced by $M_{R_*(A,B,C)}$. This leaves invariant the conjugacy class of A, and it is also easy to check that $M_{A,B,C}$ and $M_{R_*(A,B,C)}$ are symplectically conjugated. The first space in the GIT sequence is then the GIT quotient $\mathrm{Sp}^{\mathcal{I}}(2n)//GL_n(\mathbb{R})$ corresponding to this action.

B-**signature.** We will explain how the B-sign is defined in the hyperbolic case, for $n = 2$, which is one of the novelties of this paper. Suppose that $M_{A,B,C} \in \mathrm{Sp}^{\mathcal{I}}(4)$ and A has two distinct real eigenvalues. If μ be one of the eigenvalues of A, its eigenspace E_μ is then one-dimensional. Since $AB = BA$, B leaves E_μ invariant. Then there exists a real number b_μ such that for any

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[4]This algebraic fact has a geometric realization in the case of a symmetric orbit, where every symplectic matrix of any point in the orbit is symplectically conjugated to the ones corresponding to the symmetric points.
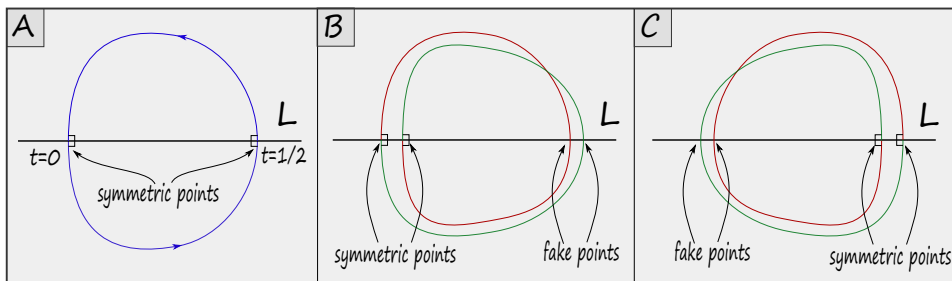
Figure 1: (A) A simple symmetric orbit x; we indicate the intersection with $L =$ Fix(ρ) with right angles. The fixed point locus of the involution ρ is a Lagrangian submanifold, and the intersection points of the orbit with this locus are the *symmetric* points. (B) A period doubling bifurcation, only intersecting L near $x(0)$. (C) The alternative version of (B), intersecting L only near $x(1/2)$. The B-signature jumps at the symmetric point where the new doubled orbit is *not* symmetric (a "fake" intersection point). Invariance of the Euler characteristic of the local Lagrangian Floer homology at x implies that only *one* of (B) or (C) can happen; see [7].

eigenvector v to the eigenvalue μ of A we have

$$Bv = b_\mu v.$$

Definition 1.1. *The eigenvalue μ of A is called B-**positive** if b_μ is positive and B-**negative** if b_μ is negative.*

Moreover, this definition is independent of the eigenbasis. In the presence of a period doubling bifurcation of symmetric orbit, the B-signature jumps either at $t = 0$ or at $t = 1/2$ (as in Figure 1). The orbit arising from a period doubling bifurcations will be symmetric near the point where the B-sign did *not* jump (this is illustrated in [7] with numerical examples).

The GIT sequence. The GIT sequence is then

$$(5) \qquad \mathrm{Sp}^{\mathcal{I}}(2n)//\mathrm{GL}_n(\mathbb{R}) \to \mathrm{Sp}(2n)//\mathrm{Sp}(2n) \to \mathrm{M}_{n \times n}(\mathbb{R})//\mathrm{GL}_n(\mathbb{R})$$

given by

$$[M_{A,B,C}] \mapsto [[M_{A,B,C}]] \mapsto [A].$$

Here, brackets denote the corresponding equivalence class. Wonenburger's aforementioned result implies that the first map is surjective, and the second one is easily seen to be so. By mapping the equivalence class of a matrix

$A \in M_{n \times n}(\mathbb{R})$ to the coefficients of its characteristic polynomial, we get the identification $M_{n \times n}(\mathbb{R})//GL_n(\mathbb{R}) \cong \mathbb{R}^n$. We shall review this nice fact in Appendix A.

Orbit cylinders and bifurcations in the GIT sequence. The GIT quotients in the GIT sequence are *labeled n-dimensional branched manifolds* (LBMs), in such a way that the maps in the GIT sequence preserve this structure. Namely, the branches of these LBMs are equipped with positive/negative labels, keeping track of the B-signs of the corresponding matrices, and the maps collapse branches together (see Figure 2 and 5[5]). Crucially, this data stays invariant in the presence of an orbit cylinder. Namely, an orbit gives a point in the base of the GIT sequence, and hence a family of orbits gives a path. This path corresponds to a bifurcation if it crosses suitable "walls" in the base of the GIT sequence. For instance, if the path crosses the line with slope 1 in Figure 4, it means that there is a bifurcation. If it crosses the line with slope -1, the *double-cover* of the orbit bifurcates. What is more, a bifurcation of the k-fold cover is captured by a line of slope $\cos(2\pi k/l)$, for some l, in the same figure. This can all be implemented easily and visually (we will provide simple and explicit formulas).

The topological obstructions. The topology of the spaces $Sp^{\mathcal{I}}(2n)//GL_n(\mathbb{R})$ and $Sp(2n)//Sp(2n)$, together with the labels, provide precisely the obstruction to the existence of orbit cylinders. Namely, if two orbits correspond to points lying in different component of the diagram in Figure 3, then one should expect a bifurcation on a family that joins them (if any). Moreover, this is refined with the B-signs. Namely, even if two orbits lie in the same component of the base diagram in $\mathbb{R}^2$, but have *different B-signature*, they also cannot be joined by a regular family (see again Figure 5).

Example. If $n = 2$, and if the orbit is symmetric with monodromy matrix $M_{A,B,C}$, and it is *not* non-real[6], we assign to the orbit the tuple (p, ϵ), where the B-signature data is $\epsilon = (\epsilon_1, \epsilon_2)$ (corresponding to its pair of two eigenvalues, with $\epsilon_i \in \{\pm\}$), and $p = (\mathrm{tr}(A), \det(A)) \in \mathbb{R}^2$. This is how the orbit is meant to be stored in a "data base". Note that the data to be stored is cheap and resource-efficient, thus addressing Guiding Question 2.

Krein theory, B-signature, and stability. Krein theory, roughly speaking, is a refinement of the spectrum of a given symplectic matrix, equipping

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[5]The only branch with no well-defined signature is the *non-real* one $\mathcal{N}$, see footnote below.

[6]We say that a 4×4 symplectic matrix is non-real if it has as eigenvalues $\lambda, \lambda^{-1}, \bar{\lambda}.\bar{\lambda}^{-1}$ with $\lambda \notin S^1 \cup \mathbb{R}$.

the eigenvalues with suitable signs. This data completely characterizes the strong stability of the matrix, as proved independently by Krein and Moser; we shall review this in Appendix B (Theorem B.3).

We will also show in Appendix B that in the elliptic case the B-sign and the Krein sign of a symmetric orbit agree. This has the advantage of incorporating Krein theory in a much simpler and efficient way, for the purposes of practical implementations, as is rather straightforward to check whether a matrix is B-positive/negative (provided it is presented in its Wonenburger form $M_{A,B,C}$). Also, this notion is motivated from the theory of symmetric spaces, i.e. $\mathrm{Sp}^{\mathcal{I}}(2n)$ can be identified with the space of linear antisymplectic involutions, a symmetric space where the product of two such involutions is given by conjugating one with the other.

The information carried by the topology of the LBM $\mathrm{Sp}(2n)//\mathrm{Sp}(2n)$ coincides precisely with the information carried by Krein theory; its labels only apply for the elliptic case. On the other hand, the information carried by the topology of the LBM $\mathrm{Sp}^{\mathcal{I}}(2n)//\mathrm{GL}_n(\mathbb{R})$ is given by the more refined B-signature, as it allows to distinguish more symmetric orbits via the associated labels. Topologically, this means that $\mathrm{Sp}^{\mathcal{I}}(2n)//\mathrm{GL}_n(\mathbb{R})$ has more branches than $\mathrm{Sp}(2n)//\mathrm{Sp}(2n)$, some of which get collapsed under the natural map in the GIT sequence; see Figure 5.

Organization of the paper. In Section 2 we give the general dynamical setup with focus on symmetric orbits, and explicitly describe the spatial circular restricted three-body problem. In Section 3: we explain how the linearization at a symmetric point is a Wonenburger matrix; we prove the basic algebraic relation between the characteristic polynomial of a Wonenburger matrix and its first block (Lemma 3.2); we study the GIT sequence in more detail, and completely determine the cases $n = 1, 2$; and finally topologically relate the cases $n = 1$ with $n = 2$, i.e. we show that the $n = 2$ case is a product of two copies of the $n = 1$ case away from the non-real region, up to a suitable identification by $\mathbb{Z}_2$ over a parabola in $\mathbb{R}^2$. In Section 4, we explain how the GIT sequence encodes every type of bifurcation in the form of a pencil of lines, for $n = 2$. In Section 5, we give a preferred representative of each regular and singular component of the base of the GIT sequence for $n = 2$. In Appendix A, we give a proof that $\mathrm{M}_{n \times n}(\mathbb{R})//\mathrm{GL}_n(\mathbb{R}) \cong \mathbb{R}^{2n}$. In Appendix B, we review Krein theory, its relation to stability of orbits, and show that the Krein-signature coincides with the B-signature for symmetric elliptic orbits (Lemma 3.4).

2. Geometric and dynamical setup

We now describe the general setup, which motivates the linear algebra of the sections to come. We assume that (M, ω) is a symplectic manifold and $H \colon M \to \mathbb{R}$ a smooth function. The Hamiltonian vector field X_H of H is defined by the requirement that

$$dH = \omega(\cdot, X_H).$$

Abbreviate by $S^1 = \mathbb{R}/\mathbb{Z}$ the circle. A periodic orbit $x \in C^\infty(S^1, M)$ is a solution of the ODE

$$\partial_t x(t) = \tau X_H(x(t)), \quad t \in S^1,$$

where τ is a positive real number referred to as the period of the periodic orbit. If ϕ_H^t denotes the flow of the Hamiltonian vector field of H, we can characterize the periodic orbit equivalently by

$$x(t) = \phi_H^{\tau t}(x(0)).$$

Abbreviating $x_0 = x(0)$ the differential

$$d\phi^\tau(x_0) \colon T_{x_0} M \to T_{x_0} M$$

is a linear symplectic map of the symplectic vector space $(T_{x_0} M, \omega_{x_0})$ called the *monodromy*.

We assume now that the periodic orbit x is nonconstant which is equivalent to the requirement that $X_H(x(t))$ is never zero or in other words $x(t)$ is no critical point of H for every $t \in S^1$. Since H is autonomous, i.e.,

does not depend on time, it follows that

$$d\phi_H^\tau(x_0)X_H(x_0) = X_H(x_0),$$

i.e., $X_H(x_0)$ is an eigenvector to the eigenvalue 1 of the monodromy. Moreover, by preservation of energy H is constant along the periodic orbit and therefore the monodromy maps the codimension one subspace $\ker dH(x_0)$ of the tangent space $T_{x_0}M$ into itself. Therefore the monodromy induces a linear map on the quotient space

$$\overline{d\phi_H^\tau(x_0)}\colon \ker dH(x_0)/\langle X_H(x_0)\rangle \to \ker dH(x_0)/\langle X_H(x_0)\rangle$$

which we refer to as the *reduced monodromy*. The kernel of the restriction of the symplectic form to the tangent space $\ker dH(x_0)$ of the regular level set $\Sigma := H^{-1}(c)$ containing x_0 is precisely spanned by the Hamiltonian vector field. Therefore the reduced monodromy is itself symplectic with respect to the induced symplectic form on the quotient space. Hence if the dimension of the symplectic manifold is $2n$ we can associate to the periodic orbit x an element in $\mathrm{Sp}(2n-2)//\mathrm{Sp}(2n-2)$, namely the equivalence class of its reduced monodromy. It is interesting to remark that this class does not depend on the starting point of the periodic orbit. In fact, if we translate our periodic orbit in time

$$r_*x(t) = x(t+r), \quad t \in S^1$$

for $r \in S^1$, we obtain different (parametrized) periodic orbits r_*x whose reduced monodromy gives rise to the same element in $\mathrm{Sp}(2n-2)//\mathrm{Sp}(2n-2)$, since the reduced monodromies at different starting points of the periodic orbit are symplectically conjugated to each other via the differential of the flow of the Hamiltonian vector field.

We now consider a *real symplectic manifold* (M, ω, ρ). This is a symplectic manifold (M, ω) together with an antisymplectic involution ρ, i.e., a diffeomorphism of M satisfying

$$\rho^2 = \mathrm{id}_M, \quad \rho^*\omega = -\omega.$$

We assume that H is invariant under ρ, i.e.,

$$H \circ \rho = H.$$

This implies that the Hamiltonian vector field is antiinvariant, meaning that

$$\rho^* X_H = -X_H.$$

In particular, the flow is conjugated to its inverse by the antisymplectic involution, in formulas

(6) $$\rho \phi_H^t \rho = \phi_H^{-t}.$$

A periodic orbit x of H is called *symmetric* if it satisfies

$$x(t) = \rho(x(-t)), \quad t \in S^1.$$

In particular, we have for a symmetric periodic orbit that

$$x(0), \; x\left(\tfrac{1}{2}\right) \in \text{Fix}(\rho).$$

The fixed point set of an antisymplectic involution $\text{Fix}(\rho)$ is a Lagrangian submanifold of M, and in particular, if we look just at half the symmetric periodic orbit we get a chord $\text{Fix}(\rho)$ to itself. Hence we can think of a symmetric periodic orbit in two ways, either as a closed string or as an open string from the Lagrangian $\text{Fix}(\rho)$ to itself.

