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T-duality for transitive Courant algebroids

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We develop a theory of T-duality for transitive Courant algebroids. We show that T-duality between transitive Courant algebroids $E \to M$ and $\tilde{E} \to \tilde{M}$ induces a map between the spaces of sections of the corresponding canonical weighted spinor bundles $\mathbb{S}_E$ and $\mathbb{S}_{\tilde{E}}$ intertwining the canonical Dirac generating operators. The map is shown to induce an isomorphism between the spaces of invariant spinors, compatible with an isomorphism between the spaces of invariant sections of the Courant algebroids. The notion of invariance is defined after lifting the vertical parallelisms of the underlying torus bundles $M \to B$ and $\tilde{M} \to B$ to the Courant algebroids and their spinor bundles. We prove a general existence result for T-duals under assumptions generalizing the cohomological integrality conditions for T-duality in the exact case. Specializing our construction, we find that the T-dual of an exact or a heterotic Courant algebroid is again exact or heterotic, respectively.

1. Introduction

The concept of T-duality appeared first in theoretical physics as a duality between a pair of physical theories related by compactification of a common (possibly hidden) theory along circles of reciprocal radii. Examples include the famous duality between type IIA and type IIB string theories. More generally, it refers to an isomorphism between certain type of structures on a pair of torus bundles over the same manifold [9]. Already in the case of circle bundles the topology of the bundle typically changes under T-duality [7, 8].

Precise formulations of T-duality are available in the framework of generalized geometry (in the sense of Hitchin) [2, 10]. Recall that the basic idea of generalized geometry is to replace the tangent bundle TM of a manifold M by a Courant algebroid E. The first examples of Courant algebroids considered in the literature were the exact Courant algebroids. They are obtained from the generalized tangent bundle $\mathbb{T}M := T^*M \oplus TM$ by twisting the canonical Dorfman bracket with a closed 3-form. Another important class of Courant algebroids is represented by the so called heterotic Courant algebroids, which were considered by Baraglia and Hekmati in [2], with motivation from string theory. Cavalcanti and Gualtieri [10] developed a theory of T-duality for exact Courant algebroids and Baraglia and Hekmati [2] extended it to heterotic Courant algebroids. The approach of [2] is based on reduction of exact Courant algebroids and uses T-duality for the latter algebroids.

Exact and heterotic Courant algebroids are particular classes of transitive Courant algebroids. A Courant algebroid with surjective anchor is called transitive. Our aim in this article is to develop a T-duality for such Courant algebroids. Our theory applies to general transitive Courant algebroids, which might not arise from reduction of an exact Courant algebroid. In fact, it does not use reduction. Our main focus is the systematic study of the interplay of T-duality with Dirac generating operators.

Let M and $\tilde{M}$ be principal k-torus bundles over a manifold B. We call two transitive Courant algebroids E and $\tilde{E}$ over M and $\tilde{M}$, respectively, T-dual if there exists a certain type of isomorphism between the pullbacks of E and $\tilde{E}$ to the fiber product $N = M \times_B \tilde{M}$ (see Definition 71 for details). We show that T-duality gives rise to a map between the spaces of sections of canonical weighted spinor bundles $\mathbb{S}_E$ and $\mathbb{S}_{\tilde{E}}$ of E and $\tilde{E}$ intertwining the canonical Dirac generating operators, see Theorem 79. More specifically, we obtain compatible isomorphisms between the spaces of (appropriately defined) invariant sections of E and $\tilde{E}$ as well as between the spaces of invariant sections of $\mathbb{S}_E$ and $\mathbb{S}_{\tilde{E}}$. This implies, in particular, that any invariant geometric structure on the Courant algebroid E gives rise to a corresponding invariant 'T-dual' geometric structure on $\tilde{E}$. A structure solving a system of partial differential equations on E will be mapped to a solution of the corresponding system on $\tilde{E}$. Examples include integrability equations as considered in [12] and equations of motion of physical theories such as supergravity. For instance, it was shown in [13, Section 7] that the Hull-Strominger system is invariant under T-duality. We plan to investigate these type of applications in the future.

In Theorem 85 we prove the existence of a T-dual $\tilde{E}$ for a class of transitive Courant algebroids E over a principal torus bundle $M \to B$ under the assumption that certain cohomology classes in $H^2(B, \mathbb{R})$ are integral. The result generalizes a theorem of Bouwknegt, Hannabuss, and Mathai [9] in the exact case, see Section 6.4.1. In the heterotic case we show that the T-dual Courant algebroids obtained from our construction are again heterotic, see Proposition 88.

Note that for a given transitive Courant algebroid a T-dual (if it exists) is in general not uniquely determined. A topological classification in the spirit of [4] does not lie within the scope of this paper.

A natural continuation of this work is to study how invariant geometric structures behave under the T-duality for transitive Courant algebroids, developed in this paper. Generalized metrics on arbitrary (not necessarily exact or heterotic Courant algebroids) were already considered in the literature (see e.g. [13]). It is expected that their behavior under our T-duality will be described by formulae analogous to the Busher's rules from the exact or heterotic T-duality. Other geometric structures (like generalized complex, generalized Kähler, etc) on arbitrary Courant algebroids were defined in our previous work [12]. We hope to use our T-duality in the construction of new examples of such structures. These questions are left for future work.

In this paper we only considered Courant algebroids with scalar product of neutral signature. It would be interesting to develop T-duality for other classes of Courant algebroids including, in particular, the 'odd exact' Courant algebroids studied in [18]. A first step in this direction would be to develop a theory of Dirac generating operators for such Courant algebroids.

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2. Preliminary material

To keep the text reasonably self-contained, we recall, following [11, 12], basic facts we need on transitive Courant algebroids and their canonical Dirac generating operator. We assume that the reader is familiar with the definition of Courant algebroids, Dirac generating operators, generalized connections and E-connections. Basic facts on these notions can be found e.g. in [12], the approach and notation of which we preserve along the paper. In this paper we always assume that the Courant algebroids have scalar product of neutral signature. For the definition of densities we keep the conventions from our previous work [12] which coincide with those from [3]. Namely, if V is a vector space of dimension n and $s \in \mathbb{R}$, then the one-dimensional oriented vector space $|\mathrm{det} V^*|^s$ of s-densities on V consists of all maps $\omega : \Lambda^n V \setminus \{0\} \to \mathbb{R}$ (called s-densities) which satisfy $\omega(\lambda \vec{v}) = |\lambda|^s \omega(\vec{v})$, for any $\vec{v} \in \Lambda^n V \setminus \{0\}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. We will often use the notation $|\mathrm{det} V^*| := |\mathrm{det} V^*|^1$ for the vector space of 1-densities. Note that, when s is an integer, $|\mathrm{det} V^*|^s$ is canonically isomorphic to $|\mathrm{det} V^*|^{\otimes s}$ and $|\mathrm{det} V^*|^{2s}$ to $(\mathrm{det} V^*)^{\otimes 2s}$. Any non-zero form $\omega \in \Lambda^n V^*$ defines an s-density $|\omega|^s(\vec{v}) = |\omega(v_1, \cdots, v_n)|^s$, where $\vec{v} := v_1 \wedge \cdots \wedge v_n \in \Lambda^n V \setminus \{0\}$. When V is oriented we will identify $\Lambda^n V^*$ with $|\det V^*|$ by the isomorphism which assigns to a positively oriented volume form $\omega \in \Lambda^n V^*$ the density $|\omega|$ and we will denote $|\mathrm{det} V^*|^s$ by $(\mathrm{det} V^*)^s$. The same notation will be used when V is replaced by a vector bundle.

Many results from the next sections (on Dirac generating operators, bilinear pairings, induced isomorphisms on spinor bundles etc) hold in the larger setting of regular (rather than transitive) Courant algebroids. For the purpose of this paper they will be formulated for transitive Courant algebroids.

2.1. The canonical Dirac generating operator

Let $(E, \pi, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$ be a transitive Courant algebroid over a manifold M, with anchor $\pi : E \to TM$, Dorfman bracket $[\cdot, \cdot]$ and scalar product (of neutral signature) $\langle \cdot, \cdot \rangle$. Let S be an irreducible $\mathrm{Cl}(E)$-bundle (sometimes called a spinor bundle of E). We denote by $E \ni v \mapsto \gamma_v$ the Clifford action of E on S. We assume that S is $\mathbb{Z}_2$-graded and that the grading is compatible with the Clifford multiplication (this always holds when E is oriented). Let

$|\det S^*|^{1/r}$ be the line bundle of $1/r$-densities on S, where $r := \mathrm{rk}\, S$. An E-connection D^S on S induces an E-connection on any density line bundle, in particular on $|\det S^*|^{1/r}$: if $\mathrm{vol}_S \in \Gamma(\Lambda^r S^*)$ is a local volume form on S and $D^S_e \mathrm{vol}_S = \omega(e)\mathrm{vol}_S$ then the induced connection on $|\det S^*|^{1/r}$ satisfies $D^S_e |\mathrm{vol}_S|^{1/r} = \frac{1}{r}\omega(e)|\mathrm{vol}_S|^{1/r}$, for any $e \in E$. By taking the tensor product we obtain an E-connection induced by D on the canonical spinor bundle of S, defined by $\mathcal{S} := S \otimes |\det S^*|^{1/r}$.

The canonical Dirac generating operator $\not{d}$ of E acts on sections of the canonical weighted spinor bundle of E determined by S. The latter is defined by

$$\mathbb{S} := S \otimes |\det S^*|^{1/r} \otimes |\det T^*M|^{1/2} = \mathcal{S} \otimes L, \tag{1}$$

where $L := |\det T^*M|^{1/2}$. The operator $\not{d} : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ is given by

$$\not{d} = \not{D} + \frac{1}{4}\gamma_{T^D}, \tag{2}$$

where $\not{D} := \frac{1}{2}\sum_i \gamma_{\tilde{e}_i} D^{\mathbb{S}}_{e_i}$ is the Dirac operator computed with $D^{\mathbb{S}} := D^{\mathcal{S}} \otimes D^L$, (e_i) is a frame of E, $(\tilde{e}_i)$ the metrically dual frame (i.e. $\langle e_i, \tilde{e}_j\rangle = \delta_{ij}$), $D^{\mathcal{S}}$ is the E-connection on $\mathcal{S}$ induced by an arbitrary E-connection D^S on S compatible with a given generalized connection D on E, D^L is the E-connection on L defined by D by the rule

$$D^L_v(\mu) = \mathcal{L}_{\pi(v)}\mu - \frac{1}{2}\mathrm{div}_D(v)\mu,\ \forall v \in E,\ \mu \in \Gamma(L), \tag{3}$$

where $\mathrm{div}_D(v) := \mathrm{tr}\,(Dv)$, $T^D \in \Gamma(\Lambda^3 E^*)$ is the torsion of D, viewed as a section of the Clifford bundle $\mathrm{Cl}(E)$ and acting by Clifford multiplication on $\mathbb{S}$, and $\mathcal{L}_{\pi(v)}\mu$ is the Lie derivative of the density μ in the direction of the vector field $\pi(v)$. The definition of $\not{d}$ is independent of the choice of generalized connection D and compatible (with respect to D) E-connection D^S. In particular, $D^{\mathcal{S}}$ is independent of the choice of D^S, as long as D^S is compatible with a given generalized connection D (a statement for which the neutral signature of $\langle\cdot,\cdot\rangle$ plays a key role). We recall that the connection D^S is compatible with D if, by definition,

$$D^S_v(u \cdot s) = (D_v u)\cdot s + u \cdot D^S_v s,\ \forall u, v \in \Gamma(E),\ s \in \Gamma(S), \tag{4}$$

where $u \cdot s = \gamma_u s$ denotes the Clifford action of u on s.

2.2. Transitive Courant algebroids

2.2.1. Basic properties. Recall that a scalar product on a Lie algebra is called invariant, if the adjoint representation acts by skew-symmetric endomorphisms. A Lie algebra endowed with an invariant (non-degenerate) scalar product is called a quadratic Lie algebra.

Similarly, a vector bundle $\mathcal{G} \to M$ endowed with a tensor field $[\cdot,\cdot] \in \Gamma(\wedge^2\mathcal{G}^* \otimes \mathcal{G})$ satisfying the Jacobi identity is called a Lie algebra bundle if in a neighborhood of every point $p \in M$ the tensor field has constant coefficients with respect to some local frame. A bundle of quadratic Lie algebras (or, shortly, a quadratic Lie algebra bundle) is a Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot])$ endowed with an invariant metric $\langle\cdot,\cdot\rangle \in \Gamma(\mathrm{Sym}^2\mathcal{G}^*)$, which we assume of neutral signature.

Let $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ be a quadratic Lie algebra bundle over a manifold M and E a Courant algebroid with underlying bundle $T^*M \oplus \mathcal{G} \oplus TM$, anchor the natural projection $\mathrm{pr}_{TM} : E \to TM$, scalar product

$$\langle \xi + r_1 + X, \eta + r_2 + Y\rangle = \frac{1}{2}(\eta(Y) + \xi(X)) + \langle r_1, r_2\rangle_{\mathcal{G}},$$

for any $\xi, \eta \in T^*M$, $r_1, r_2 \in \mathcal{G}$, $X, Y \in TM$, and whose Dorfman bracket satisfies

$$\mathrm{pr}_{\mathcal{G}}[r_1, r_2] = [r_1, r_2]_{\mathcal{G}},\ \forall r_1, r_2 \in \Gamma(\mathcal{G}),$$

where $\mathrm{pr}_{\mathcal{G}} : E \to \mathcal{G}$ is the natural projection. As proved in [11], the Dorfman bracket of E restricted to various components of E is given by

$$\begin{aligned}
&[X, Y] = \mathcal{L}_X Y + R(X, Y) + i_Y i_X H\\
&[X, r] = \nabla_X r - 2\langle i_X R, r\rangle_{\mathcal{G}}\\
&[r_1, r_2] = [r_1, r_2]_{\mathcal{G}} + 2\langle \nabla r_1, r_2\rangle_{\mathcal{G}}\\
&[X, \eta] = \mathcal{L}_X \eta,\ [\eta_1, \eta_2] = [r, \eta] = 0,
\end{aligned} \tag{5}$$

where ∇ is a connection on the vector bundle $\mathcal{G}$, $R \in \Omega^2(M, \mathcal{G})$ and $H \in \Omega^3(M)$, such that ∇ preserves $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ and $[\cdot,\cdot]_{\mathcal{G}}$, the curvature R^∇ of ∇ is given by

$$R^\nabla(X, Y)r = [R(X, Y), r]_{\mathcal{G}},\ \forall X, Y \in \mathfrak{X}(M),\ r \in \Gamma(\mathcal{G}), \tag{6}$$

and the following relations hold:

$$d^{\nabla} R = 0, \tag{7}$$

$$dH = \langle R \wedge R \rangle_{\mathcal{G}}. \tag{8}$$

We recall that

$$(d^{\nabla} R)(X,Y,Z) := \sum_{\mathfrak{S}(X,Y,Z)} (\nabla_X (R(Y,Z)) - R(\mathcal{L}_X Y, Z))$$
$$\langle R \wedge R \rangle_{\mathcal{G}}(X,Y,Z,W) := 2 \sum_{\mathfrak{S}(X,Y,Z)} \langle R(X,Y), R(Z,W) \rangle_{\mathcal{G}},$$

where $X,Y,Z,W \in \mathfrak{X}(M)$ and $\mathfrak{S}(X,Y,Z)$ denotes cyclic permutations over X,Y,Z. The Dorfman bracket is uniquely determined by the triple (∇, R, H) by relations (5) and the additional condition

$$[u,v] + [v,u] = 2d\langle u, v\rangle,\ \forall u,v \in \Gamma(E).$$

The above properties of (∇, R, H) are equivalent to the defining properties of the Dorfman bracket $[\cdot,\cdot]$. Conversely, a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ together with a triple (∇, R, H) with the above properties give rise to a transitive Courant algebroid.

Definition 1. *A Courant algebroid E as above is called the* standard Courant algebroid defined by the quadratic Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ and the data (∇, R, H).

As proved in [11], any transitive Courant algebroid is isomorphic to a standard Courant algebroid. A dissection of a transitive Courant algebroid E is an isomorphism from E to a standard Courant algebroid. The quadratic Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ which is a summand in a standard Courant algebroid isomorphic to E is isomorphic to $\operatorname{Ker}\pi/(\operatorname{Ker}\pi)^{\perp}$ (with scalar product and Lie bracket induced from E), where $\pi : E \to TM$ is the anchor of E. The following simple lemma holds.

Lemma 2. *Let E be a transitive Courant algebroid with anchor $\pi : E \to TM$. Let $(\mathcal{G}_0, [\cdot,\cdot]_0, \langle\cdot,\cdot\rangle_0)$ be a quadratic Lie algebra bundle, isomorphic to $\operatorname{Ker}\pi/(\operatorname{Ker}\pi)^{\perp}$. Then E admits a dissection $I_0 : E \to T^*M \oplus \mathcal{G}_0 \oplus TM$.*

Proof. Start with an arbitrary dissection $I : E \to E_M = T^*M \oplus \mathcal{G} \oplus TM$, where the target is defined by a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$

and data (∇, R, H). Then $(\mathcal{G}_0, [\cdot,\cdot]_0, \langle\cdot,\cdot\rangle_0)$ and data $(\tilde{\nabla}, \tilde{R}, \tilde{H})$ where

$$\tilde{\nabla}_X := K\nabla_X K^{-1},\ \tilde{R}(X,Y) := KR(X,Y),\ \tilde{H}(X,Y,Z) := H(X,Y,Z),$$

for any $X, Y, Z \in \mathfrak{X}(M)$, where $K : \mathcal{G} \to \mathcal{G}_0$ is an isomorphism of quadratic Lie algebra bundles, define a standard Courant algebroid isomorphic to E_M (use relations (10) below with $\Phi := 0$ and $\beta := 0$). By composing this isomorphism with I we obtain the required dissection of E. $\square$

Let $E_i := T^*M \oplus \mathcal{G}_i \oplus TM$ $(i = 1, 2)$ be two standard Courant algebroids over a manifold M, defined by quadratic Lie algebra bundles $(\mathcal{G}_i, [\cdot,\cdot]_{\mathcal{G}_i}, \langle\cdot,\cdot\rangle_{\mathcal{G}_i})$ and data $(\nabla^{(i)}, R_i, H_i)$. As proved in Proposition 2.7 of [11], any fiber preserving Courant algebroid isomorphism $I_E : E_1 \to E_2$ is of the form

$$\begin{aligned} I_E(\eta) &= \eta,\\ I_E(r) &= -2\Phi^*K(r) + K(r),\\ I_E(X) &= i_X\beta - \Phi^*\Phi(X) + \Phi(X) + X, \end{aligned} \tag{9}$$

for any $X \in TM$, $r \in \mathcal{G}_1$ and $\eta \in T^*M$. Above $\beta \in \Omega^2(M)$, $K \in \Gamma\,\mathrm{Isom}(\mathcal{G}_1, \mathcal{G}_2)$ is an isomorphism of quadratic Lie algebra bundles, $\Phi \in \Omega^1(M, \mathcal{G}_2)$,

$$\Phi^*\Phi : TM \to T^*M,\ (\Phi^*\Phi)(X)(Y) := \langle \Phi(X), \Phi(Y)\rangle_{\mathcal{G}_2},$$
$$\Phi^*K : \mathcal{G}_1 \to T^*M,\ (\Phi^*K)(r)(X) := \langle K(r), \Phi(X)\rangle_{\mathcal{G}_2},$$

for any $X, Y \in TM$ and $r \in \mathcal{G}_1$, and the next relations are satisfied:

$$\begin{aligned} \nabla^{(2)}_X r &= K\nabla^{(1)}_X(K^{-1}r) + [r, \Phi(X)]_{\mathcal{G}_2},\\ KR_1(X,Y) - R_2(X,Y) &= (d^{\nabla^{(2)}}\Phi)(X,Y) + [\Phi(X), \Phi(Y)]_{\mathcal{G}_2},\\ H_1 - H_2 &= d\beta + \langle (KR_1 + R_2) \wedge \Phi\rangle_{\mathcal{G}_2} - c_3, \end{aligned} \tag{10}$$

for any $X, Y \in \mathfrak{X}(M)$ and $r \in \Gamma(\mathcal{G}_2)$, where

$$c_3(X,Y,Z) := \langle \Phi(X), [\Phi(Y), \Phi(Z)]_{\mathcal{G}_2}\rangle_{\mathcal{G}_2},$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

Remark 3. The second and third relations (10) are equivalent to relations (46) and (47) of [11] (easy check) but are written in a simpler form. (We decomposed $\mathrm{pr}_{T^*M}(I_E|_{TM})$, which in the notation of [11] is denoted by β, into its symmetric part $-\langle\Phi(\cdot),\Phi(\cdot)\rangle_{\mathcal{G}_2}$ and skew-symmetric part β, see relation (44) of [11]).

Notation 4. In the next lemma (and along the entire paper) we shall denote by $\mathrm{Der}\,(\mathcal{G})$ the bundle of skew-symmetric derivations of a quadratic Lie algebra bundle $(\mathcal{G},[\cdot,\cdot]_{\mathcal{G}},\langle\cdot,\cdot\rangle_{\mathcal{G}})$.

Lemma 5. *Assume that the adjoint actions* $\mathrm{ad}_{\mathcal{G}_i}:\mathcal{G}_i\to\mathrm{Der}(\mathcal{G}_i)$ *of the Lie algebra bundles* $(\mathcal{G}_i,[\cdot,\cdot]_{\mathcal{G}_i})$ *of the standard Courant algebroids* E_i *are injective. Then the second relation (10) follows from the first.*

Proof. From the injectivity of $\mathrm{ad}_{\mathcal{G}_2}$, the second relation (10) holds if and only if

$$[KR_1(X,Y)-R_2(X,Y),r]_{\mathcal{G}_2}=[(d^{\nabla^{(2)}}\Phi)(X,Y)+[\Phi(X),\Phi(Y)]_{\mathcal{G}_2},r]_{\mathcal{G}_2} \tag{11}$$

for any $X,Y\in\mathfrak{X}(M)$ and $r\in\Gamma(\mathcal{G}_2)$. Taking the covariant derivative with respect to $\nabla^{(2)}$ of the first relation (10) we obtain

$$\begin{aligned}[\nabla^{(2)}_Y(\Phi(X)),r]_{\mathcal{G}_2}=&[R_2(X,Y),r]_{\mathcal{G}_2}+\nabla^{(2)}_{\mathcal{L}_XY}r+\nabla^{(2)}_Y(K\nabla^{(1)}_X(K^{-1}r))\\&-K\nabla^{(1)}_X(K^{-1}\nabla^{(2)}_Yr),\end{aligned}\tag{12}$$

where we used $R^{\nabla^{(2)}}(X,Y)r=[R_2(X,Y),r]_{\mathcal{G}_2}$, which follows from (6). Now, a straightforward computation which uses the first relation (10), relation (12), and

$$(d^{\nabla^{(2)}}\Phi)(X,Y)=\nabla^{(2)}_X(\Phi(Y))-\nabla^{(2)}_Y(\Phi(X))-\Phi(\mathcal{L}_XY)$$

shows that

$$\begin{aligned}[(d^{\nabla^{(2)}}\Phi)(X,Y),r]_{\mathcal{G}_2}=&[KR_1(X,Y)-R_2(X,Y),r]_{\mathcal{G}_2}\\&+\left((\nabla_XK)(\nabla_YK^{-1})-(\nabla_YK)(\nabla_XK^{-1})\right)(r),\end{aligned}\tag{13}$$

where ∇ denotes the connection on $\mathrm{End}(\mathcal{G}_1,\mathcal{G}_2)$ induced by $\nabla^{(1)}$ and $\nabla^{(2)}$. On the other hand, using the Jacobi identity for $[\cdot,\cdot]_{\mathcal{G}_2}$ and the first relation (10), we can compute

$$[[\Phi(X),\Phi(Y)]_{\mathcal{G}_2},r]_{\mathcal{G}_2}=\left((\nabla_YK)(\nabla_XK^{-1})-(\nabla_XK)(\nabla_YK^{-1})\right)(r). \tag{14}$$

Adding (13) with (14) we obtain (11). □

The proof of the following proposition is straightforward.

Proposition 6. *If $I_1 : E_1 \to E_2$ and $I_2 : E_2 \to E_3$ are isomorphisms between standard Courant algebroids $E_i = T^*M \oplus \mathcal{G}_i \oplus TM$, defined, according to (9), by (β_1, K_1, Φ_1) and (β_2, K_2, Φ_2) respectively, then $I_2 \circ I_1 : E_1 \to E_3$ is defined by (β_3, K_3, Φ_3) where*

$$K_3 := K_2K_1,\ \Phi_3 := \Phi_2 + K_2\Phi_1 \tag{15}$$

and, for any $X, Y \in TM$,

$$\begin{aligned} \beta_3(X,Y) := (\beta_1 + \beta_2)(X,Y) + \langle \Phi_2(X), K_2\Phi_1(Y)\rangle_{\mathcal{G}_2} \\ - \langle \Phi_2(Y), K_2\Phi_1(X)\rangle_{\mathcal{G}_2}. \end{aligned} \tag{16}$$

In particular,

$$\begin{aligned} (\beta_3 - \Phi_3^*\Phi_3)(X,Y) = (\beta_1 - \Phi_1^*\Phi_1)(X,Y) + (\beta_2 - \Phi_2^*\Phi_2)(X,Y) \\ - 2\langle K_2\Phi_1(X), \Phi_2(Y)\rangle_{\mathcal{G}_3}. \end{aligned} \tag{17}$$

Definition 7. *We say that two dissections $I_i : E \to T^*M \oplus \mathcal{G}_i \oplus TM$ of a transitive Courant algebroid E are* related *by (β, K, Φ), where $\beta \in \Omega^2(M)$, $K \in \Gamma\,\mathrm{Isom}(\mathcal{G}_1, \mathcal{G}_2)$ and $\Phi \in \Omega^1(M, \mathcal{G}_2)$, if the isomorphism $I_2 \circ I_1^{-1}$ of standard Courant algebroids is given by (9).*

2.2.2. The canonical Dirac generating operator of a standard Courant algebroid. Let $E = T^*M \oplus \mathcal{G} \oplus TM$ be a standard Courant algebroid defined by a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data (∇, R, H). Let $S_{\mathcal{G}}$ be an irreducible $\mathrm{Cl}(\mathcal{G})$-bundle (always assumed to be graded), with canonical spinor bundle $\mathcal{S}_{\mathcal{G}} = S_{\mathcal{G}} \otimes |\det S_{\mathcal{G}}^*|^{1/r}$, where $r := \mathrm{rk}\, S_{\mathcal{G}}$. Then

$$S := \Lambda(T^*M)\hat{\otimes} S_{\mathcal{G}}$$

is an irreducible spinor bundle of E, with Clifford action

$$\gamma_{\xi+r+X}(\omega \otimes s) = (i_X\omega + \xi \wedge \omega) \otimes s + (-1)^{|\omega|}\omega \otimes (r \cdot s), \tag{18}$$

for any $\xi \in T^*M$, $r \in \mathcal{G}$, $X \in TM$, $\omega \in \Lambda(T^*M)$, $s \in S_{\mathcal{G}}$, where $r \cdot s$ denotes the Clifford action of r on s. The canonical weighted spinor bundle of E

determined by S, as defined in (1), is canonically isomorphic to

$$\mathbb{S} = \Lambda(T^*M)\hat{\otimes}\mathcal{S}_{\mathcal{G}} \tag{19}$$

owing to the canonical isomorphism

$$|\det(\Lambda(TM) \otimes S^*_{\mathcal{G}})|^{\frac{1}{Nr}} \otimes |\det T^*M|^{1/2} \cong |\det S^*_{\mathcal{G}}|^{1/r} \tag{20}$$

given by

$$\begin{aligned} &|(Z_1 \otimes s_1^*) \wedge \cdots \wedge (Z_N \otimes s_r^*)|^{\frac{1}{Nr}} \otimes |\alpha_1 \wedge \cdots \wedge \alpha_m|^{1/2} \\ &\quad \mapsto |s_1^* \wedge \cdots \wedge s_r^*|^{1/r}, \end{aligned} \tag{21}$$

where $N := \mathrm{rk}\,\Lambda(TM)$, (s_i^*) is a local frame of $S^*_{\mathcal{G}}$, (α_i) is a local frame of T^*M and (Z_i) is the local frame of $\Lambda(TM)$ induced by the local frame (X_i) of TM dual to (α_i).