The differential of the antisymplectic involution

$$d\rho(x_0) \colon T_{x_0} M \to T_{x_0} M$$

gives rise to a linear antisymplectic involution on the symplectic vector space $(T_{x_0} M, \omega_{x_0})$ which induces a linear antisymplectic involution on the quotient space

$$\overline{d\rho(x_0)} \colon \ker dH(x_0)/\langle X_H(x_0) \rangle \to \ker dH(x_0)/\langle X_H(x_0) \rangle.$$

Differentiating (6) we get

$$\overline{d\rho(x_0)} \circ \overline{d\phi_H^\tau(x_0)} \circ \overline{d\rho(x_0)} = \left(\overline{d\phi_H^\tau(x_0)}\right)^{-1},$$

i.e., equation (7) for

$$\mathcal{I} = \overline{d\rho(x_0)}, \quad M = \overline{d\phi_H^\tau(x_0)}.$$

Hence to a symmetric periodic orbit the reduced monodromy associates an element in the quotient space $\text{Sp}^\mathcal{I}(2n-2)//\text{GL}_{2n-2}(\mathbb{R})$, which we describe

in detail in the next section. The reason why we have to mod out the general linear group lies in the ambiguity of choosing a basis on the linear Lagrangian subspace $\text{Fix}(\mathcal{I})$ in $\ker dH(x_0)/\langle X_H(x_0)\rangle$. To obtain this map we have to choose the starting point of the periodic orbit on the Lagrangian $\text{Fix}(\rho)$.

Example. The motivating example is the well-known circular restricted three-body problem, concerning the motion of a massless particle in $\mathbb{R}^3$ under the influence of two heavy primaries with mass μ and $1 - \mu$, where $\mu \in (0, 1)$ is a mass parameter. By a time-dependent rotation, we can fix these primaries at $\vec{m} = (\mu - 1, 0, 0)$, which we will call Moon, and $\vec{e} = (\mu, 0, 0)$, which we will call Earth. The corresponding Hamiltonian, defined on $(T^*\mathbb{R}^3 \setminus \{\vec{m}, \vec{e}\}, d\vec{p} \wedge d\vec{q})$, is

$$H(\vec{q}, \vec{p}) = \frac{1}{2}\|\vec{p}\|^2 - \frac{\mu}{\|\vec{q} - \vec{m}\|} - \frac{1 - \mu}{\|\vec{q} - \vec{e}\|} + p_1 q_2 - p_2 q_1.$$

The planar case of this problem is obtained by setting $q_3 = p_3 = 0$.

The Hamiltonian is preserved by the anti-symplectic involutions

$$\rho_1 : (q_1, q_2, q_3, p_1, p_2, p_3) \mapsto (q_1, -q_2, -q_3, -p_1, p_2, p_3)$$

$$\rho_2 : (q_1, q_2, q_3, p_1, p_2, p_3) \mapsto (q_1, -q_2, q_3, -p_1, p_2, -p_3).$$

In the most symmetric case $\mu = 1/2$, there is a further anti-symplectic involution ρ_3 preserving H with a similar expression.

3. The symplectic group, symmetries, and GIT quotients

Although we later restrict to dimension four we start our discussion for the general case. Our starting point is a fascinating theorem due to Wonenburger [22] which tells us that every element $M \in \text{Sp}(2n)$ can be written as the product of two linear antisymplectic involutions

$$M = \mathcal{I}_1 \mathcal{I}_2.$$

Since $\mathcal{I}_1$ and $\mathcal{I}_2$ are involutions, it follows that

$$M^{-1} = \mathcal{I}_2 \mathcal{I}_1 = \mathcal{I}_1 M \mathcal{I}_1,$$

i.e. M is conjugated to its inverse via an antisymplectic involution. All linear antisymplectic involutions are conjugated to each other, and in particular

to the standard antisymplectic involution

$$\mathcal{I} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

where I is the $n \times n$ identity matrix. Hence there exists $G \in \mathrm{Sp}(2n)$ such that

$$G^{-1}\mathcal{I}_1 G = \mathcal{I}.$$

With this notion we rewrite

$$M^{-1} = G\mathcal{I}G^{-1}MG\mathcal{I}G^{-1}$$

and therefore one has

$$G^{-1}M^{-1}G = \mathcal{I}G^{-1}MG\mathcal{I}.$$

Hence after conjugation we can assume that

(7)
$$M^{-1} = \mathcal{I}M\mathcal{I}.$$

If we write M as a block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for $n \times n$-matrices $A, B, C,$ and D, it follows since M is symplectic that these matrices satisfy

$$AB^T = BA^T, \quad CD^T = DC^T, \quad AD^T - BC^T = I.$$

Moreover, the inverse of M is given by

$$M^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}.$$

It follows from (7) that

$$\begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}.$$

Therefore $D = A^T$, and so M can be written as

(8)
$$M = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}$$

where A, B, C satisfy the equations

$$(9) \quad B = B^T, \quad C = C^T, \quad AB = BA^T, \quad A^T C = CA, \quad A^2 - BC = I.$$

We summarize this discussion in the following proposition.

Proposition 3.1. *Every symplectic matrix $M \in \mathrm{Sp}(2n)$ is symplectically conjugated to a matrix of the form (8) with A, B, C satisfying (9).*

As the above discussion shows, symplectic matrices of the from (8) with A, B, C satisfying (9) are precisely the symplectic matrices M satisfying (7). We abbreviate this submanifold of $\mathrm{Sp}(2n)$ by

$$\mathrm{Sp}^{\mathcal{I}}(2n) = \left\{ M \in \mathrm{Sp}(2n) : M^{-1} = \mathcal{I} M \mathcal{I} \right\}.$$

This space itself has an interesting structure. Note that it follows from (7) that

$$(\mathcal{I} M)^2 = \mathcal{I} M \mathcal{I} M = M^{-1} M = I$$

so that $\mathcal{I} M$ is itself an antisymplectic involution. Therefore $M = \mathcal{I}(\mathcal{I} M)$ is precisely a Wonenburger decomposition of M into the product of two antisymplectic involutions. On the other hand if $\mathcal{J}$ is an antisymplectic involution, then $\mathcal{I}\mathcal{J}$ lies in $\mathrm{Sp}^{\mathcal{I}}(2n)$, since

$$(\mathcal{I}\mathcal{J})^{-1} = \mathcal{J}\mathcal{I} = \mathcal{I}(\mathcal{I}\mathcal{J})\mathcal{I}.$$

Consequently one can identify the space $\mathrm{Sp}^{\mathcal{I}}(2n)$ via the map $M \mapsto \mathcal{I} M$ with the space of linear antisymplectic involutions, which itself corresponds to the tangent bundle of the Lagrangian Grassmannian [2].

In the following we will freely identify the space $\mathrm{Sp}^{\mathcal{I}}(2n)$ as the moduli space

$$\left\{ (A, B, C) \in \mathrm{M}_{n \times n}(\mathbb{R}) : (A, B, C) \text{ solution of } (9) \right\} = \mathrm{Sp}^{\mathcal{I}}(2n),$$

via the map

$$(A, B, C) \mapsto M_{A,B,C} := \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}.$$

Since every symplectic matrix is symplectically conjugated to one in $\mathrm{Sp}^{\mathcal{I}}(2n)$ it suffices to restrict one's attention to this submanifold in order to understand the GIT quotient $\mathrm{Sp}(2n)//\mathrm{Sp}(2n)$. Although it might be in general

cumbersome to find for a general matrix $M \in \mathrm{Sp}(2n)$ a matrix conjugated to M in $\mathrm{Sp}^{\mathcal{I}}(2n)$, we explained in the previous section that for reduced monodromy matrices of symmetric periodic orbits in the restricted three-body problem there is a simple geometric way to do that. The message of this paper which we want to convey is that given a symmetric form (8) of a symplectic matrix in its similarity class, i.e., the equivalence class of a symplectic matrix up to conjugation by symplectic matrices, it is advantageous to keep it for further exploration of the similarity class. For instance for a symmetric periodic orbit we recommend to look at the reduced monodromy at a point in the fixed point set of the antisymplectic involution and not an arbitrary point on the periodic orbit. A first hint of this philosophy is provided by the following lemma, which tells us that the characteristic polynomial $p_{A,B,C}$ of the symplectic matrix $M_{A,B,C}$ is completely determined by the matrix A and does not depend on the matrices B and C.

Lemma 3.2. *The characteristic polynomial of a matrix $M_{A,B,C} \in \mathrm{Sp}^{\mathcal{I}}(2n)$ is given by*

$$p_{A,B,C}(t) = t^n p_{-2A}\left(-t - \tfrac{1}{t} \right),$$

where p_{-2A} is the characteristic polynomial of the matrix $-2A$.

Proof. For a matrix $M_{A,B,C} \in \mathrm{Sp}^{\mathcal{I}}(2n)$ its characteristic polynomial is given by

$$p_{A,B,C}(t) = \det \begin{pmatrix} A - tI & B \\ C & A^T - tI \end{pmatrix}.$$

Note that in view of (9) we have

$$\begin{pmatrix} A - tI & B \\ C & A^T - tI \end{pmatrix} \begin{pmatrix} A - tI & 0 \\ -C & I \end{pmatrix} = \begin{pmatrix} (A - tI)^2 - BC & B \\ CA - A^T C & A^T - tI \end{pmatrix}$$
$$= \begin{pmatrix} t^2 I - 2tA + I & B \\ 0 & A^T - tI \end{pmatrix}.$$

Taking determinants on both sides we obtain

$$p_{A,B,C}(t) \cdot \det(A - tI) = \det(t^2 I - 2tA + I) \cdot \det(A^T - tI)$$
$$= \det(t^2 I - 2tA + I) \cdot \det(A - tI)$$

and hence

$$p_{A,B,C}(t) = \det(t^2 I - 2tA + I) = t^n p_{-2A}\left(-t - \tfrac{1}{t} \right)$$

where p_{-2A} is the characteristic polynomial of the matrix $-2A$. This proves Lemma 3.2. □

Note that in the case $2n = 4$ we have

$$p_{-2A}(s) = s^2 + 2(\mathrm{tr}A)s + 4\det A$$

and therefore

$$p_{A,B,C}(t) = t^4 - 2(\mathrm{tr}A)t^3 + 2(1 + 2\det A)t^2 - 2(\mathrm{tr}A)t + 1.$$

The group $\mathrm{GL}_n(\mathbb{R})$ acts on $\mathrm{Sp}^{\mathcal{I}}(2n)$ by

(10) $$R_*(A, B, C) = \left(RAR^{-1}, RBR^T, (R^T)^{-1}CR^{-1}\right),$$

where $R \in \mathrm{GL}_n(\mathbb{R})$ and $(A, B, C) \in \mathrm{Sp}^{\mathcal{I}}(2n)$. As explained in the Introduction, this action comes from the ambiguity in choosing a basis for the tangent space to the Lagrangian fixed-point locus, at an endpoint of a chord. Note that A transforms as a linear map, whereas B, C transform as bilinear forms. This corresponds to the conjugation action by a linear symplectomorphism:

$$M_{R_*(A,B,C)} = \begin{pmatrix} R & 0 \\ 0 & (R^T)^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ C & A^T \end{pmatrix} \begin{pmatrix} R^{-1} & 0 \\ 0 & R^T \end{pmatrix}.$$

We therefore obtain a sequence of maps between GIT quotients

(11) $$\mathrm{Sp}^{\mathcal{I}}(2n)//\mathrm{GL}_n(\mathbb{R}) \to \mathrm{Sp}(2n)//\mathrm{Sp}(2n) \to \mathrm{M}_{n \times n}(\mathbb{R})//\mathrm{GL}_n(\mathbb{R})$$

given by

$$[M_{A,B,C}] \mapsto [[M_{A,B,C}]] \mapsto [A],$$

as explained in the Introduction.

Warm-up: two dimensional case. Let us first describe the sequence (11) in the simplest possible case, i.e. $n = 1$. This case has also been studied in [23, Appendix B], where it plays an important role when trying to define a real version of Embedded Contact Homology (ECH). The identification

$$\mathrm{M}_{1 \times 1}(\mathbb{R})//\mathrm{GL}_1(\mathbb{R}) \cong \mathbb{R}$$

is tautological, and the relevant maps are $[M_{A,B,C}] \mapsto [[M_{A,B,C}]] \mapsto [A] = A = \mathrm{tr}(M_{A,B,C})/2$. The action of $GL_1(\mathbb{R}) = \mathbb{R}^+$ on $\mathrm{Sp}^{\mathcal{I}}(2)$ is simply
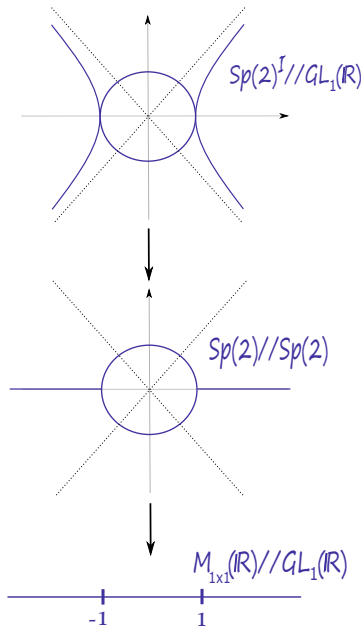
Figure 2: The GIT sequence in the case $n = 1$.

$$\epsilon \cdot \begin{pmatrix} A & B \\ C & A \end{pmatrix} = \begin{pmatrix} A & \epsilon^2 B \\ \frac{1}{\epsilon^2} C & A \end{pmatrix},$$

where $A^2 - BC = 1$, $\epsilon > 0$. We have $\mathrm{Sp}(2) = SL(2, \mathbb{R})$, and a matrix $A \in \mathrm{Sp}(2)$ is either hyperbolic (i.e. $|\mathrm{tr}(A)| > 2$, in which case it has two real eigenvalues $r, 1/r$ with $|r| > 1$), elliptic (i.e. $|\mathrm{tr}(A)| < 2$, in which case it has two conjugate complex eigenvalues in the unit circle), or parabollic (i.e. $|\mathrm{tr}(A)| = 2$, in which case it has eigenvalue ± 1 with algebraic multiplicity two). From the discussion in [8, Section 10.5], we gather that $\mathrm{Sp}(2)//\mathrm{Sp}(2)$ admits a homeomorphism

$$\mathrm{Sp}(2)//\mathrm{Sp}(2) = \{z \in \mathbb{C} : |z| = 1\} \cup \{r \in \mathbb{R} : |r| \geq 1\} \subset \mathbb{C},$$

via the identification

$$s(e^{i\theta}) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \; s(r) = \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix}.$$

The hyperbolic locus consists of closed orbits and corresponds to $\{|r| > 1\}$; the elliptic locus also consists of closed orbits, and corresponds to $\{|z| =$

$1\}\backslash\{\pm 1\}$; and the parabollic locus is $\{\pm 1\}$, where $\{+1\}$ corresponds to the three different Jordan forms with eigenvalue 1 of algebraic multiplicity two, and, similarly $\{-1\}$ corresponds to the three Jordan forms with eigenvalue -1 of algebraic multiplicity two.