As shown in Theorem 67 of [12], the canonical Dirac generating operator $\not{d} : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ takes the form

$$\begin{aligned} \not{d}(\omega \otimes s) &= (d\omega - H \wedge \omega) \otimes s + \nabla^{\mathcal{S}_{\mathcal{G}}}(s) \wedge \omega \\ &\quad + \frac{1}{4}(-1)^{|\omega|+1}\omega \otimes (C_{\mathcal{G}} \cdot s) + (-1)^{|\omega|+1}\bar{R}^E(\omega \otimes s), \end{aligned} \tag{22}$$

where $\omega \in \Omega(M)$ and $s \in \Gamma(\mathcal{S}_{\mathcal{G}})$. Above $C_{\mathcal{G}} \in \Gamma(\Lambda^3\mathcal{G}^*) \subset \Gamma(\mathrm{Cl}(\mathcal{G}))$ is the Cartan form $C_{\mathcal{G}}(u,v,w) := \langle [u,v]_{\mathcal{G}}, w\rangle_{\mathcal{G}}$ which acts on s by Clifford multiplication, $\nabla^{\mathcal{S}_{\mathcal{G}}}$ is the connection on $\mathcal{S}_{\mathcal{G}}$ induced by a connection $\nabla^{S_{\mathcal{G}}}$ on $S_{\mathcal{G}}$ compatible with ∇,

$$\nabla^{\mathcal{S}_{\mathcal{G}}}(s) \wedge \omega := \sum_i (\alpha_i \wedge \omega) \otimes (\nabla^{\mathcal{S}_{\mathcal{G}}}_{X_i} s)$$

and

$$\bar{R}^E(\omega \otimes s) := \frac{1}{2}\sum_{i,j,k} \langle R(X_i, X_j), r_k\rangle_{\mathcal{G}} (\alpha_i \wedge \alpha_j \wedge \omega) \otimes (\tilde{r}_k \cdot s),$$

where (r_k) is a local frame of $\mathcal{G}$ and $(\tilde{r}_k)$ the metrically dual frame (i.e. $\langle r_i, \tilde{r}_j\rangle_{\mathcal{G}} = \delta_{ij}$ for any i, j). The connection $\nabla^{\mathcal{S}_{\mathcal{G}}}$ is independent of the choice of connection $\nabla^{S_{\mathcal{G}}}$ compatible with ∇ and preserves the grading of $\mathcal{S}_{\mathcal{G}}$. Similar to (4), the compatibility of $\nabla^{S_{\mathcal{G}}}$ with ∇ means that

$$\nabla^{S_{\mathcal{G}}}_X(r \cdot s) = (\nabla_X r) \cdot s + r \cdot \nabla^{S_{\mathcal{G}}}_X s, \tag{23}$$

for any $X \in \mathfrak{X}(M)$, $r \in \Gamma(\mathcal{G})$ and $s \in \Gamma(S_{\mathcal{G}})$. Sometimes it will be convenient to write the canonical Dirac generating operator in the equivalent form

$$\begin{aligned} \not{d}(\omega \otimes s) &= (d\omega) \otimes s + \sum_i (\alpha_i \wedge \omega) \otimes (\nabla^{S_{\mathcal{G}}}_{X_i} s) - H \cdot (\omega \otimes s) \\ &- \frac{1}{4} C_{\mathcal{G}} \cdot (\omega \otimes s) - \frac{1}{2} \sum_{i,j,k} \langle R(X_i, X_j), r_k \rangle_{\mathcal{G}} \tilde{r}_k \cdot \alpha_i \cdot \alpha_j \cdot (\omega \otimes s), \end{aligned} \tag{24}$$

where the dots denote the Clifford action of $\mathrm{Cl}(E) \cong \Lambda E$ on $\mathbb{S}$.

3. The bilinear pairing on spinors

Let $(E, \pi, [\cdot, \cdot], \langle \cdot, \cdot \rangle_E)$ be a rank $2n \geq 2$ transitive Courant algebroid over a manifold M, S an irreducible spinor bundle of E of rank r and $\mathcal{S} = S \otimes |\det S^*|^{1/r}$ the canonical spinor bundle of S. Before we state the next proposition we need to define the determinant of a bilinear pairing $\langle \cdot, \cdot \rangle$ on $\mathcal{S}$. For this we consider $\langle \cdot, \cdot \rangle$ as a map $\mathcal{S} \to \mathcal{S}^*$, $v \mapsto \langle v, \cdot \rangle$. Its determinant $\det \mathcal{S} \to \det \mathcal{S}^*$ defines a nowhere vanishing section $\det \langle \cdot, \cdot \rangle \in \Gamma((\det \mathcal{S}^*)^{\otimes 2})$. Since $\mathcal{S} = S \otimes |\det S^*|^{1/r}$, $\det \mathcal{S} = \det S \otimes |\det S^*|$ and $(\det \mathcal{S})^2 \cong (\det S)^2 \otimes |\det S^*|^2 \cong (\det S)^2 \otimes (\det S^*)^2$ is canonically identified with the trivial line bundle, which means that $\det \langle \cdot, \cdot \rangle$ is simply a real-valued function. This function can be computed as follows: let (s_i) be a local frame of S defined on some open set $U \subset M$ and $l := |s_1 \wedge \cdots \wedge s_r|^{-1/r}$. Then

$$\det \langle \cdot, \cdot \rangle|_U = \det(a_{ij}),\ a_{ij} := \langle s_i \otimes l, s_j \otimes l \rangle. \tag{25}$$

Note that $\det \lambda \langle \cdot, \cdot \rangle = \lambda^r \det \langle \cdot, \cdot \rangle$, for any $\lambda \in \mathbb{R}^*$.

Proposition 8. *i) For any $U \subset M$ open and sufficiently small, there is a pairing*

$$\langle \cdot, \cdot \rangle_{\mathcal{S}|_U} : \Gamma(\mathcal{S}|_U) \times \Gamma(\mathcal{S}|_U) \to C^\infty(U) \tag{26}$$

which is $C^\infty(U)$-linear, satisfies

$$\langle u \cdot s, u \cdot \tilde{s} \rangle_{\mathcal{S}|_U} = \langle u, u \rangle_E \langle s, \tilde{s} \rangle_{\mathcal{S}|_U},\ \forall u \in \Gamma(E|_U), s, \tilde{s} \in \Gamma(\mathcal{S}|_U) \tag{27}$$

and has determinant 1 *if* $n > 1$ *and* -1 *if* $n = 1$*. Any two such pairings differ by multiplication by* ± 1*.*

ii) If n *is even then the even and odd parts* $\mathcal{S}^0|_U$ *and* $\mathcal{S}^1|_U$ *of* $\mathcal{S}|_U$ *are orthogonal with respect to* $\langle\cdot,\cdot\rangle_{\mathcal{S}|_U}$*. If* n *is odd then* $\mathcal{S}^0|_U$ *and* $\mathcal{S}^1|_U$ *are isotropic with respect to* $\langle\cdot,\cdot\rangle_{\mathcal{S}|_U}$*.*

iii) The pairing $\langle\cdot,\cdot\rangle_{\mathcal{S}|_U}$ *is symmetric if* $n \equiv 0, 1 \pmod 4$ *and skew-symmetric if* $n \equiv 2, 3 \pmod 4$*.*

iv) Let D *be a generalized connection on* E*. The pairing* $\langle\cdot,\cdot\rangle_{\mathcal{S}|_U}$ *is preserved by the* E*-connection* $D^{\mathcal{S}}$ *induced by (any)* E*-connection* D^S *on* S*, compatible with* D*.*

Definition 9. *The pairings* $\langle\cdot,\cdot\rangle_{\mathcal{S}|_U}$ *are called* canonical pairings of $\mathcal{S}|_U$*.*

The remaining part of this section is devoted to the proof of Proposition 8 and to various corollaries. Let V be an n-dimensional vector space. We begin by considering the irreducible $\mathrm{Cl}(V \oplus V^*)$-module ΛV^* where $V \oplus V^*$ is endowed with its standard metric of neutral signature

$$\langle X + \xi, Y + \eta\rangle = \frac{1}{2}(\xi(Y) + \eta(X)),\ \forall X, Y \in V,\ \xi, \eta \in V^*$$

and the Clifford action is given by

$$(X + \xi)\cdot\omega := i_X\omega + \xi\wedge\omega,\ \forall X \in V,\ \xi \in V^*,\ \omega \in \Lambda V^*.$$

It is well known that the vector valued bilinear pairing

$$\langle\cdot,\cdot\rangle : \Lambda V^* \otimes \Lambda V^* \to \Lambda^n V^*,\ \langle\omega, \tilde{\omega}\rangle := \left(\omega^t \wedge \tilde{\omega}\right)_{\mathrm{top}}, \tag{28}$$

where ${}^t : \Lambda V^* \to \Lambda V^*$ is defined on decomposable forms by $(\alpha_1 \wedge \cdots \wedge \alpha_k)^t := \alpha_k \wedge \cdots \wedge \alpha_1$ and, for a form $\omega \in \Lambda V^*$, $\omega_{\mathrm{top}} \in \Lambda^n V^*$ denotes its component of maximal degree, satisfies (27) (see e.g. [15]). Since the metric of $V \oplus V^*$ has neutral signature, we obtain that (28) is determined (up to multiplication by a non-zero constant), by this property. Note that $\Lambda^{\mathrm{even}} V^*$ and $\Lambda^{\mathrm{odd}} V^*$ are orthogonal with respect to the pairing (28) when n is even and are isotropic when n is odd. Also it is easy to check that (28) is non-degenerate, symmetric if $n \equiv 0, 1 \pmod 4$ and skew-symmetric if $n \equiv 2, 3 \pmod 4$. By choosing a volume form on V, we obtain an $\mathbb{R}$-valued pairing with the same properties. These considerations hold for any irreducible Clifford module in neutral signature and lead to the following lemma.

Lemma 10. *Let* W *be an irreducible* $\mathrm{Cl}(n, n)$*-module and* $\mathcal{W} := W \otimes |\det W^*|^{1/r}$ *where* $r := \dim W$*.*

i) There is an $\mathbb{R}$-valued pairing $\langle\cdot,\cdot\rangle_{\mathcal{W}}$ on $\mathcal{W}$ which satisfies

$$\langle u\cdot w, u\cdot \tilde{w}\rangle_{\mathcal{W}} = \langle u,u\rangle\langle w,\tilde{w}\rangle_{\mathcal{W}},\ \forall u\in\mathbb{R}^{2n},\ w,\tilde{w}\in\mathcal{W} \tag{29}$$

and $\det\langle\cdot,\cdot\rangle_{\mathcal{W}} = 1$ *if* $n>1$, *respectively* $\det\langle\cdot,\cdot\rangle_{\mathcal{W}} = -1$ *if* $n=1$. *Such a pairing is unique up to multiplication by* ± 1.

ii) The pairing $\langle\cdot,\cdot\rangle_{\mathcal{W}}$ is symmetric if $n\equiv 0,1 \pmod 4$ and skew-symmetric if $n\equiv 2,3 \pmod 4$. The even and odd parts $\mathcal{W}^0$ and $\mathcal{W}^1$ of $\mathcal{W}$ are orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathcal{W}}$ when n is even and are isotropic when n is odd.

Proof. It remains to prove that we can rescale $\langle\cdot,\cdot\rangle_{\mathcal{W}}$ appropriately in order to have $\det\langle\cdot,\cdot\rangle_{\mathcal{W}} = 1$ or -1. Assume that $n>1$. Using $\det(\lambda\langle\cdot,\cdot\rangle_{\mathcal{W}}) = \lambda^r\det(\langle\cdot,\cdot\rangle_{\mathcal{W}})$, this reduces to showing that $\det\langle\cdot,\cdot\rangle_{\mathcal{W}}>0$ for any bilinear pairing $\langle\cdot,\cdot\rangle_{\mathcal{W}}$ which satisfies (29). We compute the determinant $\det\langle\cdot,\cdot\rangle_{\mathcal{W}}$ using a formula similar to (25), with a basis of W of the form $(w_1,\cdots,w_{r/2}, v\cdot w_1,\cdots,v\cdot w_{r/2})$ where $v\in\mathbb{R}^{2n}$ is of norm one (with respect to the standard scalar product of neutral signature of $\mathbb{R}^{2n}$) and $(w_i)_{1\leq i\leq r/2}$ is a basis of the even part W^0 of W. We obtain $\det(\langle\cdot,\cdot\rangle_{\mathcal{W}}) = (\det A)^2$, where $A=(A_{ij})\in M_{r/2\times r/2}(\mathbb{R})$ with $A_{ij} = \langle w_i\otimes l, w_j\otimes l\rangle_{\mathcal{W}}$ when n is even, $A_{ij} = \langle w_i\otimes l, v\cdot w_j\otimes l\rangle_{\mathcal{W}}$ when $n>1$ is odd and $l := |w_1\wedge\cdots\wedge w_{r/2}\wedge v\cdot w_1\wedge\cdots\wedge v\cdot w_{r/2}|^{-1/r}$ in both cases. For $n=1$ we obtain instead $\det(\langle\cdot,\cdot\rangle_{\mathcal{W}}) = -(\det A)^2$. □

Definition 11. *The pairings $\langle\cdot,\cdot\rangle_{\mathcal{W}}$ are called* canonical pairings of $\mathcal{W}$.

Remark 12. There is the following tautological way to express $\langle\cdot,\cdot\rangle_{\mathcal{W}}$. Consider a bilinear pairing $\langle\cdot,\cdot\rangle_W$ on W, which satisfies (29). Take a basis (w_i) of W and let $l := |w_1\wedge\cdots\wedge w_r|^{-1/r}$. Then

$$\langle w\otimes l, \tilde{w}\otimes l\rangle_{\mathcal{W}} = |\det C|^{-1/r}\langle w,\tilde{w}\rangle_W,\ C := (\langle w_i, w_j\rangle_W)_{i,j}. \tag{30}$$

The next lemma concludes the proof of Proposition 8. It shows that the pairings $\langle\cdot,\cdot\rangle_{\mathcal{S}|_U}$ exist, whenever $E|_U$ is trivial (and U is open).

Lemma 13. *Let $U\subset M$ be an open subset such that $E|_U$ is trivial. The section of $(\mathcal{S}^*\otimes\mathcal{S}^*)/\pm 1$ defined by canonical pairings $\langle\cdot,\cdot\rangle_{\mathcal{S}_p}$ $(p\in U)$ lifts to a smooth section $\langle\cdot,\cdot\rangle_{\mathcal{S}|_U}$ of $\mathcal{S}^*|_U\otimes\mathcal{S}^*|_U$, which is preserved by the E-connection $D^{\mathcal{S}}$ on $\mathcal{S}$ induced by any generalized connection D on E.*

Proof. Consider a local frame $(e_i)_{1\leq i\leq 2n}$ of $E|_U$ with $\langle e_i, e_j\rangle_E = \epsilon_i\delta_{ij}$, where $\epsilon_i = 1$ for $i\leq n$ and -1 for $i\geq n+1$. On $\mathbb{R}^{2n}$ we consider the standard basis $(v_i)_{1\leq i\leq 2n}$ and metric $\langle\cdot,\cdot\rangle_{\mathbb{R}^{2n}}$ defined by $\langle v_i, v_j\rangle_{\mathbb{R}^{2n}} = \epsilon_i\delta_{ij}$. Let W be an irreducible $\mathrm{Cl}(\mathbb{R}^{2n}, \langle\cdot,\cdot\rangle_{\mathbb{R}^{2n}})$-module and $\Sigma := U \times W$ the trivial bundle over U with fiber W, which is an irreducible $\mathrm{Cl}(E|_U)$-bundle with Clifford action $\gamma_{e_i}(p, w) := (p, v_i \cdot w)$, for any $(p, w) \in U \times W$. Since E has neutral signature, $S|_U = \Sigma \otimes L$ where L is a line bundle and

$$\mathcal{S}|_U = \Sigma \otimes |\det \Sigma^*|^{1/r} \otimes L \otimes |L^*|,$$

where $r = \dim W = \operatorname{rank} S$. The bilinear pairing $\langle\cdot,\cdot\rangle_{\mathcal{S}|_U}$ we are looking for is given by

$$\langle s \otimes l, \tilde{s} \otimes l\rangle_{\mathcal{S}|_U} = \langle s, \tilde{s}\rangle_{\Sigma\otimes|\det\Sigma^*|^{1/r}} l^2,$$

for any $s, \tilde{s} \in \Sigma \otimes |\det \Sigma^*|^{1/r}$ and $l \in L \otimes |L^*|$, where $\langle s, \tilde{s}\rangle_{\Sigma\otimes|\det\Sigma^*|^{1/r}}$ is the constant pairing on Σ defined by a canonical pairing on $W \otimes |\det W^*|^{1/r}$ (see Definition 11) and $l^2 \in C^\infty(U)$ under the canonical isomorphism $(L \otimes |L^*|)^2 = U \times \mathbb{R}$. We now prove that $\langle\cdot,\cdot\rangle_{\mathcal{S}|_U}$ is preserved by the E-connection $D^{\mathcal{S}}$. If

$$D_u(e_k) = 2\sum_{j<p} \omega_{pj}(u)(e_p \wedge e_j)(e_k),\ \forall u \in \Gamma(E|_U),$$

where $\omega_{pj} \in \Gamma(E^*|_U)$, then the E-connection D^Σ on Σ defined by

$$D^\Sigma_u(\sigma_\alpha) := \frac{1}{2}\sum_{i<j} \omega_{ji}(u) e_j e_i \cdot \sigma_\alpha,\ 1 \leq \alpha \leq r,$$

where (σ_α) is a constant frame of Σ, is compatible with D (see e.g. [12]). From $\mathrm{trace}(\, e_i e_j \cdot) = 0$, we deduce that $D^\Sigma(\sigma_1 \wedge \cdots \wedge \sigma_r) = 0$ and that the E-connection induced by D^Σ on $\Sigma \otimes |\det \Sigma^*|^{1/r}$, also denoted by D^Σ, satisfies

$$D^\Sigma_u(\sigma_\alpha \otimes l_\Sigma) = \frac{1}{2}\sum_{i<j} \omega_{ji}(u)(e_j e_i \cdot \sigma_\alpha) \otimes l_\Sigma,$$

where $l_\Sigma := |\sigma_1 \wedge \cdots \wedge \sigma_r|^{-1/r}$. Since $\langle\cdot,\cdot\rangle_{\Sigma\otimes|\det\Sigma^*|^{1/r}}$ is constant in the frame $(\sigma_\alpha \otimes l_\Sigma)$ and the Clifford action of $e_i e_j$ is skew-symmetric with respect to $\langle\cdot,\cdot\rangle_{\Sigma\otimes|\det\Sigma^*|^{1/r}}$ (from the property (27) of $\langle\cdot,\cdot\rangle_{\Sigma\otimes|\det\Sigma^*|^{1/r}}$), we obtain that D^Σ preserves $\langle\cdot,\cdot\rangle_{\Sigma\otimes|\det\Sigma^*|^{1/r}}$. Let D^L be an E-connection on L. Then $D^\Sigma \otimes D^L$ is an E-connection on $S|_U$, compatible with D, with the property that the induced connection on $\mathcal{S}|_U$ preserves $\langle\cdot,\cdot\rangle_{\mathcal{S}|_U}$ (easy check). The latter coincides with $D^{\mathcal{S}}|_U$. $\square$

Remark 14. Let $E = T^*M \oplus \mathcal{G} \oplus TM$ be a standard Courant algebroid defined by a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data

(∇, R, H). Let $S_{\mathcal{G}}$ be an irreducible $\mathrm{Cl}(\mathcal{G})$-bundle. The same argument as in Proposition 8 shows that for any $U \subset M$ open and sufficiently small, there is a smooth $C^\infty(U)$-bilinear pairing

$$\langle \cdot, \cdot \rangle_{\mathcal{S}_{\mathcal{G}}|_U} : \Gamma(\mathcal{S}_{\mathcal{G}}|_U) \times \Gamma(\mathcal{S}_{\mathcal{G}}|_U) \to C^\infty(U) \tag{31}$$

of normalized determinant (i.e. equal to 1 if $\mathrm{rk}\, \mathcal{G} > 2$ and equal to -1 if $\mathrm{rk}\, \mathcal{G} = 2$), which satisfies

$$\langle u \cdot s, u \cdot \tilde{s} \rangle_{\mathcal{S}_{\mathcal{G}}|_U} = \langle u, u \rangle_{\mathcal{G}} \langle s, \tilde{s} \rangle_{\mathcal{S}_{\mathcal{G}}|_U}, \tag{32}$$

for any $s, \tilde{s} \in \Gamma(\mathcal{S}_{\mathcal{G}}|_U)$ and $u \in \Gamma(\mathcal{G}|_U)$. Such a pairing is unique up to multiplication by ± 1 and is preserved by the connection $\nabla^{\mathcal{S}_{\mathcal{G}}}$ induced by any connection $\nabla^{S_{\mathcal{G}}}$ on $S_{\mathcal{G}}$ compatible with ∇.

Definition 15. *The pairings $\langle \cdot, \cdot \rangle_{\mathcal{S}_{\mathcal{G}}|_U}$ are called* canonical pairings of $\mathcal{S}_{\mathcal{G}}|_U$.

As a consequence of Proposition 8 we obtain, for any $U \subset M$ open and sufficiently small, a (unique modulo ± 1) $C^\infty(U)$-bilinear pairing on the canonical weighted spinor bundle of $\mathbb{S}$ of E determined by S (see relation (1))

$$\langle \cdot, \cdot \rangle_{\mathbb{S}|_U} : \Gamma(\mathbb{S}|_U) \times \Gamma(\mathbb{S}|_U) \to |\det T^*U|,\ \langle s \otimes l, \tilde{s} \otimes l \rangle_{\mathbb{S}|_U} := \langle s, \tilde{s} \rangle_{\mathcal{S}|_U} l^2, \tag{33}$$

where $s, \tilde{s} \in \Gamma(\mathcal{S}|_U)$ and $l \in \Gamma(|\det T^*U|^{1/2})$. It satisfies

$$\langle u \cdot (s \otimes l), u \cdot (\tilde{s} \otimes \tilde{l}) \rangle_{\mathbb{S}|_U} = \langle u, u \rangle_E \langle s \otimes l, \tilde{s} \otimes \tilde{l} \rangle_{\mathbb{S}|_U}, \tag{34}$$

for any $u \in \Gamma(E|_U)$ and $s \otimes l, \tilde{s} \otimes \tilde{l} \in \Gamma(\mathbb{S}|_U)$.

Definition 16. *The pairings $\langle \cdot, \cdot \rangle_{\mathbb{S}|_U}$ are called* canonical pairings of $\mathbb{S}|_U$.

Remark 17. When M is oriented, $\langle \cdot, \cdot \rangle_{\mathbb{S}|_U}$ takes values in the bundle $\det T^*U$ of forms of top degree on U. A pairing with similar properties (but with values in $\det(T^*U) \otimes \mathbb{C}$) was constructed in Proposition 3.14 of [16].

Lemma 18. *Assume that $E = T^*M \oplus \mathcal{G} \oplus TM$ is a standard Courant algebroid over an oriented manifold M, defined by a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}}, \langle \cdot, \cdot \rangle_{\mathcal{G}})$ and data (∇, R, H). Let $S_{\mathcal{G}}$ be an irreducible $\mathrm{Cl}(\mathcal{G})$-bundle, $\mathcal{S}_{\mathcal{G}} = S_{\mathcal{G}} \otimes |\det S^*_{\mathcal{G}}|^{1/r}$ the canonical spinor bundle of $S_{\mathcal{G}}$ and $\mathbb{S} = \Lambda(T^*M) \hat{\otimes} \mathcal{S}_{\mathcal{G}}$ the canonical weighted spinor bundle of E determined by* *the spinor bundle* $\Lambda(T^*M)\hat{\otimes}S_{\mathcal{G}}$. *For any* $U \subset M$ *open and sufficiently small, a canonical pairing* $\langle \cdot, \cdot \rangle_{\mathbb{S}|_U}$ *is given by*

$$\langle \omega \otimes s, \tilde{\omega} \otimes \tilde{s} \rangle_{\mathbb{S}|_U} = (-1)^{|s|(|\omega|+|\tilde{\omega}|)} (\omega^t \wedge \tilde{\omega})_{\mathrm{top}} \langle s, \tilde{s} \rangle_{\mathcal{S}_{\mathcal{G}}|_U}, \tag{35}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{S}_{\mathcal{G}}|_U}$ *is a canonical bilinear pairing of* $\mathcal{S}_{\mathcal{G}}|_U$.

Proof. The claim is a consequence of the following general statement: if $(V_i, \langle \cdot, \cdot \rangle_i)$ are pseudo-Euclidean vector spaces with metrics of neutral signature and S_i are irreducible $\mathbb{Z}_2$-graded $\mathrm{Cl}(V_i)$-modules of ranks r_i, with canonical pairings $\langle \cdot, \cdot \rangle_{\mathcal{S}_i}$ on $\mathcal{S}_i = S_i \otimes |\det S_i^*|^{1/r_i}$, then the graded tensor product $S := S_1 \hat{\otimes} S_2$ is an irreducible $\mathrm{Cl}(V_1 \oplus V_2)$-module, with canonical spinor module $\mathcal{S} = \mathcal{S}_1 \hat{\otimes} \mathcal{S}_2$ and a canonical bilinear pairing on $\mathcal{S}$ is given by

$$\langle s_1 \otimes s_2, \tilde{s}_1 \otimes \tilde{s}_2 \rangle_{\mathcal{S}} = (-1)^{|s_2|(|s_1|+|\tilde{s}_1|)} \langle s_1, \tilde{s}_1 \rangle_{\mathcal{S}_1} \langle s_2, \tilde{s}_2 \rangle_{\mathcal{S}_2}. \tag{36}$$

Indeed, the scalar product (36) satisfies (27) (easy check). In order to show that it has normalized determinant, we remark that

$$\det \langle \cdot, \cdot \rangle_{\mathcal{S}} = \det \langle \cdot, \cdot \rangle'_{\mathcal{S}} = (\det \langle \cdot, \cdot \rangle_{\mathcal{S}_1})^{r_2} (\det \langle \cdot, \cdot \rangle_{\mathcal{S}_2})^{r_1} = 1, \tag{37}$$

where

$$\langle s_1 \otimes s_2, \tilde{s}_1 \otimes \tilde{s}_2 \rangle'_{\mathcal{S}} := \langle s_1, \tilde{s}_1 \rangle_{\mathcal{S}_1} \langle s_2, \tilde{s}_2 \rangle_{\mathcal{S}_2}.$$

The first relation in (37) can be checked using that the scalar products $\langle \cdot, \cdot \rangle_{\mathcal{S}}$ and $\langle \cdot, \cdot \rangle'_{\mathcal{S}}$ differ only by a sign (dependent on degrees) when restricted to tensor products of homogeneous elements. (Recall that the even and odd parts of $\mathcal{S}_i$ are orthogonal or isotropic with respect to $\langle \cdot, \cdot \rangle_{\mathcal{S}_i}$.) ☐

The next corollary will be used in Section 4.3.

Corollary 19. *In the setting of Lemma 18, let* $\nabla^{\mathcal{S}_{\mathcal{G}}}$ *be the connection on* $\mathcal{S}_{\mathcal{G}}$ *induced by an arbitrary connection* $\nabla^{S_{\mathcal{G}}}$ *on* $S_{\mathcal{G}}$, *compatible with* ∇. *For any* $U \subset M$ *open and sufficiently small, define* $\mathcal{E} \in \mathrm{End}\,\Gamma(\mathbb{S}|_U)$ *by*

$$\mathcal{E}(\omega \otimes s) := (d\omega) \otimes s + \sum_i (\alpha_i \wedge \omega) \otimes \nabla^{\mathcal{S}_{\mathcal{G}}}_{X_i} s, \tag{38}$$

for any $\omega \in \Omega(U)$ *and* $s \in \Gamma(\mathcal{S}_{\mathcal{G}}|_U)$, *where* (X_i) *is a local frame of* TU, *with dual frame* (α_i). *Then, for any products* $\omega \otimes s, \tilde{\omega} \otimes \tilde{s} \in \Gamma(\mathbb{S}|_U)$ *of homogeneous elements,*

$$\begin{aligned} &\langle \mathcal{E}(\omega \otimes s), \tilde{\omega} \otimes \tilde{s}\rangle_{\mathbb{S}|_U} + \langle \omega \otimes s, \mathcal{E}(\tilde{\omega} \otimes \tilde{s})\rangle_{\mathbb{S}|_U} \\ &\quad = (-1)^{|s|(|\omega|+|\tilde{\omega}|+1)+|\omega|} d\left(\langle s, \tilde{s}\rangle_{\mathcal{S}_{\mathcal{G}}|_U} (\omega^t \wedge \tilde{\omega})_{m-1}\right). \end{aligned} \tag{39}$$

Here m *is the dimension of* M *and* ω_{m-1} *denotes the degree* $(m-1)$-*component of a form* $\omega \in \Omega(U)$.