Similarly, the GIT quotient $\mathrm{Sp}^{\mathcal{I}}(2)//\mathrm{GL}_1(\mathbb{R})$ admits an identification [23, Appendix B]

$$\mathrm{Sp}^{\mathcal{I}}(2)//\mathrm{GL}_1(\mathbb{R}) = \{z \in \mathbb{C} : |z| = 1\} \cup \{(\pm\cosh(u), \sinh(u)) : u \in \mathbb{R}\} \subset \mathbb{C},$$

via

$$t(e^{i\theta}) = s(e^{i\theta}) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \quad t(u) = \begin{pmatrix} \pm\cosh(u) & \sinh(u) \\ \sinh(u) & \pm\cosh(u) \end{pmatrix}.$$

The matrix $t(u)$ has eigenvalues $\pm e^u, \pm e^{-u}$. Moreover, the matrices $t(u)$ and $t(-u)$ are both symplectically conjugate to $\mathrm{diag}(\pm e^u, \pm e^{-u})$, hence to each other, and therefore define the same element in $\mathrm{Sp}(2)//\mathrm{Sp}(2)$. After these identifications, the GIT sequence becomes

$$e^{i\theta} \mapsto e^{i\theta} \mapsto \cos(\theta),$$

$$(\pm\cosh(u), \sinh(u)) \mapsto r = \pm e^{|u|} \mapsto r = \pm e^{|u|}.$$

This sequence is topologically depicted in Figure 2; note that it consists of branched maps between 1-dimensional branched manifolds, with branching locus $\{\pm 1\}$. The covering degree of the first map is two over the hyperbolic locus, and one everywhere else. For the second map, it is two over the elliptic locus, and one elsewhere.

Four dimensional case. We now describe the sequence (11) in the four dimensional case, where $n = 2$. In this case the identification

$$\mathrm{M}_{2\times 2}(\mathbb{R})//\mathrm{GL}_2(\mathbb{R}) \cong \mathbb{R}^2$$

is obtained via the map

$$\mathrm{M}_{2\times 2}(\mathbb{R})//\mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^2, \quad [A] \mapsto (\mathrm{tr}(A), \det(A)).$$

The spaces $\mathrm{Sp}^{\mathcal{I}}(4)//\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{Sp}(4)//\mathrm{Sp}(4)$ are not manifolds but have some branch points. The branch points lie over three curves in $\mathrm{M}_{2\times 2}(\mathbb{R})//\mathrm{GL}_2(\mathbb{R}) = \mathbb{R}^2$ which we describe next. We abbreviate coordinates

on $\mathbb{R}^2$ by (τ, δ), where τ stands for trace and δ for determinant. The first branch locus is the graph of the parabola $\delta = \frac{1}{4}\tau^2$

$$\Gamma_d = \left\{ \left(\tau, \tfrac{1}{4}\tau^2\right) : \tau \in \mathbb{R} \right\}$$

at which the characteristic polynomial of A has a double root. The two other branch loci are the straight lines

$$\Gamma_1 = \left\{ (\tau, \tau - 1) : \tau \in \mathbb{R} \right\}, \quad \Gamma_{-1} = \left\{ (\tau, -\tau - 1) : \tau \in \mathbb{R} \right\}$$

at which the characteristic polynomial has a root at 1 respectively -1, i.e., the matrix A has 1 respectively -1 as an eigenvalue. We shall refer to the preimage of $\Gamma_1 \cup \Gamma_{-1}$ under the map $\mathrm{Sp}(4)//\mathrm{Sp}(4) \to \mathbb{R}^2$ as the *bifurcation locus* of $\mathrm{Sp}(4)//\mathrm{Sp}(4)$. In the same way we refer to the preimage of $\Gamma_1 \cup \Gamma_{-1}$ under the map $\mathrm{Sp}^{\mathcal{I}}(4)//\mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^2$ as the bifurcation locus of $\mathrm{Sp}^{\mathcal{I}}(4)//\mathrm{GL}_2(\mathbb{R})$. Note that the map $\mathrm{Sp}^{\mathcal{I}}(4)//\mathrm{GL}_2(\mathbb{R}) \to \mathrm{Sp}(4)//\mathrm{Sp}(4)$ maps the latter to the former.

The branch locus Γ_d touches the branching loci Γ_1 and Γ_{-1} in the points

$$\Gamma_d \cap \Gamma_1 = \left\{(2, 1)\right\}, \quad \Gamma_d \cap \Gamma_{-1} = \left\{(-2, 1)\right\}$$

at which the characteristic polynomial has a double root at 1 respectively -1. Finally the two branch loci Γ_1 and Γ_{-1} intersect in the point

$$\Gamma_1 \cap \Gamma_{-1} = \left\{(0, -1)\right\}$$

at which the characteristic polynomial has a root at 1 and -1, i.e., the matrix A has eigenvalues 1 and -1. We abbreviate by

$$\Gamma = \Gamma_d \cup \Gamma_1 \cup \Gamma_{-1}$$

the full branch locus. Its complement decomposes into seven connected components

$$(12) \qquad \mathbb{R}^2 \setminus \Gamma = \mathcal{E}^2 \cup \mathcal{E}\mathcal{H}^+ \cup \mathcal{E}\mathcal{H}^- \cup \mathcal{H}^{++} \cup \mathcal{H}^{-+} \cup \mathcal{H}^{--} \cup \mathcal{N}$$

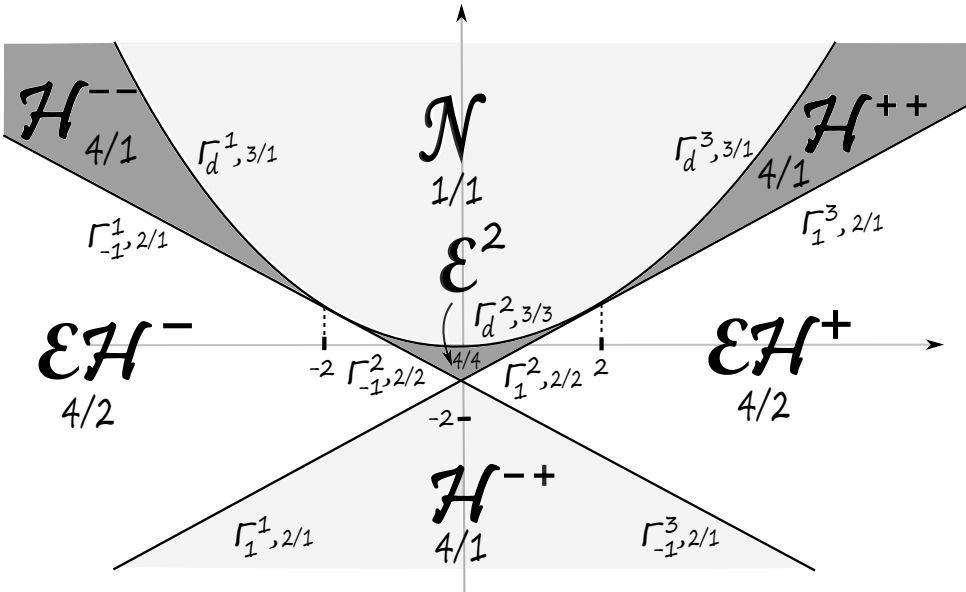which we describe next; see Figure 3.

Figure 3: The branching locus and the seven components of $M_{2\times2}(\mathbb{R})//GL_2(\mathbb{R})$. We denote the number of sheets on the interior of each component as a/b, where a is the number of sheets of $\mathrm{Sp}^{\mathcal{I}}(4)//\mathrm{GL}_2(\mathbb{R})$, and b, that of $\mathrm{Sp}(4)//\mathrm{Sp}(4)$. Similarly, we indicate the number of sheets over the branch locus; cf. Figure 5. This information gives a refinement of the "Broucke stability diagram" [4].

The only bounded component in the decomposition the *doubly elliptic component*

$$\mathcal{E}^2 = \Big\{(\tau,\delta)\in\mathbb{R}^2 : -2 < \tau < 2,\ \max\{-\tau-1,\tau-1\} < \delta < \tfrac{1}{4}\tau^2\Big\}.$$

A matrix A corresponding to this component has two distinct real eigenvalues $-1 < \mu_1 < \mu_2 < 1$, while a matrix $M_{A,B,C}\in\mathrm{Sp}^{\mathcal{I}}(4)$ corresponding to such a matrix A has two pairs of eigenvalues on the unit circle $(e^{i\theta_1},e^{-i\theta_1})$ and $(e^{i\theta_2},e^{-i\theta_2})$ with $0 < \theta_2 < \theta_1 < \pi$ which are related to the eigenvalues of A in view of Lemma 3.2 by

$$e^{i\theta_1} = \mu_1 + i\sqrt{1-\mu_1^2}, \quad e^{i\theta_2} = \mu_2 + i\sqrt{1-\mu_2^2}.$$

The *elliptic/positive hyperbolic component* $\mathcal{EH}^+$ is given by

$$\mathcal{EH}^+ = \Big\{ (\tau, \delta) \in \mathbb{R}^2 : \tau > 0, \ |\delta + 1| < \tau \Big\}.$$

A matrix A for this component has two distinct real eigenvalues $-1 < \mu_1 < 1 < \mu_2$, while a matrix $M_{A,B,C} \in \mathrm{Sp}^{\mathcal{I}}(4)$ corresponding to A has one pair of eigenvalues on the unit circle $(e^{i\theta}, e^{-i\theta})$ for $0 < \theta < \pi$ and a pair of positive real eigenvalues $(\lambda, \frac{1}{\lambda})$ with $\lambda > 1$ such that

$$e^{i\theta} = \mu_1 + i\sqrt{1 - \mu_1^2}, \quad \lambda = \mu_2 + \sqrt{\mu_2^2 - 1}.$$

The *elliptic/negative hyperbolic component* $\mathcal{EH}^-$ is given by

$$\mathcal{EH}^- = \Big\{ (\tau, \delta) \in \mathbb{R}^2 : \tau < 0, \ |\delta + 1| < -\tau \Big\}.$$

A matrix A for this component has two distinct real eigenvalues $\mu_1 < -1 < \mu_2 < 1$, while a matrix $M_{A,B,C} \in \mathrm{Sp}^{\mathcal{I}}(4)$ corresponding to A has one pair of eigenvalues on the unit circle $(e^{i\theta}, e^{-i\theta})$ for $0 < \theta < \pi$ and a pair of negative real eigenvalues $(\lambda, \frac{1}{\lambda})$ with $\lambda < -1$ such that

$$e^{i\theta} = \mu_2 + i\sqrt{1 - \mu_2^2}, \quad \lambda = \mu_1 - \sqrt{\mu_1^2 - 1}.$$

The *negative/positive hyperbolic component* $\mathcal{H}^{-+}$ is given by

$$\mathcal{H}^{-+} = \Big\{ (\tau, \delta) \in \mathbb{R}^2 : \delta < -1, \ |\tau| < -\delta - 1 \Big\}.$$

A matrix A for this component has two distinct real eigenvalues $\mu_1 < -1 < 1 < \mu_2$, while a matrix $M_{A,B,C} \in \mathrm{Sp}^{\mathcal{I}}(4)$ corresponding to A has one pair negative real eigenvalues $(\lambda_1, \frac{1}{\lambda_1})$ with $\lambda_1 < -1$ and one pair of positive real eigenvalues $(\lambda_2, \frac{1}{\lambda_2})$ such that

$$\lambda_1 = \mu_1 - \sqrt{\mu_1^2 - 1}, \quad \lambda_2 = \mu_2 + \sqrt{\mu_2^2 - 1}.$$

The *doubly positive hyperbolic component* $\mathcal{H}^{++}$ is given by

$$\mathcal{H}^{++} = \Big\{ (\tau, \delta) \in \mathbb{R}^2 : \tau > 2, \ \tau - 1 < \delta < \tfrac{1}{4}\tau^2 \Big\}.$$

A matrix A for this component has two distinct real eigenvalues $1 < \mu_1 < \mu_2$, while a matrix $M_{A,B,C} \in \mathrm{Sp}^{\mathcal{I}}(4)$ corresponding to A has two pairs of positive

real eigenvalues $\left(\lambda_1, \frac{1}{\lambda_1}\right)$ and $\left(\lambda_2, \frac{1}{\lambda_2}\right)$ with $1 < \lambda_1 < \lambda_2$ such that

$$\lambda_1 = \mu_1 + \sqrt{\mu_1^2 - 1}, \quad \lambda_2 = \mu_2 + \sqrt{\mu_2^2 - 1}.$$

The *doubly negative hyperbolic component* $\mathcal{H}^{--}$ is given by

$$\mathcal{H}^{--} = \left\{ (\tau, \delta) \in \mathbb{R}^2 : \tau < -2, \; -\tau - 1 < \delta < \tfrac{1}{4}\tau^2 \right\}.$$

A matrix A for this component has two distinct real eigenvalues $\mu_1 < \mu_2 < -1$, while a matrix $M_{A,B,C} \in \mathrm{Sp}^{\mathcal{I}}(4)$ corresponding to A has two pairs of negative real eigenvalues $\left(\lambda_1, \frac{1}{\lambda_1}\right)$ and $\left(\lambda_2, \frac{1}{\lambda_2}\right)$ with $\lambda_1 < \lambda_2 < -1$ such that

$$\lambda_1 = \mu_1 - \sqrt{\mu_1^2 - 1}, \quad \lambda_2 = \mu_2 - \sqrt{\mu_2^2 - 1}.$$

Finally, the *nonreal component* $\mathcal{N}$ is given by the region above the graph of the parabola $\delta = \frac{1}{4}\tau^2$

$$\mathcal{N} = \left\{ (\tau, \delta) \in \mathbb{R}^2 : \delta > \tfrac{1}{4}\tau^2 \right\}.$$

A matrix A for this component has no real eigenvalues but a pair of two nonreal complex conjugated eigenvalues $(\mu, \overline{\mu})$. A matrix $M_{A,B,C} \in \mathrm{Sp}^{\mathcal{I}}(4)$ has than a quadruple of complex eigenvalues $\left(\lambda, \overline{\lambda}, \frac{1}{\lambda}, \frac{1}{\overline{\lambda}}\right)$ which are neither real nor lie on the unit circle where

$$\lambda = \mu + \sqrt{\mu^2 - 1}$$

where in this case $\sqrt{\mu^2 - 1}$ is the choice of a complex root of the complex number $\mu^2 - 1$.

The union of the first six connected components in the decomposition (12) we abbreviate by

$$\mathcal{R} = \mathcal{E}^2 \cup \mathcal{E}\mathcal{H}^+ \cup \mathcal{E}\mathcal{H}^- \cup \mathcal{H}^{++} \cup \mathcal{H}^{-+} \cup \mathcal{H}^{--}$$

and refer to it as the *real part* of $\mathbb{R}^2 \setminus \Gamma$. With this notion we have a decomposition

$$\mathbb{R}^2 \setminus \Gamma = \mathcal{R} \cup \mathcal{N}$$

into real and nonreal part. An equivalence class of matrices $[A]$ in the real part has two distinct real eigenvalues, while in the nonreal part it has two

nonreal eigenvalues which are related to each other by complex conjugation.

If $V \subset \mathbb{R}^2$ is an open subset we denote by $\mathrm{Sp}(4)//\mathrm{Sp}(4)\big|_V$ the subset of $\mathrm{Sp}(4)//\mathrm{Sp}(4)$ consisting of all $[[M_{A,B,C}]] \in \mathrm{Sp}(4)//\mathrm{Sp}(4)$ such that $[A]$ lies in V and similarly $\mathrm{Sp}^{\mathcal{I}}(4)//\mathrm{GL}_2(\mathbb{R})\big|_V$. Outside the branch locus Γ the maps

$$\mathrm{Sp}^{\mathcal{I}}(4)//\mathrm{GL}_2(\mathbb{R})\big|_{\mathbb{R}^2 \setminus \Gamma} \to \mathrm{Sp}(4)//\mathrm{Sp}(4)\big|_{\mathbb{R}^2 \setminus \Gamma} \to \mathbb{R}^2 \setminus \Gamma$$

are smooth coverings where the number of sheets however depends on the connected component in $\mathbb{R}^2 \setminus \Gamma$. On this set it actually does not matter if one is considering the GIT quotient or just the topological quotient.

Suppose now that $M_{A,B,C} \in \mathrm{Sp}^{\mathcal{I}}(4)$ and A has two distinct real eigenvalues, i.e., $[A] \in \mathcal{R}$. Let μ be one of the eigenvalues of A. Its eigenspace $E_\mu \subset \mathbb{R}^2$ is then one-dimensional. In an eigenbasis the matrix A is diagonal (so $A = A^T$) and hence in view of the equation $AB = BA$ it follows that B leaves E_μ invariant. In particular, there exists a real number b_μ such that for any eigenvector v to the eigenvalue μ of A we have

$$Bv = b_\mu v.$$

Definition 3.3. *The eigenvalue μ of A is called B-positive if b_μ is positive and B-negative if b_μ is negative.*

Note that positivity and negativity of b_μ does not depend on the choice of the eigenbasis, since B transforms as a symmetric form and therefore under change of the eigenbasis b_μ gets multiplied by a positive number. We now consider the elliptic case, i.e., $-1 < \mu < 1$. In particular, we must have $[A] \in \mathcal{E}^2 \cup \mathcal{EH}^+ \cup \mathcal{EH}^-$. Then

$$\lambda = \mu + i\sqrt{1 - \mu^2}$$

is an eigenvalue of the symplectic matrix $M_{A,B,C}$.

Lemma 3.4. *In the elliptic case the eigenvalue μ of A is B-positive (negative) if and only if the eigenvalue λ of $M_{A,B,C}$ is Krein-positive (negative).*

We shall prove this lemma in Appendix B. Note that it is crucial to take the positive sign for the imaginary part of the eigenvalue λ. Its complex

conjugate

$$\overline{\lambda} = \mu - i\sqrt{1 - \mu^2}$$

is then another eigenvalue of the symplectic matrix $M_{A,B,C}$ of opposite Krein-type than λ. That means that if μ is B-positive, then $\overline{\lambda}$ is Krein-negative and if μ is B-negative, then $\overline{\lambda}$ is Krein-positive. We further point out that the Krein-type of an eigenvalue of a symplectic matrix only depends on the conjugation class of the symplectic matrix.

In the hyperbolic case where a real eigenvalue μ of A satisfies $|\mu| > 1$ there is no analogon of the Krein-type. On the other hand the B-type of an eigenvalue of a symplectic matrix $M_{A,B,C} \in \mathrm{Sp}^{\mathcal{I}}(4)$ is defined in the hyperbolic case as well and independent of the action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathrm{Sp}^{\mathcal{I}}(4)$. This is the reason why in the hyperbolic case there are more sheets in the covering $\mathrm{Sp}^{\mathcal{I}}(4)//\mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^2$, than in the branched covering $\mathrm{Sp}(4)//\mathrm{Sp}(4) \to \mathbb{R}^2$.

Planar vs. spatial GIT quotients. We now explain the relationship between the GIT quotients for the two dimensional case (or planar case, i.e. $n = 1$), and the four dimensional case (or spatial case, i.e $n = 2$). Intuitively speaking, the spatial case behaves as a product of two planar cases (i.e. when two pairs of eigenvalues are independent of each other), except for the case where a non-real quadruple arises. Topologically, this means that the product of two copies of the GIT spaces for $n = 1$ corresponds to the GIT space for $n = 2$ with the non-real locus removed (although one needs to take a quotient by a $\mathbb{Z}_2$ action which forgets the order of the matrices which lie over the locus Γ_d). The details are as follows.

Note that the product of two copies of the base of GIT sequence for $n = 1$ is a copy of $\mathbb{R}^2$, and an element in this space corresponds to an ordered list of eigenvalues. For $n = 2$, the base is parametrized by the trace and determinant of a 2×2-matrix. Therefore the map to consider is

$$F \colon \mathbb{R}^2 \to \mathbb{R}^2, (a, b) \mapsto (a + b, ab),$$

i.e. the map which associates to an ordered list of the two eigenvalues the trace and determinant of the matrix. In view of the inequality

$$(a + b)^2 \geq (a + b)^2 - (a - b)^2 = 4ab,$$

the image of F precisely misses the non-real component $\mathcal{N}$ of the base of the GIT sequence for $n = 2$. Moreover, on $\mathbb{R}^2$ we have the involution

$$I \colon \mathbb{R}^2 \to \mathbb{R}^2, (a, b) \mapsto (b, a)$$

interchanging the two eigenvalues. The map F is invariant under the involution I, i.e.

$$F \circ I = F.$$

This reflects the fact that for the trace and determinant the order of the eigenvalues does not play a role. The fixed point set of I is the diagonal Δ in $\mathbb{R}^2$ which is mapped under F precisely to the branching locus Γ_d. If $p_0 : \mathrm{Sp}^{\mathcal{I}}(2n)//\mathrm{GL}_n(\mathbb{R}) \to \mathrm{Sp}(2n)//\mathrm{Sp}(2n)$ and $p_1 : \mathrm{Sp}(2n)//\mathrm{Sp}(2n) \to \mathrm{M}_{n \times n}(\mathbb{R})//\mathrm{GL}_n(\mathbb{R})$ are the maps in the GIT sequence, and $p_2 = p_1 \circ p_0$, we conclude that

$$(\mathrm{Sp}^{\mathcal{I}}(4)//\mathrm{GL}_2(\mathbb{R}))\backslash p_2^{-1}(\mathcal{N}) \simeq (\mathrm{Sp}^{\mathcal{I}}(2)//\mathrm{GL}_1(\mathbb{R})) \times (\mathrm{Sp}^{\mathcal{I}}(2)//\mathrm{GL}_1(\mathbb{R}))/\sim,$$

$$(\mathrm{Sp}(4)//\mathrm{Sp}(4))\backslash p_1^{-1}(\mathcal{N}) \simeq (\mathrm{Sp}(2)//\mathrm{Sp}(2)) \times (\mathrm{Sp}(2)//\mathrm{Sp}(2))/\sim,$$

where the quotient identifies a pair $(M_1 = M_{A_1, B_1, C_1}, M_2 = M_{A_2, B_2, C_2})$ with first blocks satisfying $A_1 = A_2 \in \mathbb{R}$, with the pair $I(M_1, M_2) = (M_2, M_1)$.

4. Higher-order bifurcations and pencils of lines

In this section, for $n = 2$, we consider the complete bifurcation locus, i.e. the locus of matrices having one eigenvalue which is a k-th root of unity for some k, and therefore corresponding to a k-fold bifurcation. We will see that this locus projects to the base of the GIT sequence as a line whose slope depends on k. More generally, we will consider the locus of matrices having a fixed eigenvalue. It turns out that the collection of such loci gives a pencil of lines in the plane, tangent to a parabola. This already appears e.g. in [12, p. 49] and [18, Figure 3]. For completeness, the details are as follows.

Let λ be an eigenvalue of $M = M_{A,B,C}$, which is a k-fold root of unity, i.e. it satisfies $\lambda^k = 1$. By Lemma 3.2, we have $a = a(\lambda) = \frac{1}{2}(\lambda + \frac{1}{\lambda})$ is an eigenvalue of A. If we write $\lambda = e^{2\pi i\ell/k}$, we have $a = \cos(2\pi\ell/k)$. Moreover, if b is the remaining eigenvalue of A, its trace is $\tau = a + b$, and its determinant

is $\delta = ab$, and we have the following equation

$$\delta = a\tau - a^2 = \cos(2\pi\ell/k)\tau - \cos(2\pi\ell/k)^2,$$

which is a linear relation between δ and τ, depending on k and ℓ. Note that the resulting lines in $\mathbb{R}^2$, denoted $\Gamma_{k,\ell}$, are all tangent to the parabola Γ_d, as no bifurcation can occur over the nonreal component $\mathcal{N}$. Moreover, two of such lines intersect at a point, consisting of the loci of curves whose two eigenvalues bifurcate with the orders corresponding to the lines. More generally, one can consider the locus Γ_θ consisting of matrices having $e^{2\pi i\theta}$ as an eigenvalue. The same computation as above shows that this is the line

$$\delta = \cos(2\pi\theta)\tau - \cos(2\pi\theta)^2,$$

tangent to Γ_d at $\tau = 2\cos(2\pi\theta)$. The collection $\{\Gamma_\theta : \theta \in [0, 2\pi)\}$ is a pencil of lines, tangent to the parabola Γ_d, but with slopes varying only in $[-1, 1]$. The intersection of Γ_θ with $\Gamma_{\theta'}$ lies in $\mathcal{E}^2$, and is the locus of matrices having eigenvalues $e^{\pm 2\pi i\theta}, e^{\pm 2\pi i\theta'}$. Analogously, we can consider the case where λ is real, i.e. hyperbolic, in which case the resulting line $\Gamma_\lambda = \{\delta = a(\lambda)\tau - a(\lambda)^2\}$, consisting of the locus of matrices with eigenvalue λ, is tangent to Γ_d at $\tau = 2a$; note that $\Gamma_\lambda = \Gamma_{1/\lambda}$. The slope $a(\lambda)$ is greater than 1 (resp. smaller than -1) if and only if λ is positive (resp. negative) hyperbolic. The intersections between any of the lines $\Gamma_\theta, \Gamma_\lambda$ again have the obvious interpretation. See Figure 4.

5. Normal forms

In this section we describe normal forms for matrices $M_{A,B,C} \in \mathrm{Sp}^{\mathcal{I}}(4)$ under the action of $\mathrm{GL}_2(\mathbb{R})$ given by (10). Our normal forms still lie in $\mathrm{Sp}^{\mathcal{I}}(4)$, therefore with the exception of the doubly elliptic case they differ rather from standard normal forms of symplectic matrices as for example explained in [19].