Proof. We use the expression (35) of the canonical pairing $\langle \cdot, \cdot \rangle_{\mathbb{S}|_U}$. Since $d(\omega^t) = (-1)^{|\omega|}(d\omega)^t$ we obtain

$$\begin{aligned} &\langle (d\omega) \otimes s, \tilde{\omega} \otimes \tilde{s}\rangle_{\mathbb{S}|_U} = \\ &(-1)^{|s|(|\omega|+|\tilde{\omega}|+1)+|\omega|} \langle s, \tilde{s}\rangle_{\mathcal{S}_{\mathcal{G}}|_U} \left(d((\omega^t \wedge \tilde{\omega})_{m-1}) + (-1)^{|\omega|+1} (\omega^t \wedge d\tilde{\omega})_{\mathrm{top}} \right). \end{aligned} \tag{40}$$

Similarly, since $(\alpha_i \wedge \omega)^t = \omega^t \wedge \alpha_i$ and using that $\nabla^{\mathcal{S}_{\mathcal{G}}}$ preserves $\langle \cdot, \cdot \rangle_{\mathcal{S}_{\mathcal{G}}|_U}$ and the grading of $\mathcal{S}|_{\mathcal{G}}$, we obtain

$$\begin{aligned} &\langle (\alpha_i \wedge \omega) \otimes \nabla^{\mathcal{S}_{\mathcal{G}}}_{X_i} s, \tilde{\omega} \otimes \tilde{s}\rangle_{\mathbb{S}|_U} = \\ &(-1)^{|s|(|\omega|+|\tilde{\omega}|+1)} (\omega^t \wedge \alpha_i \wedge \tilde{\omega})_{\mathrm{top}} \left(X_i \langle s, \tilde{s}\rangle_{\mathcal{S}_{\mathcal{G}}|_U} - \langle s, \nabla^{\mathcal{S}_{\mathcal{G}}}_{X_i} \tilde{s}\rangle_{\mathcal{S}_{\mathcal{G}}|_U} \right). \end{aligned} \tag{41}$$

By adding (40) with (41) we obtain (39). □

4. Dirac generating operator and operations on spinors

4.1. Behavior of canonical Dirac generating operators under isomorphisms

Lemma 20. *Let* $I_E : E_1 \to E_2$ *be an isomorphism of transitive Courant algebroids over a manifold* M *and* S_i *irreducible* $\mathrm{Cl}(E_i)$*-bundles* ($i = 1, 2$). *Then, for any* $U \subset M$ *open and sufficiently small, there is a unique (up to multiplication by a smooth non-vanishing function) isomorphism* $I_{S|_U} : S_1|_U \to S_2|_U$ *such that*

$$I_{S|_U} \circ \gamma_u = \gamma_{I_E(u)} \circ I_{S|_U}, \ \forall u \in E_1|_U. \tag{42}$$

The map $I_{S|_U}$ is homogeneous (i.e. even or odd). If $\not{d}_1 \in \mathrm{End}\,\Gamma(S_1|_U)$ is a Dirac generating operator of $E_1|_U$ then

$$\not{d}_2 := I_{S|_U} \circ \not{d}_1 \circ I_{S|_U}^{-1} \in \mathrm{End}\,\Gamma(S_2|_U) \tag{43}$$

is a Dirac generating operator of $E_2|_U$.

Proof. Assume that $E_1|_U$ admits an orthonormal frame (e_i) and let $(\tilde{e}_i) := (I_E(e_i))$ be the corresponding orthonormal frame of $E_2|_U$. Like in the proof of Lemma 13, $S_i|_U = \Sigma_i \otimes L_i$ where $\Sigma_i := U \times W$ are $\mathrm{Cl}(E_i)$-bundles, constructed using an irreducible $\mathrm{Cl}(\mathbb{R}^{2n}, \langle\cdot,\cdot\rangle_{\mathbb{R}^{2n}})$-module W and the orthonormal frames (e_i) and $(\tilde{e}_i)$ respectively, and L_i are line bundles over U. Restricting U if necessary, we may assume that L_i are isomorphic. Let $I_L : L_1 \to L_2$ be an isomorphism. Then $I_{S|_U} : S_1|_U \to S_2|_U$ defined by $I_{S|_U}(\sigma \otimes l) := \sigma \otimes I_L(l)$, for any $\sigma \in \Sigma_1$ and $l \in L_1$, satisfies (42). The even and odd parts of S_1 are given by $S_1^0 = \frac{1}{2}(1 + \epsilon\gamma_\omega)S$ and $S_1^1 = \frac{1}{2}(1 - \epsilon\gamma_\omega)S$, where $\epsilon \in \{\pm 1\}$ and $\omega = e_1 \cdots e_{2n}$, and similarly for the even and odd parts of S_2 (with ω replaced by $\tilde{\omega} = \tilde{e}_1 \cdots \tilde{e}_n$). Therefore the statement that $I_{S|_U}$ is homogeneous follows from (42), which implies that $I_{S|_U} \circ \gamma_\omega = \gamma_{\tilde{\omega}} \circ I_{S|_U}$. Since $I_{S|_U}$ is homogeneous and $\not{d}_2$ is odd, we obtain that also $\not{d}_1$ is odd. It is clear that $\not{d}_2^2 \in C^\infty(U)$, as $\not{d}_1^2 \in C^\infty(U)$. The statement that $\not{d}_2$ satisfies the condition

$$[[\not{d}, \gamma_{e_1}], \gamma_{e_2}] = \gamma_{[e_1, e_2]}, \ \forall e_1, e_2 \in \Gamma(E)$$

from the definition of Dirac generating operators (see e. g. Definition 39 of [12]) follows from (42) and (43), which imply

$$[\not{d}_2, \gamma_{I_E(u)}] = I_{S|_U} \circ [\not{d}_1, \gamma_u] \circ I_{S|_U}^{-1}, \ \forall u \in \Gamma(E_1|_U),$$

and from the properties of $\not{d}_1$. The remaining condition

$$[[\not{d}_2, f], \gamma_e] = \pi_2(e)(f), \ \forall e \in \Gamma(E_2), \ f \in C^\infty(M)$$

follows similarly. □

Remark 21. i) In general, the isomorphisms $I_{S|_U}$ do not glue together to give an isomorphism $I_S : S_1 \to S_2$ compatible with I_E. However, assume that $E_1 = E_2 = E$ and let $S_1 = S_2 = S$ be an irreducible spinor bundle over $\mathrm{Cl}(E)$. If $I_E \in \mathrm{Aut}(E)$ is of the form $I_E(u) = \alpha \cdot u \cdot \alpha^{-1}$, where $\alpha \in \Gamma(\mathrm{Pin}(E))$, then $I_S \in \mathrm{Aut}(S)$ defined by $I_S(s) := \alpha \cdot s$, $s \in \Gamma(S)$, satisfies (42) (and is globally defined).

ii) For example, if $E = T^*M \oplus \mathcal{G} \oplus TM$ is in the standard form, $S_{\mathcal{G}}$ is an irreducible $\mathrm{Cl}(\mathcal{G})$-bundle and $\beta \in \Omega^2(M)$, then

$$I_E(\xi + r + X) = \xi + i_X\beta + r + X$$

can be written as $I_E(u) = \alpha \cdot u \cdot \alpha^{-1}$ for $\alpha := e^{-\beta}$ and the induced action on the spinor bundle $S := \Lambda(T^*M)\hat{\otimes}S_{\mathcal{G}}$ is given by $I_S(\omega \otimes r) := (e^{-\beta} \wedge \omega) \otimes r$ and is globally defined. If, moreover, $d\beta = 0$ then I_E is a Courant algebroid automorphism (see relations (10)). Similarly, any automorphism $K \in \Gamma\,\mathrm{Aut}(\mathcal{G})$ of the quadratic Lie algebra bundle $\mathcal{G}$, which belongs to the connected component $\Gamma\,\mathrm{Aut}(\mathcal{G}, \langle\cdot,\cdot\rangle_{\mathcal{G}})_0$ and is parallel with respect to the connection ∇ from the data which defines E, defines a Courant algebroid automorphism of E whose action on spinors is globally defined.

An isomorphism $I_{S|_U} : S_1|_U \to S_2|_U$ like in Lemma 20 induces an isomorphism $I_{\mathcal{S}|_U} : \mathcal{S}_1|_U \to \mathcal{S}_2|_U$ between the canonical spinor bundles of $S_1|_U$ and $S_2|_U$, given by

$$I_{\mathcal{S}|_U}(s \otimes |s_1 \wedge \cdots \wedge s_r|^{-1/r}) := (I_S s) \otimes |I_S s_1 \wedge \cdots \wedge I_S s_r|^{-1/r}, \tag{44}$$

where $s_1 \wedge \cdots \wedge s_r \in \Gamma(\Lambda^r(S_1|_U))$ is non-vanishing.

Lemma 22. *For any $U \subset M$ open and sufficiently small, the isomorphism $I_{\mathcal{S}|_U}$ preserves the canonical pairings $\langle\cdot,\cdot\rangle_{\mathcal{S}_i|_U}$ of $\mathcal{S}_i|_U$, i.e.*

$$\langle I_{\mathcal{S}}s, I_{\mathcal{S}}\tilde{s}\rangle_{\mathcal{S}_2|_U} = \epsilon\langle s, \tilde{s}\rangle_{\mathcal{S}_1|_U},$$

for all $s, \tilde{s} \in \Gamma(\mathcal{S}_1|_U)$, where $\epsilon \in \{\pm 1\}$ is independent of $s, \tilde{s}$.

Proof. From relation (42) and the fact that I_E is an isometry, we obtain that bilinear pairing $\langle s, \tilde{s}\rangle'_{\mathcal{S}_1|_U} := \langle I_{\mathcal{S}}(s), I_{\mathcal{S}}(\tilde{s})\rangle_{\mathcal{S}_2|_U}$ on $\mathcal{S}_1|_U$ satisfies (27). Also, $\det\langle\cdot,\cdot\rangle'_{\mathcal{S}_1|_U} = \det\langle\cdot,\cdot\rangle_{\mathcal{S}_2|_U} = 1$, if $\mathrm{rk}\,E_1 > 2$ (and $= -1$ if $\mathrm{rk}\,E_1 = 2$). □

From Lemma 22, $I_{\mathbb{S}|_U} := I_{\mathcal{S}|_U} \otimes \mathrm{Id}_{|\det T^*U|^{1/2}} : \mathbb{S}_1|_U \to \mathbb{S}_2|_U$ satisfies

$$\langle I_{\mathbb{S}|_U}(s), I_{\mathbb{S}|_U}(\tilde{s})\rangle_{\mathbb{S}_2|_U} = \epsilon\langle s, \tilde{s}\rangle_{\mathbb{S}_1|_U},\ \forall s, \tilde{s} \in \Gamma(\mathbb{S}_1|_U) \tag{45}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{S}_i|_U}$ are canonical pairings of the canonical weighted spinor bundles $\mathbb{S}_i|_U$ of $E_i|_U$ determined by $S_i|_U$ and $\epsilon \in \{\pm 1\}$.

Definition 23. *The isomorphisms I_S, $I_{\mathcal{S}|_U}$ and $I_{\mathbb{S}|_U}$ are called* isomorphisms induced by I (or compatible with I) on the spinor bundle S, canonical spinor bundle $\mathcal{S}$ *and* canonical weighted spinor bundle $\mathbb{S}$.

Notation 24. Since the isomorphism $I_{S|_U}$ is unique up to a multiplicative factor, the isomorphism $I_{\mathcal{S}|_U}$ (hence also $I_{\mathbb{S}}|_U$) is independent of the choice of $I_{S|_U}$, modulo multiplication by ± 1 (see also Remark 55 of [12]). In our computations we will often choose (without repeating it each time) one $I_{\mathcal{S}|_U}$, or $I_{\mathbb{S}|_U}$, and refer to it as the isomorphism induced by I (or compatible with I) on the canonical spinor bundle and canonical weighted spinor bundle respectively.

Remark 25. In the setting of Lemma 20, assume that $E_i = T^*M \oplus \mathcal{G}_i \oplus TM$ $(i = 1, 2)$ are standard Courant algebroids. Let $S_{\mathcal{G}_i}$ be irreducible $\mathrm{Cl}(\mathcal{G}_i)$-bundles of rank r and $S_i := \Lambda(T^*M)\hat{\otimes} S_{\mathcal{G}_i}$, which are irreducible $\mathrm{Cl}(E_i)$-bundles. Recall that $\mathbb{S}_i := \Lambda(T^*M)\hat{\otimes}\mathcal{S}_{\mathcal{G}_i}$ are the canonical weighted spinor bundles of E_i determined by S_i, where $\mathcal{S}_{\mathcal{G}_i}$ are the canonical spinor bundles of $S_{\mathcal{G}_i}$. For simplicity, we assume that $I_S : S_1 \to S_2$ is defined globally. Using (20), one can show that the isomorphism $I_{\mathbb{S}} : \mathbb{S}_1 \to \mathbb{S}_2$ induced by $I : E_1 \to E_2$ is given in terms of $I_S : S_1 \to S_2$ by

$$I_{\mathbb{S}}(\omega \otimes s \otimes |s_1^* \wedge \cdots \wedge s_r^*|^{\frac{1}{r}}) = |\det(I_S)|^{-\frac{1}{Nr}} I_S(\omega \otimes s) \otimes |\tilde{s}_1^* \wedge \cdots \wedge \tilde{s}_r^*|^{\frac{1}{r}}, \tag{46}$$

where $\omega \otimes s \in \Gamma(S_1)$, (s_i^*) and $(\tilde{s}_i^*)$ are local frames of $S^*_{\mathcal{G}_1}$ and $S^*_{\mathcal{G}_2}$ respectively, $N := \mathrm{rk}\,\Lambda(TM)$, $r := \mathrm{rk}\, S_{\mathcal{G}_i}$, and $\det(I_S)$ is the determinant of the representation matrix of I_S in the local frames $(\Omega_i \otimes s_j)$ and $(\Omega_i \otimes \tilde{s}_j)$ respectively, where (Ω_i) is the local frame of $\Lambda(T^*M)$ induced by a local frame of TM and (s_i), $(\tilde{s}_i)$ are the frames dual to (s_i^*) and $(\tilde{s}_i^*)$ respectively. We shall refer to $\det(I_S)$ as the determinant of I_S with respect to the local frames (s_i) and $(\tilde{s}_i)$.

Proposition 26. *In the setting of Lemma 20, let $\mathbb{S}_i$ be the canonical weighted spinor bundles of E_i determined by S_i. If $\not{d}_1 \in \mathrm{End}\,\Gamma(\mathbb{S}_1|_U)$ is the canonical Dirac generating operator of $E_1|_U$ then*

$$\not{d}_2 = I_{\mathbb{S}|_U} \circ \not{d}_1 \circ I^{-1}_{\mathbb{S}|_U} \in \mathrm{End}\,\Gamma(\mathbb{S}_2|_U) \tag{47}$$

is the canonical Dirac generating operator of $E_2|_U$.

Proof. Let $\nabla^{(1)}$ be a metric connection on $E_1|_U$ and ∇^{S_1} a connection on $S_1|_U$ compatible with $\nabla^{(1)}$. Let $D^{(1)}$ and D^{S_1} be the generalized connection on $E_1|_U$ and the $E_1|_U$-connection on $S_1|_U$ defined by $\nabla^{(1)}$ and ∇^{S_1} respectively. Let $\nabla^{(2)} := I_E \circ \nabla^{(1)} \circ I_E^{-1}$ and $\nabla^{S_2} := I_{S|_U} \circ \nabla^{S_1} \circ (I_{S|_U})^{-1}$. Then $\nabla^{(2)}$ is a metric connection on $E_2|_U$ and ∇^{S_2} is compatible with $\nabla^{(2)}$. Let $D^{(2)}$ and D^{S_2} be the generalized connection on $E_2|_U$ and the (compatible)

$E_2|_U$-connection on $S_2|_U$, defined by $\nabla^{(2)}$ and ∇^{S_2} respectively. As formula (2) for the canonical Dirac generating operator is independent of the choice of generalized connection and compatible E-connection, we can (and will) choose to compute $\not{d}_1$ and $\not{d}_2$ using the pairs $(D^{(1)}, D^{S_1})$ and $(D^{(2)}, D^{S_2})$ respectively.

A straightforward computation using (3) shows that

$$(D^{(2)})^L_{I_E(u)}\mu = (D^{(1)})^L_u \mu, \ \forall u \in \Gamma(E_1|_U), \ \mu \in \Gamma(L|_U) \tag{48}$$

and

$$(D^{\mathcal{S}_2} \otimes (D^{(2)})^L)_u = I_{\mathbb{S}|_U} \circ (D^{\mathcal{S}_1} \otimes (D^{(1)})^L)_{I_E^{-1}(u)} \circ (I_{\mathbb{S}|_U})^{-1}, \ \forall u \in \Gamma(E_1|_U), \tag{49}$$

where $D^{\mathcal{S}_i}$ are the $E_i|_U$-connections on $\mathcal{S}_i|_U$ induced by D^{S_i}. Relation (49) implies that the Dirac operators $\not{D}^{(2)}$ on $\mathbb{S}_2|_U$ and $\not{D}^{(1)}$ on $\mathbb{S}_1|_U$ computed with $D^{\mathcal{S}_2} \otimes (D^{(2)})^L$ and $D^{\mathcal{S}_1} \otimes (D^{(1)})^L$ respectively, are related by

$$\not{D}^{(2)} = I_{\mathbb{S}|_U} \circ \not{D}^{(1)} \circ (I_{\mathbb{S}|_U})^{-1}. \tag{50}$$

On the other hand, it is easy to see that

$$T^{D^{(2)}}(u, v, w) = T^{D^{(1)}}(I_E^{-1}u, I_E^{-1}v, I_E^{-1}w), \ \forall u, v, w \in \Gamma(E_2|_U)$$

which implies that

$$T^{D^{(2)}} = I_E(T^{D^{(1)}}) \tag{51}$$

where $T^{D^{(i)}} \in \Gamma(\Lambda^3 E_i|_U) \subset \Gamma\,\mathrm{Cl}(E_i|_U)$ and $I_E : \mathrm{Cl}(E_1|_U) \to \mathrm{Cl}(E_2|_U)$ denotes the map induced by the isometry I_E. Relations (42) and (51) imply that

$$\gamma_{T^{D^{(2)}}} = I_{\mathbb{S}|_U} \circ \gamma_{T^{D^{(1)}}} \circ (I_{\mathbb{S}|_U})^{-1}$$

which, together with (2) and (50), implies our claim. □

4.2. Pullback of spinors

Following [17], we recall the definition of pullback Courant algebroid. Let $f : M \to N$ be a submersion and E a transitive Courant algebroid over N. Let $\mathbb{T}M := T^*M \oplus TM$ be the generalized tangent bundle with its standard

Courant algebroid structure, i.e. anchor the natural projection from $\mathbb{T}M$ to TM, Dorfman bracket given by

$$[\xi + X, \eta + Y] := \mathcal{L}_X \eta - i_Y d\xi + \mathcal{L}_X Y, \tag{52}$$

for any $X, Y \in \mathfrak{X}(M)$, $\xi, \eta \in \Omega^1(M)$ and scalar product

$$\langle \xi + X, \eta + Y \rangle := \frac{1}{2}(\xi(Y) + \eta(X)).$$

Consider the direct product Courant algebroid $E \times \mathbb{T}M$ and let $a : E \times \mathbb{T}M \to T(N \times M)$ be its anchor. Then $C := a^{-1}(TM_f)$ is a coisotropic subbundle of $E \times \mathbb{T}M$ over the graph $M_f \subset N \times M$ of f, which we identify with M. Its fiber over $p \in M$ is given by

$$C_p := \{(u, \xi + X) \in E_{f(p)} \times \mathbb{T}_p M \mid \pi(u) = (d_p f)(X)\} \tag{53}$$

and its orthogonal complement (with respect to the scalar product of $E \times \mathbb{T}M$) is

$$C_p^{\perp} := \{(\frac{1}{2}\pi^* \gamma, -(d_p f)^* \gamma) \mid \gamma \in T^*_{f(p)} N\} \subset C_p, \tag{54}$$

where $\pi^* : T^*N \to E$ is the dual of the anchor $\pi : E \to TN$ composed with the natural identification $E^* \stackrel{\sim}{\to} E$ induced by the scalar product $\langle \cdot, \cdot \rangle$ of E. The quotient $C/C^{\perp}$ is a Courant algebroid over $M \cong M_f$ with anchor, scalar product and Courant bracket induced from $E \times \mathbb{T}M$ (see [17]). The Courant algebroid $C/C^{\perp}$ was called in [17] the pullback of E by the map f. We denote it by $f^! E$.

Lemma 27. *i) Let $E = T^*N \oplus \mathcal{G} \oplus TN$ be a standard Courant algebroid, defined by a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}}, \langle \cdot, \cdot \rangle_{\mathcal{G}})$ and data (∇, R, H). Then $f^! E$ is isomorphic to the standard Courant algebroid defined by the quadratic Lie algebra bundle*

$$(f^* \mathcal{G},\ [\cdot, \cdot]_{f^* \mathcal{G}} := f^* [\cdot, \cdot]_{\mathcal{G}},\ \langle \cdot, \cdot \rangle_{f^* \mathcal{G}} := f^* \langle \cdot, \cdot \rangle_{\mathcal{G}})$$

and data $(f^* \nabla, f^* R, f^* H)$.

ii) Let $I : E_1 \to E_2$ be an isomorphism between two transitive Courant algebroids over N and $a_i : E_i \times \mathbb{T}M \to T(N \times M)$ the anchors of the direct *product Courant algebroids $E_i \times \mathbb{T}M$ $(i = 1, 2)$. Then I induces an isomorphism between the pullback Courant algebroids $I^f : f^! E_1 \to f^! E_2$ defined by*

$$I^f[(u, \xi + X)] := [(I(u), \xi + X)],\ \forall (u, \xi + X) \in (C_1)_p, \tag{55}$$

where $C_i = (a_i)^{-1}(TM_f)$ and $[(I(u), \xi + X)]$ denotes the class of $(I(u), \xi + X) \in (C_2)_p$ modulo $(C_2)_p^{\perp}$.

iii) Let E be a transitive Courant algebroid over N. Any dissection of E induces a dissection of $f^! E$. Moreover, if $I_i : E \to T^*N \oplus \mathcal{G}_i \oplus TN$ are two dissections of E, related by (β, K, Φ), then the induced dissections of $f^! E$ are related by $(f^*\beta, f^*K, f^*\Phi)$.

Proof. i) We claim that the quadratic Lie algebra bundle $(f^*\mathcal{G}, [\cdot,\cdot]_{f^*\mathcal{G}}, \langle\cdot,\cdot\rangle_{f^*\mathcal{G}})$ together with $(f^*\nabla, f^*R, f^*H)$ define a standard Courant algebroid. The proof reduces to the verification of the conditions stated in Section 2.2.1. The form f^*R is defined by $(f^*R)(X, Y) = R((d_p f)X, (d_p f)Y) \in \mathcal{G}_{f(p)} = (f^*\mathcal{G})_p$, for any $X, Y \in T_pM$ and $p \in M$. To show, for instance, that

$$[d^{f^*\nabla}(f^*R)](X, Y, Z) = 0,\ \forall X, Y, Z \in \mathfrak{X}(M), \tag{56}$$

cf. equation (7), we notice that it holds for any projectable vector fields $X, Y, Z \in \mathfrak{X}(M)$, since

$$\begin{aligned}(f^*\nabla)_X[(f^*R)(Y, Z)] &= f^*[\nabla_{f_*X} R(f_*Y, f_*Z)],\\ (f^*R)(\mathcal{L}_X Y, Z) &= f^*[R(\mathcal{L}_{f_*X} f_*Y, f_*Z)]\end{aligned}$$

and that it is $C^\infty(M)$-linear in all arguments X, Y, Z. Here $f_*X \in \mathfrak{X}(N)$ is the projection of a projectable vector field $X \in \mathfrak{X}(M)$. Relation (56) follows. In a similar way we prove that $(f^*\mathcal{G}, [\cdot,\cdot]_{f^*\mathcal{G}}, \langle\cdot,\cdot\rangle_{f^*\mathcal{G}})$ together with $(f^*\nabla, f^*R, f^*H)$ satisfy the remaining conditions for standard Courant algebroids.

One can show that the map

$$F : T^*M \oplus f^*\mathcal{G} \oplus TM \to f^! E,\ F(\xi + r + X) := [(r + (d_p f)X, \xi + X)] \tag{57}$$

where $\xi \in T_p^*M$, $r \in \mathcal{G}_{f(p)}$, $X \in T_pM$ and $p \in M$ is arbitrary, is a Courant algebroid isomorphism between the standard Courant algebroid defined by the quadratic Lie algebra bundle $(f^*\mathcal{G}, [\cdot,\cdot]_{f^*\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ and $(f^*\nabla, f^*R, f^*H)$, and $f^! E$.

ii), iii) Claim ii) is an easy check and claim iii) follows by combining claims i) and ii) and using (9). $\square$

Notation 28. Owing to Lemma 27 i), we shall often identify (sometimes, when it is clear from the context, without stating it explicitly), the pullback $f^! E$ of a standard Courant algebroid E defined by a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data (∇, R, H), with the standard Courant algebroid defined by the quadratic Lie algebra bundle $(f^*\mathcal{G}, f^*[\cdot,\cdot]_{\mathcal{G}}, f^*\langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data $(f^*\nabla, f^*R, f^*H)$.

Our next aim is to define a pullback from spinors of E to spinors of $f^! E$. At first, we assume that $E = T^*N \oplus \mathcal{G} \oplus TN$ is a standard Courant algebroid, defined by a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data (∇, R, H). As mentioned above, we identify $f^! E$ with the standard Courant algebroid $T^*M \oplus f^*\mathcal{G} \oplus TM$ defined by the quadratic Lie algebra bundle $(f^*\mathcal{G}, f^*[\cdot,\cdot]_{\mathcal{G}}, f^*\langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data $(f^*\nabla, f^*R, f^*H)$. We fix an irreducible $\mathrm{Cl}(\mathcal{G})$-bundle $S_{\mathcal{G}}$. Then $S_{f^*\mathcal{G}} := f^* S_{\mathcal{G}}$ is an irreducible $\mathrm{Cl}(f^*\mathcal{G})$-bundle and its canonical spinor bundle is the pullback of the canonical spinor bundle of $S_{\mathcal{G}}$: $\mathcal{S}_{f^*\mathcal{G}} = f^*\mathcal{S}_{\mathcal{G}}$. The canonical weighted spinor bundles of E and $f^! E$ determined by the irreducible $\mathrm{Cl}(E)$ and $\mathrm{Cl}(f^! E)$-bundles $\Lambda(T^*N)\hat{\otimes} S_{\mathcal{G}}$ and $\Lambda(T^*M)\hat{\otimes} f^* S_{\mathcal{G}}$ are given by

$$\mathbb{S}_N := \Lambda(T^*N)\hat{\otimes}\mathcal{S}_{\mathcal{G}},\ \mathbb{S}_M := \Lambda(T^*M)\hat{\otimes} f^*\mathcal{S}_{\mathcal{G}}. \tag{58}$$

Definition 29. *The (degree-preserving) natural map*

$$f^* : \Gamma(\mathbb{S}_N) \to \Gamma(\mathbb{S}_M),\ \omega \otimes s \mapsto f^*(\omega) \otimes f^*(s),$$

where $\omega \in \Omega(N)$ and $s \in \Gamma(\mathcal{S}_{\mathcal{G}})$, is called the pullback on spinors.

Remark 30. In the above setting, assume that $f : M \to N$ is endowed with a horizontal distribution. For any $X \in \mathfrak{X}(N)$, we denote by $\widehat{X} \in \mathfrak{X}(M)$ the horizontal lift of X. We define

$$f^* : \Gamma(E) \to \Gamma(f^! E),\ f^*(\xi + r + X) := f^*(\xi + r) + \widehat{X}. \tag{59}$$

Let $\langle\cdot,\cdot\rangle_E$ and $\langle\cdot,\cdot\rangle_{f^! E}$ be the scalar products of E and $f^! E$. As

$$\langle f^*(u), f^*(v)\rangle_{f^! E} = \langle u, v\rangle_E \circ f,\ \forall u, v \in \Gamma(E),$$

we obtain an induced map $f^* : \Gamma\,\mathrm{Cl}(E) \to \Gamma\,\mathrm{Cl}(f^! E)$, which satisfies

$$f^*(u \cdot v) = f^*(u) \cdot f^*(v),\ \forall u, v \in \Gamma\,\mathrm{Cl}(E) \tag{60}$$

and

$$f^*(u \cdot s) = f^*(u) \cdot f^*(s), \ \forall u \in \Gamma\mathrm{Cl}(E), \ s \in \Gamma(\mathbb{S}_N). \tag{61}$$

Assume now that E is a transitive, but not necessarily standard, Courant algebroid and let $\mathbb{S}_E$ and $\mathbb{S}_{f^!E}$ be canonical weighted spinor bundles of E and $f^!E$, determined by irreducible spinor bundles S_E and $S_{f^!E}$ respectively. In order to define a pullback map from $\Gamma(\mathbb{S}_E)$ to $\Gamma(\mathbb{S}_{f^!E})$, the spinor bundles S_E and $S_{f^!E}$ need to be related in a suitable way. This relation is expressed in the next definition.

Definition 31. *An* admissible pair *for* $\mathbb{S}_E$ *and* $\mathbb{S}_{f^!E}$ *is a pair* $(I, S_{\mathcal{G}})$ *formed by a dissection* $I : E \to E_N = T^*N \oplus \mathcal{G} \oplus TN$ *of* E *and an irreducible* $\mathrm{Cl}(\mathcal{G})$*-bundle* $S_{\mathcal{G}}$*, such that* I *and the induced dissection*

$$I^f : f^!E \to E_M = T^*M \oplus f^*\mathcal{G} \oplus TM$$

of $f^!E$ *(see Lemma 27 iii)) induce* global *isomorphisms*

$$I_S : S_E \to \Lambda(T^*N)\hat{\otimes} S_{\mathcal{G}}, \ I_S^f : S_{f^!E} \to \Lambda(T^*M)\hat{\otimes} f^*S_{\mathcal{G}}$$

between spinor bundles.

Assume that $\mathbb{S}_E$ and $\mathbb{S}_{f^!E}$ admit an admissible pair $(I, S_{\mathcal{G}})$. In the notation of Definition 31 let

$$I_{\mathbb{S}} : \mathbb{S}_E \to \mathbb{S}_N = \Lambda(T^*N)\hat{\otimes}\mathcal{S}_{\mathcal{G}}, \ I_{\mathbb{S}}^f : \mathbb{S}_{f^!E} \to \mathbb{S}_M = \Lambda(T^*M)\hat{\otimes} f^*\mathcal{S}_{\mathcal{G}} \tag{62}$$

be the (global) isomorphisms induced by I and I^f on the canonical weighted spinor bundles determined by S_E, $\Lambda(T^*N)\hat{\otimes}S_{\mathcal{G}}$, $S_{f^!E}$ and $\Lambda(T^*M)\hat{\otimes}f^*S_{\mathcal{G}}$.