5.1. The regular cases

In this section we assume that our symplectic matrix does not lie over the branch locus. Under this assumption the orbits are closed and there is no difference between the equivalence class in the GIT quotient and the topological quotient.
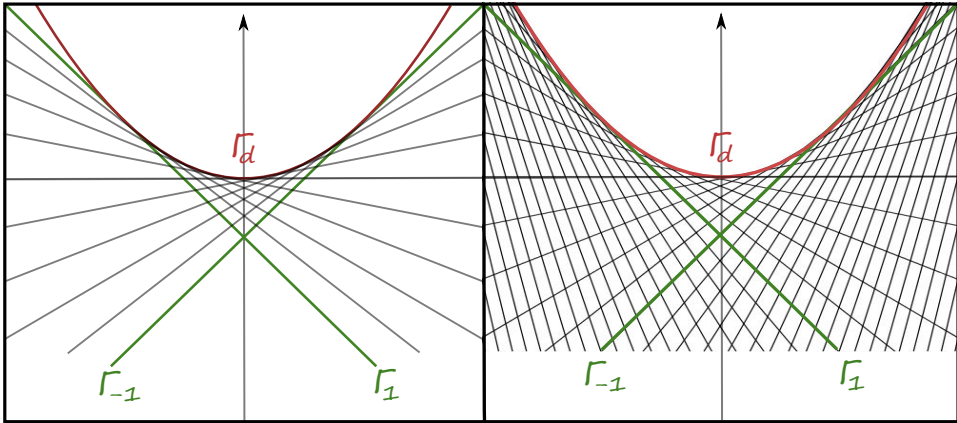
Figure 4: On the left, the *elliptic* pencil of lines $\{\Gamma_\theta, \theta \in [0, 2\pi)\}$. On the right, the complete pencil, also containing the *hyperbolic* pencil $\{\Gamma_\lambda : \lambda \in \mathbb{R} \setminus [-1, 1]\}$.

We first consider the case where A has two different real eigenvalues, i.e., $[A] \in \mathcal{R}$. In this case A is diagonalizable and after acting with $\mathrm{GL}_2(\mathbb{R})$ we can assume without loss of generality that A is actually diagonal

$$A = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$$

while we order the eigenvalues in increasing order $\mu_1 < \mu_2$. In particular, we have that $A = A^T$ and therefore

$$AB = BA, \quad AC = CA,$$

i.e., A commutes with both matrices B and C. Since the two eigenvalues of A are different, this implies that B and C are diagonal as well

$$B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}.$$

From the equation $A^2 - BC = I$ we obtain

$$b_1 c_1 = \mu_1^2 - 1, \quad b_2 c_2 = \mu_2^2 - 1.$$

Since we are considering the regular case the squares of μ_1 and μ_2 are different from one and therefore in particular, b_1, b_2, c_1, c_2 are all nonzero. After

acting by the matrix

$$R = \begin{pmatrix} \left|\frac{b_1}{c_1}\right|^{-1/4} & 0 \\ 0 & \left|\frac{b_2}{c_2}\right|^{-1/4} \end{pmatrix}$$

according to (10) we can achieve in addition that

$$|b_1| = |c_1|, \quad |b_2| = |c_2|$$

so that we have

$$b_1^2 = |\mu_1^2 - 1|, \quad b_2^2 = |\mu_2^2 - 1|.$$

In the following we discuss the six connected components of the real part $\mathcal{R}$ one by one. We start by assuming that $[A]$ lies in the double elliptic component $\mathcal{E}^2$. In this case the two real eigenvalues of A satisfy $-1 < \mu_1 < \mu_2 < 1$. In particular, there exist unique angles $\theta_1, \theta_2 \in (0, \pi)$ such that

$$\mu_1 = \cos\theta_1, \quad \mu_2 = \cos\theta_2.$$

We then have

$$|b_1| = |c_1| = \sqrt{1 - \cos^2\theta_1} = \sin\theta_1, \quad |b_2| = |c_2| = \sqrt{1 - \cos^2\theta_2} = \sin\theta_2$$

and the signs of b_1 and c_1 as well as those of b_2 and c_2 have to be opposite. We therefore get the following four normal forms

$$\begin{pmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & -\sin\theta_2 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}, \begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & -\sin\theta_2 \\ -\sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix},$$

$$\begin{pmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & \sin\theta_2 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}, \begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & \sin\theta_2 \\ -\sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}.$$

We therefore see that the fiber of $\mathrm{Sp}^{\mathcal{I}}(4)//\mathrm{GL}_2(\mathbb{R})$ over $[A]$ consists of four points. Moreover, in view of Lemma 3.4 all these matrices are distinguished symplectically by the Krein-type, namely the B-sign, of their eigenvalues. Therefore the fiber of $\mathrm{Sp}(4)//\mathrm{Sp}(4)$ consists of four points as well.

We next discuss the case that $[A]$ lies in the elliptic/positive hyperbolic component $\mathcal{EH}^+$. In this case the two real eigenvalues of A satisfy

$-1 < \mu_1 < 1 < \mu_2$. Hence there exist unique $\theta_1 \in (0, \pi)$ and $\theta_2 \in (0, \infty)$ such that

$$\mu_1 = \cos \theta_1, \quad \mu_2 = \cosh \theta_2.$$

Hence

$$|b_1| = |c_1| = \sqrt{1 - \cos^2 \theta_1} = \sin \theta_1, \quad |b_2| = |c_2| = \sqrt{\cosh^2 \theta_2 - 1} = \sinh \theta_2.$$

Where the signs of b_1 and c_1 are opposite the signs of b_2 and c_2 agree. We obtain the following four normal forms.

$$\begin{pmatrix} \cos \theta_1 & 0 & -\sin \theta_1 & 0 \\ 0 & \cosh \theta_2 & 0 & -\sinh \theta_2 \\ \sin \theta_1 & 0 & \cos \theta_1 & 0 \\ 0 & -\sinh \theta_2 & 0 & \cosh \theta_2 \end{pmatrix}, \quad \begin{pmatrix} \cos \theta_1 & 0 & -\sin \theta_1 & 0 \\ 0 & \cosh \theta_2 & 0 & \sinh \theta_2 \\ \sin \theta_1 & 0 & \cos \theta_1 & 0 \\ 0 & \sinh \theta_2 & 0 & \cosh \theta_2 \end{pmatrix},$$

$$\begin{pmatrix} \cos \theta_1 & 0 & \sin \theta_1 & 0 \\ 0 & \cosh \theta_2 & 0 & -\sinh \theta_2 \\ -\sin \theta_1 & 0 & \cos \theta_1 & 0 \\ 0 & -\sinh \theta_2 & 0 & \cosh \theta_2 \end{pmatrix}, \quad \begin{pmatrix} \cos \theta_1 & 0 & \sin \theta_1 & 0 \\ 0 & \cosh \theta_2 & 0 & \sinh \theta_2 \\ -\sin \theta_1 & 0 & \cos \theta_1 & 0 \\ 0 & \sinh \theta_2 & 0 & \cosh \theta_2 \end{pmatrix}.$$

The fiber of $\mathrm{Sp}^{\mathcal{I}}(4)//\mathrm{GL}_2(\mathbb{R})$ over $[A]$ consists again of four points. However, the two matrices on the first line are symplectically conjugated via the symplectic matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and the same holds for the two matrices on the second line. On the other hand the matrices on the first line and the ones on the second line are symplectically still distinguished by the Krein-type of the first eigenvalue $\mu_1 = \cos \theta_1$. Hence the fiber of $\mathrm{Sp}(4)//\mathrm{Sp}(4)$ consists over $[A]$ consists of two points.

In the elliptic/negative hyperbolic case the eigenvalues satisfy $\mu_1 < -1 < \mu_2 < 1$ so that we write

$$\mu_1 = -\cosh \theta_1, \quad \mu_2 = \cos \theta_2$$

for $\theta_1 \in (0, \infty)$ and $\theta_2 \in (0, \pi)$. We have the following four normal forms

$$\begin{pmatrix} -\cosh\theta_1 & 0 & -\sinh\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & -\sin\theta_2 \\ -\sinh\theta_1 & 0 & -\cosh\theta_1 & 0 \\ 0 & \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}, \begin{pmatrix} -\cosh\theta_1 & 0 & \sinh\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & -\sin\theta_2 \\ \sinh\theta_1 & 0 & -\cosh\theta_1 & 0 \\ 0 & \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix},$$

$$\begin{pmatrix} -\cosh\theta_1 & 0 & -\sinh\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & \sin\theta_2 \\ -\sinh\theta_1 & 0 & -\cosh\theta_1 & 0 \\ 0 & -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}, \begin{pmatrix} -\cosh\theta_1 & 0 & \sinh\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & \sin\theta_2 \\ \sinh\theta_1 & 0 & -\cosh\theta_1 & 0 \\ 0 & -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}.$$

Again on each line the matrices are symplectically conjugated while on different lines they are symplectically distinguished by the Krein-type of the second eigenvalue. Therefore the fiber of $\mathrm{Sp}^{\mathcal{I}}(4)//\mathrm{GL}_2(\mathbb{R})$ over $[A]$ has again four points while the one of $\mathrm{Sp}(4)//\mathrm{Sp}(4)$ has just two points.

In the doubly positive hyperbolic case $\mathcal{H}^{++}$ we obtain the following four normal forms

$$\begin{pmatrix} \cosh\theta_1 & 0 & -\sinh\theta_1 & 0 \\ 0 & \cosh\theta_2 & 0 & -\sinh\theta_2 \\ -\sinh\theta_1 & 0 & \cosh\theta_1 & 0 \\ 0 & -\sinh\theta_2 & 0 & \cosh\theta_2 \end{pmatrix}, \begin{pmatrix} \cosh\theta_1 & 0 & \sinh\theta_1 & 0 \\ 0 & \cosh\theta_2 & 0 & -\sinh\theta_2 \\ \sinh\theta_1 & 0 & \cosh\theta_1 & 0 \\ 0 & -\sinh\theta_2 & 0 & \cosh\theta_2 \end{pmatrix},$$

$$\begin{pmatrix} \cosh\theta_1 & 0 & -\sinh\theta_1 & 0 \\ 0 & \cosh\theta_2 & 0 & \sinh\theta_2 \\ -\sinh\theta_1 & 0 & \cosh\theta_1 & 0 \\ 0 & \sinh\theta_2 & 0 & \cosh\theta_2 \end{pmatrix}, \begin{pmatrix} \cosh\theta_1 & 0 & \sinh\theta_1 & 0 \\ 0 & \cosh\theta_2 & 0 & \sinh\theta_2 \\ \sinh\theta_1 & 0 & \cosh\theta_1 & 0 \\ 0 & \sinh\theta_2 & 0 & \cosh\theta_2 \end{pmatrix}.$$

which now are all symplectically conjugated. Similarly are the negative/positive hyperbolic case $\mathcal{H}^{-+}$ and the doubly negative hyperbolic case $\mathcal{H}^{--}$. The only difference is that in the negative/positive hyperbolic case $\cosh\theta_1$ has to be replaced by $-\cosh\theta_1$ and in the doubly negative hyperbolic case both $\cosh\theta_1$ and $\cosh\theta_2$ get a minus sign.

A different treatment is needed in the nonreal case, where $[A] \in \mathcal{N}$. In this case A has two nonreal eigenvalues which are complex conjugate of each other

$$\mu = re^{i\theta}, \quad \overline{\mu} = re^{-i\theta}$$

for $r > 0$ and $\theta \in (0, \pi)$. After conjugation we can assume that the matrix A is the composition of a dilation by r and a rotation by θ

$$A = \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix}.$$

The matrix B is symmetric and transforms as a bilinear form. Hence after a further rotation which does not affect the matrix A, since A commutes with every rotation, we can assume that B is diagonal

$$B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}.$$

The equation $AB = BA^T$ implies that $b_2 = -b_1$ and after a further dilation which still does not affect A we can assume that B has the form

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

i.e., B is just an orthogonal reflection at the first axis. The equation $A^2 - BC = I$ implies that

$$C = B(A^2 - I) = \begin{pmatrix} r^2 \cos 2\theta - 1 & -r^2 \sin 2\theta \\ -r^2 \sin 2\theta & -r^2 \cos 2\theta + 1 \end{pmatrix}.$$

Therefore we obtain the unique canonical form

$$\begin{pmatrix} r \cos \theta & -r \sin \theta & 1 & 0 \\ r \sin \theta & r \cos \theta & 0 & -1 \\ r^2 \cos 2\theta - 1 & -r^2 \sin 2\theta & r \cos \theta & r \sin \theta \\ -r^2 \sin 2\theta & -r^2 \cos 2\theta + 1 & -r \sin \theta & r \cos \theta \end{pmatrix}.$$

In particular, since the canonical form is unique we see that both fibers of $\mathrm{Sp}^{\mathcal{I}}(4)//\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{Sp}(4)//\mathrm{Sp}(4)$ over $[A]$ consist of a single point, that means over the nonreal component the coverings are just trivial, i.e., homeomorphisms.

5.2. The branch locus

In this section we discuss normal forms over the branch locus. Over the branch locus not all orbits are closed and there is can be a difference between normal forms for the GIT quotient and the usual quotient. The branch locus itself has three singular points at $(2, 1)$, $(-2, 1)$, and $(0, -1)$. On the complement of the singularities the branch locus consists of nine connected components all homeomorphic to an open interval.