Lemma 32. *The map*

$$f^! : \Gamma(\mathbb{S}_E) \to \Gamma(\mathbb{S}_{f^!E}), \ f^! := (I_{\mathbb{S}}^f)^{-1} \circ f^* \circ I_{\mathbb{S}} \tag{63}$$

is well defined (i.e. independent on the choice of admissible pair) up to multiplication by ± 1.

Proof. Consider an isomorphism

$$I : E_1 = T^*N \oplus \mathcal{G}_1 \oplus TN \to E_2 = T^*N \oplus \mathcal{G}_2 \oplus TN \tag{64}$$

between standard Courant algebroids and

$$I^f : f^! E_1 = T^*M \oplus f^*\mathcal{G}_1 \oplus TM \to f^! E_2 = T^*M \oplus f^*\mathcal{G}_2 \oplus T^*M$$

the induced isomorphism between their pullbacks. Let $S_{\mathcal{G}_i}$ be irreducible $\mathrm{Cl}(\mathcal{G}_i)$-bundles, such that I and I^f induce global isomorphisms

$$I_{S_N} : S^1_N \to S^2_N,\ I^f_{S_M} : S^1_M \to S^2_M$$

between spinor bundles

$$S^i_N := \Lambda(T^*N)\hat{\otimes} S_{\mathcal{G}_i},\ S^i_M := \Lambda(T^*M)\hat{\otimes} f^* S_{\mathcal{G}_i}. \tag{65}$$

By considering two admissible pairs for $\mathbb{S}_E$ and $\mathbb{S}_{f^! E}$ the claim reduces to showing that

$$I^f_{\mathbb{S}_{\mathbb{M}}} \circ f^* = \epsilon f^* \circ I_{\mathbb{S}_N} \tag{66}$$

where $\epsilon \in \{\pm 1\}$,

$$I_{\mathbb{S}_N} : \mathbb{S}^1_N \to \mathbb{S}^2_N,\ I^f_{\mathbb{S}_M} : \mathbb{S}^1_M \to \mathbb{S}^2_M \tag{67}$$

are the isomorphisms induced by I_{S_N} and $I^f_{S_M}$ on the canonical weighted spinor bundles

$$\mathbb{S}^i_N = \Lambda(T^*N)\hat{\otimes}\mathcal{S}_{\mathcal{G}_i},\ \mathbb{S}^i_M = \Lambda(T^*M)\hat{\otimes} f^*\mathcal{S}_{\mathcal{G}_i}$$

determined by S^i_N and S^i_M and $f^* : \mathbb{S}^i_N \to \mathbb{S}^i_M$ are defined by (59). In order to prove (66) we fix a distribution $\mathcal{D} \subset TM$ complementary to $\mathrm{Ker}\, df$ and we decompose orthogonally $f^! E_i = V_i^+ \oplus V^-$, where V_i^+ and V^- are given by

$$\begin{aligned}(V_i^+)_p &= \mathcal{D}^*_p \oplus (\mathcal{G}_i)_{f(p)} \oplus \mathcal{D}_p\\ (V^-)_p &= (\mathrm{Ker}\, d_p f)^* \oplus \mathrm{Ker}\, d_p f,\end{aligned}$$

for any $p \in M$. Consider the spinor bundles

$$S_i^+ := \Lambda\, \mathcal{D}^* \hat{\otimes} f^* S_{\mathcal{G}_i},\ S^- = \Lambda\, (\mathrm{Ker}\, df)^* \tag{68}$$

of V_i^+ and V^-. Then

$$\bar{S}^i_M := S_i^+ \hat{\otimes} S^- \tag{69}$$

is a spinor bundle of $f^! E_i = V_i^+ \oplus V_-$, isomorphic to the spinor bundle S^i_M (defined in (65)) via the $\mathrm{Cl}(f^! E_i)$- bundle isomorphism

$$T_i : \bar{S}^i_M \to S^i_M,\ T_i((\omega \otimes s) \otimes \eta) = (-1)^{|s||\eta|}(\omega \wedge \eta) \otimes s \tag{70}$$

where $\omega \in \Lambda\, \mathcal{D}^*$ and $s \in f^* S_{\mathcal{G}_i}$, $\eta \in S^-$ are homogeneous.

Assume that I is defined by (β, K, Φ) as in Section 2.2.1. Then, from Lemma 27 iii), I^f is defined by $(f^*\beta, f^*K, f^*\Phi)$ and acts as the identity on V^- while its restriction $I^{f_+} := I^f|_{V_1^+} : V_1^+ \to V_2^+$ satisfies

$$(I^{f_+})_p(f^* u) = f^*(I_{f(p)}(u)),\ \forall u \in (E_1)_{f(p)},\ p \in N, \tag{71}$$

where $f^* : (E_i)_{f(p)} \to (V_i^+)_p$ are given by (59), constructed using the distribution $\mathcal{D}$. We deduce that the isomorphism $I^f_{\bar{S}_M} : \bar{S}^1_M \to \bar{S}^2_M$ induced by I^f is given by

$$I^f_{\bar{S}_M}(s \otimes \eta) = (-1)^{|\eta||I^{f_+}_{S_+}|} I^{f_+}_{S_+}(s) \otimes \eta,\ \forall s \in S_1^+,\ \eta \in S^- \tag{72}$$

where $I^{f_+}_{S_+} : S_1^+ \to S_2^+$ is the isomorphism induced by $I^{f_+} : V_1^+ \to V_2^+$. Combining (70) and (72) we obtain an isomorphism

$$I^f_{S_M} : S^1_M \to S^2_M,\ I^f_{S_M} := T_2 \circ I^f_{\bar{S}_M} \circ T_1^{-1}$$

compatible with I^f. Since it maps S_1^+ onto S_2^+ we can define

$$I_{S_N} := (f^*)^{-1} \circ I^f_{S_M} \circ f^* : S^1_N \to S^2_N. \tag{73}$$

where $f^* : S^1_N \to S_1^+$ and $(f^*)^{-1} : S_2^+ \to S^2_N$ are induced by pullback. It is easy to check that I_{S_N} is compatible with I. We will show that the isomorphisms $I^f_{\mathbb{S}_M} : \mathbb{S}^1_M \to \mathbb{S}^2_M$ and $I_{\mathbb{S}_N} : \mathbb{S}^1_N \to \mathbb{S}^2_N$ induced by $I^f_{S_M}$ and I_{S_N} are related by (66). For this, we use Remark 25. Let (s_i), $(\tilde{s}_i)$ be local frames of $\mathcal{S}_{\mathcal{G}_1}$, $\mathcal{S}_{\mathcal{G}_2}$ and (s_i^*), $(\tilde{s}_i^*)$ the dual frames. From Remark 25,

$$\begin{aligned} & I^f_{\mathbb{S}_M}((\omega \otimes s) \otimes |f^* s_1^* \wedge \cdots \wedge f^* s_r^*|^{1/r}) \\ & = I^f_{S_M}(\omega \otimes s) \otimes |f^* \tilde{s}_1^* \wedge \cdots \wedge f^* \tilde{s}_r^*|^{1/r} |\det(I^f_{S_M})|^{-\frac{1}{r N_h N_v}} \end{aligned} \tag{74}$$

where $\omega \in \Lambda(T^*M)$, $s \in f^* S_{\mathcal{G}_1}$, $N_h := \mathrm{rk}\,(\Lambda \mathcal{D})$, $N_v := \mathrm{rk}\,(\Lambda\,\mathrm{Ker}\, df)$, $r := \mathrm{rk}\, S_{\mathcal{G}_i}$ and $\det(I^f_{S_M})$ denotes the determinant of $I^f_{S_M}$ with respect to the local frames $(f^* s_i)$ and $(f^* \tilde{s}_i)$. Similarly,

$$
\begin{aligned}
& I_{\mathbb{S}_N}((\omega \otimes s) \otimes |s_1^* \wedge \cdots \wedge s_r^*|^{1/r}) \\
(75) \qquad & \quad = I_{S_N}(\omega \otimes s) \otimes |\tilde{s}_1^* \wedge \cdots \wedge \tilde{s}_r^*|^{1/r} |\det(I_{S_N})|^{-\frac{1}{rN_h}}
\end{aligned}
$$

where $\omega \in \Lambda(T^* N)$, $s \in S_{\mathcal{G}_1}$ and $\det(I_{S_N})$ denotes the determinant of I_{S_N} with respect to the local frames (s_i) and $(\tilde{s}_i)$. Relation (66) follows from (74) and (75), by noticing that

$$
(76) \qquad \det(I^f_{S_M}) = \det(I_{S_N})^{N_v} \circ f
$$

In order to prove the latter relation we remark, from $I^f|_{T^*M} = \mathrm{Id}_{T^*M}$, the compatibility of $I^f_{S_M}$ with I^f and relation (73), that

(77)

$$
I^f_{S_M}(\eta \wedge f^*(\omega \otimes s)) = \eta \wedge f^* I_{S_N}(\omega \otimes s), \ \forall \eta \in \Lambda \,(\mathrm{Ker}\, df)^*, \ \omega \otimes s \in S^1_N.
$$

The above relation implies that $I^f_{S_M}$ is block diagonal when we decompose $S^i_M = \sum_{k=0}^{n_v} \Lambda^k \,(\mathrm{Ker}\, df)^* \otimes f^* S^i_N$ (where $n_v := \mathrm{rk}\,\mathrm{Ker}(df)$) and, restricted to each block $\Lambda^k \,(\mathrm{Ker}\, df)^* \otimes f^* S^1_N$, it is given by $\mathrm{Id}_{\Lambda^k\,(\mathrm{Ker}\, df)^*} \otimes f^* I_{S_N}$. Relation (76) follows. □

Definition 33. *The map $f^! : \Gamma(\mathbb{S}_E) \to \Gamma(\mathbb{S}_{f^!E})$ defined by (63) is called the* pullback on spinors*.*

Proposition 34. *Let $f : M \to N$ be a submersion, E a transitive Courant algebroid over N and $\mathbb{S}_E$, $\mathbb{S}_{f^!E}$ canonical weighted spinor bundles of E and $f^!E$ such that the pullback $f^! : \Gamma(\mathbb{S}_E) \to \Gamma(\mathbb{S}_{f^!E})$ is defined. Let $\not{d}_E \in \mathrm{End}\,\Gamma(\mathbb{S}_E)$ and $\not{d}_{f^!E} \in \mathrm{End}\,\Gamma(\mathbb{S}_{f^!E})$ be the canonical Dirac generating operators of E and $f^!E$. Then*

$$
(78) \qquad f^! \circ \not{d}_E = \not{d}_{f^!E} \circ f^!.
$$

Proof. From the invariance of the canonical Dirac generating operators under isomorphisms (see Proposition 26) and the definition (63) of the map $f^!$, it is sufficient to prove (78) in the setting of standard Courant algebroids. With the notation before Definition 29, we need to show that

$$
(79) \qquad f^* \circ \not{d}_N = \not{d}_M \circ f^* : \Gamma(\mathbb{S}_N) \to \Gamma(\mathbb{S}_M),
$$

where $\not{d}_N$ and $\not{d}_M$ are the canonical Dirac generating operators of the standard Courant algebroids E and $f^!E$, which can be computed using (22). Let

m and n be the dimensions of M and N respectively. Let $(X_i)_{1\leq i\leq m}$ be a local frame of TM such that $(X_i)_{1\leq i\leq n}$ are projectable and their projections $(f_*X_i)_{i\leq n}$ form a local frame of TN and $(X_i)_{n+1\leq i\leq m}$ are vertical. Let $(\alpha_i)_{1\leq i\leq n}$ be the dual frame of $(f_*X_i)_{1\leq i\leq n}$. Then, using $f_*X_i=0$ for any $i\geq n+1$,

$$\begin{aligned}&\bar{R}^{f^!E}(f^*(\omega\otimes s))\\&\quad=\frac{1}{2}\sum_{i,j\leq n}\langle f^*(R(f_*X_i,f_*X_j)),f^*r_k\rangle_{f^*\mathcal{G}}f^*(\alpha_i\wedge\alpha_j\wedge\omega)\otimes(f^*(r_k)\cdot f^*(s))\end{aligned}$$

that is,

$$(\bar{R}^{f^!E}\circ f^*)(\omega\otimes s)=(f^*\circ\bar{R}^E)(\omega\otimes s),\ \forall\omega\otimes s\in\Gamma(\mathbb{S}_N).\tag{80}$$

On the other hand, if $\nabla^{S_\mathcal{G}}$ is compatible with ∇ then $\nabla^{S_{f^*\mathcal{G}}}:=f^*\nabla^{S_\mathcal{G}}$ is compatible with $f^*\nabla$. The induced connections on $\mathcal{S}_{f^*\mathcal{G}}$ and $\mathcal{S}_\mathcal{G}$ are related similarly by pullback

$$\nabla^{\mathcal{S}_{f^*\mathcal{G}}}=f^*\nabla^{\mathcal{S}_\mathcal{G}}.\tag{81}$$

Relations (80), (81), $C_{f^*\mathcal{G}}=f^*C_\mathcal{G}$ and the expression of the canonical Dirac generating operator (22) imply (79). □

4.3. Pushforward on spinors

Let $f:M\to N$ be a fiber bundle with compact fibers and M, N oriented. Let E be a transitive Courant algebroid over N. In this section we define a pushforward from spinors of $f^!E$ to spinors of E. As for the pullback, we assume first that E is a standard Courant algebroid over N defined by a quadratic Lie algebra bundle $(\mathcal{G},[\cdot,\cdot]_\mathcal{G},\langle\cdot,\cdot\rangle_\mathcal{G})$ and data (∇,R,H). We preserve the notation introduced before Definition 29. As we did there, we choose an irreducible $\mathrm{Cl}(\mathcal{G})$-bundle $S_\mathcal{G}$, with canonical spinor bundle $\mathcal{S}_\mathcal{G}$. Consider an open cover $\mathcal{U}=\{U_i\}$ of N and, for any $U_i\in\mathcal{U}$, a canonical pairing $\langle\cdot,\cdot\rangle_{\mathcal{S}_\mathcal{G}|_{U_i}}$ on $\mathcal{S}_\mathcal{G}|_{U_i}$. We define $\langle\cdot,\cdot\rangle_{f^*\mathcal{S}_\mathcal{G}|_{f^{-1}(U_i)}}:=f^*\langle\cdot,\cdot\rangle_{\mathcal{S}_\mathcal{G}|_{U_i}}$, which is a canonical pairing on $\mathcal{S}_{f^*\mathcal{G}}|_{f^{-1}(U_i)}$. We denote by $\langle\cdot,\cdot\rangle_{\mathbb{S}_N|_{U_i}}$ and $\langle\cdot,\cdot\rangle_{\mathbb{S}_M|_{f^{-1}(U_i)}}$ the corresponding canonical pairings on $\mathbb{S}_N|_{U_i}$ and $\mathbb{S}_M|_{f^{-1}(U_i)}$, see Lemma 18. For any $U_i\in\mathcal{U}$, let

$$f_*^{U_i}:\Gamma(\mathbb{S}_M|_{f^{-1}(U_i)})=\Omega(f^{-1}(U_i),f^*\mathcal{S}_\mathcal{G})\to\Gamma(\mathbb{S}_N|_{U_i})=\Omega(U_i,\mathcal{S}_\mathcal{G})\tag{82}$$

be defined by

$$\int_{U_i} \langle f_*^{U_i} s_1, s_2 \rangle_{\mathbb{S}_N|_{U_i}} = \int_{f^{-1}(U_i)} \langle s_1, f^* s_2 \rangle_{\mathbb{S}_M|_{f^{-1}(U_i)}}, \tag{83}$$

for all $s_1 \in \Gamma(\mathbb{S}_M|_{f^{-1}(U_i)})$ and $s_2 \in \Gamma_c(\mathbb{S}_N|_{U_i})$, where $\Gamma_c(V)$ denotes the space of compactly supported sections of a vector bundle V. Using the maps $f_*^{U_i}$ and a partition of unity $\{\lambda_i\}$ of $\mathcal{U}$ we obtain a map

$$f_* : \Gamma(\mathbb{S}_M) \to \Gamma(\mathbb{S}_N), \;\; f_* s := \sum_i \lambda_i f_*^{U_i}(s|_{f^{-1}(U_i)}), \tag{84}$$

for any $s \in \Gamma(\mathbb{S}_M)$. This map is well-defined (i.e. independent of choices, see Lemma 37 below).

Definition 35. *The map (84) is called the* pushforward *on spinors.*

Remark 36. Recall that the pushforward on forms $f_* : \Omega(M) \to \Omega(N)$ has the properties

$$f_* \circ d = d \circ f_*, \;\; f_*((f^*\alpha) \wedge \beta) = \alpha \wedge f_*\beta, \;\; \int_M (f^*\alpha) \wedge \beta = \int_N \alpha \wedge f_*\beta. \tag{85}$$

Let U be a local chart over which the fiber bundle $f : M \to N$ is trivial. Then we can identify $f^{-1}(U)$ with $U \times F$, where F is the compact fiber. The decomposition $U \times F$ induces a bigrading on $\Lambda T_p^* M = \Lambda T_x^* U \otimes \Lambda T_t^* F = \bigoplus_{k,\ell} \Lambda^k T_x^* U \otimes \Lambda^\ell T_t^* F$ for all $p = (x,t) \in U \times F$. Then $f_*\omega = 0$ for every differential form ω on $U \times F$ of type (k,ℓ), $\ell \neq r = \dim F$. Choosing a positively oriented volume form vol_F on the fiber F, we can write every differential form of type (k,r) as $\omega = h\omega_U \wedge \mathrm{vol}_F$, where h is a function on $U \times F$ and ω_U is a differential form on U. Then

$$f_*\omega = \omega_U \int_F h(x,t)\mathrm{vol}_F(t). \tag{86}$$

So f_* is simply integration over the fibers.

The next lemma provides a concrete formulation for the pushforward on spinors in terms of the pushforward on forms.

Lemma 37. *For any $\omega \otimes f^*s \in \Gamma(\mathbb{S}_M)$ such that s is homogenous,*

$$f_*(\omega \otimes f^*s) = (-1)^{r|s|+nr+\frac{r(r-1)}{2}}(f_*\omega) \otimes s, \tag{87}$$

where n and r are the dimensions of N and the fibers of f, respectively. In particular, the map f_* is well-defined (i.e. independent on the choice of $\mathcal{U}$, partition of unity $\{\lambda_i\}$ and canonical pairings $\langle\cdot,\cdot\rangle_{\mathcal{S}_{\mathcal{G}}|_{U_i}}$).

Proof. We show that for any $\omega \otimes f^*s \in \Gamma(\mathbb{S}_M|_{f^{-1}(U_i)})$ with s homogeneous and $\tilde{\omega} \otimes \tilde{s} \in \Gamma_c(\mathbb{S}_N|_{U_i})$,

$$\begin{aligned}\int_{U_i} &\langle (f_*\omega) \otimes s, \tilde{\omega} \otimes \tilde{s}\rangle_{\mathbb{S}_N|_{U_i}} \\ &= (-1)^{r|s|+nr+\frac{r(r-1)}{2}} \int_{f^{-1}(U_i)} \langle \omega \otimes f^*s, f^*(\tilde{\omega} \otimes \tilde{s})\rangle_{\mathbb{S}_M|_{f^{-1}(U_i)}}.\end{aligned} \tag{88}$$

In order to prove (88), we assume, without loss of generality, that ω, $\tilde{\omega}$ and $\tilde{s}$ are also homogeneous. If $|\omega| + |\tilde{\omega}| \neq m$ (where $m := n + r$) both terms in (88) vanish. Assume now that $|\omega| + |\tilde{\omega}| = m$. Then, applying (35),

$$\begin{aligned}\int_{f^{-1}(U_i)} &\langle \omega \otimes f^*s, f^*(\tilde{\omega} \otimes \tilde{s})\rangle_{\mathbb{S}_M|_{f^{-1}(U_i)}} \\ &= (-1)^{|s|m+|\omega||\tilde{\omega}|} \int_{f^{-1}(U_i)} f^*(\langle s, \tilde{s}\rangle_{\mathcal{S}_{\mathcal{G}}}\tilde{\omega}) \wedge \omega^t \\ &= (-1)^{|s|m+|\omega||\tilde{\omega}|} \int_{U_i} \langle s, \tilde{s}\rangle_{\mathcal{S}_{\mathcal{G}}}\tilde{\omega} \wedge f_*(\omega^t) \\ &= (-1)^{r(m-\frac{r+1}{2}-|s|)} \int_{U_i} \langle (f_*\omega) \otimes s, \tilde{\omega} \otimes \tilde{s}\rangle_{\mathbb{S}_N|_{U_i}},\end{aligned}$$

where we used $f_*(\omega^t) = (f_*\omega)^t(-1)^{\frac{r(r-1)}{2}+r(|\omega|-r)}$, which can be checked using (86) and the third property (85) of the pushforward on forms. Relation (88) is proved and implies (87). □

Remark 38. In the above setting, assume that f is endowed with a horizontal distribution, like in Remark 30. Then the pullback map $f^* : \Gamma\mathrm{Cl}(E) \to \Gamma\mathrm{Cl}(f^!E)$ is defined (see relation (59)) and

$$f_*(f^*(u) \cdot s) = u \cdot f_*s,\ \forall u \in \Gamma\mathrm{Cl}(E),\ s \in \Gamma(\mathbb{S}_M). \tag{89}$$

Relation (89) with $u \in \Gamma(E)$ follows from the definition (18) of the Clifford action and for arbitrary $u \in \Gamma\,\mathrm{Cl}(E)$ by iteration. Relation (89) with $u \in \Gamma\Lambda(T^*N \oplus \mathcal{G})$ is independent of the horizontal distribution.

Assume now that E is a transitive, but not necessarily standard, Courant algebroid. Then we can define the pushforward $f_! : \Gamma(\mathbb{S}_{f^! E}) \to \Gamma(\mathbb{S}_E)$ for any canonical weighted spinor bundles $\mathbb{S}_E$ and $\mathbb{S}_{f^! E}$, for which the pullback $f^! : \Gamma(\mathbb{S}_E) \to \Gamma(\mathbb{S}_{f^! E})$ is defined.

Definition 39. *Assume there is an admissible pair $(I : E \to T^*N \oplus \mathcal{G} \oplus TN, S_{\mathcal{G}})$ for the canonical weighted spinor bundles $\mathbb{S}_E$ and $\mathbb{S}_{f^! E}$. The map*

$$f_! : \Gamma(\mathbb{S}_{f^! E}) \to \Gamma(\mathbb{S}_E),\ f_! := (I_{\mathbb{S}})^{-1} \circ f_* \circ I^f_{\mathbb{S}} \tag{90}$$

is called the pushforward on spinors. *Above $f_* : \Gamma(\mathbb{S}_M) \to \Gamma(\mathbb{S}_N)$ is the map (84) and*

$$I_{\mathbb{S}} : \mathbb{S}_E \to \Lambda\,(T^*N)\hat{\otimes}\mathcal{S}_{\mathcal{G}},\ I^f_{\mathbb{S}} : \mathbb{S}_{f^! E} \to \Lambda\,(T^*M)\hat{\otimes} f^*\mathcal{S}_{\mathcal{G}}$$

are the (globally defined) isomorphisms induced by I and I^f.

In particular, (90) is well defined, up to multiplication by ± 1.

Proposition 40. *The pushforward $f_! : \Gamma(\mathbb{S}_{f^! E}) \to \Gamma(\mathbb{S}_E)$ commutes with the canonical Dirac generating operators $\not{d}_E \in \mathrm{End}\,\Gamma(\mathbb{S}_E)$ and $\not{d}_{f^! E} \in \mathrm{End}\,\Gamma(\mathbb{S}_{f^! E})$, i. e.*

$$f_! \circ \not{d}_{f^! E} = \not{d}_E \circ f_!.$$

Proof. Like in the proof of Proposition 34, it is sufficient to show that

$$f_* \circ \not{d}_M = \not{d}_N \circ f_*, \tag{91}$$

where we preserve the notation from the proof of that proposition. Using the expression (24) of the canonical Dirac generating operator and relation (89) with $u := H$, $C_{\mathcal{G}}$ and $\tilde{r}_k \cdot \alpha_i \cdot \alpha_j$, we obtain that

$$\not{d}_N f_*(\tilde{\omega} \otimes f^* s) = f_* \not{d}_M(\tilde{\omega} \otimes f^* s),\ \forall \tilde{\omega} \in \Omega(M),\ s \in \Gamma(\mathcal{S}_{\mathcal{G}}) \tag{92}$$

reduces to

$$f_* \mathcal{E}_M(\tilde{\omega} \otimes f^* s) = \mathcal{E}_N f_*(\tilde{\omega} \otimes f^* s),\ \forall \tilde{\omega} \in \Omega(M),\ s \in \Gamma(\mathcal{S}_{\mathcal{G}}), \tag{93}$$

where

$$\mathcal{E}_N(\omega \otimes s) := (d\omega) \otimes s + \sum_i (\alpha_i \wedge \omega) \otimes \nabla^{\mathcal{S}_{\mathcal{G}}}_{X_i} s$$
$$\mathcal{E}_M(\tilde{\omega} \otimes f^* s) := (d\tilde{\omega}) \otimes f^* s + \sum_i ((f^*\alpha_i) \wedge \tilde{\omega}) \otimes f^*(\nabla^{\mathcal{S}_{\mathcal{G}}}_{X_i} s),$$

for any $\omega \in \Omega(N)$, $\tilde{\omega} \in \Omega(M)$ and $s \in \Gamma(\mathcal{S}_{\mathcal{G}})$, where (X_i) is a local frame of TN with dual frame (α_i). In order to show (93) it is sufficient to show that for any $U \subset N$ open and sufficiently small and $\beta \in \Gamma_c(\mathbb{S}_N|_U)$,

$$\int_U \langle \mathcal{E}_N f^U_*(\tilde{\omega} \otimes f^* s), \beta \rangle_{\mathbb{S}_N|_U} = \int_{f^{-1}(U)} \langle \mathcal{E}_M(\tilde{\omega} \otimes f^* s), f^*\beta \rangle_{\mathbb{S}_M|_{f^{-1}(U)}}. \tag{94}$$

From Corollary 19 and $f^* \circ \mathcal{E}_N = \mathcal{E}_M \circ f^*$ we have

$$\begin{aligned}
\int_U \langle \mathcal{E}_N f^U_*(\tilde{\omega} \otimes f^* s), \beta \rangle_{\mathbb{S}_N|_U} &= -\int_U \langle f^U_*(\tilde{\omega} \otimes f^* s), \mathcal{E}_N(\beta) \rangle_{\mathbb{S}_N|_U} \\
&= -\int_{f^{-1}(U)} \langle \tilde{\omega} \otimes f^* s, f^* \mathcal{E}_N(\beta) \rangle_{\mathbb{S}_M|_{f^{-1}(U)}} \\
&= -\int_{f^{-1}(U)} \langle \tilde{\omega} \otimes f^* s, \mathcal{E}_M(f^*\beta) \rangle_{\mathbb{S}_M|_{f^{-1}(U)}} \\
&= \int_{f^{-1}(U)} \langle \mathcal{E}_M(\tilde{\omega} \otimes f^* s), f^*\beta \rangle_{\mathbb{S}_M|_{f^{-1}(U)}},
\end{aligned}$$

which proves (94). Here $f^U : f^{-1}(U) \to U$ denotes the restriction of f to $f^{-1}(U)$. □

5. Actions on transitive Courant algebroids

5.1. Basic properties

In this section we consider a class of actions on a transitive Courant algebroid which generalizes torus actions on exact and, more generally, on heterotic Courant algebroids. For the latter types of Courant algebroids, a notion of T-duality has been developed in [10] and [2] respectively.

Let E be a transitive Courant algebroid over a manifold M, with anchor $\pi : E \to TM$, Dorfman bracket $[\cdot, \cdot]$ and scalar product $\langle \cdot, \cdot \rangle$. Recall that the automorphism group $\mathrm{Aut}(E)$ of E is the group of orthogonal automorphisms

$F : E \to E$ which cover a diffeomorphism $f : M \to M$, such that

$$\pi(F(u)) = (d_p f)\pi(u),\ \forall u \in E_p,\ p \in M$$

and the natural map induced by F on the space of sections of E preserves the Dorfman bracket. The Lie algebra of $\mathrm{Aut}(E)$ is the Lie algebra $\mathrm{Der}(E)$ of derivations of E. This is the subalgebra of $\mathrm{End}\,\Gamma(E)$ of first order linear differential operators $D : \Gamma(E) \to \Gamma(E)$ which satisfy, for any $s, s_1, s_2 \in \Gamma(E)$,

$$\begin{aligned} &D[s_1, s_2] = [Ds_1, s_2] + [s_1, Ds_2]\\ &X\langle s_1, s_2\rangle = \langle Ds_1, s_2\rangle + \langle s_1, Ds_2\rangle\\ &\pi \circ D(s) = \mathcal{L}_X \pi(s), \end{aligned} \tag{95}$$

where $X \in \mathfrak{X}(M)$ is a vector field on M, uniquely determined by D (from the second relation (95)) and usually denoted by $\pi(D)$.

Let $\mathfrak{g}$ be a Lie algebra acting on M by an infinitesimal action

$$\psi : \mathfrak{g} \to \mathfrak{X}(M),\ a \mapsto \psi(a) = X_a.$$

We will always assume (without repeating it each time) that all the infinitesimal actions considered are free, which means that the fundamental vector fields X_a are non-vanishing, for all $a \in \mathfrak{g} \setminus \{0\}$.

Definition 41. *i) An* (infinitesimal) action *of $\mathfrak{g}$ on E which lifts ψ is an algebra homomorphism $\Psi : \mathfrak{g} \to \mathrm{Der}(E)$ which satisfies $\pi\Psi(a) = X_a$ for any $a \in \mathfrak{g}$.*

ii) Let $\Psi : \mathfrak{g} \to \mathrm{Der}(E)$ be an action which lifts ψ. An invariant dissection *of E is a dissection $I : E \to T^*M \oplus \mathcal{G} \oplus TM$ for which the action*

$$\mathfrak{g} \ni a \mapsto I \circ \Psi(a) \circ I^{-1} \in \mathrm{Der}(T^*M \oplus \mathcal{G} \oplus TM)$$

preserves the summands T^*M, $\mathcal{G}$ and TM.

We will only consider (without repeating it each time) infinitesimal actions on Courant algebroids for which there is an invariant dissection. The next proposition shows that this is automatically the case if the infinitesimal action is induced from an action of a compact group.

Proposition 42. *Let $\Psi : G \to \mathrm{Aut}(E)$ be an action of a compact group G by automorphisms of a Courant algebroid E over M, covering a group action $\psi : G \to \mathrm{Diff}(M)$. Then E admits a dissection invariant under Ψ.*

Proof. By compactness of G there exists a G-invariant positive definite metric h in E. Using the auxiliary metric h we can define a G-invariant splitting $\sigma_0 : TM \to E$ of the anchor $\pi : E \to TM$, where $\sigma_0(TM)$ is the h-orthogonal complement of $\operatorname{Ker}\pi$. The section σ_0 of π can be modified to a G-invariant totally isotropic section σ defined by

$$\langle \sigma(X), v\rangle = \langle \sigma_0(X), v - \frac{1}{2}\sigma_0(\pi(v))\rangle$$

for all $X \in T_pM$, $v \in E_p$, $p \in M$. Let $\pi^* : T^*M \to E^*$ be the adjoint of the anchor π. Identifying E with E^* using the scalar product of E, we consider π^*T^*M as a subbundle of E. If we define $\mathcal{G}$ as the $\langle\cdot,\cdot\rangle$-orthogonal complement of $\pi^*T^*M \oplus \sigma(TM)$, then the natural isomorphism between $E = \pi^*T^*M \oplus \mathcal{G} \oplus \sigma(TM)$ and $T^*M \oplus \mathcal{G} \oplus TM$ is a G-invariant dissection. $\square$

In the remaining part of this section we assume that

$$E = T^*M \oplus \mathcal{G} \oplus TM \tag{96}$$

is a standard Courant algebroid, defined by a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data (∇, R, H) and we consider in detail the class of actions $\Psi : \mathfrak{g} \to \operatorname{Der}(E)$ which lift $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ and preserve the factors T^*M, $\mathcal{G}$ and TM of E. From the third condition (95), the restriction of Ψ to TM is given by

$$\Psi(a)(X) = \mathcal{L}_{X_a}X,\ \forall a \in \mathfrak{g},\ X \in \mathfrak{X}(M). \tag{97}$$

Since X_a (with $a \in \mathfrak{g} \setminus \{0\}$) are nowhere vanishing we can define

$$\nabla^{\Psi}_{X_a(p)} r := \left(\Psi(a)(r)\right)(p),\ \forall a \in \mathfrak{g},\ r \in \Gamma(\mathcal{G}),\ p \in M, \tag{98}$$

which is a partial connection on $\mathcal{G}$.

Lemma 43. *There is a one to one correspondence between actions $\Psi : \mathfrak{g} \to \operatorname{Der}(E)$ which lift ψ and preserve the factors T^*M, $\mathcal{G}$, TM of E and partial connections ∇^{Ψ} on $\mathcal{G}$ such that the following conditions are satisfied:*

i) ∇^{Ψ} is flat and preserves $[\cdot,\cdot]_{\mathcal{G}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{G}}$;

ii) H and R are invariant, i.e. for any $a \in \mathfrak{g}$,

$$\mathcal{L}_{X_a} H = 0,\ \mathcal{L}_{\Psi(a)} R = 0 \tag{99}$$

where

$$(\mathcal{L}_{\Psi(a)} R)(X, Y) := \nabla^{\Psi}_{X_a}(R(X, Y)) - R(\mathcal{L}_{X_a} X, Y) - R(X, \mathcal{L}_{X_a} Y) \tag{100}$$

for any $X, Y \in \mathfrak{X}(M)$;

iii) for any $a \in \mathfrak{g}$, the endomorphism $A_a := \nabla^{\Psi}_{X_a} - \nabla_{X_a}$ satisfies

$$(\nabla_X A_a)(r) = [R(X_a, X), r]_{\mathcal{G}}, \tag{101}$$

for any $X \in \mathfrak{X}(M)$ and $r \in \Gamma(\mathcal{G})$.

If i), ii) and iii) are satisfied, then the corresponding action Ψ acts naturally (by Lie derivative) on the subbundle $T^*M \oplus TM$ of E, i.e.

$$\Psi(a)(\xi + X) = \mathcal{L}_{X_a}(\xi + X), \tag{102}$$

for any $a \in \mathfrak{g}$, $\xi \in \Omega^1(M)$ and $X \in \mathfrak{X}(M)$, and on $\mathcal{G}$ by

$$\Psi(a)(r) = \nabla^{\Psi}_{X_a} r,\ \forall r \in \Gamma(\mathcal{G}). \tag{103}$$

Moreover, for any $a \in \mathfrak{g}$, the endomorphism A_a is a skew-symmetric derivation of $\mathcal{G}$.

Proof. Let Ψ be an action as in the statement of the lemma. From the second relation (95) applied to $D := \Psi(a)$, we obtain

$$X_a \langle X, \eta \rangle = \langle \Psi(a)(X), \eta \rangle + \langle X, \Psi(a)(\eta) \rangle \tag{104}$$

for any $a \in \mathfrak{g}$, $X \in \mathfrak{X}(M)$ and $\eta \in \Omega^1(M)$. Using (97), (104), and $\Psi(a)(\eta) \in \Omega^1(M)$ we obtain that $\Psi(a)(\eta) = \mathcal{L}_{X_a}\eta$. Relation (102) follows. From our comments above, ∇^{Ψ} defined by (103) is a partial connection on $\mathcal{G}$. Using (5) we obtain that the relations (95) satisfied by Ψ are equivalent to the following conditions: R and H are invariant, ∇^{Ψ} is flat, preserves $[\cdot, \cdot]_{\mathcal{G}}$ and

$\langle\cdot,\cdot\rangle_{\mathcal{G}}$, and

$$\begin{aligned}
&\nabla^{\Psi}_{X_a}\nabla_X r-\nabla_X\nabla^{\Psi}_{X_a}r-\nabla_{\mathcal{L}_{X_a}X}r=0\\
&\mathcal{L}_{X_a}\langle i_X R, r\rangle_{\mathcal{G}}=\langle i_{\mathcal{L}_{X_a}X}R, r\rangle_{\mathcal{G}}+\langle i_X R,\nabla^{\Psi}_{X_a}r\rangle_{\mathcal{G}}\\
&\mathcal{L}_{X_a}\langle\nabla r,\tilde r\rangle_{\mathcal{G}}=\langle\nabla(\nabla^{\Psi}_{X_a}r),\tilde r\rangle_{\mathcal{G}}+\langle\nabla r,\nabla^{\Psi}_{X_a}\tilde r\rangle_{\mathcal{G}},
\end{aligned}\tag{105}$$

for any $a\in\mathfrak{g}$, $X\in\mathfrak{X}(M)$ and $r,\tilde r\in\Gamma(\mathcal{G})$. Using $R^{\nabla}(X_a,X)r=[R(X_a,X),r]_{\mathcal{G}}$ we obtain that the first relation (105) is equivalent to (101). Since both ∇ and ∇^{Ψ} preserve $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ and $[\cdot,\cdot]_{\mathcal{G}}$, the endomorphism A_a is a skew-symmetric derivation. The second relation (105) follows from the invariance of R and the fact that ∇^{Ψ} preserves $\langle\cdot,\cdot\rangle_{\mathcal{G}}$. The third relation (105) follows by writing $\nabla^{\Psi}_{X_a}=\nabla_{X_a}+A_a$ and using that ∇ preserves $\langle\cdot,\cdot\rangle_{\mathcal{G}}$, relation (101), and again $R^{\nabla}(X_a,X)r=[R(X_a,X),r]_{\mathcal{G}}$. ☐

Corollary 44. *The skew-symmetric derivations A_a from Lemma 43 satisfy*

$$\nabla^{\Psi}_{X_b}(A_a)=A_{[b,a]},\ \forall a,b\in\mathfrak{g}.\tag{106}$$

Proof. From $\nabla^{\Psi}_{X_a}=\nabla_{X_a}+A_a$, the flatness of ∇^{Ψ} and the expression (6) of R^{∇}, we obtain, for any $r\in\Gamma(\mathcal{G})$,

$$[R(X_a,X_b),r]_{\mathcal{G}}+(d^{\nabla}A)(X_a,X_b)(r)+[A_a,A_b](r)=0,\ \forall a,b\in\mathfrak{g},\tag{107}$$

where $[A_a,A_b]:=A_aA_b-A_bA_a$ is the commutator of A_a and A_b. But

$$\begin{aligned}
(d^{\nabla}A)(X_a,X_b)(r)&=\left(\nabla_{X_a}A_b-\nabla_{X_b}A_a-A_{[a,b]}\right)r\\
&=2[R(X_b,X_a),r]_{\mathcal{G}}-A_{[a,b]}(r)
\end{aligned}$$

where we used relation (101) and $[X_a,X_b]=X_{[a,b]}$. From (107) we obtain

$$[R(X_b,X_a),r]_{\mathcal{G}}-A_{[a,b]}r+[A_a,A_b](r)=0,\ \forall a,b\in\mathfrak{g},\ r\in\mathcal{G}.\tag{108}$$

On the other hand,

$$\begin{aligned}
\nabla^{\Psi}_{X_b}(A_a)(r)&=\nabla^{\Psi}_{X_b}(A_a(r))-A_a(\nabla^{\Psi}_{X_b}r)\\
&=\nabla_{X_b}(A_a)(r)+[A_b,A_a](r)=[R(X_a,X_b),r]_{\mathcal{G}}-[A_a,A_b](r)
\end{aligned}\tag{109}$$

where in the second equality we used $\nabla^{\Psi}_{X_b}=\nabla_{X_b}+A_b$ and in the third equality we used again relation (101). Combining (108) with (109) we obtain (106). ☐

Remark 45. i) The first relation (105) implies that ∇ is invariant, i.e. its Lie derivative $\mathcal{L}_{\Psi(a)}\nabla$, defined by

$$(\mathcal{L}_{\Psi(a)}\nabla)_X r := \Psi(a)(\nabla_X r) - \nabla_{\mathcal{L}_{X_a}X} r - \nabla_X(\Psi(a)r),$$

for any $X \in \mathfrak{X}(M)$, $r \in \Gamma(\mathcal{G})$, vanishes, for any $a \in \mathfrak{g}$.

ii) Like for R, we define the Lie derivative

$$\begin{aligned}(\mathcal{L}_{\Psi(a)}\alpha)(X_1, \cdots, X_k) &:= \Psi(a)(\alpha(X_1, \cdots, X_k)) \\ &\quad - \alpha(\mathcal{L}_{X_a}X_1, \cdots, X_k) - \cdots - \alpha(X_1, \cdots, \mathcal{L}_{X_a}X_k),\end{aligned}$$

for any form $\alpha \in \Omega^k(M, \mathcal{G})$. The Lie derivative so defined can be extended in the usual way to forms with values in the tensor bundle $\mathcal{T}(\mathcal{G})$ of $\mathcal{G}$. In particular, for $\alpha \in \Omega^k(M)$ we simply define $\mathcal{L}_{\Psi(a)}\alpha := \mathcal{L}_{X_a}\alpha$. A $\mathcal{T}(\mathcal{G})$-valued form α is called invariant if $\mathcal{L}_{\Psi(a)}\alpha = 0$ for any $a \in \mathfrak{g}$.

iii) Relation (106) can be written in the equivalent way

$$\mathcal{L}_{\Psi(b)}(A_a) = A_{[b,a]},\ \forall a, b \in \mathfrak{g}. \tag{110}$$

In particular, the endomorphisms A_a are invariant when $\mathfrak{g}$ is abelian.

Let E_i $(i = 1, 2)$ be two transitive Courant algebroids over M and $\Psi_i : \mathfrak{g} \to \mathrm{Der}(E_i)$ actions which lift ψ. A fiber preserving Courant algebroid isomorphism $F : E_1 \to E_2$ is called invariant if

$$\Psi_2(a)(F(u)) = F\Psi_1(a)(u),\ \forall a \in \mathfrak{g},\ u \in \Gamma(E_1). \tag{111}$$

Lemma 46. *Let $E_i = T^*M \oplus \mathcal{G}_i \oplus TM$ $(i = 1, 2)$ be standard Courant algebroids over M defined by quadratic Lie algebra bundles $(\mathcal{G}_i, [\cdot,\cdot]_{\mathcal{G}_i}, \langle\cdot,\cdot\rangle_{\mathcal{G}_i})$ and data $(\nabla^{(i)}, R_i, H_i)$. Assume that E_i are endowed with actions $\Psi_i : \mathfrak{g} \to \mathrm{Der}(E_i)$ which lift $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ and preserve the factors T^*M, $\mathcal{G}_i$ and TM of E_i, and that a fiber preserving Courant algebroid isomorphism $F : E_1 \to E_2$, defined by (β, Φ, K), where $\beta \in \Omega^2(M)$, $\Phi \in \Omega^1(M, \mathcal{G}_2)$ and $K \in \Gamma\,\mathrm{Isom}(\mathcal{G}_1, \mathcal{G}_2)$, is given, as in (9). Let ∇^{Ψ_i} $(i = 1, 2)$ be the partial connections defined by Ψ_i. Then F is invariant if and only if K maps ∇^{Ψ_1} to ∇^{Ψ_2} (i.e. $\nabla^{\Psi_2} = K \circ \nabla^{\Psi^1} \circ K^{-1}$) and the forms β and Φ are invariant.*

Proof. The proof uses the expression (5) for the Dorfman bracket. □

5.1.1. A class of T^k-actions. Let $(E = T^*M \oplus \mathcal{G} \oplus TM, \langle\cdot,\cdot\rangle, [\cdot,\cdot])$ be a standard Courant algebroid over the total space of a principal T^k-bundle

$\pi : M \to B$, where $T^k = \mathbb{R}^k/\mathbb{Z}^k$ denotes the k-dimensional torus. We assume that E is defined by a quadratic Lie algebra bundle $(\mathcal{G}, \langle\cdot,\cdot\rangle_{\mathcal{G}}, [\cdot,\cdot]_{\mathcal{G}})$ and data (∇, R, H), where ∇ is a connection on the vector bundle $\mathcal{G}$ compatible with the tensor fields $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ and $[\cdot,\cdot]_{\mathcal{G}}$, $R \in \Omega^2(M, \mathcal{G})$ and $H \in \Omega^3(M)$. Recall that these data satisfy the compatibility equations

$$dH = \langle R \wedge R\rangle_{\mathcal{G}} \ , \quad d^{\nabla} R = 0 \ , \quad R^{\nabla} = \mathrm{ad}_R. \tag{112}$$

The Dorfman bracket, scalar product and anchor of E are then expressed by the usual formulas (5) in terms of the above data.

We assume that the vertical parallellism of π is lifted to an action of $\mathfrak{t}^k := \mathrm{Lie}(T^k)$ on E,

$$\Psi : \mathfrak{t}^k = \mathbb{R}^k \to \mathrm{Der}(E) \ ,$$
$$a \mapsto \Psi(a) = (\xi + r + X \mapsto \mathcal{L}_{X_a}\xi + \nabla^{\Psi}_{X_a} r + \mathcal{L}_{X_a} X),$$

where X_a is the fundamental vector field of π determined by $a \in \mathfrak{t}^k$ and ∇^{Ψ} is a partial flat connection on $\mathcal{G}$. We recall (see Lemma 43) that ∇^{Ψ} preserves $[\cdot,\cdot]_{\mathcal{G}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ and that, for any $a \in \mathfrak{t}^k$, $X \in \mathfrak{X}(M)$ and $r \in \Gamma(\mathcal{G})$,

$$\mathcal{L}_{X_a} R = 0, \ \mathcal{L}_{X_a} H = 0, \ (\nabla_X A_a)(r) = [R(X_a, X), r]_{\mathcal{G}}, \tag{113}$$

where $A_a := \nabla^{\Psi}_{X_a} - \nabla_{X_a} \in \mathrm{End}(\mathcal{G})$ is a skew-symmetric derivation, which is invariant since $\mathfrak{t}^k$ is abelian (see Remark 45 iii)). Recall also that $\mathcal{L}_{\Psi(a)}\nabla = 0$ (from Remark 45 i)).

We consider

$$\Omega^s_b(M, \mathcal{G}) := \{\alpha \in \Omega^s(M, \mathcal{G}) \mid \mathcal{L}_{\Psi(a)}\alpha = 0 \ , i_{X_a}\alpha = 0, \ \forall a \in \mathfrak{t}^k\},$$

the space of basic $\mathcal{G}$-valued s-forms on M. The space $\Omega^s_b(M)$ of basic scalar valued s-forms on M can be defined similarly and coincides with $\pi^*\Omega^s(B) \cong \Omega^s(B)$. The analogous fact for $\Omega^s_b(M, \mathcal{G})$ is stated in the next proposition.

Proposition 47. *$\Omega^s_b(M, \mathcal{G}) \cong \pi^*\Omega^s(B) \otimes \Gamma_{t^k}(\mathcal{G})$, where $\Gamma_{t^k}(\mathcal{G})$ denotes the space of $\mathfrak{t}^k$-invariant (i.e. ∇^{Ψ}-parallel) sections.*

Proof. Let $U \subset B$ be an open set such that $\Lambda^s T^*B|_U$ is trivial. Then any horizontal form $\alpha \in \Omega^s(\pi^{-1}(U), \mathcal{G})$ (i.e. $i_{X_a}\alpha = 0$ for any $a \in \mathfrak{t}^k$) can be written as $\alpha = \sum_i (\pi^*\beta_i) \otimes s_i$ where (β_i) is a basis of $\Lambda^s T^*B|_U$ and $s_i \in \Gamma(\mathcal{G}|_{\pi^{-1}(U)})$. Then $\mathcal{L}_{\Psi(a)}\alpha = \sum_i (\pi^*\beta_i) \otimes \mathcal{L}_{\Psi(a)} s_i$ from where we deduce that $\Omega^s_b(\pi^{-1}(U), \mathcal{G}) = \pi^*\Omega^s(U) \otimes \Gamma_{t^k}(\mathcal{G}|_{\pi^{-1}(U)})$. Using a partition of unity in B one can deduce that the same holds globally for $U = B$. $\square$

In the following we will always identify $\Lambda^s T^*M \otimes \mathcal{G}$ with $\mathcal{G} \otimes \Lambda^s T^*M$, which allows to freely write decomposable elements as $\omega \otimes r$ or as $r \otimes \omega$. Let (e_i) be a basis of $\mathfrak{t}^k$, $X_i := X_{e_i} \in \mathfrak{X}(M)$ the associated fundamental vector fields and $A_i := A_{e_i} = \nabla^{\Psi}_{X_i} - \nabla_{X_i} \in \mathrm{End}(\mathcal{G})$. We choose a connection $\mathcal{H}$ on the principal bundle $\pi : M \to B$, with connection form $\theta = \sum_{i=1}^{k} \theta_i e_i$. We introduce the connection

$$\nabla^{\theta} := \nabla + \sum_{i=1}^{k} \theta_i \otimes A_i \tag{114}$$

on the vector bundle $\mathcal{G}$. Since ∇ preserves $[\cdot,\cdot]_{\mathcal{G}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ and A_i are skew-symmetric derivations, we obtain that ∇^{θ} preserves $[\cdot,\cdot]_{\mathcal{G}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{G}}$. The curvature R^{∇} of ∇ and R^{θ} of ∇^{θ} are related by

$$R^{\nabla} = R^{\theta} - \sum_{i=1}^{k} (d\theta_i) \otimes A_i + \sum_{i=1}^{k} \theta_i \wedge \mathrm{ad}_{R(X_i,\cdot)} - \frac{1}{2} \sum_{i,j} (\theta_i \wedge \theta_j) \otimes [A_i, A_j], \tag{115}$$

where for any form $\omega \in \Omega^s(M, \mathcal{G})$ (in particular $\omega := R(X_i, \cdot)$) we define $\mathrm{ad}_{\omega} \in \Omega^s(M, \mathrm{End}\, \mathcal{G})$ by

$$(\mathrm{ad}_{\omega})(Y_1, \cdots, Y_s)(r) := [\omega(Y_1, \cdots, Y_s), r]_{\mathcal{G}},\ \forall Y_i \in \mathfrak{X}(M),\ r \in \Gamma(\mathcal{G}).$$

Lemma 48. *For any invariant section $r \in \Gamma_{\mathfrak{t}^k}(\mathcal{G})$, the $\mathcal{G}$-valued 1-form $\nabla^{\theta} r$ is basic.*

Proof. The form $\nabla^{\theta} r$ is horizontal, since

$$\nabla^{\theta}_{X_i} r = \nabla^{\Psi}_{X_i} r = \mathcal{L}_{\Psi(e_i)} r = 0,\ \forall 1 \le i \le k.$$

On the other hand, as ∇, θ and A_i are $\mathfrak{t}^k$- invariant, so is ∇^{θ} and

$$\mathcal{L}_{\Psi(a)}(\nabla^{\theta} r) = \mathcal{L}_{\Psi(a)}(\nabla^{\theta}) r + \nabla^{\theta} \mathcal{L}_{\Psi(a)} r = 0, \tag{116}$$

which implies that $\nabla^{\theta} r$ is $\mathfrak{t}^k$-invariant. □

Assumption 49. From now on we will assume that the partial connection ∇^{Ψ} has trivial holonomy. Then we can define a bundle $\mathcal{G}_B \to B$ whose fiber over $p \in B$ is

$$\mathcal{G}_B|_p := \Gamma_{\mathfrak{t}^k}(\mathcal{G}|_{\pi^{-1}(p)}),$$

the vector space of ∇^{Ψ}-parallel sections of $\mathcal{G}$ over the torus $\pi^{-1}(p)$. Note that by working locally in a flow box for the vertical foliation of $M \to B$, we can always assume that ∇^{Ψ} has trivial holonomy. (We recall that a flow box is a domain $V \subset M$ such that for all $p \in \pi(V) \subset B$ the manifolds $\pi^{-1}(p) \cap V$ are diffeomorphic to $\mathbb{R}^k$).

Notation 50. We will identify $\mathcal{G} = \pi^*\mathcal{G}_B$, $\Gamma_{\mathfrak{t}^k}(\mathcal{G}) = \pi^*\Gamma(\mathcal{G}_B)$ and

$$\Omega^s_b(M, \mathcal{G}) = \pi^*\Omega^s(B) \otimes \pi^*\Gamma(\mathcal{G}_B) \cong \Omega^s(B, \mathcal{G}_B). \tag{117}$$

For a basic form $\alpha \in \Omega^s_b(M, \mathcal{G})$ we will denote by $\alpha^B \in \Omega^s(B, \mathcal{G}_B)$ the corresponding form in the identification (117).

Lemma 51. *i) The bundle $\mathcal{G}_B$ inherits a bracket $[\cdot,\cdot]_{\mathcal{G}_B}$ and a scalar product $\langle\cdot,\cdot\rangle_{\mathcal{G}_B}$ which make $(\mathcal{G}_B, [\cdot,\cdot]_{\mathcal{G}_B}, \langle\cdot,\cdot\rangle_{\mathcal{G}_B})$ into a quadratic Lie algebra bundle.*

ii) The connection ∇^{θ} induces a connection $\nabla^{\theta,B}$ on $\mathcal{G}_B$, which preserves $[\cdot,\cdot]_{\mathcal{G}_B}$ and $\langle\cdot,\cdot\rangle_{\mathcal{G}_B}$. The curvature R^{θ} of ∇^{θ} is the pullback of the curvature $R^{\theta,B}$ of $\nabla^{\theta,B}$, i.e. $(R^{\theta})^B = R^{\theta,B}$.

Proof. i) The claim follows from Lemma 43 i) and the definition of $\mathcal{G}_B$.

ii) From Lemma 48, ∇^{θ} induces a connection $\nabla^{\theta,B}$ on $\mathcal{G}_B$, defined by

$$\nabla^{\theta,B} r = (\nabla^{\theta}\pi^* r)^B, \ \forall r \in \Gamma(\mathcal{G}_B).$$

Since A_i are skew-symmetric derivations, and ∇ preserves $[\cdot,\cdot]_{\mathcal{G}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{G}}$, we obtain that ∇^{θ} preserves these tensor fields as well. We deduce that $\nabla^{\theta,B}$ preserves $[\cdot,\cdot]_{\mathcal{G}_B}$ and $\langle\cdot,\cdot\rangle_{\mathcal{G}_B}$. The statement on curvatures is trivial. □

Notation 52. We shall denote by $d^{\theta,B} : \Omega^*(B, \mathcal{G}_B) \to \Omega^{*+1}(B, \mathcal{G}_B)$ the exterior covariant derivative defined by $\nabla^{\theta,B}$.

In order to describe the Courant algebroid E together with the action $\Psi : \mathfrak{t}^k \to \Gamma(E)$ in terms of structures on the base manifold B of the torus bundle, we need to interpret equations (112) and (113) on B. The first two equations (113) mean that H and R are invariant, i.e. they are of the form

$$\begin{aligned} H &= H_{(3)} + \theta_i \wedge H^i_{(2)} + \theta_i \wedge \theta_j \wedge H^{ij}_{(1)} + H^{ijs}_{(0)} \theta_i \wedge \theta_j \wedge \theta_s \\ R &= R_{(2)} + \theta_i \wedge R^i_{(1)} + R^{ij}_{(0)} \theta_i \wedge \theta_j \end{aligned} \tag{118}$$

where $H_{(3)}$, $H^i_{(2)}$, $H^{ij}_{(1)}$, $H^{ijk}_{(0)}$, $R_{(2)}$, $R^i_{(1)}$, $R^{ij}_{(0)}$ are basic and for simplicity of notation we omit the summation signs.

Let A^B_a be the section of $\mathrm{End}(\mathcal{G}_B)$ defined by the invariant section $A_a \in \mathrm{End}(\mathcal{G})$, where $a \in \mathfrak{t}^k$.