Recall that we abbreviated the first branch locus by Γ_d which is given by

$\Gamma_d = \left\{ \left(\tau, \frac{1}{4}\tau^2 \right) : \tau \in \mathbb{R} \right\}$. It contains the two singularities $(2, 1)$ and $(2, -1)$. Its complement decomposes into three connected components

$$\Gamma_d \setminus \left\{ (2, 1) \cup (-2, 1) \right\} = \Gamma_d^1 \cup \Gamma_d^2 \cup \Gamma_d^3$$

where

$$\Gamma_d^1 = \left\{ \left(\tau, \tfrac{1}{4}\tau^2 \right) : \tau < -2 \right\},$$
$$\Gamma_d^2 = \left\{ \left(\tau, \tfrac{1}{4}\tau^2 \right) : -2 < \tau < 2 \right\},$$
$$\Gamma_d^3 = \left\{ \left(\tau, \tfrac{1}{4}\tau^2 \right) : \tau > 2 \right\}.$$

Similarly we have the decomposition

$$\Gamma_1 \setminus \left\{ (2, 1), (0, -1) \right\} = \Gamma_1^1 \cup \Gamma_1^2 \cup \Gamma_1^3$$

with

$$\Gamma_1^1 = \left\{ (\tau, \tau - 1) : \tau < 0 \right\}$$
$$\Gamma_1^2 = \left\{ (\tau, \tau - 1) : 0 < \tau < 2 \right\}$$
$$\Gamma_1^3 = \left\{ (\tau, \tau - 1) : \tau > 2 \right\}$$

as well as

$$\Gamma_{-1} \setminus \left\{ (-2, 1), (0, -1) \right\} = \Gamma_{-1}^1 \cup \Gamma_{-1}^2 \cup \Gamma_{-1}^3$$

with

$$\Gamma_{-1}^1 = \left\{ (\tau, -\tau - 1) : \tau < -2 \right\}$$
$$\Gamma_{-1}^2 = \left\{ (\tau, -\tau - 1) : -2 < \tau < 0 \right\}$$
$$\Gamma_{-1}^3 = \left\{ (\tau, -\tau - 1) : \tau > 0 \right\}.$$

With these notions the nonsingular part of the branch locus decomposes into connected components as follows

$$\Gamma \setminus \left\{ (2, 1), (-2, 1), (0, -1) \right\}$$
$$= \Gamma_d^1 \cup \Gamma_d^2 \cup \Gamma_d^3 \cup \Gamma_1^1 \cup \Gamma_1^2 \cup \Gamma_1^3 \cup \Gamma_{-1}^1 \cup \Gamma_{-1}^2 \cup \Gamma_{-1}^3.$$

We first discuss the normal forms over the nonsingular part of the branch locus. Hence we assume that $M_{A,B,C} \in \mathrm{Sp}^{\mathcal{I}}(4)$ with $[A] \in \Gamma \setminus$

$\{(2,1),(-2,1),(0,-1)\}$. We first assume that $[A] \in \Gamma_d \setminus \{(2,1),(-2,-1)\}$, i.e., A has one real eigenvalue μ different from ± 1 of algebraic multiplicity two. The geometric multiplicity of the eigenvalue μ is one or two. We first explain that by going over to the GIT quotient we can assume that its geometric multiplicity is two as well. To see that suppose that the geometric multiplicity of μ is one. After acting with $\mathrm{GL}_2(\mathbb{R})$ we can assume that A is a Jordan block

$$A = \begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix}.$$

From the equation $AB = BA^T$ we infer that the symmetric matrix B has the form

$$B = \begin{pmatrix} b_2 & b_1 \\ b_1 & 0 \end{pmatrix}$$

where from the equation $A^T C = CA$ we deduce that C has the form

$$C = \begin{pmatrix} 0 & c_1 \\ c_1 & c_2 \end{pmatrix}.$$

For $\epsilon > 0$ we consider the matrix

$$R_\epsilon = \begin{pmatrix} \epsilon & 0 \\ 0 & \frac{1}{\epsilon} \end{pmatrix}.$$

We have

$$R_\epsilon A R_\epsilon^{-1} = \begin{pmatrix} \mu & \epsilon^2 \\ 0 & \mu \end{pmatrix}$$

$$R_\epsilon B R_\epsilon^T = \begin{pmatrix} \epsilon^2 b_2 & b_1 \\ b_1 & 0 \end{pmatrix}$$

$$(R_\epsilon^T)^{-1} C R_\epsilon^{-1} = \begin{pmatrix} 0 & c_1 \\ c_1 & \epsilon^2 c_2 \end{pmatrix}$$

and therefore

$$\lim_{\epsilon \to 0} R_\epsilon A R_\epsilon^{-1} = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$$

$$\lim_{\epsilon \to 0} R_\epsilon B R_\epsilon^T = \begin{pmatrix} 0 & b_1 \\ b_1 & 0 \end{pmatrix}$$

$$\lim_{\epsilon \to 0} (R_\epsilon^T)^{-1} C R_\epsilon^{-1} = \begin{pmatrix} 0 & c_1 \\ c_1 & 0 \end{pmatrix}.$$

This shows that by going over to the GIT-quotient we can assume without loss of generality that the geometric multiplicity of the eigenvalue μ is two as well. In this case the matrix A is diagonal

$$A = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}.$$

In fact A is just a scalar multiple of the identity matrix and therefore a fixed point of the action of $\mathrm{GL}_2(\mathbb{R})$ by conjugation. The matrix B transforms as a bilinear form and since any bilinear form can be diagonalized we can assume after conjugation that B is also diagonal. Since $\mu \neq \pm 1$ the formula $A^2 - BC = I$ implies that B has to be nonsingular and therefore C has to be diagonal as well. The discussion of normal forms is now analogous to the real regular case. Different from the real regular case there are only three normal forms and not four. These are in one to one correspondence with the signature of B. If the signature of B is one in the real regular case there were still two different normal forms which were distinguished on which eigenspace of A the matrix B was positive and on which it was negative. Since the two eigenvalues now coincide this distinction is not possible anymore.

If $[A]$ lies in Γ_d^2, i.e., in the intersection of the closure of the double elliptic component and the nonreal component, we have the following three normal forms for the eigenvalue $\mu = \cos\theta$ with $\theta \in (0, \pi)$

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix}, \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix},$$

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

Hence over Γ_d^2 the branch cover $\mathrm{Sp}^{\mathcal{I}}(4)//\mathrm{GL}_2(\mathbb{R})$ has three branches. Moreover, the three normal forms are distinguished symplectically by their Krein-type so that $\mathrm{Sp}(4)//\mathrm{Sp}(4)$ over Γ_d^2 has three branches as well.

If $[A]$ lies in Γ_d^3, i.e., the intersection of the closures of the doubly positive hyperbolic component and the nonreal component we have the

following three normal forms for $\mu = \cosh\theta$ with $\theta \in (0, \infty)$

$$
\begin{pmatrix}
\cosh\theta & 0 & -\sinh\theta & 0 \\
0 & \cosh\theta & 0 & -\sinh\theta \\
-\sinh\theta & 0 & \cosh\theta & 0 \\
0 & -\sinh\theta & 0 & \cosh\theta
\end{pmatrix},
$$

$$
\begin{pmatrix}
\cosh\theta & 0 & \sinh\theta & 0 \\
0 & \cosh\theta & 0 & -\sinh\theta \\
\sinh\theta & 0 & \cosh\theta & 0 \\
0 & -\sinh\theta & 0 & \cosh\theta
\end{pmatrix},
$$

$$
\begin{pmatrix}
\cosh\theta & 0 & \sinh\theta & 0 \\
0 & \cosh\theta & 0 & \sinh\theta \\
\sinh\theta & 0 & \cosh\theta_1 & 0 \\
0 & \sinh\theta & 0 & \cosh\theta
\end{pmatrix}.
$$

In particular, over Γ_d^3 the branched cover $\mathrm{Sp}^{\mathcal{I}}(4)//\mathrm{GL}_2(\mathbb{R})$ has three branches. On the other hand the above three normal forms are symplectically conjugated and therefore $\mathrm{Sp}(4)//\mathrm{Sp}(4)$ has just one branch. A similar picture happens over Γ_d^1, i.e., the intersection of the closures of the negative hyperbolic component and the nonreal component. There one just needs to replace $\cosh\theta$ by $-\cosh\theta$ in the previous discussion.

We next assume that $[A] \in \Gamma_1 \setminus \{(2,1), (0,-1)\}$, i.e., A has one eigenvalue equal to 1 and another real eigenvalue $\mu \neq \pm 1$. In particular, A is diagonalizable and after conjugation we can assume that A has the form

$$
A = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}.
$$

In particular, we have $A = A^T$ and therefore B and C commute with A. This implies that they are diagonal as well

$$
B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}.
$$

In view of $A^2 - BC = I$ we obtain

$$
b_1 c_1 = 0, \qquad b_2 c_2 = \mu^2 - 1.
$$

The first equation implies that b_1 or c_1 is zero. We next explain that by going over to the GIT quotient we can assume that both b_1 and c_1 are zero.

To see that we first assume that $c_1 = 0$ but $b_1 \neq 0$ so that we have

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}.$$

For $\epsilon > 0$ we consider the family of matrices

$$R_\epsilon = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}.$$

Acting with R_ϵ on the triple of matrices above we obtain

$$R_\epsilon A R_\epsilon^{-1} = A$$
$$R_\epsilon B R_\epsilon^T = \begin{pmatrix} \epsilon^2 b_1 & 0 \\ 0 & b_2 \end{pmatrix}$$
$$(R_\epsilon^T)^{-1} C R_\epsilon^{-1} = C$$

and therefore

$$\lim_{\epsilon \to 0} R_\epsilon A R_\epsilon^{-1} = A$$
$$\lim_{\epsilon \to 0} R_\epsilon B R_\epsilon^T = \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}$$
$$\lim_{\epsilon \to 0} (R_\epsilon^T)^{-1} C R_\epsilon^{-1} = C.$$

This shows that we can assume that $b_1 = 0$. Similarly we see that we can assume as well that $c_1 = 0$, by using instead in the above argument the family of matrices

$$R_\epsilon = \begin{pmatrix} \frac{1}{\epsilon} & 0 \\ 0 & 1 \end{pmatrix}.$$

After using the action of $\mathrm{GL}_2(\mathbb{R})$ once more we can additionally assume that

$$|b_2| = |c_2| = \sqrt{|\mu^2 - 1|}.$$

If $[A] \in \Gamma_1^2$, i.e., if $[A]$ lies in the intersection of the closures of the doubly elliptic component and the elliptic/positive hyperbolic component, we have

the following two normal forms for $\mu = \cos\theta$ with $\theta \in (0, \pi)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

In particular, over Γ_1^2 the branched cover $\mathrm{Sp}^{\mathcal{I}}(4)//\mathrm{GL}_2(\mathbb{R})$ has two branches. This two branches are distinguished by their Krein-type and therefore not symplectically conjugated, so that $\mathrm{Sp}(4)//\mathrm{Sp}(4)$ has two branches as well.

If $[A] \in \Gamma_1^3$, i.e., if $[A]$ lies in the intersection of the closures of the doubly positive hyperbolic component and the elliptic/positive hyperbolic component, the two normal forms are for $\mu = \cosh\theta$ with $\theta \in (0, \infty)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\theta & 0 & -\sinh\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sinh\theta & 0 & \cosh\theta \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\theta & 0 & \sinh\theta \\ 0 & 0 & 1 & 0 \\ 0 & \sinh\theta & 0 & \cosh\theta \end{pmatrix}.$$

Again $\mathrm{Sp}^{\mathcal{I}}(4)/\mathrm{GL}_2(\mathbb{R})$ has two branches over Γ_1^3 but now the two branches are symplectically conjugated and there is just one branch of $\mathrm{Sp}(4)//\mathrm{Sp}(4)$. The case where $[A] \in \Gamma_1^1$, i.e., where $[A]$ lies in the intersection of the closures of the elliptic/negative hyperbolic component and the negative/positive hyperbolic component is similar, one just needs to replace $\cosh\theta$ by $-\cosh\theta$.

Finally the discussion where $[A] \in \Gamma_{-1} \setminus \{(-2, 1), (0, -1)\}$ is analogous to the one where $[A] \in \Gamma_1 \setminus \{(2, 1), (0, -1)\}$. The only difference is that one has to replace 1 by -1. This finishes the description of the branched covers over the nonsingular part of the branch locus.

It remains to consider the singular part of the branch locus namely the three points $(2, 1)$, $(-2, 1)$, and $(0, -1)$. We start with the point $(2, 1)$. In this case A has only 1 as eigenvalue with algebraic multiplicity two. If the geometric multiplicity is two as well, then A is the identity matrix. If the geometric multiplicity is one, then A is conjugated to the 2×2-Jordan block with 1 on the diagonal. We explain that in either case the 4×4-identity matrix I_4 lies in the closure of the $\mathrm{GL}_2(\mathbb{R})$-orbit of $M_{A,B,C}$. For that

purpose we first consider the case where A is the Jordan block

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

From the equations $AB = BA^T$ and $A^T C = CA$ we infer that the symmetric matrices B and C simplify to

$$B = \begin{pmatrix} b_2 & b_1 \\ b_1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & c_1 \\ c_1 & c_2 \end{pmatrix}.$$

From the equation $A^2 - BC = I$ we obtain

$$b_1 c_1 = 0, \qquad c_1 b_2 + c_2 b_1 = 2.$$

From the first equation we see that b_1 or c_1 vanishes. We first consider the case where $b_1 = 0$, so that our triple of matrices reads

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} b_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & c_1 \\ c_1 & c_2 \end{pmatrix}.$$

For $\epsilon > 0$ we consider the family of matrices

$$R_\epsilon = \begin{pmatrix} \epsilon & 0 \\ 0 & \frac{1}{\epsilon^2} \end{pmatrix}.$$

Acting with this family of matrices on the triple (A, B, C) we obtain

$$R_\epsilon A R_\epsilon^{-1} = \begin{pmatrix} 1 & \epsilon^3 \\ 0 & 1 \end{pmatrix}$$

$$R_\epsilon B R_\epsilon^T = \begin{pmatrix} \epsilon^2 b_2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$(R_\epsilon^T)^{-1} C R_\epsilon^{-1} = \begin{pmatrix} 0 & \epsilon c_1 \\ \epsilon c_1 & \epsilon^4 c_2 \end{pmatrix}$$

and therefore

$$\lim_{\epsilon \to 0} R_\epsilon A R_\epsilon^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\lim_{\epsilon \to 0} R_\epsilon B R_\epsilon^T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\lim_{\epsilon \to 0} (R_\epsilon^T)^{-1} C R_\epsilon^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We see that in this case I_4 lies in the closure of the $\mathrm{GL}_2(\mathbb{R})$-orbit of $M_{A,B,C}$. The case where $c_1 = 0$ is analogous. One just needs to use the family of matrices

$$R_\epsilon = \begin{pmatrix} \epsilon^2 & 0 \\ 0 & \frac{1}{\epsilon} \end{pmatrix}.$$