Lemma 53. *i) The compatibility equations listed in (112) are satisfied if and only if the following conditions hold:*

(119) $dH^{B}_{(3)} + H^{i,B}_{(2)} \wedge (d\theta_i)^B = \langle R^{B}_{(2)} \wedge R^{B}_{(2)} \rangle_{\mathcal{G}_B}$

(120) $dH^{p,B}_{(2)} + 2H^{pi,B}_{(1)} \wedge (d\theta_i)^B = -2\langle R^{B}_{(2)} \wedge R^{p,B}_{(1)} \rangle_{\mathcal{G}_B}$

(121) $dH^{pq,B}_{(1)} + 3H^{ipq,B}_{(0)}(d\theta_i)^B = 2\langle R^{pq,B}_{(0)}, R^{B}_{(2)} \rangle_{\mathcal{G}_B} - \langle R^{p,B}_{(1)}, R^{q,B}_{(1)} \rangle_{\mathcal{G}_B}$

(122) $3dH^{pqs,B}_{(0)} + 2(\langle R^{pq,B}_{(0)}, R^{s,B}_{(1)} \rangle_{\mathcal{G}_B} + \langle R^{sp,B}_{(0)}, R^{q,B}_{(1)} \rangle_{\mathcal{G}_B} + \langle R^{qs,B}_{(0)}, R^{p,B}_{(1)} \rangle_{\mathcal{G}_B}) = 0$

(123) $\langle R^{ij,B}_{(0)}, R^{pq,B}_{(0)} \rangle \theta_i \wedge \theta_j \wedge \theta_p \wedge \theta_q = 0$

(124) $d^{\theta,B} R^{B}_{(2)} + R^{i,B}_{(1)} \wedge (d\theta_i)^B = 0$

(125) $d^{\theta,B} R^{p,B}_{(1)} + A^{B}_{p} \wedge R^{B}_{(2)} + 2R^{pi,B}_{(0)}(d\theta_i)^B = 0$

(126) $A^{B}_{p} \wedge R^{q,B}_{(1)} - A^{B}_{q} \wedge R^{p,B}_{(1)} = 2\nabla^{\theta,B} R^{pq,B}_{(0)}$

(127) $A^{B}_{s} R^{pq,B}_{(0)} + A^{B}_{q} R^{sp,B}_{(0)} + A^{B}_{p} R^{qs,B}_{(0)} = 0$

(128) $R^{\theta,B} = (d\theta_i)^B \otimes A^{B}_{i} + \mathrm{ad}_{R^{B}_{(2)}}$

(129) $\mathrm{ad}_{R^{ij,B}_{(0)}} = \frac{1}{2}[A^{B}_{i}, A^{B}_{j}],$

where

$$\mathrm{ad} : \mathcal{G}_B \to \mathrm{Der}(\mathcal{G}_B),\ \mathrm{ad}_u(v) = [u, v]_{\mathcal{G}_B} \tag{130}$$

is the adjoint representation into the bundle of skew-symmetric derivations of the quadratic Lie algebra bundle bundle $(\mathcal{G}_B, [\cdot,\cdot]_{\mathcal{G}_B})$ *(see Notation 4) and* $1 \leq p, q, s \leq k$ *are arbitrary.*

ii) If the compatibility relations (112) are satisfied, then the third equation (113) is satisfied as well if and only if

$$\nabla^{\theta,B}_{X} A^{B}_{i} = [R^{i,B}_{(1)}(X), r]_{\mathcal{G}_B},\ \forall X \in \mathfrak{X}(B). \tag{131}$$

Proof. i) The equations (119)-(123) are obtained from the first relation (112), by comparing

$$\begin{aligned} dH &= dH_{(3)} + (d\theta_i) \wedge H^{i}_{(2)} - \theta_i \wedge dH^{i}_{(2)} + 2(d\theta_i) \wedge \theta_j \wedge H^{ij}_{(1)} \\ &\quad + \theta_i \wedge \theta_j \wedge dH^{ij}_{(1)} + 3H^{ijs}_{(0)}(d\theta_i) \wedge \theta_j \wedge \theta_s + (dH^{ijs}_{(0)}) \wedge \theta_i \wedge \theta_j \wedge \theta_s \end{aligned}$$

with

$$\begin{aligned}\langle R\wedge R\rangle_{\mathcal{G}} &= \langle R_{(2)}\wedge R_{(2)}\rangle_{\mathcal{G}} + 2\theta_i\wedge\langle R_{(2)}\wedge R^i_{(1)}\rangle_{\mathcal{G}} + 2\theta_i\wedge\theta_j\wedge\langle R_{(2)}, R^{ij}_{(0)}\rangle_{\mathcal{G}}\\ &\quad - \theta_i\wedge\theta_j\wedge\langle R^i_{(1)}\wedge R^j_{(1)}\rangle_{\mathcal{G}} + 2\theta_i\wedge\theta_j\wedge\theta_s\wedge\langle R^{ij}_{(0)}, R^s_{(1)}\rangle_{\mathcal{G}}\\ &\quad + \langle R^{ij}_{(0)}, R^{pq}_{(0)}\rangle_{\mathcal{G}}\theta_i\wedge\theta_j\wedge\theta_p\wedge\theta_q,\end{aligned}$$

using $d\theta_i\in\Omega^2(B)$, that the exterior derivative maps basic forms to basic forms, that the operation $(\alpha,\beta)\mapsto\langle\alpha\wedge\beta\rangle_{\mathcal{G}}$ maps a pair of $\mathcal{G}$-valued basic forms to a basic scalar valued form and then interpreting the resulting relations on B. The equations (124)-(127) are obtained from the second relation (112), by computing

$$\begin{aligned} 0 &= d^{\nabla}R\\ &= d^{\nabla}R_{(2)} + (d\theta_i)\wedge R^i_{(1)} - \theta_i\wedge d^{\nabla}R^i_{(1)} + (\nabla R^{ij}_{(0)})\wedge\theta_i\wedge\theta_j\\ &\quad + 2R^{ij}_{(0)}\otimes(d\theta_i)\wedge\theta_j\\ &= d^{\nabla^\theta}R_{(2)} - (\theta_i\otimes \mathrm{A}_i)\wedge R_{(2)} + R^i_{(1)}\wedge d\theta_i\\ &\quad - \theta_j\wedge\Big(d^{\nabla^\theta}R^j_{(1)} - (\theta_i\otimes A_i)\wedge R^j_{(1)}\Big)\\ &\quad + (\nabla^\theta R^{ij}_{(0)})\wedge\theta_i\wedge\theta_j - A_i(R^{js}_{(0)})\theta_i\wedge\theta_j\wedge\theta_s + 2R^{ij}_{(0)}\otimes(d\theta_i)\wedge\theta_j, \end{aligned} \tag{132}$$

identifying the horizontal and vertical parts in the last expression of (132) and interpreting the result on B. The remaining equations (128) and (129) are obtained from the third relation (112), by writing R^{∇} in terms of R^{θ} as in (115) and identifying the horizontal and vertical parts.

ii) The third equation (113) is equivalent to relation (131), together with relation (129). □

Since $\nabla^{\theta,B}$ preserves $[\cdot,\cdot]_{\mathcal{G}_B}$ and $\langle\cdot,\cdot\rangle_{\mathcal{G}_B}$, the endomorphism $R^{\theta,B}(X,Y)$ of $\mathcal{G}_B$ is a skew-symmetric derivation, for any $X,Y\in\mathfrak{X}(B)$. Recall that $A^B_i\in\Gamma\,\mathrm{End}(\mathcal{G}_B)$ is also a skew-symmetric derivation. Therefore, if the adjoint representation (130) is an isomorphism, then

$$A^B_i = \mathrm{ad}_{r^B_i},\ \ R^{\theta,B}(X,Y) = \mathrm{ad}_{\mathfrak{r}^{\theta,B}(X,Y)} \tag{133}$$

for $r^B_i\in\Gamma(\mathcal{G}_B)$ and $\mathfrak{r}^{\theta,B}\in\Omega^2(B,\mathcal{G}_B)$. From the Bianchi identity we obtain that $\mathfrak{r}^{\theta,B}$ is $d^{\theta,B}$-closed. In Corollary 55 below we will prove that the conditions from Lemma 53 simplify considerably when the adjoint representation (130) is an isomorphism.

Remark 54. i) Consider the class $\mathcal{C}$ of quadratic Lie algebras $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ for which the adjoint representation is an isomorphism onto the Lie algebra of skew-symmetric derivations of $\mathfrak{g}$. Every semi-simple Lie algebra endowed with its Killing form (or any other invariant scalar product) belongs to this class. Since the center of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ coincides with $[\mathfrak{g}, \mathfrak{g}]^{\perp}$, there is no non-zero solvable Lie algebra in $\mathcal{C}$. Nevertheless, the class $\mathcal{C}$ is strictly larger than the class of semi-simple quadratic Lie algebras. For instance, the affine Lie algebra $\mathfrak{so}(3) \ltimes \mathfrak{so}(3)^* \cong \mathfrak{so}(3) \ltimes \mathbb{R}^3$ can be endowed with the invariant scalar product of neutral signature defined by duality. The adjoint representation is faithful and one can easily check that all skew-symmetric derivations are inner. There exist solvable Lie algebras with faithful adjoint representation for which all derivations are inner [19]. However these do not admit any invariant scalar product as we have already remarked.

ii) The adjoint representation of $(\mathcal{G}_B, [\cdot, \cdot]_{\mathcal{G}_B}, \langle \cdot, \cdot \rangle_{\mathcal{G}_B})$ is an isomorphism onto the bundle $\mathrm{Der}(\mathcal{G}_B)$ of skew-symmetric derivations if and only if the same is true for the quadratic Lie algebra bundle $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}}, \langle \cdot, \cdot \rangle_{\mathcal{G}})$ of the Courant algebroid E. Courant algebroids with this property will be described in Proposition 90.

Corollary 55. *Let $\pi : M \to B$ be a principal T^k-bundle and $\mathcal{H}$ a principal connection on π, with connection form $\theta = \sum_{i=1}^{k} \theta_i e_i \in \Omega^1(M, \mathfrak{t}^k)$, where (e_i) is a basis of $\mathfrak{t}^k$. There is a one to one correspondence between*

1) *standard Courant algebroids $E = T^*M \oplus \mathcal{G} \oplus TM$ for which the adjoint representation of the quadratic Lie algebra bundle $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}}, \langle \cdot, \cdot \rangle_{\mathcal{G}})$ is an isomorphism onto the bundle of skew-symmetric derivations, together with an action $\Psi : \mathfrak{t}^k \to \mathrm{Der}(E)$ which lifts the vertical parallelism of π, preserves the factors T^*M, $\mathcal{G}$ and TM of E, and for which the flat partial connection ∇^{Ψ} has trivial holonomy*

   and

2) *quadratic Lie algebra bundles $(\mathcal{G}_B, \langle \cdot, \cdot \rangle_{\mathcal{G}_B}, [\cdot, \cdot]_{\mathcal{G}_B})$ over B, whose adjoint action is an isomorphism onto the bundle of skew-symmetric derivations, together with a connection ∇^B on the vector bundle $\mathcal{G}_B$ which preserves $\langle \cdot, \cdot \rangle_{\mathcal{G}_B}$ and $[\cdot, \cdot]_{\mathcal{G}_B}$, sections $r_i^B \in \Gamma(\mathcal{G}_B)$ $(1 \leq i \leq k)$, a 3-form $H_{(3)}^B \in \Omega^3(B)$, 2-forms $H_{(2)}^{i,B} \in \Omega^2(B)$, 1-forms $H_{(1)}^{ij,B} \in \Omega^1(B)$ and constants $c_{ijp} \in \mathbb{R}$ $(1 \leq i, j, p \leq k)$ such that*

$$
\begin{aligned}
dH^B_{(3)} &= \langle \mathfrak{r}^B \wedge \mathfrak{r}^B\rangle_{\mathcal{G}_B} - (H^{i,B}_{(2)} + 2\langle \mathfrak{r}^B, r^B_i\rangle_{\mathcal{G}_B}\\
&\quad - \langle r^B_i, r^B_j\rangle_{\mathcal{G}_B}(d\theta_j)^B) \wedge (d\theta_i)^B,\\
dH^{p,B}_{(2)} &= 2(\langle \nabla^B r^B_p, r^B_i\rangle_{\mathcal{G}_B} - H^{pi,B}_{(1)}) \wedge (d\theta_i)^B - 2\langle \mathfrak{r}^B \wedge \nabla^B r^B_p\rangle_{\mathcal{G}_B},\\
dH^{pq,B}_{(1)} &= -3c_{ipq}(d\theta_i)^B + \langle \mathfrak{r}^B, [r^B_p, r^B_q]_{\mathcal{G}_B}\rangle_{\mathcal{G}_B} - \langle \nabla^B r^B_p \wedge \nabla^B r^B_q\rangle_{\mathcal{G}_B},
\end{aligned}
\tag{134}
$$

where $\mathfrak{r}^B \in \Omega^2(B, \mathcal{G}_B)$ *is related to the curvature* R^B *of the connection* ∇^B *by* $R^B(X,Y) = \mathrm{ad}_{\mathfrak{r}^B(X,Y)}$ *for any* $X, Y \in \mathfrak{X}(B)$.

Proof. The claim follows from Lemma 53, by letting $\nabla^B := \nabla^{\theta,B}$, $A^B_i = \mathrm{ad}_{r^B_i}$ for some sections $r^B_i \in \Gamma(\mathcal{G}_B)$ and simplifying the relations from this lemma. More precisely, relations (129), (131) and (128) determine $R^{ij,B}_{(0)}$, $R^{i,B}_{(1)}$ and $R^B_{(2)}$ respectively by

$$
R^{ij,B}_{(0)} = \frac{1}{2}[r^B_i, r^B_j]_{\mathcal{G}_B},\ R^{i,B}_{(1)} = \nabla^B r^B_i,\ R^B_{(2)} = \mathfrak{r}^B - (d\theta_i)^B \otimes r^B_i. \tag{135}
$$

Relation (122) with $R^{ij,B}_{(0)}$ and $R^{i,B}_{(1)}$ given by (135) implies that

$$
H^{pqs,B}_{(0)} = -\frac{1}{3}\langle [r^B_p, r^B_q]_{\mathcal{G}_B}, r^B_s\rangle_{\mathcal{G}_B} + c_{pqs} \tag{136}
$$

for some constants c_{pqs}. Written in terms of $\mathfrak{r}^B$ rather than $R^B_{(2)}$, relations (119), (120), (121) become relations (134). The remaining relations from Lemma 53, with $R^{ij,B}_{(0)}, R^{i,B}_{(1)}, R^B_{(2)}$ and $H^{pqs,B}_{(0)}$ as above and $A_i = \mathrm{ad}_{r^B_i}$ are satisfied. □

Example 56. Let $\pi : M \to B$ be a principal T^k-bundle and $\mathcal{H}$ a principal connection on π, with connection form $\theta = \sum_{i=1}^k \theta_i e_i$, like in Corollary 55. Let $(\mathcal{G}_B, \langle\cdot,\cdot\rangle_{\mathcal{G}_B}, [\cdot,\cdot]_{\mathcal{G}_B})$ be a quadratic Lie algebra bundle over B, whose adjoint action is an isomorphism onto the bundle of skew-symmetric derivations, together with a connection ∇^B on the vector bundle $\mathcal{G}_B$ which preserves $\langle\cdot,\cdot\rangle_{\mathcal{G}_B}$ and $[\cdot,\cdot]_{\mathcal{G}_B}$. Choose arbitrary sections $r^B_i \in \Gamma(\mathcal{G}_B)$ $(1 \leq i \leq k)$ and define, for any i, j, s,

$$
c_{ijp} := 0, \tag{137}
$$

and

$$
H^{ijs,B}_{(0)} := -\frac{1}{3}\langle [r^B_i, r^B_j]_{\mathcal{G}_B}, r^B_s\rangle_{\mathcal{G}_B} \tag{138}
$$

and

$$H_{(1)}^{ij,B} := \frac{1}{2}\left(\langle \nabla^B r_i^B, r_j^B\rangle_{\mathcal{G}_B} - \langle \nabla^B r_j^B, r_i^B\rangle_{\mathcal{G}_B}\right). \tag{139}$$

With these choices, the third relation (134) is satisfied. For any forms $H_{(3)}^B$ and $H_{(2)}^{i,B}$, such that

$$\mathcal{K}_i := H_{(2)}^{i,B} + 2\langle \mathfrak{r}^B, r_i^B\rangle_{\mathcal{G}_B} - \langle r_i^B, r_j^B\rangle_{\mathcal{G}_B}(d\theta_j)^B \tag{140}$$

is closed and

$$dH_{(3)}^B = \langle \mathfrak{r}^B \wedge \mathfrak{r}^B\rangle_{\mathcal{G}_B} - \mathcal{K}_i \wedge (d\theta_i)^B \tag{141}$$

the relations (134) are satisfied and we thus obtain a standard Courant algebroid on M together with an action $\Psi : \mathfrak{t}^k \to \mathrm{Der}(E)$ lifting the vertical parallelism of the principal torus bundle $\pi : M \to B$. Note that 2-forms $H_{(2)}^{i,B}$ as required in the above construction do always exist and are unique up to addition of closed forms whereas $H_{(3)}^B$ exists if and only if the closed form $\langle \mathfrak{r}^B \wedge \mathfrak{r}^B\rangle_{\mathcal{G}_B} - \mathcal{K}_i \wedge (d\theta_i)^B$ is exact. It is also unique up to addition of a closed form.

5.2. Invariant spinors

Let E be a transitive Courant algebroid over an oriented manifold M and $\Psi : \mathfrak{g} \to \mathrm{Der}(E)$ an action on E, which lifts an action $\psi : \mathfrak{g} \to \mathfrak{X}(M)$, $a \mapsto X_a$ of $\mathfrak{g}$ on M. Let $\mathbb{S}$ be the canonical weighted spinor bundle of E determined by an irreducible $\mathrm{Cl}(E)$-bundle S. Our aim in this section is to define an action of $\mathfrak{g}$ on $\Gamma(\mathbb{S})$. In order to find a proper definition we assume that Ψ integrates to a Lie group action

$$G \to \mathrm{Aut}(E),\ g \mapsto I_E^g : E \to E$$

such that I_E^g induces a globally defined isomorphism $I_{\mathbb{S}}^g : \mathbb{S} \to \mathbb{S}$, for any $g \in \mathfrak{g}$. Recall that

$$I_{\mathbb{S}}^g \circ \gamma_u = \gamma_{I_E^g(u)} \circ I_{\mathbb{S}}^g,\ \forall g \in G,\ u \in E.$$

Consider a curve $g = g(t)$ of G with $g(0) = e$ and $\dot{g}(0) = a$. We choose $I_{\mathbb{S}}^{g(t)}$ depending smoothly on t and such that $I_{\mathbb{S}}^{g(0)} = \mathrm{Id}_{\mathbb{S}}$. Replacing in the above relation g by $g(t)$ and taking the derivative at $t=0$ we obtain that $\Psi^{\mathbb{S}}(a) := \frac{d}{dt}\big|_{t=0} I_{\mathbb{S}}^{g(t)} \in \mathrm{End}\,\Gamma(\mathbb{S})$ satisfies

$$\Psi^{\mathbb{S}}(a)\gamma_u s = \gamma_{\Psi(a)(u)} s + \gamma_u \Psi^{\mathbb{S}}(a)s,\ \forall u \in \Gamma(E),\ s \in \Gamma(\mathbb{S}),\ a \in \mathfrak{g}. \tag{142}$$

In the following we do not assume that Ψ integrates to an action of G.

Proposition 57. *i) There is a unique linear map*

$$\Psi^{\mathbb{S}} : \mathfrak{g} \to \mathrm{End}\,\Gamma(\mathbb{S})$$

which satisfies relation (142), the Leibniz rule

$$\Psi^{\mathbb{S}}(a)(fs) = f\Psi^{\mathbb{S}}(a)(s) + X_a(f)s \tag{143}$$

for any $f \in C^\infty(M)$, $s \in \Gamma(\mathbb{S})$ and $a \in \mathfrak{g}$, and, for any $U \subset M$ open and sufficiently small, preserves the canonical pairing $\langle \cdot, \cdot \rangle_{\mathbb{S}|_U}$ of $\mathbb{S}|_U$, i.e.

$$\mathcal{L}_{X_a} \langle s, \tilde{s} \rangle_{\mathbb{S}|_U} = \langle \Psi^{\mathbb{S}}(a)s, \tilde{s} \rangle_{\mathbb{S}|_U} + \langle s, \Psi^{\mathbb{S}}(a)\tilde{s} \rangle_{\mathbb{S}|_U}, \tag{144}$$

for any $s, \tilde{s} \in \Gamma(\mathbb{S}|_U)$ and $a \in \mathfrak{g}$.

ii) The map $\Psi^{\mathbb{S}} : \mathfrak{g} \to \mathrm{End}\,\Gamma(\mathbb{S})$ satisfies

$$[\Psi^{\mathbb{S}}(a), \Psi^{\mathbb{S}}(b)] = \Psi^{\mathbb{S}}[a,b],\ \forall a, b \in \mathfrak{g}. \tag{145}$$

It is called the action on spinors *induced by Ψ.*

The remaining part of this section is devoted to the proof of Proposition 57. For uniqueness, let $\Psi^{\mathbb{S}}$ and $\tilde{\Psi}^{\mathbb{S}}$ be two maps which satisfy the required conditions. Then $F(a) := \Psi^{\mathbb{S}}(a) - \tilde{\Psi}^{\mathbb{S}}(a) \in \mathrm{End}\,\Gamma(\mathbb{S})$ is $C^\infty(M)$-linear and commutes with the Clifford action. Hence $F(a) = \lambda(a)\mathrm{Id}_{\mathbb{S}}$, for $\lambda(a) \in C^\infty(M)$. Since $F(a)$ is skew-symmetric with respect to $\langle \cdot, \cdot \rangle_{\mathbb{S}|_U}$, we obtain $\lambda(a) = 0$. The uniqueness follows. To prove the existence we start with the next lemma.

Lemma 58. *Let E_i $(i = 1, 2)$ be two transitive Courant algebroids over M with canonical weighted spinor bundles $\mathbb{S}_i$ and actions $\Psi_i : \mathfrak{g} \to \mathrm{Der}(E_i)$ which lift ψ. Assume there is an invariant isomorphism $I : E_1 \to E_2$ and a map $\Psi^{\mathbb{S}_1} : \mathfrak{g} \to \mathrm{End}\,\Gamma(\mathbb{S}_1)$ which satisfies the properties from Proposition 57.*

Then the map $\Psi^{\mathbb{S}_2} : \mathfrak{g} \to \operatorname{End}\Gamma(\mathbb{S}_2)$ *given by*

$$\Psi^{\mathbb{S}_2}(a)(s)|_U = I_{\mathbb{S}|_U} \circ \Psi^{\mathbb{S}_1}(a) \circ (I_{\mathbb{S}|_U})^{-1}(s|_U), \tag{146}$$

for any $a \in \mathfrak{g}$ *and* $s \in \Gamma(\mathbb{S}_2)$, *is well-defined and satisfies the properties of Proposition 57. Above* $U \subset M$ *is any sufficiently small open subset of* M *and* $I_{\mathbb{S}|_U} : \mathbb{S}_1|_U \to \mathbb{S}_2|_U$ *is the isomorphism induced by* I.

Proof. In order to show that $\Psi^{\mathbb{S}_2}$ is well defined we use that $I_{\mathbb{S}|_U}$ are uniquely determined up to multiplication by ± 1 and that if $s \in \Gamma(\mathbb{S}_1)$ satisfies $s|_U = 0$, then $\Psi^{\mathbb{S}_1}(a)(s)|_U = 0$ (from relation (143) satisfied by $\Psi^{\mathbb{S}_1}$). The map $\Psi^{\mathbb{S}_2}$ obviously satisfies (143) and (145) and, from (45), it satisfies (144) as well. Using $I_{\mathbb{S}|_U} \circ \gamma_u = \gamma_{I_E(u)} \circ I_{\mathbb{S}|_U}$ and relation (142) satisfied by $\Psi^{\mathbb{S}_1}$, we obtain

$$\Psi^{\mathbb{S}_2}(a)\gamma_u(s) = \gamma_{I_E \Psi_1(a) I_E^{-1}(u)} s + \gamma_u \Psi^{\mathbb{S}_2}(a)s, \tag{147}$$

for any $a \in \mathfrak{g}$, $u \in \Gamma(E_2|_U)$ and $s \in \Gamma(\mathbb{S}_2|_U)$. Since I_E is invariant, $\Psi_2(a) = I_E \circ \Psi_1(a) \circ I_E^{-1}$ and we obtain that $\Psi^{\mathbb{S}_2}$ satisfies (142) too. $\square$

Owing to Lemma 58, we can assume in Proposition 57 (by choosing an invariant dissection of E), that $E = E_M = T^*M \oplus \mathcal{G} \oplus TM$ is a standard Courant algebroid defined by a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data (∇, R, H), with action

$$\Psi : \mathfrak{g} \mapsto \operatorname{Der}(E_M),\ \Psi(a)(\xi + r + X) := \mathcal{L}_{X_a}\xi + \nabla^{\Psi}_{X_a} r + \mathcal{L}_{X_a} X \tag{148}$$

which lifts an action

$$\psi : \mathfrak{g} \mapsto \mathfrak{X}(M),\ a \mapsto X_a$$

of $\mathfrak{g}$ on M. We can also assume that $\mathbb{S} = \mathbb{S}_M := \Lambda(T^*M)\hat{\otimes}\mathcal{S}_{\mathcal{G}}$ is the canonical weighted spinor bundle of E determined by $S = S_M := \Lambda(T^*M)\hat{\otimes}S_{\mathcal{G}}$, where $S_{\mathcal{G}}$ is an irreducible $\mathrm{Cl}(\mathcal{G})$-bundle. The next lemma concludes the proof of Proposition 57.

Lemma 59. *The map* $\Psi^{\mathbb{S}_M} : \mathfrak{g} \to \operatorname{End}\Gamma(\mathbb{S}_M)$ *defined by*

$$\Psi^{\mathbb{S}_M}(a)(\omega \otimes s) := (\mathcal{L}_{X_a}\omega) \otimes s + \omega \otimes \nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}_{X_a} s, \tag{149}$$

for any $a \in \mathfrak{g}$, $\omega \in \Omega(M)$ *and* $s \in \Gamma(\mathcal{S}_{\mathcal{G}})$, *satisfies the conditions from Proposition 57. Above* $\nabla^{\Psi,\mathcal{S}_{\mathcal{G}}}$ *is the partial connection on* $\mathcal{S}_{\mathcal{G}}$ *induced by any partial connection* $\nabla^{\Psi,S_{\mathcal{G}}}$ *on* $S_{\mathcal{G}}$, *compatible with the partial connection* ∇^{Ψ}.

Proof. Relation (143) is obviously satisfied. To prove relation (144) we recall that $\langle\cdot,\cdot\rangle_{\mathbb{S}_M|_U}$ is given by (35), where $\langle\cdot,\cdot\rangle_{\mathcal{S}_\mathcal{G}|_U}$ is a canonical pairing of $\mathcal{S}_\mathcal{G}|_U$. We remark that

$$X_a\langle s,\tilde{s}\rangle_{\mathcal{S}_\mathcal{G}|_U} = \langle\nabla^{\Psi,\mathcal{S}_\mathcal{G}}_{X_a} s,\tilde{s}\rangle_{\mathcal{S}_\mathcal{G}|_U} + \langle s,\nabla^{\Psi,\mathcal{S}_\mathcal{G}}_{X_a}\tilde{s}\rangle_{\mathcal{S}_\mathcal{G}|_U}, \tag{150}$$

which follows from the fact that ∇^Ψ preserves $\langle\cdot,\cdot\rangle_\mathcal{G}$, which is of neutral signature, and $\nabla^{\Psi,\mathcal{S}_\mathcal{G}}$ is induced by $\nabla^{\Psi,\mathcal{S}_\mathcal{G}}$. (The argument is similar to the one used in the proof of Lemma 13). Relation (144) follows from (35), (150) and the fact that $\nabla^{\Psi,\mathcal{S}_\mathcal{G}}$ preserves the degree of sections of $\mathcal{S}_\mathcal{G}$. In order to prove (142), let $u\in\Gamma(E_M)$ and decompose it as $u=\xi+r+X$. Then

$$\Psi(a)(u) = \mathcal{L}_{X_a}\xi + \nabla^\Psi_{X_a} r + \mathcal{L}_{X_a}X,$$

from where we deduce that

$$\begin{aligned}\gamma_{\Psi(a)(u)}(\omega\otimes s) &= \gamma_{\mathcal{L}_{X_a}(\xi+X)}(\omega\otimes s) + \gamma_{\nabla^\Psi_{X_a}r}(\omega\otimes s)\\ &= (i_{\mathcal{L}_{X_a}X}\omega + (\mathcal{L}_{X_a}\xi)\wedge\omega)\otimes s + (-1)^{|\omega|}\omega\otimes(\nabla^\Psi_{X_a}r)\cdot s. \end{aligned} \tag{151}$$

Similar computations show that

$$\begin{aligned}\Psi^{\mathbb{S}_M}(a)\gamma_u(\omega\otimes s) &= \mathcal{L}_{X_a}(i_X\omega+\xi\wedge\omega)\otimes s + (i_X\omega+\xi\wedge\omega)\otimes\nabla^{\Psi,\mathcal{S}_\mathcal{G}}_{X_a}s\\ &\quad + (-1)^{|\omega|}(\mathcal{L}_{X_a}\omega\otimes(r\cdot s) + \omega\otimes\nabla^{\Psi,\mathcal{S}_\mathcal{G}}_{X_a}(r\cdot s))\end{aligned} \tag{152}$$

$$\begin{aligned}\gamma_u\Psi^{\mathbb{S}_M}(a)(\omega\otimes s) &= (i_X\mathcal{L}_{X_a}\omega + \xi\wedge\mathcal{L}_{X_a}\omega)\otimes s + (-1)^{|\omega|}(\mathcal{L}_{X_a}\omega)\otimes(r\cdot s)\\ &\quad + (i_X\omega+\xi\wedge\omega)\otimes\nabla^{\Psi,\mathcal{S}_\mathcal{G}}_{X_a}s + (-1)^{|\omega|}\omega\otimes(r\cdot\nabla^{\Psi,\mathcal{S}_\mathcal{G}}_{X_a}s).\end{aligned} \tag{153}$$

Combining relations (151), (152) and (153) and using that $\nabla^{\Psi,\mathcal{S}_\mathcal{G}}$ is compatible with ∇^Ψ we obtain (142). Relation (145) follows from the definition of the map $\Psi^{\mathbb{S}_M}$ and the flatness of $\nabla^{\Psi,\mathcal{S}_\mathcal{G}}$ (which is a consequence of the flatness of ∇^Ψ). □

Definition 60. *In the setting of Proposition 57, a section of the canonical weighted spinor bundle* $\mathbb{S}$ *is an* invariant spinor *if it is annihilated by the operators* $\Psi^\mathbb{S}(a)$, *for all* $a\in\mathfrak{g}$.

Notation 61. Given an action $\Psi:\mathfrak{g}\to\mathrm{Der}(E)$ on a transitive Courant algebroid E, we shall denote by $\Gamma_\mathfrak{g}(\mathbb{S})$ the vector space of invariant spinors and by $\Gamma_\mathfrak{g}(E)$ the vector space of invariant sections of E.

The next corollary is a consequence of Lemma 58.

Corollary 62. *Let E_i $(i = 1, 2)$ be two transitive Courant algebroids over M with canonical weighted spinor bundles $\mathbb{S}_i$ and actions $\Psi_i : \mathfrak{g} \to \mathrm{Der}(E_i)$ which lift ψ. Any invariant isomorphism $I : E_1 \to E_2$ (if it exists) induces a isomorphism $I_{\mathbb{S}} : \Gamma_{\mathfrak{g}}(\mathbb{S}_1) \to \Gamma_{\mathfrak{g}}(\mathbb{S}_2)$.*

We end this section with a compatibility property between Dirac generating operators and actions on spinors.

Lemma 63. *In the setting of Proposition 57,*

$$\not{d} \circ \Psi^{\mathbb{S}}(a) = \Psi^{\mathbb{S}}(a) \circ \not{d},\ \forall a \in \mathfrak{g}, \tag{154}$$

where $\not{d} \in \mathrm{End}\,\Gamma(\mathbb{S})$ is the canonical Dirac generating operator of E.

Proof. From Proposition 26 and Lemma 58, it is sufficient to prove the statement for $E = E_M$ a standard Courant algebroid defined by a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data (∇, R, H), with action (148), canonical weighted spinor bundle $\mathbb{S}_M = \Lambda(T^*M)\hat{\otimes}\mathcal{S}_{\mathcal{G}}$ and action on spinors given by (149). We need to show that for any $a \in \mathfrak{g}$, $\omega \in \Omega(M)$ and $s \in \Gamma(\mathcal{S}_{\mathcal{G}})$

$$\not{d}_M \Psi^{\mathbb{S}_M}(a)(\omega \otimes s) = \Psi^{\mathbb{S}_M}(a)\not{d}_M(\omega \otimes s) \tag{155}$$

where $\not{d}_M \in \mathrm{End}\,\Gamma(\mathbb{S}_M)$ is the Dirac generating operator of E_M, which can be computed using (22). We consider an invariant local frame (X_i) of TM. Since ∇^{Ψ} is flat we may (and will) take the local frame (r_k) of $\mathcal{G}$ to be ∇^{Ψ}-parallel. Since ∇^{Ψ} preserves the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{G}}$, the $\langle\cdot,\cdot\rangle_{\mathcal{G}}$- dual frame $(\tilde{r}_k)$ is also ∇^{Ψ}-parallel. Since ∇^{Ψ} preserves the Lie bracket $[\cdot,\cdot]_{\mathcal{G}}$, the Cartan form $C_{\mathcal{G}} \in \Gamma(\Lambda^3\mathcal{G}^*)$ is ∇^{Ψ}-parallel.

Since R, X_i and r_k are invariant, from the second relation (95) applied to $D := \Psi(a)$ we obtain

$$\mathcal{L}_{X_a}\langle R(X_i, X_j), r_k\rangle_{\mathcal{G}} = 0,\ \forall a \in \mathfrak{g}. \tag{156}$$

From (156), $\nabla^{\Psi} C_{\mathcal{G}} = 0$, the fact that $\nabla^{\Psi,\mathcal{S}_{\mathcal{G}}}$ is compatible with ∇^{Ψ} and the expressions (22), (149) for $\not{d}_M$ and $\Psi^{\mathbb{S}}$, we see that relation (155) reduces to

$$\nabla^{\Psi,\mathcal{S}_{\mathcal{G}}}_{X_a}\nabla^{\mathcal{S}_{\mathcal{G}}}_{X_i} s = \nabla^{\mathcal{S}_{\mathcal{G}}}_{X_i}\nabla^{\Psi,\mathcal{S}_{\mathcal{G}}}_{X_a} s,\ \forall a \in \mathfrak{g},\ s \in \Gamma(\mathcal{S}_{\mathcal{G}}), \tag{157}$$

where, we recall, $\nabla^{\mathcal{S}_{\mathcal{G}}}$ is the connection on $\mathcal{S}_{\mathcal{G}}$ induced by any connection on $S_{\mathcal{G}}$ compatible with ∇ and $\nabla^{\Psi,\mathcal{S}_{\mathcal{G}}}$ is the partial connection on $\mathcal{S}_{\mathcal{G}}$ induced by any partial connection on $S_{\mathcal{G}}$ compatible with ∇^{Ψ}. For any $a \in \mathfrak{g}$, let $A_a := \nabla^{\Psi}_{X_a} - \nabla_{X_a}$. Then

$$\nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}_{X_a} s = \nabla^{\mathcal{S}_{\mathcal{G}}}_{X_a} s - \frac{1}{2} A_a \cdot s \tag{158}$$

where $A_a \cdot s$ denotes the Clifford action of $A_a \in \Gamma(\Lambda^2 \mathcal{G}) \subset \Gamma\,\mathrm{Cl}(\mathcal{G})$ on $s \in \Gamma(\mathcal{S}_{\mathcal{G}})$ (see e.g. Proposition 53 of [12] for more details). From (158), (101) and $\mathcal{L}_{X_a} X_i = 0$ we deduce that (157) is equivalent to

$$R^{\nabla^{\mathcal{S}_{\mathcal{G}}}}(X_a, X_i)s + \frac{1}{2}(\mathrm{ad}_{R(X_a, X_i)}) \cdot s = 0, \tag{159}$$

where $(\mathrm{ad}_{R(X_a,X_i)}) \cdot s$ means the Clifford action of $\mathrm{ad}_{R(X_a,X_i)} := [\mathrm{ad}_{R(X_a,X_i)}, \cdot]_{\mathcal{G}} \in \Gamma(\Lambda^2 \mathcal{G}) \subset \Gamma\,\mathrm{Cl}(\mathcal{G})$ on s. In order to prove (159) we remark first that both endomorphisms $R^{\nabla^{\mathcal{S}_{\mathcal{G}}}}(X_a, X_i)$ and $(\mathrm{ad}_{R(X_a,X_i)})$ of $\mathcal{S}_{\mathcal{G}}$ are trace free (the statement for $R^{\nabla^{\mathcal{S}_{\mathcal{G}}}}(X_a, X_i)$ follows from the fact that $\nabla^{\mathcal{S}_{\mathcal{G}}}$ is induced by a connection $\nabla^{S_{\mathcal{G}}}$ on $S_{\mathcal{G}}$). On the other hand, since $\nabla^{\mathcal{S}_{\mathcal{G}}}$ is compatible with ∇, we obtain that $T := R^{\nabla^{\mathcal{S}_{\mathcal{G}}}}(X_a, X_i) \in \Gamma\,\mathrm{End}(\mathcal{S}_{\mathcal{G}})$ satisfies

$$T(r \cdot s) = (R^{\nabla}(X_a, X_i) r) \cdot s + r \cdot T(s), \forall r \in \Gamma(\mathcal{G}),\ s \in \Gamma(\mathcal{S}_{\mathcal{G}}). \tag{160}$$

The same relation is satisfied by $T := -\frac{1}{2}\mathrm{ad}_{R(X_a,X_i)}$ acting by Clifford multiplication (here we use that $R^{\nabla}(X_a, X_i) r = \mathrm{ad}_{R(X_a,X_i)}(r)$ and relation $\omega(r) = -\frac{1}{2}[\omega, r]_{\mathrm{Cl}}$, for any $\omega \in \Lambda^2 \mathcal{G} \subset \mathrm{Cl}(\mathcal{G})$, where $[\omega, r]_{\mathrm{Cl}} = \omega \cdot r - r \cdot \omega$ denotes the commutator of ω and r in the Clifford algebra and $\omega(r)$ the action of $\omega \in \Lambda^2 \mathcal{G} \cong \mathfrak{so}(\mathcal{G})$ on $r \in \mathcal{G}$).

To summarize: both $R^{\nabla^{\mathcal{S}_{\mathcal{G}}}}(X_a, X_i)$ and $-\frac{1}{2}(\mathrm{ad}_{R(X_a,X_i)})$ are trace-free and satisfy (160). Since $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ has neutral signature, they coincide. □

5.3. Pullback actions and spinors

Let $f : M \to N$ be a submersion and

$$\psi^M : \mathfrak{g} \to \mathfrak{X}(M),\ a \mapsto X_a^M$$
$$\psi^N : \mathfrak{g} \to \mathfrak{X}(N),\ a \mapsto X_a^N$$

be f-related infinitesimal actions, i.e. $X_a^N \circ f = df X_a^M$ for all $a \in \mathfrak{g}$. Let E be a transitive Courant algebroid over N with anchor $\pi : E \to TN$ and

$$\mathfrak{g} \ni a \mapsto \Psi(a) \in \mathrm{Der}(E)$$

be an action on E which lifts ψ^N. Recall that the pullback Courant algebroid $f^!E$ is the quotient bundle $C/C^\perp$ over M (identified with the graph M_f of f), where, for any $p \in M$,

$$\begin{aligned} C_p &:= \{(u, \xi + X) \in E_{f(p)} \times \mathbb{T}_p M \mid \pi(u) = (d_p f)(X)\} \\ C_p^\perp &:= \{(\frac{1}{2}\pi^*\gamma, -(d_p f)^*\gamma) \mid \gamma \in T^*_{f(p)}N\} \subset C_p \end{aligned}$$

with the Courant algebroid structure defined at the beginning of Section 4.2. For $U \subset N$ open, a section of $C|_{f^{-1}(U)}$ (and the induced section of $(f^!E)|_{f^{-1}(U)}$) of the form $(f^*u, \xi + X)$ where $u \in \Gamma(E|_U)$, $X \in \mathfrak{X}(f^{-1}(U))$ is f-projectable with $f_*X = \pi(u)$ and $\xi \in \Omega^1(f^{-1}(U))$, will be called distinguished. Let $\mathcal{U} = \{U_i\}$ be an open cover of N, with sufficiently small sets U_i. Any section of $C|_{f^{-1}(U_i)}$ is a $C^\infty(f^{-1}(U_i))$-linear combination of distinguished sections. For each U_i we define

$$\widehat{\Psi}^{U_i} : \mathfrak{g} \to \operatorname{End}\Gamma(C|_{f^{-1}(U_i)}), \tag{161}$$

which satisfies the Leibniz rule

$$\widehat{\Psi}^{U_i}(a)(fs) = X_a^M(f)s + f\widehat{\Psi}^{U_i}(a)(s), \tag{162}$$

for any $a \in \mathfrak{g}$, $f \in C^\infty(f^{-1}(U_i))$, $s \in \Gamma(C|_{f^{-1}(U_i)})$ and on distinguished sections is given by

$$\widehat{\Psi}^{U_i}(a)(f^*u, \xi + X) := (f^*(\Psi(a)u), \mathcal{L}_{X_a^M}(\xi + X)). \tag{163}$$

Lemma 64. *The map $\Psi : \mathfrak{g} \to \operatorname{End}\Gamma(C)$ given by*

$$\widehat{\Psi}(a)(s)|_{f^{-1}(U_i)} = \widehat{\Psi}^{U_i}(a)(s|_{f^{-1}(U_i)}), \tag{164}$$

for any $a \in \mathfrak{g}$ and $s \in \Gamma(C)$, is well defined, preserves $\Gamma(C^\perp)$, and induces an action

$$f^!\Psi : \mathfrak{g} \to \operatorname{Der}(f^!E) \tag{165}$$

which lifts ψ^M. It is called the pullback action of Ψ.

Proof. The statement that $\widehat{\Psi}$ is well defined reduces to showing that for any $U_k, U_p \in \mathcal{U}$, if

$$\sum_i \lambda_i(f^*u_i, \xi_i + X_i) = 0$$

where $\lambda_i \in C^\infty(f^{-1}(U_k \cap U_p))$ and $(f^*u_i, \xi_i + X_i)$ are distinguished sections on $f^{-1}(U_k \cap U_p)$, then

$$\sum_i \big(X_a^M(\lambda_i) f^*u_i + \lambda_i f^*\Psi(a)(u_i)\big) = 0. \tag{166}$$

This follows by writing $u_i \in \Gamma(E|_{U_k \cap U_p})$ in terms of a frame of $E|_{U_k \cap U_p}$ and using the Leibniz rule for $\Psi(a)$ and that X_a^M projects to X_a^N. For any distinguished section $(f^*u, \xi + X)$ of C,

$$\hat{\Psi}(a)(f^*(u), \xi + X) = (f^*(\Psi(a)u), \mathcal{L}_{X_a^M}(\xi + X))$$

is also a section of C, because

$$f_*\mathcal{L}_{X_a^M} X = \mathcal{L}_{X_a^N}(f_*X) = \mathcal{L}_{X_a^N}(\pi(u)) = \pi(\Psi(a)u),$$

where in the last equality we used the third relation (95) applied to the derivation $\Psi(a)$. We proved that $\hat{\Psi}(a)$ is an endomorphism of $\Gamma(C)$. For any distinguished sections $w_1 := (f^*u, \xi + X)$ and $w_2 := (f^*v, \eta + Y)$ of C, the following relation

$$\langle \hat{\Psi}(a)w_1, w_2\rangle_{E\times\mathbb{T}M} + \langle w_1, \hat{\Psi}(a)w_2\rangle_{E\times\mathbb{T}M} = X_a^M\langle w_1, w_2\rangle_{E\times\mathbb{T}M} \tag{167}$$

holds, where $\langle\cdot,\cdot\rangle_{E\times\mathbb{T}M}$ is the scalar product of the direct product Courant algebroid $E \times \mathbb{T}M$. The above relation implies that $\hat{\Psi}(a)$ preserves $\Gamma(C^\perp)$. We obtain an induced endomorphism $(f^!\Psi)(a)$ of $\Gamma(C/C^\perp)$. It is easy to see that the map $a \mapsto (f^!\Psi)(a)$ is an action of $\mathfrak{g}$ on $f^!E$. □

Assume now that the Courant algebroid E from the beginning of this section is a standard Courant algebroid defined by a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data (∇, R, H) and that the action Ψ preserves the factors of E. Recall that $f^!E$ is isomorphic, by means of the isomorphism F given by (57), to the standard Courant algebroid E_M defined by $(f^*\mathcal{G}, f^*[\cdot,\cdot]_{\mathcal{G}}, f^*\langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data $(f^*\nabla, f^*R, f^*H)$ (see Lemma 27 i)). The pullback action $f^!\Psi$ induces an action

$$\Psi^M : \mathfrak{g} \to \mathrm{Der}(E_M),\ \Psi^M(a) := F^{-1} \circ (f^!\Psi)(a) \circ F$$

of $\mathfrak{g}$ on E_M. It turns out that Ψ^M preserves the factors of E_M and that the partial connection which corresponds to Ψ^M (according to Lemma 43) is just the pullback of the partial connection which corresponds to Ψ. These statements are proved in the next lemma.

Lemma 65. *In the above setting, let*

$$\Psi(a)(\xi + r + X) := \mathcal{L}_{X_a^N}\xi + \nabla^{\Psi}_{X_a^N} r + \mathcal{L}_{X_a^N} X, \tag{168}$$

where $\xi \in \Omega^1(N)$, $r \in \Gamma(\mathcal{G})$ *and* $X \in \mathfrak{X}(N)$. *Then*

$$\Psi^M(a)(\xi + r + X) = \mathcal{L}_{X_a^M}\xi + (f^*\nabla^{\Psi})_{X_a^M} r + \mathcal{L}_{X_a^M} X, \tag{169}$$

where $\xi \in \Omega^1(M)$, $r \in \Gamma(f^*\mathcal{G})$ *and* $X \in \mathfrak{X}(M)$. *Above* $f^*\nabla^{\Psi}$ *is the partial connection on* $f^*\mathcal{G}$ *defined along* $\mathrm{span}_{C^\infty(M)}\{X_a^M\}$ *by*

$$(f^*\nabla^{\Psi})_{X_a^M}(f^*r) = f^*(\nabla^{\Psi}_{X_a^N} r),\ \forall a \in \mathfrak{g},\ r \in \Gamma(\mathcal{G}). \tag{170}$$

Proof. The isomorphism F given by (57) induces an isomorphism $F : \Gamma(E_M) \to \Gamma(f^!E)$ which satisfies

$$F(\xi + f^*r + X) = [(f^*(r + f_*X), \xi + X)] \tag{171}$$

where $r \in \Gamma(\mathcal{G})$, $X \in \mathfrak{X}(M)$ is f-projectable and $\xi \in \Omega^1(M)$. (In the right hand side of (171) $r + f_*X \in \Gamma(\mathcal{G} \oplus TN) \subset \Gamma(E)$ and $f^*(r + f_*X) \in \Gamma(f^*E)$). Then

$$(f^!\Psi)(a) \circ F(\xi + f^*r + X) = [(f^*(\nabla^{\Psi}_{X_a^N} r + \mathcal{L}_{X_a^N} f_*X), \mathcal{L}_{X_a^M}(\xi + X))],$$

and, applying F^{-1}, we obtain (169). □

Remark 66. In terms of the partial connections ∇^{Ψ} and $(f^*\nabla)^{\Psi^M}$ on $\mathcal{G}$ and $f^*\mathcal{G}$ which correspond to the actions Ψ and Ψ^M (according to Lemma 43), relation (169) can be written in the equivalent way

$$(f^*\nabla)^{\Psi^M} = f^*\nabla^{\Psi}, \tag{172}$$

where $f^*\nabla^{\Psi}$ is defined by (170). We deduce a similar relation for the corresponding compatible partial connections on an irreducible $\mathrm{Cl}(\mathcal{G})$-bundle

$S_{\mathcal{G}}$ and its pullback $S_{f^*\mathcal{G}} := f^* S_{\mathcal{G}}$, or between the partial connections which they induce on the canonical spinor bundles $\mathcal{S}_{\mathcal{G}}$ and $\mathcal{S}_{f^*\mathcal{G}} = f^*\mathcal{S}_{\mathcal{G}}$ of $S_{\mathcal{G}}$ and $S_{f^*\mathcal{G}}$ respectively:

$$(f^*\nabla)^{\Psi^M, \mathcal{S}_{f^*\mathcal{G}}} = f^*\nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}. \tag{173}$$

The next proposition states several compatibilities between pullback actions, isomorphisms, pullback and pushforward on spinors.

Proposition 67. *Let $f : M \to N$ be a submersion and $\psi^M : \mathfrak{g} \to \mathfrak{X}(M)$, $\psi^N : \mathfrak{g} \to \mathfrak{X}(N)$ f-related actions.*

i) Let (E_i, Ψ_i) $(i = 1, 2)$ be transitive Courant algebroids over N with actions $\Psi_i : \mathfrak{g} \to \mathrm{Der}(E_i)$ which lift ψ^N. If $I : E_1 \to E_2$ is an isomorphism which is invariant with respect to Ψ_i, then the isomorphism $I^f : f^! E_1 \to f^! E_2$ is invariant with respect to $f^!\Psi_i$.

ii) Let E be a transitive Courant algebroid over N with an action $\Psi : \mathfrak{g} \to \mathrm{Der}(E)$ which lifts ψ^N. Let $\Psi^{\mathbb{S}} : \mathfrak{g} \to \mathrm{End}\,\Gamma(\mathbb{S}_E)$ and $(f^!\Psi)^{\mathbb{S}} : \mathfrak{g} \to \mathrm{End}\,\Gamma(\mathbb{S}_{f^!E})$ be the actions on canonical weighted spinor bundles, induced by the actions $\Psi : \mathfrak{g} \to \mathrm{Der}(E)$ and $f^!\Psi : \mathfrak{g} \to \mathrm{Der}(f^!E)$. Assume that there is an admissible pair $(I : E \to T^*N \oplus \mathcal{G} \oplus TN, S_{\mathcal{G}})$ for $\mathbb{S}_E$ and $\mathbb{S}_{f^!E}$ (see Definition 31) such that I is invariant, cf. Section 4.2. Then

$$f^! \circ \Psi^{\mathbb{S}}(a) = (f^!\Psi)^{\mathbb{S}}(a) \circ f^!,\ \forall a \in \mathfrak{g}. \tag{174}$$

If, in addition, $f : M \to N$ has compact fibers and M, N are oriented, then also the pushforward $f_! : \Gamma(\mathbb{S}_{f^!E}) \to \Gamma(\mathbb{S}_E)$ is defined and

$$f_! \circ (f^!\Psi)^{\mathbb{S}}(a) = (-1)^{r|s| + nr + \frac{r(r-1)}{2}} \Psi^{\mathbb{S}}(a) \circ f_!,\ \forall a \in \mathfrak{g}, \tag{175}$$

where m, n and r are the dimension of M, N and the fibers of f.

Proof. i) We need to check that

$$I^f \circ f^!\Psi_1(a)[(f^*u, \xi + X)] = f^!\Psi_2(a) \circ I^f[(f^*u, \xi + X)],\ \forall a \in \mathfrak{g} \tag{176}$$

for any distinguished section $[(f^*u, \xi + X)]$ of $f^!E_1$. From the definition of I^f (see relation (55)), we obtain

$$I^f[(f^*u, \xi + X)] = [f^*I(u), \xi + X)] \tag{177}$$

Relation (176) follows from (177), the definition of $f^!\Psi_i$ and the invariance of I.

ii) The claim follows from the definition of $f^!$ and $f_!$, Lemma 59, relation (173) and $f_* X_a^M = X_a^N$ for any $a \in \mathfrak{g}$. □

6. T-duality

6.1. Definition of T-duality

Let $\pi : M \to B$ and $\tilde{\pi} : \tilde{M} \to B$ be principal bundles over the same manifold B with structure group the k-dimensional torus T^k. For notational convenience, we will denote the structure group of $\tilde{\pi}$ by $\tilde{T}^k$ and its Lie algebra by $\tilde{\mathfrak{t}}^k$. We assume that M, $\tilde{M}$ and B are oriented. Let

$$\mathrm{Lie}\,(T^k) = \mathfrak{t}^k \ni a \mapsto \psi^M(a) := X_a^M,\ \tilde{\mathfrak{t}}^k \ni a \mapsto \psi^{\tilde{M}}(a) := X_a^{\tilde{M}},$$

be the vertical parallellism of π and $\tilde{\pi}$. We denote by

$$N := M \times_B \tilde{M} := \{(m, \tilde{m}) \in M \times \tilde{M} \mid \pi(m) = \tilde{\pi}(\tilde{m})\}$$

the fiber product of M and $\tilde{M}$ and by $\pi_N : N \to M$ and $\tilde{\pi}_N : N \to \tilde{M}$ the natural projections. The actions of T^k on M and $\tilde{T}^k$ on $\tilde{M}$ induce naturally an action of $T^{2k} = T^k \times \tilde{T}^k$ on N, with infinitesimal action

$$\mathrm{Lie}\,(T^{2k}) = \mathfrak{t}^{2k} \ni a \mapsto \psi^N(a) = X_a^N,$$

where, for any $a \in \mathfrak{t}^k := \mathfrak{t}^k \oplus 0 \subset \mathfrak{t}^{2k}$,

$$(\pi_N)_* X_a^N = X_a^M, \quad (\tilde{\pi}_N)_* X_a^N = 0,$$

and for any $a \in \tilde{\mathfrak{t}}^k := 0 \oplus \mathfrak{t}^k \subset \mathfrak{t}^{2k}$,

$$(\pi_N)_* X_a^N = 0, \quad (\tilde{\pi}_N)_* X_a^N = X_a^{\tilde{M}}.$$

Let E and $\tilde{E}$ be transitive Courant algebroids over M and $\tilde{M}$, and assume they come with actions

$$\Psi : \mathfrak{t}^k \to \mathrm{Der}(E),\ \ \tilde{\Psi} : \tilde{\mathfrak{t}}^k \to \mathrm{Der}(\tilde{E}),$$

which lift ψ^M and $\psi^{\tilde{M}}$, such that there are invariant dissections

$$I : E \to E_M = T^*M \oplus \mathcal{G} \oplus TM$$
$$\tilde{I} : \tilde{E} \to \tilde{E}_{\tilde{M}} = T^*\tilde{M} \oplus \tilde{\mathcal{G}} \oplus T\tilde{M}$$

with the property that the partial connections ∇^{Ψ} and $\tilde{\nabla}^{\tilde{\Psi}}$ on $\mathcal{G}$ and $\tilde{\mathcal{G}}$, which correspond to the induced actions $\Psi \circ I \circ \Psi^{-1}$ and $\tilde{\Psi} \circ \tilde{I} \circ \tilde{\Psi}^{-1}$ on E_M and $\tilde{E}_{\tilde{M}}$ have trivial holonomy (it is easy to see that this condition is independent of the choice of invariant dissections). The pullback Courant algebroids $\pi_N^! E$ and $\tilde{\pi}_N^! \tilde{E}$ inherit the pullback actions (see Lemma 64)

$$\pi_N^! \Psi : \mathfrak{t}^k \to \mathrm{Der}(\pi_N^! E),\ \tilde{\pi}_N^! \tilde{\Psi} : \tilde{\mathfrak{t}}^k \to \mathrm{Der}(\tilde{\pi}_N^! \tilde{E}) \tag{178}$$

which lift the infinitesimal actions $\mathfrak{t}^k \ni a \mapsto X_a^N$ and $\tilde{\mathfrak{t}}^k \ni a \to X_a^N$ respectively. The situation is summarized in the following commutative diagram, in which the arrows pointing down are quotient maps with respect to principal T^k-actions: $B = M/T^k = \tilde{M}/\tilde{T}^k = N/T^{2k}$, $M = N/\tilde{T}^k$, $\tilde{M} = N/T^k$ $(T^{2k} = T^k \times \tilde{T}^k)$.

$$\begin{tikzcd}
 & \mathfrak{t}^k \curvearrowright \pi_N^! E \arrow[r] & N \arrow[dl, "\pi_N"'] \arrow[dr, "\tilde{\pi}_N"] & \tilde{\pi}_N^! \tilde{E} \curvearrowleft \tilde{\mathfrak{t}}^k \arrow[l] & \\
\mathfrak{t}^k \curvearrowright E \arrow[r] & M \arrow[dr] & & \tilde{M} \arrow[dl] & \tilde{E} \curvearrowleft \tilde{\mathfrak{t}}^k \arrow[l] \\
 & & B & &
\end{tikzcd}$$

In the next lemma we extend the $\mathfrak{t}^k$-action $\pi_N^! \Psi$ to a $\mathfrak{t}^{2k}$-action.

Lemma 68. *i) The map*

$$\pi_N^{!!} \Psi : \mathfrak{t}^{2k} \to \mathrm{Der}(\pi_N^! E)$$

which on $\mathfrak{t}^k$ coincides with $\pi_N^! \Psi$ and the evaluation of which on any $b \in \tilde{\mathfrak{t}}^k$ satisfies the Leibniz rule

$$(\pi_N^{!!} \Psi)(b)(fs) = f(\pi_N^{!!} \Psi)(b)(s) + X_b^N(f)s,\ \forall f \in C^{\infty}(N),\ s \in \Gamma(\pi_N^! E)$$

and on distinguished sections $[(\pi_N^*(u), \xi + X)]$ of $\pi_N^! E$ is given by

$$(\pi_N^{!!} \Psi)(b)[(\pi_N^*(u), \xi + X)] = [(0, \mathcal{L}_{X_b^N}(\xi + X))] \tag{179}$$

is a well defined action on $\pi_N^! E$.

ii) Let (E_1, Ψ_1) be another transitive Courant algebroid over M with an action $\Psi_1 : \mathfrak{t}^k \to \mathrm{Der}(E_1)$ which lifts ψ^M. If $I : E \to E_1$ is an isomorphism invariant with respect to Ψ and Ψ_1, then the pullback isomorphism $I^{\pi_N} : \pi_N^! E \to \pi_N^! E_1$ is invariant with respect to the $\mathfrak{t}^{2k}$-actions $\pi_N^{!!} \Psi$ and $\pi_N^{!!} \Psi_1$.

Proof. Claim i) follows from arguments similar to the proof of Lemma 64. To prove claim ii), we need to show that

$$I^{\pi_N} \circ (\pi_N^{!!}\Psi)(a)(s) = (\pi_N^{!!}\Psi_1)(a) \circ I^{\pi_N}(s),\ \forall a \in \mathfrak{t}^{2k},\ s \in \Gamma(\pi_N^! E). \tag{180}$$

Relation (180) with $a \in \mathfrak{t}^k$, follows from Proposition 67 i). Relation (180) with $a \in \tilde{\mathfrak{t}}^k$ follows by taking $s = [(\pi_N^*(u), \xi + X)]$ a distinguished section of $\pi_N^! E$. Then both sides of (180) equal $[(0, \mathcal{L}_{X_a^N}(\xi + X)]$ (we use (179) together with (177)). □

Since ∇^Ψ has trivial holonomy, the quadratic Lie algebra bundle $\mathcal{G}_B$ from Assumption 49 is defined.

Lemma 69. *i) If $E = T^*M \oplus \mathcal{G} \oplus TM$ is a standard Courant algebroid and Ψ preserves the summands of E, then $\pi_N^{!!}\Psi$ preserves the summands of $\pi_N^! E = T^*N \oplus \pi_N^*\mathcal{G} \oplus TN$. The partial connection $(\pi_N^*\nabla)^{\pi_N^{!!}\Psi}$ on $\pi_N^*\mathcal{G}$ which corresponds to $\pi_N^{!!}\Psi$ (according to Lemma 43) is the pullback of the partial connection ∇^Ψ on $\mathcal{G}$ which corresponds to Ψ:*

$$(\pi_N^*\nabla)^{\pi_N^{!!}\Psi}_{X_a^N}(\pi_N^* r) = \pi_N^* \nabla^\Psi_{X_a^M} r,\ (\pi_N^*\nabla)^{\pi_N^{!!}\Psi}_{X_b^N}(\pi_N^* r) = 0, \tag{181}$$

for any $r \in \Gamma(\mathcal{G})$, $a \in \mathfrak{t}^k$ and $b \in \tilde{\mathfrak{t}}^k$.

ii) A section of $\pi_N^*\mathcal{G}$ is $(\pi_N^*\nabla)^{\pi_N^{!!}\Psi}$-parallel if and only if it is the pullback by π_N of a ∇^Ψ-parallel section of $\mathcal{G}$ or the pullback by $\Pi := \pi \circ \pi_N$ of a section of $\mathcal{G}_B$.