It remains to discuss the case where A is the identity matrix

$$A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The equation $A^2 - BC = I$ implies in this case that

$$BC = 0.$$

We first consider the case $B = 0$. In this case the family of matrices $\frac{1}{\epsilon} I$ acts on the triple (A, B, C) by

$$\left(\tfrac{1}{\epsilon} I \right)_* (I, 0, C) = (I, 0, \epsilon^2 C)$$

with limit

$$\lim_{\epsilon \to 0} \left(\tfrac{1}{\epsilon} I \right)_* (I, 0, C) = (I, 0, 0)$$

which shows that in this case I_4 lies in the closure of the $\mathrm{GL}_2(\mathbb{R})$-orbit $M_{A,B,C}$. A similar argument holds in the case where $C = 0$ by using the family of matrices ϵI instead. It remains to discuss the case where neither B nor C are the zero-matrix. Since B is symmetric and transforms as a bilinear form we can diagonalize B so that we can assume without loss of generality that

$$B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$$

with $b_1 \neq 0$. Since C is symmetric as well we obtain from the equation that C has to be of the form

$$C = \begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}.$$

Since C is not the zero-matrix we must have $c_2 \neq 0$ which implies in view of $BC = 0$ that $b_2 = 0$ so that our triple becomes

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}.$$

We consider the family of matrices

$$R_\epsilon = \begin{pmatrix} \epsilon & 0 \\ 0 & \frac{1}{\epsilon} \end{pmatrix}.$$

We act with this family of matrices on the triple (A, B, C) to obtain

$$R_\epsilon A R_\epsilon^{-1} = A$$
$$R_\epsilon B R_\epsilon^T = \begin{pmatrix} \epsilon^2 b_1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$(R_\epsilon^T)^{-1} C R_\epsilon^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^2 c_2 \end{pmatrix}$$

and hence

$$\lim_{\epsilon \to 0} R_\epsilon A R_\epsilon^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\lim_{\epsilon \to 0} R_\epsilon B R_\epsilon^T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\lim_{\epsilon \to 0} (R_\epsilon^T)^{-1} C R_\epsilon^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This proves that as well in this last case I_4 lies in the closure of the $GL_2(\mathbb{R})$-orbit of $M_{A,B,C}$ and therefore over $(2, 1)$ the branched covers $Sp^{\mathcal{I}}(4)//GL_2(\mathbb{R})$ as well as $Sp(4)//Sp(4)$ consists of a single point namely the equivalence class of the matrix I_4.

The story over $(-2, 1)$ is completely analogous. Over this point the two branched covers just consist of the equivalence class of the matrix $-I_4$.

We are left with a last point, namely $(0, -1)$. If a matrix A lies over this point it has 1 and -1 as eigenvalues. In particular, it is diagonalizable and hence after taking advantage of the $GL_2(\mathbb{R})$-action we can assume that A has the form

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In particular, we have $A = A^T$ implying that the symmetric matrices B and C commute with A. In particular, they have to keep invariant the eigenspaces
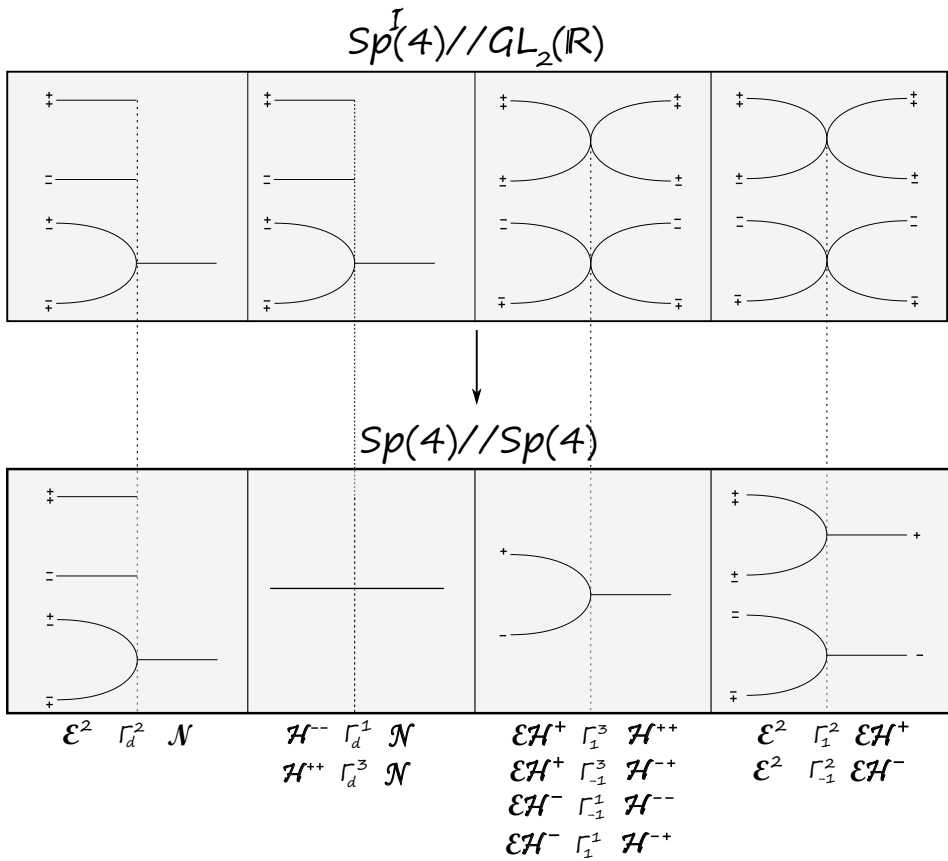
$$Sp^{\mathcal{I}}(4)//GL_2(\mathbb{R})$$

$$Sp(4)//Sp(4)$$

$\mathcal{E}^2$ Γ_d^2 $\mathcal{N}$	$\mathcal{H}^{--}$ Γ_d^1 $\mathcal{N}$	$\mathcal{EH}^+$ Γ_1^3 $\mathcal{H}^{++}$	$\mathcal{E}^2$ Γ_1^2 $\mathcal{EH}^+$
	$\mathcal{H}^{++}$ Γ_d^3 $\mathcal{N}$	$\mathcal{EH}^+$ Γ_{-1}^3 $\mathcal{H}^{-+}$	$\mathcal{E}^2$ Γ_{-1}^2 $\mathcal{EH}^-$
		$\mathcal{EH}^-$ Γ_{-1}^1 $\mathcal{H}^{--}$	
		$\mathcal{EH}^-$ Γ_1^1 $\mathcal{H}^{-+}$	

Figure 5: Different (2-dimensional) branches of $Sp^{\mathcal{I}}(4)//GL_2(\mathbb{R})$ and $Sp(4)//Sp(4)$. The signs on each branch correspond to B-positivity/negativity of the corresponding eigenvalue (a priori there are 4 possibilities). The first vignette shows how they come together when crossing from $\mathcal{E}^2$ to $\mathcal{N}$ along Γ_d^2. On the second, when crossing from $\mathcal{H}^{--}$ to $\mathcal{N}$ along Γ_d^1. The picture is the same for $\mathcal{H}^{++}$ to $\mathcal{N}$ along Γ_d^3, and so on. All branches come together to a single point along each of the three singular points $(2,1), (0,-1), (-2,1)$. The natural map $Sp^{\mathcal{I}}(4)//GL_2(\mathbb{R}) \to Sp(4)//Sp(4)$ in the GIT sequence collapses branches together, as shown in the picture. For example, above, B-positivity/negativity over the hyperbolic eigenspace of matrices of type $\mathcal{EH}^+$ is not invariant under symplectic conjugation, and hence the corresponding branches come together in $Sp(4)//Sp(4)$; we may still distinguish the elliptic eigenvalues via Krein theory, however.

of A and are therefore themselves diagonal matrices

$$B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}.$$

The formula $A^2 - BC = I$ implies that

$$b_1 c_1 = 0, \qquad b_2 c_2 = 0,$$

i.e., b_1 or c_1 has to vanish or b_2 or c_2 has to vanish. By going over to the GIT-quotient we can arrange that all of them vanish. For example if b_1 does not vanish, then c_1 has to vanish and we use the sequence of matrices

$$R_\epsilon = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$$

to arrange that in the limit as ϵ goes to zero b_1 vanishes as well. Similarly, if c_1 does not vanish, then b_1 has to vanish and in this case we use the sequence of matrices

$$R_\epsilon = \begin{pmatrix} \frac{1}{\epsilon} & 0 \\ 0 & 1 \end{pmatrix}$$

and similarly for b_2 and c_2. Hence we can assume that $B = C = 0$ and therefore we have the unique normal form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

In particular, the two branched covers consists as well of a single point over $(0, -1)$. This finishes the description of normal forms in all cases.

Appendix A. The GIT quotient

In this appendix, we review the definition of the GIT quotient, and some nice general facts about the GIT quotient corresponding to the conjugation action of the general linear group on the space of matrices.

Assume that G is a Lie group which acts on a manifold X. The space of orbits X/G is in general not a Hausdorff space. In order to remedy this

situation in some cases we consider the *orbit closure relation* on X, namely

$$x \sim y \quad :\Longleftrightarrow \quad \overline{Gx} \cap \overline{Gy} \neq \emptyset,$$

i.e., the closures of the orbits of x and of y intersect. This relation is obviously reflexive and symmetric. If it is in addition transitive, it is an equivalence relation and in this case we define the GIT quotient as

$$X//G := X/\sim.$$

The following example plays an important role in our story.

Proposition A.1. *The group* $\mathrm{GL}_n(\mathbb{R})$ *acts on the space of real* $n \times n$-*matrices* $\mathrm{M}_{n \times n}(\mathbb{R})$ *by conjugation*

$$R_* A = RAR^{-1}, \quad R \in \mathrm{GL}_n(\mathbb{R}), \ A \in \mathrm{M}_{n \times n}(\mathbb{R}).$$

For this action the orbit closure relation is transitive and the GIT quotient $\mathrm{M}_{n \times n}(\mathbb{R})//\mathrm{GL}_n(\mathbb{R})$ *is homeomorphic to* $\mathbb{R}^n$. *If for* $A \in \mathrm{M}_{n \times n}(\mathbb{R})$ *the characteristic polynomial is written as*

$$p_A(t) = (-1)^n t^n + c_{n-1} t^{n-1} + \ldots + c_1 t + c_0,$$

then a homeomorphism is given by mapping the equivalence class of a matrix to the coefficients of its characteristic polynomial

$$\mathrm{M}_{n \times n}(\mathbb{R})//\mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^n, \quad [A] \mapsto (c_{n-1}, \ldots, c_1, c_0).$$

Proof. Suppose that $A \in \mathrm{M}_{n \times n}(\mathbb{R})$. Then A is conjugated by a matrix $R \in \mathrm{GL}_n(\mathbb{R})$ to a matrix in real Jordan form. In case all eigenvalues of A are real, the real Jordan form does not differ from the complex Jordan form. In case A has as well nonreal eigenvalues its real Jordan form does not agree with its complex Jordan form and needs some explanation. We first note that nonreal eigenvalues of A appear in pairs $(\lambda, \overline{\lambda})$, since the matrix A is real. In order to avoid double counting we restrict our attention to nonreal eigenvalues in the upper halfplane $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$.

If $\lambda \in \mathbb{R}$ and $m \in \mathbb{N}$ we define the Jordan block $J_{\lambda,m}$ as in the complex case as the $m \times m$-matrix whose diagonal entries are all λ, and whose superdiagonal entries are all 1, while all other entries are zero like. For

example,

$$J_{\lambda,3} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

If $\lambda = a + bi \in \mathbb{H}$ we first define the 2×2-matrix

$$C_\lambda = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Then different from the complex case we define for $m \in \mathbb{N}$ the Jordan block $J_{\lambda,m}$ as the $2m \times 2m$-matrix consisting of $m \times m$ blocks of 2×2-matrices whose diagonal entries are all C_λ, and whose superdiagonal entries are all I_2, i.e., the 2×2-identity matrix, while all other entries are zero. For instance we have

$$J_{\lambda,3} = \begin{pmatrix} C_\lambda & I_2 & 0 \\ 0 & C_\lambda & I_2 \\ 0 & 0 & C_\lambda \end{pmatrix}.$$

A real Jordan matrix is then as usual a block matrix having Jordan blocks on the diagonal and zeros elsewhere.

Each Jordan block is similar to one where the superdiagonal is scaled by $\epsilon > 0$. We illustrate this paradigmatically for the Jordan block $J_{\lambda,3}$ for real λ, for which we have

$$\begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} \frac{1}{\epsilon^2} & 0 & 0 \\ 0 & \frac{1}{\epsilon} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & \epsilon & 0 \\ 0 & \lambda & \epsilon \\ 0 & 0 & \lambda \end{pmatrix}.$$

In particular, in the orbit closure of the matrix A there lies a block diagonal matrix, namely

$$D_A = \bigoplus_{\lambda \in \mathfrak{S}(A) \cap \mathbb{R}} \lambda^{\oplus \mathfrak{a}(\lambda)} \oplus \bigoplus_{\lambda \in \mathfrak{S}(A) \cap \mathbb{H}} C_\lambda^{\oplus \mathfrak{a}(\lambda)},$$

where $\mathfrak{S}(A)$ denotes the spectrum of A, i.e., the set of all eigenvalues of A, and $\mathfrak{a}(\lambda)$ denotes the algebraic multiplicity of an eigenvalue. Strictly speaking we need to specify an order on the eigenvalues, in order to make D_A well-defined as a matrix. We choose the lexicographic order with real value as the first letter and imaginary value as the second one. Since however different ordering conventions lead to conjugated matrices the reader is free to choose his own preferred convention which will not influence the following

arguments. We note that D_A is uniquely determined by the characteristic polynomial of A. In particular, we see that if two matrices A and B have the same characteristic polynomial $p_A = p_B$ we have

$$D_A \in \overline{\mathrm{GL}_n(\mathbb{R})A} \cap \overline{\mathrm{GL}_n(\mathbb{R})B}$$

implying that $A \sim B$. On the other hand, suppose that $A \sim B$. This means that there exists a matrix

$$D \in \overline{\mathrm{GL}_n(\mathbb{R})A} \cap \overline{\mathrm{GL}_n(\mathbb{R})B}.$$

In particular, there exist a sequence $R_\nu \in \mathrm{GL}_n(\mathbb{R})$ such that

$$\lim_{\nu \to \infty} R_\nu A R_\nu^{-1} = D$$

as well as a sequence $S_\nu \in \mathrm{GL}_n(\mathbb{R})$ with

$$\lim_{\nu \to \infty} S_\nu B S_\nu^{-1} = D.$$

Since conjugated matrices have the same characteristic polynomial we have $p_A = p_{R_\nu A R_\nu^{-1}}$ for every ν and therefore

$$p_D = \lim_{\nu \to \infty} p_{R_\nu A R_\nu^{-1}} = p_A$$

and similarly

$$p_D = p_B$$

implying that

$$p_A = p_B.$$

We have proved that

$$A \sim B \iff p_A = p_B.$$

Since every polynomial with leading coefficient $(-1)^n$ arises as the characteristic polynomial of a matrix $A \in \mathrm{M}_{n \times n}(\mathbb{R})$, the proposition follows. In particular, each equivalence class has a canonical representative consisting of a real Jordan matrix with no superdiagonal blocks. $\square$

Appendix B. Krein theory and strong stability for Hamiltonian systems

In this appendix, we review some basic facts about Krein theory, its relationship with stability for orbits of Hamiltonian systems, and compare it to our notion of B-positivity in the case of symmetric orbits. We follow the exposition in Ekeland's book [6] (see also Abbondandolo's book [1]).

Consider a linear symplectic ODE

$$\dot{x} = M(t)x,$$

where $M(t) = JA(t) \in \mathfrak{sp}(2n)$, with $A(t)$ symmetric, and T-periodic, i.e. $A(t + T) = A(t)$ for all t, and $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is the standard complex multiplication. The solutions are given by $x(t) = R(t)x(0)$, where $R(t) \in \mathrm{Sp}(2n)$ is symplectic and solves $\dot{R}(t) = M(t)R(t)$, $R(0) = I$.

Definition B.1. **(stability)** *The ODE $\dot{x} = JA(t)x$ is called* stable *if all solutions remain bounded for all $t \in \mathbb{R}$. It is* strongly *stable if there exists $\epsilon > 0$ such that, if $B(t)$ is symmetric and satisfies $\|A(t) - B(t)\| < \epsilon$, then the ODE $\dot{x} = JB(t)x$ is stable. Similarly, a symplectic matrix R is* stable *if all its iterates R^k remain bounded for $k \in \mathbb{Z}$, and it is* strongly *stable if there exists $\epsilon > 0$ such that all symplectic matrices S with $\|R - S\| < \epsilon$ are also stable.*

Appealing to Floquet theory, one can show that the ODE $\dot{x} = JA(t)x$ is (strongly) stable if and only if $R(T)$ is (strongly) stable; see [6, Section 2, Proposition 3]. Moreover, stability is equivalent to $R(T)$ being diagonalizable (i.e. all eigenvalues are semi-simple, meaning that their algebraic and geometric multiplicities agree), with its spectrum lying in the unit circle [6, Section 1, Proposition 1]. Questions about the strong stability of Hamiltonian systems are therefore reduced to questions about the strong stability of symplectic matrices.

Now, recall that the spectrum of a symplectic matrix R satisfies special symmetries. Concretely, its eigenvalues come in families of the form $\{\lambda, \overline{\lambda}, \lambda^{-1}, \overline{\lambda}^{-1}\}$. Therefore, if ± 1 are eigenvalues, then they have even multiplicity. In addition, if all its eigenvalues are simple, different from ± 1, and lie in the unit circle, then they come in pairs $\{\lambda, \overline{\lambda}\}$. In this case, this implies that any other symplectic matrix close to R will also have simple eigenvalues in the unit circle different from ± 1 (otherwise an eigenvalue would have to

bifurcate into two, which is not possible if eigenspaces are 1-dimensional). Therefore in this case, R is strongly stable. The case of eigenvalues with higher multiplicity is the subject of Krein theory, which we now review.

Consider the nondegenerate bilinear form $G(x, y) = \langle -iJx, y \rangle$ on $\mathbb{C}^{2n}$, associated to the Hermitian matrix $-iJ$. Every real symplectic matrix R preserves G. Moreover, if λ, μ are eigenvalues of R which satisfy $\lambda \overline{\mu} \neq 1$, then the corresponding eigenspaces are G-orthogonal, since

$$G(x, y) = G(Rx, Ry) = \lambda \overline{\mu} G(x, y),$$

if x, y are the corresponding eigenvectors. Moreover, if we consider the generalized eigenspaces

$$E_\lambda = \bigcup_{m \geq 1} \ker(R - \lambda I)^m,$$

then it also holds that E_λ, E_μ are G-orthogonal if $\lambda \overline{\mu} \neq 1$ [6, Section 2, Proposition 5]. This, in particular, implies that if $|\lambda| \neq 1$, then E_λ is G-isotropic, i.e. $G|_{E_\lambda} = 0$. If $\sigma(R)$ denotes the spectrum of R, we have a G-orthogonal decomposition

$$\mathbb{C}^{2n} = \bigoplus_{\substack{\lambda \in \sigma(R) \\ |\lambda| \geq 1}} F_\lambda,$$

where $F_\lambda = E_\lambda$ if $|\lambda| = 1$, and $F_\lambda = E_\lambda \oplus E_{\overline{\lambda}^{-1}}$ if $|\lambda| > 1$. Since G is non-degenerate, and the above splitting is G-orthogonal, the restriction $G_\lambda = G|_{F_\lambda}$ is also non-degenerate. Recall that the signature of a non-degenerate bilinear form G is the pair (p, q), where p is the dimension of a maximal subspace where G is positive definite, and q is the dimension of a maximal subspace where G is negative definite. Note that if $|\lambda| \neq 1$, with algebraic multiplicity d, then the $2d$-dimensional space F_λ has E_λ as a d-dimensional isotropic subspace, and hence the signature of G_λ is (d, d). On the other hand, if $|\lambda| = 1$, then the non-degenerate form G_λ can have any signature. This justifies the following:

Definition B.2. *(Krein-positivity/negativity) If λ is an eigenvalue of the symplectic matrix R with $|\lambda| = 1$, then the signature (p, q) of G_λ is called the* Krein-type *or* Krein signature *of λ. If $q = 0$, i.e. G_λ is positive definite, λ is said to be* Krein-positive. *If $p = 0$, i.e. G_λ is negative definite, λ is said to be* Krein-negative. *If λ is either Krein-negative or Krein-positive, we say that it is* Krein-definite. *Otherwise, we say that it is* Krein-indefinite.

If λ is of Krein-type (p, q), then $\overline{\lambda}$ is of Krein-type (q, p) [6, Section 2, Lemma 9]. If λ satisfies $|\lambda| = 1$ and it is not semi-simple, then it is easy to show that it is Krein-indefinite [6, Section 2, Proposition 7]. Moreover, ± 1 are always Krein-indefinite if they are eigenvalues, as they have real eigenvectors x, which are therefore G-isotropic, i.e. $G(x, x) = 0$. The following, originally proved by Krein in [14–17] and independently rediscovered by Moser in [20], gives a characterization of strong stability in terms of Krein theory:

Theorem B.3. *R is strongly stable if and only if it is stable and all its eigenvalues are Krein-definite.*

See [6, Section 2, Theorem 3] for a proof. Note that this generalizes the case where all eigenvalues are simple, different from ± 1 and in the unit circle, as discussed above.

We now prove Lemma 3.4.

Proof of Lemma 3.4. Since the notion of Krein-type is invariant under symplectic conjugation and only involves the eigenspaces it suffices to show that for the matrix

$$M = \left(\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right)$$

the eigenvalue $e^{i\theta}$ is Krein-negative; the positive case is analogous. An eigenvector is given by

$$v = \left(\begin{array}{c} 1 \\ -i \end{array} \right).$$

We have

$$G(v, v) = -2$$

and this shows that $e^{i\theta}$ is Krein-negative, concluding the proof. $\square$

Stability in the GIT sequence. In the setup of the GIT sequence for $n = 2$, the (linearly) stable orbits are the ones whose matrices lie over the closure of the $\mathcal{E}^2$ component of the GIT quotient $M_{2\times2}(\mathbb{R})//GL_2(\mathbb{R})$. The strongly (linearly) stable orbits correspond to those lying over $\mathcal{E}^2$, i.e. they cannot be perturbed to lie away from $\mathcal{E}^2$. However, there are also matrices which are strongly stable and lying over Γ_d^2, corresponding to the boundary of the $++$ and $--$ branches (see Figure 5). We remark that the relationship between linear stability and the diagram of Figure 5 was already observed in [4].

References

[1] A. Abbondandolo, Morse theory for Hamiltonian systems. Chapman & Hall/CRC Research Notes in Mathematics, 425. Chapman & Hall/CRC, Boca Raton, FL, 2001. xii+189 pp. ISBN `1-58488-202-6`.

[2] P. Albers, U. Frauenfelder, *The space of linear anti-symplectic involutions is a homogeneous space*, Arch. Math. (Basel) **99** (2012), no. 6.

[3] R. Abraham, J. Marsden, *Foundations of Mechanics*, 2nd ed. Addison-Wesley, New York (1978).

[4] R. Broucke, Stability of periodic orbits in the elliptic, restricted three-body problem. AIAA J. 7,1003 (1969).

[5] A. D. Bruno, The restricted 3-body problem: plane periodic orbits. Translated from the Russian by Bálint Érdi. With a preface by Victor G. Szebehely. De Gruyter Expositions in Mathematics, 17. Walter de Gruyter & Co., Berlin, 1994. xiv+362 pp. ISBN `3-11-013703-8`.

[6] I. Ekeland, Convexity methods in Hamiltonian mechanics. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 19. Springer-Verlag, Berlin, 1990. x+247 pp. ISBN `3-540-50613-6`.

[7] U. Frauenfelder, D. Koh, and A. Moreno, *Symplectic methods in the numerical search of orbits in real-life planetary systems*, Preprint `arXiv:2206.00627`.

[8] U. Frauenfelder, O. van Koert, The Restricted Three-Body Problem and Holomorphic Curves, Pathways in Mathematics. Birkhäuser/Springer, Cham, 2018. xi+374 pp. ISBN `978-3-319-72277-1; 978-3-319-72278-8`.

[9] U. Frauenfelder, O. van Koert, The Hörmander index of symmetric periodic orbits. Geom. Dedicata **168** (2014), 197–205.

[10] M. Hénon, Generating Families in The Restricted Three-Body Problem. II. Quantitative Study of Bifurcations. Lecture Notes in Physics. New Series m: Monographs, 65. Springer-Verlag, Berlin, 2001. xii+301 pp. ISBN `3-540-41733-8`.

[11] J. E. Howard, R. S. MacKay, *Linear stability of symplectic maps*, J. Math. Phys. **28** (1987), 1038–1051.

[12] C. E. Thomas, K. C. Howell, *Bifurcations from families of periodic solutions in the circular restricted problem with application to trajectory design*, Master's Thesis, Purdue University, West Lafayette, IN, USA, 1999; p. 186.

[13] V. Kalantonis, *Numerical investigation for periodic orbits in the Hill three-body problem*, Universe. **2020** 6. 72.

[14] M. Krein, *Generalization of certain investigations of A.M. Liapunov on linear differential equations with periodic coefficients*, Doklady Akad. Nauk USSR **73** (1950) 445–448.

[15] M. Krein, *On the application of an algebraic proposition in the theory of monodromy matrices*, Uspekhi Math. Nauk **6** (1951), 171–177.

[16] M. Krein, *On the theory of entire matrix-functions of exponential type*, Ukrainian Math. Journal **3** (1951), 164–173.

[17] M. Krein, *On some maximum and minimum problems for characteristic numbers and Liapunov stability zones*, Prikl. Math. Mekh. **15** (1951), 323–348.

[18] Q. Li, Y. Tao, F. Jiang, *Orbital stability and invariant manifolds on distant retrograde orbits around Ganymede and nearby higher-period orbits* Aerospace. **2022** 9 (8): 454.

[19] Y. Long, Index Theory for Symplectic Paths with Applications, Progress in Mathematics **207**, Birkhäuser Verlag, Basel (2002).

[20] J. Moser, *New aspects in the theory of stability of Hamiltonian systems*, Comm. Pure Appl. Math. **11** (1958), 81–114.

[21] V. Szebehely, *Theory of orbits: The restricted problem of three bodies*, American Journal of Physics **36**, 375 (1968).

[22] M. Wonenburger, *Transformations which are products of two involutions*, J. Math. Mech. **16** (1996), 327–338.

[23] Beijia Zhou, *Iteration formulae for brake orbit and index inequalities for real pseudoholomorphic curves*, Preprint `arXiv:2011.07958`, 2020.

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