Proof. The first relation (181) follows from relation (172). The second relation (181) follows by noticing that the section $\pi_N^*(r) \in \Gamma(\pi_N^*\mathcal{G})$ of $\pi_N^*\mathcal{G} \subset T^*N \oplus \pi_N^*\mathcal{G} \oplus TN$ corresponds, under the isomorphism (171) to the distinguished section $[(\pi_N^*(r), 0)]$ of $\pi_N^! E$ and using relation (179). Claim ii) follows from claim i). □

In a similar way, we construct an action $\tilde{\pi}_N^{!!}\tilde{\Psi} : \mathfrak{t}^{2k} \to \mathrm{Der}(\tilde{\pi}_N^! \tilde{E})$ which extends $\tilde{\pi}_N^! \tilde{\Psi}$. Analogous considerations as above hold for $\tilde{\pi} : \tilde{M} \to B$ and $\tilde{E}$.

Assumption 70. From now on the Courant algebroids $\pi_N^! E$ and $\tilde{\pi}_N^! \tilde{E}$ will be considered with the $\mathfrak{t}^{2k}$-actions $\pi_N^{!!}\Psi$ and $\tilde{\pi}_N^{!!}\tilde{\Psi}$. In particular, in the next definition the invariance is meant with respect to these $\mathfrak{t}^{2k}$-actions.

Definition 71. *The Courant algebroids E and $\tilde{E}$ are called T-*dual *if there is an invariant fiber preserving Courant algebroid isomorphism*

$$F : \pi_N^! E \to \tilde{\pi}_N^! \tilde{E}$$

such that the following non-degeneracy condition is satisfied. Let

$$I : E \to T^*M \oplus \mathcal{G} \oplus TM,$$
$$\tilde{I} : \tilde{E} \to T^*\tilde{M} \oplus \tilde{\mathcal{G}} \oplus T\tilde{M},$$

be dissections of E and $\tilde{E}$, and

$$I^{\pi_N} : \pi_N^! E \to T^*N \oplus \pi_N^* \mathcal{G} \oplus TN,$$
$$\tilde{I}^{\tilde{\pi}_N} : \tilde{\pi}_N^! \tilde{E} \to T^*N \oplus \tilde{\pi}_N^* \tilde{\mathcal{G}} \oplus TN$$

the induced dissections of $\pi_N^! E$ and $\tilde{\pi}_N^! \tilde{E}$ (according to Lemma 27 iii)). Let (β, Φ, K), where $\beta \in \Omega^2(N)$, $\Phi \in \Omega^1(N, \tilde{\pi}_N^* \tilde{\mathcal{G}})$ and $K \in \Gamma \operatorname{Isom}(\pi_N^* \mathcal{G}, \tilde{\pi}_N^* \tilde{\mathcal{G}})$, be the data which defines the Courant algebroid isomorphism

$$\tilde{I}^{\tilde{\pi}_N} \circ F \circ (I^{\pi_N})^{-1} : T^*N \oplus \pi_N^* \mathcal{G} \oplus TN \to T^*N \oplus \tilde{\pi}_N^* \tilde{\mathcal{G}} \oplus TN \tag{182}$$

(according to relation (9) from Section 2.2.1). Then

$$\beta - \Phi^* \Phi : \operatorname{Ker}\,(d\pi_N) \times \operatorname{Ker}\,(d\tilde{\pi}_N) \to \mathbb{R} \tag{183}$$

is non-degenerate.

Definition 72. *The above definition is independent of the choice of dissections.*

Proof. Let $I_i : E \to T^*M \oplus \mathcal{G}_i \oplus TM$ $(i = 1, 2)$ be two dissections of E and $\hat{F}_i$ the maps (182) constructed using the dissections I_i rather than I, i. e.

$$\hat{F}_i := \tilde{I}^{\tilde{\pi}_N} \circ F \circ (I_i^{\pi_N})^{-1} : T^*N \oplus \pi_N^* \mathcal{G}_i \oplus TN \to T^*N \oplus \tilde{\pi}_N^* \tilde{\mathcal{G}} \oplus TN.$$

Then

$$\hat{F}_2 = \hat{F}_1 \circ (I_1 \circ I_2^{-1})^{\pi_N}. \tag{184}$$

Assume that the dissections I_1 and I_2 are related by (β, K, Φ). Then, from Lemma 27 iii), the induced dissections of $\pi_N^! E$ are related by

$(\pi_N^*\beta, \pi_N^* K, \pi_N^*\Phi)$, i.e. the Courant algebroid isomorphism

$$(I_1 \circ I_2^{-1})^{\pi_N} : T^*N \oplus \pi_N^*\mathcal{G}_2 \oplus TN \to T^*N \oplus \pi_N^*\mathcal{G}_1 \oplus TN$$

is given by (9), with (β, K, Φ) replaced by $(\pi_N^*\beta, \pi_N^* K, \pi_N^*\Phi)$. The independence of the non-degeneracy condition (183) on the dissection of E follows from (17). The independence on the dissection of $\tilde{E}$ can be proved similarly. □

Remark 73. As opposed to the T-duality for exact Courant algebroids, the definition of T-dual transitive Courant algebroids E and $\tilde{E}$ is not symmetric with respect to E and $\tilde{E}$, in general. This follows from the lack of symmetry in the non-degeneracy condition from Definition 71.

Lemma 74. *Let (E_1, Ψ_1) and $(\tilde{E}_1, \tilde{\Psi}_1)$ be transitive Courant algebroids over M and $\tilde{M}$, together with actions*

$$\Psi_1 : \mathfrak{t}^k \to \mathrm{Der}(E_1),\ \ \tilde{\Psi}_1 : \tilde{\mathfrak{t}}^k \to \mathrm{Der}(\tilde{E}_1)$$

which lift ψ^M and $\psi^{\tilde{M}}$ respectively. Assume that

$$G : E_1 \to E,\ \tilde{G} : \tilde{E}_1 \to \tilde{E} \tag{185}$$

are invariant, fiber preserving Courant algebroid isomorphisms. If E and $\tilde{E}$ are T-dual, then also E_1 and $\tilde{E}_1$ are T-dual.

Proof. Let $F : \pi_N^! E \to \tilde{\pi}_N^! \tilde{E}$ be an isomorphism with the properties from Definition 71. Then the isomorphism

$$F_1 := (\tilde{G}^{\tilde{\pi}_N})^{-1} \circ F \circ G^{\pi_N} : \pi_N^! E_1 \to \tilde{\pi}_N^! \tilde{E}_1. \tag{186}$$

has the same properties. (For the invariance of $\tilde{G}^{\tilde{\pi}_N}$ and G^{π_N} we use Lemma 68 ii)). □

The next lemma states the conditions that two standard Courant algebroids are T-dual. Let $E = T^*M \oplus \mathcal{G} \oplus TM$ and $\tilde{E} = T^*\tilde{M} \oplus \tilde{\mathcal{G}} \oplus T\tilde{M}$ be standard Courant algebroids over M and $\tilde{M}$, defined by a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data (∇, R, H) and, respectively, a quadratic Lie algebra bundle $(\tilde{\mathcal{G}}, [\cdot,\cdot]_{\tilde{\mathcal{G}}}, \langle\cdot,\cdot\rangle_{\tilde{\mathcal{G}}})$ and data $(\tilde{\nabla}, \tilde{R}, \tilde{H})$. Let $\Psi : \mathfrak{t}^k \to \mathrm{Der}\,(E)$ and $\tilde{\Psi} : \tilde{\mathfrak{t}}^k \to \mathrm{Der}\,(\tilde{E})$ be actions which lift ψ^M and $\psi^{\tilde{M}}$ and preserve the factors of E and $\tilde{E}$.

Lemma 75. *The standard Courant algebroids E and $\tilde{E}$ are T-dual if and only if there are invariant forms $\beta \in \Omega^2(N)$ and $\Phi \in \Omega^1(N, \tilde{\pi}_N^* \tilde{\mathcal{G}})$ and a quadratic Lie algebra bundle isomorphism $K \in \Gamma \operatorname{Isom}(\pi_N^* \mathcal{G}, \tilde{\pi}_N^* \tilde{\mathcal{G}})$ which maps $(\pi_N^* \nabla)^{\pi_N^{!!} \Psi}$ to $(\tilde{\pi}_N^* \tilde{\nabla})^{\tilde{\pi}_N^{!!} \tilde{\Psi}}$ such that the non-degeneracy condition (183) and the following relations hold:*

$$(\tilde{\pi}_N^* \tilde{\nabla})_X r = K(\pi_N^* \nabla)_X (K^{-1} r) + [r, \Phi(X)]_{\tilde{\pi}_N^* \tilde{\mathcal{G}}}, \tag{187}$$

$$K \pi_N^* R - \tilde{\pi}_N^* \tilde{R} = d^{\tilde{\pi}_N^* \tilde{\nabla}} \Phi + c_2, \tag{188}$$

$$\pi_N^* H - \tilde{\pi}_N^* \tilde{H} = d\beta + \langle (K \pi_N^* R + \tilde{\pi}_N^* \tilde{R}) \wedge \Phi \rangle_{\tilde{\pi}_N^* \tilde{\mathcal{G}}} - c_3, \tag{189}$$

where $c_2 \in \Omega^2(N, \tilde{\pi}_N^* \tilde{\mathcal{G}})$ and $c_3 \in \Omega^3(N)$ are defined by

$$c_2(X, Y) := [\Phi(X), \Phi(Y)]_{\tilde{\pi}_N^* \tilde{\mathcal{G}}},$$
$$c_3(X, Y, Z) := \langle \Phi(X), [\Phi(Y), \Phi(Z)]_{\tilde{\pi}_N^* \tilde{\mathcal{G}}} \rangle_{\tilde{\pi}_N^* \tilde{\mathcal{G}}},$$

for any $X, Y, Z \in \mathfrak{X}(N)$.

Proof. An invariant isomorphism $F : \pi_N^! E \to \tilde{\pi}_N^! \tilde{E}$ between the standard Courant algebroids $\pi_N^! E$ and $\tilde{\pi}_N^! \tilde{E}$ is defined by a triple (β, K, Φ) (according to relation (9)) where $\beta \in \Omega^2(N)$ and $\Phi \in \Omega^1(N, \tilde{\pi}_N^* \tilde{\mathcal{G}})$ are invariant and $K \in \Gamma \operatorname{Isom}(\pi_N^* \mathcal{G}, \tilde{\pi}_N^* \tilde{\mathcal{G}})$ satisfies $(\tilde{\pi}_N^* \tilde{\nabla})^{\tilde{\pi}_N^{!!} \tilde{\Psi}} \circ K = K \circ (\pi_N^* \nabla)^{\pi_N^{!!} \Psi}$ (see Lemma 46). The relations from the statement of the lemma coincide with relations (10), applied to F. □

We end this section with a simple lemma on the existence of preferred dissections of T-dual transitive Courant algebroids.

Lemma 76. *Let E and $\tilde{E}$ be T-dual transitive Courant algebroids. Then E and $\tilde{E}$ admit invariant dissections of the form*

$$I : E \to T^*M \oplus \pi^* \mathcal{G}_B \oplus TM,$$
$$\tilde{I} : \tilde{E} \to T^*\tilde{M} \oplus \tilde{\pi}^* \mathcal{G}_B \oplus T\tilde{M}, \tag{190}$$

where $(\mathcal{G}_B, \langle \cdot, \cdot \rangle_{\mathcal{G}_B}, [\cdot, \cdot]_{\mathcal{G}_B})$ is a quadratic Lie algebra bundle on B.

Proof. Let $I : E \to E_M = T^*M \oplus \mathcal{G} \oplus TM$ and $I' : \tilde{E} \to \tilde{E}_{\tilde{M}} = T^*\tilde{M} \oplus \tilde{\mathcal{G}} \oplus T\tilde{M}$ be invariant dissections of E and $\tilde{E}$, where E_M and $\tilde{E}_{\tilde{M}}$ are standard Courant algebroids defined by $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}}, \langle \cdot, \cdot \rangle_{\mathcal{G}})$ and (∇, R, H) and, respectively, by $(\tilde{\mathcal{G}}, [\cdot, \cdot]_{\tilde{\mathcal{G}}}, \langle \cdot, \cdot \rangle_{\tilde{\mathcal{G}}})$ and $(\tilde{\nabla}, \tilde{R}, \tilde{H})$. Let Ψ^M and $\tilde{\Psi}^{\tilde{M}}$ be the induced actions on E_M and $\tilde{E}_{\tilde{M}}$. Since the corresponding partial connections have trivial holonomy, $\mathcal{G} = \pi^*\mathcal{G}_B$ and $\tilde{\mathcal{G}} = \tilde{\pi}^*\tilde{\mathcal{G}}_B$ for some quadratic Lie algebra bundles $\mathcal{G}_B$ and $\tilde{\mathcal{G}}_B$ on B. From Lemma 74, E_M and $\tilde{E}_{\tilde{M}}$ are T-dual.

In particular, there is a quadratic Lie algebra bundle isomorphism $K : \pi_N^*\mathcal{G} \to \tilde{\pi}_N^*\tilde{\mathcal{G}}$ which maps $(\pi_N^*\nabla)^{\pi_N^{!!}\Psi^M}$ to $(\tilde{\pi}_N^*\tilde{\nabla})^{\tilde{\pi}_N^{!!}\tilde{\Psi}^{\tilde{M}}}$. From Lemma 69 ii), $K = \Pi^* K_B$ where $K_B : \mathcal{G}_B \to \tilde{\mathcal{G}}_B$ is a quadratic Lie algebra bundle isomorphism. Consider the standard Courant algebroid $\tilde{E}^{(1)}_{\tilde{M}} := T^*\tilde{M} \oplus \tilde{\pi}^*\mathcal{G}_B \oplus T\tilde{M}$ defined by the quadratic Lie algebra bundle $\tilde{\pi}^*\mathcal{G}_B$ and data

$$\tilde{\nabla}^{(1)}_X := (\tilde{\pi}^* K_B^{-1}) \circ \tilde{\nabla}_X \circ (\tilde{\pi}^* K_B),\ \tilde{R}^{(1)} := (\tilde{\pi}^* K_B^{-1}) \circ \tilde{R},\ \tilde{H}^{(1)} := H$$

for any $X \in \mathfrak{X}(\tilde{M})$. Then $\tilde{I}^{(1)} : \tilde{E}_{\tilde{M}} \to \tilde{E}^{(1)}_{\tilde{M}}$ defined by $\tilde{I}^{(1)}|_{T^*\tilde{M}\oplus T\tilde{M}} = \mathrm{Id}$ and $\tilde{I}^{(1)}|_{\tilde{\pi}^*\mathcal{G}_B} = \tilde{\pi}^* K_B^{-1}$ is a fiber preserving Courant algebroid isomorphism and $\tilde{I} := \tilde{I}^{(1)} \circ I' : \tilde{E} \to \tilde{E}^{(1)}_{\tilde{M}}$ is a dissection like in (190). $\square$

6.2. *T*-duality and spinors

Assume that E and $\tilde{E}$ are T-dual transitive Courant algebroids and let

$$F : \pi_N^! E \to \tilde{\pi}_N^! \tilde{E}$$

be an invariant isomorphism as in Definition 71. Let $\mathbb{S}_E$, $\mathbb{S}_{\tilde{E}}$, $\mathbb{S}_{\pi_N^! E}$ and $\mathbb{S}_{\tilde{\pi}_N^! \tilde{E}}$ be canonical weighted spinor bundles of E, $\tilde{E}$, $\pi_N^! E$ and $\tilde{\pi}_N^! \tilde{E}$ respectively, such that the pullbacks $\pi_N^! : \Gamma(\mathbb{S}_E) \to \Gamma(\mathbb{S}_{\pi_N^! E})$ and $\tilde{\pi}_N^! : \Gamma(\mathbb{S}_{\tilde{E}}) \to \Gamma(\mathbb{S}_{\tilde{\pi}_N^! \tilde{E}})$ are defined. We consider an admissible pair $(I, S_{\mathcal{G}})$ for $\mathbb{S}_E$ and $\mathbb{S}_{\pi_N^! E}$, and an admissible pair $(\tilde{I}, S_{\tilde{\mathcal{G}}})$ for $\mathbb{S}_{\tilde{E}}$ and $\mathbb{S}_{\tilde{\pi}_N^! \tilde{E}}$, such that the dissections

$$I : E \to E_M = T^*M \oplus \pi^*\mathcal{G}_B \oplus TM,$$
$$\tilde{I} : \tilde{E} \to \tilde{E}_{\tilde{M}} = T^*\tilde{M} \oplus \tilde{\pi}^*\tilde{\mathcal{G}}_B \oplus T\tilde{M}$$

are invariant, $S_{\mathcal{G}} = \pi^* S_B$ and $S_{\tilde{\mathcal{G}}} = \tilde{\pi}^* \tilde{S}_B$, where S_B is an irreducible $\mathrm{Cl}(\mathcal{G}_B)$-bundle and $\tilde{S}_B$ is an irreducible $\mathrm{Cl}(\tilde{\mathcal{G}}_B)$-bundle. The Courant algebroids E_M and $\tilde{E}_{\tilde{M}}$ are defined by quadratic Lie algebra bundle $\pi^*(\mathcal{G}_B, [\cdot,\cdot]_{\mathcal{G}_B}, \langle\cdot,\cdot\rangle_{\mathcal{G}_B})$ and data (∇, R, H), respectively by $\tilde{\pi}^*(\tilde{\mathcal{G}}_B, [\cdot,\cdot]_{\tilde{\mathcal{G}}_B}, \langle\cdot,\cdot\rangle_{\tilde{\mathcal{G}}_B})$ and $(\tilde{\nabla}, \tilde{R}, \tilde{H})$, where $(\mathcal{G}_B, [\cdot,\cdot]_{\mathcal{G}_B}, \langle\cdot,\cdot\rangle_{\mathcal{G}_B})$ and $(\tilde{\mathcal{G}}_B, [\cdot,\cdot]_{\tilde{\mathcal{G}}_B}, \langle\cdot,\cdot\rangle_{\tilde{\mathcal{G}}_B})$ are quadratic Lie algebra bundles over B. They will be considered with the actions $\Psi^M : \mathfrak{t}^k \to \mathrm{Der}\,(E_M)$ and $\tilde{\Psi}^{\tilde{M}} : \tilde{\mathfrak{t}}^k \to \mathrm{Der}\,(\tilde{E}_{\tilde{M}})$ induced by the $\mathfrak{t}^k$ and $\tilde{t}^k$-actions on E and $\tilde{E}$ respectively.

Assumption 77. We assume that the isomorphism $F_{\mathbb{S}} : \mathbb{S}_{\pi_N^! E} \to \mathbb{S}_{\tilde{\pi}_N^! \tilde{E}}$ induced by F is globally defined. This is equivalent to assuming that the isomorphism

$$(F_1)_{\mathbb{S}} : \mathbb{S}_N \to \tilde{\mathbb{S}}_N,\ (F_1)_{\mathbb{S}} = (\tilde{I}^{\tilde{\pi}_N})_{\mathbb{S}} \circ F_{\mathbb{S}} \circ (I^{\pi_N})_{\mathbb{S}}^{-1}$$

is globally defined, where

$$\mathbb{S}_N := \Lambda(T^*N)\hat{\otimes}\Pi^*\mathcal{S}_B,\ \tilde{\mathbb{S}}_N := \Lambda(T^*N)\hat{\otimes}\Pi^*\tilde{\mathcal{S}}_B \tag{191}$$

are the canonical weighted spinor bundles of $\pi_N^! E_M = T^*N \oplus \Pi^*\mathcal{G}_B \oplus TN$ and $\tilde{\pi}_N^! \tilde{E}_{\tilde{M}} = T^*N \oplus \Pi^*\tilde{\mathcal{G}}_B \oplus TN$ determined by the spinor bundles

$$S_N := \Lambda(T^*N)\hat{\otimes}\Pi^* S_B,\ \tilde{S}_N := \Lambda(T^*N)\hat{\otimes}\Pi^*\tilde{S}_B$$

respectively, $\Pi = \pi \circ \pi_N = \tilde{\pi} \circ \tilde{\pi}_N$, and

$$(I^{\pi_N})_{\mathbb{S}} : \mathbb{S}_{\pi_N^! E} \to \mathbb{S}_N,\ (\tilde{I}^{\tilde{\pi}_N})_{\mathbb{S}} : \mathbb{S}_{\tilde{\pi}_N^! \tilde{E}} \to \tilde{\mathbb{S}}_N$$

are induced by the isomorphisms $I^{\pi_N} : \pi_N^! E \to \pi_N^! E_M$ and $\tilde{I}^{\tilde{\pi}_N} : \tilde{\pi}_N^! \tilde{E} \to \tilde{\pi}_N^! \tilde{E}_{\tilde{M}}$ respectively. Remark that $(F_1)_{\mathbb{S}}$ is compatible with the isomorphism $F_1 := \tilde{I}^{\tilde{\pi}_N} \circ F \circ (I^{\pi_N})^{-1} : \pi_N^! E_M \to \tilde{\pi}_N^! \tilde{E}_{\tilde{M}}$.

Remark 78. When $\mathcal{G}_B = \tilde{\mathcal{G}}_B$ as quadratic Lie algebra bundles (see Lemma 76) and $S_B = \tilde{S}_B$ as $\mathrm{Cl}(\mathcal{G}_B)$-bundles, F_1 is an automorphism of the vector bundle

$$\pi_N^! E_M = \tilde{\pi}_N^! E_{\tilde{M}} = T^*N \oplus \Pi^*\mathcal{G}_B \oplus TN$$

with scalar product

$$\langle \xi + \Pi^*(r_1) + X, \eta + \Pi^*(r_2) + Y\rangle = \frac{1}{2}(\xi(Y) + \eta(X)) + \Pi^*\langle r_1, r_2\rangle_{\mathcal{G}_B} \tag{192}$$

and $(F_1)_{\mathbb{S}}$ is an automorphism of $\mathbb{S}_N = \tilde{\mathbb{S}}_N$. If F_1 belongs to the image of the exponential map $\exp : \mathfrak{so}(T^*N \oplus \Pi^*\mathcal{G}_B \oplus TN) \to \mathrm{SO}_0(T^*N \oplus \Pi^*\mathcal{G}_B \oplus TN)$, then $(F_1)_{\mathbb{S}}$ is automatically globally defined (cf. Remark 21). In fact, in that case F_1 can be lifted to $(F_1)_{\mathbb{S}}$ using the exponential map for $\mathfrak{spin}(T^*N \oplus \Pi^*\mathcal{G}_B \oplus TN)$.

Theorem 79. *i) The map*

$$\tau := (\tilde{\pi}_N)_! \circ F_{\mathbb{S}} \circ \pi_N^! : \Gamma(\mathbb{S}_E) \to \Gamma(\mathbb{S}_{\tilde{E}}) \tag{193}$$

intertwines the canonical Dirac generating operators of E and $\tilde{E}$ and maps invariant spinors to invariant spinors. In particular, there is the following *commutative diagram*

$$\begin{tikzcd} \Gamma_{\mathfrak{t}^{2k}}(\mathbb{S}_{\pi_N^! E}) \arrow[r, "F_{\mathbb{S}}"] & \Gamma_{\mathfrak{t}^{2k}}(\mathbb{S}_{\tilde{\pi}_N^! \tilde{E}}) \arrow[d, "(\tilde{\pi}_N)_!"] \\ \Gamma_{\mathfrak{t}^k}(\mathbb{S}_E) \arrow[u, "\pi_N^!"] \arrow[r, "\tau"] & \Gamma_{\tilde{\mathfrak{t}}^k}(\mathbb{S}_{\tilde{E}}) \end{tikzcd}$$

ii) There is an isomorphism $\rho : \Gamma_{\mathfrak{t}^k}(E) \to \Gamma_{\tilde{\mathfrak{t}}^k}(\tilde{E})$ *of* $C^\infty(B)$*-modules which preserves Courant brackets, scalar products and is compatible with* τ*, i.e.*

$$\tau(\gamma_u s) = \gamma_{\rho(u)}\tau(s),\ [\rho(u), \rho(v)]_{\tilde{E}} = \rho[u, v]_E,\ \langle \rho(u), \rho(v)\rangle_{\tilde{E}} = \langle u, v\rangle_E, \tag{194}$$

for any $u, v \in \Gamma_{\mathfrak{t}^k}(E)$ *and* $s \in \Gamma_{\mathfrak{t}^k}(\mathbb{S}_E)$.

The claim from Theorem 79 i) concerning the canonical Dirac generating operators follows from Propositions 26, 34 and 40. The remaining claims from Theorem 79 will be proved in the next two lemmas.

Lemma 80. *In the above setting,* E_M *and* $\tilde{E}_{\tilde{M}}$ *are* T*-dual. The assertions of Theorem 79 hold for the pair* $(E, \tilde{E})$ *and canonical weighted spinor bundles* $\mathbb{S}_E$ *and* $\mathbb{S}_{\tilde{E}}$ *if and only if they hold for the pair* $(E_M, \tilde{E}_{\tilde{M}})$ *and canonical weighted spinor bundles*

$$\mathbb{S}_M := \Lambda(T^*M)\hat{\otimes}\pi^*\mathcal{S}_B,\ \mathbb{S}_{\tilde{M}} := \Lambda(T^*\tilde{M})\hat{\otimes}\tilde{\pi}^*\tilde{\mathcal{S}}_B$$

of E_M *and* $\tilde{E}_{\tilde{M}}$ *respectively.*

Proof. The fact that E_M and $\tilde{E}_{\tilde{M}}$ are T-dual follows from Lemma 74. We now assume that the assertions of Theorem 79 hold for $(E, \tilde{E})$ and we show that they hold for $(E_M, \tilde{E}_{\tilde{M}})$ as well. The same arguments prove also the converse statement. Using relation (186), we obtain that the map τ_1 defined for the pair $(E_M, \tilde{E}_{\tilde{M}})$ as is defined τ for the pair $(E, \tilde{E})$, that is,

$$\tau_1 := (\tilde{\pi}_N)_* \circ (F_1)_{\mathbb{S}} \circ \pi_N^* : \Gamma(\mathbb{S}_M) \to \Gamma(\mathbb{S}_{\tilde{M}}), \tag{195}$$

where $\pi_N^* : \Gamma(\mathbb{S}_M) \to \Gamma(\mathbb{S}_N)$ and $(\tilde{\pi}_N)_* : \Gamma(\tilde{\mathbb{S}}_N) \to \Gamma(\mathbb{S}_{\tilde{M}})$ are the pullback and pushforward maps (59) and (84), and $\mathbb{S}_N$ and $\tilde{\mathbb{S}}_N$ are defined by (191), is related to τ by $\tau_1 = \epsilon \tilde{I}_{\mathbb{S}} \circ \tau \circ (I_{\mathbb{S}})^{-1}$. Here $\epsilon \in \{\pm 1\}$ and $I_{\mathbb{S}} : \mathbb{S}_E \to \mathbb{S}_M$, $\tilde{I}_{\mathbb{S}} : \mathbb{S}_{\tilde{E}} \to \mathbb{S}_{\tilde{M}}$ are induced by $I : E \to E_M$ and $\tilde{I} : \tilde{E} \to \tilde{E}_{\tilde{M}}$. As I and $\tilde{I}$ are invariant, $I_{\mathbb{S}}$ and $\tilde{I}_{\mathbb{S}}$ map invariant spinors to invariant spinors (from

Corollary 62). By our hypothesis, this is true also for τ. We obtain that τ_1 maps invariant spinors to invariant spinors. Define

$$\rho_1 : \Gamma_{\mathfrak{t}^k}(E_M) \to \Gamma_{\tilde{\mathfrak{t}}^k}(\tilde{E}_{\tilde{M}}),\ \rho_1 := \tilde{I} \circ \rho \circ I^{-1}.$$

Since (τ, ρ) satisfy (194), so do (τ_1, ρ_1). □

The next lemma concludes the proof of Theorem 79.

Lemma 81. *The statements from Theorem 79 hold for the pair $(E_M, \tilde{E}_{\tilde{M}})$ and canonical weighted spinor bundles $\mathbb{S}_M$ and $\mathbb{S}_{\tilde{M}}$.*

Proof. In agreement with our notation from the previous sections, various partial connections will be denoted as follows (we shall use similar conventions for their tilde analogue):

- ∇^{Ψ^M} is the partial connection on $\pi^*\mathcal{G}_B$ which corresponds to the $\mathfrak{t}^k$-action Ψ^M on E_M;
- $(\pi_N^*\nabla)^{\pi_N^{!!}\Psi^M}$ is the partial connection on $\Pi^*\mathcal{G}_B$ which corresponds to the $\mathfrak{t}^{2k}$-action $\pi_N^{!!}\Psi^M$ on $\pi_N^! E_M$;
- $\nabla^{\Psi^M, \pi^*\mathcal{S}_B}$ is the partial connection on $\pi^*\mathcal{S}_B$ induced by any partial connection on $\pi^* S_B$ compatible with ∇^{Ψ^M}.
- $(\pi_N^*\nabla)^{\pi_N^{!!}\Psi^M, \Pi^*\mathcal{S}_B}$ is the partial connection on $\Pi^*\mathcal{S}_B$ induced by any partial connection on $\Pi^* S_B$ compatible with $(\pi_N^*\nabla)^{\pi_N^{!!}\Psi^M}$.

From Lemma 69, $(\pi_N^*\nabla)^{\pi_N^{!!}\Psi^M}$ is the pullback by π_N of ∇^{Ψ^M} (see relation (181)) and similarly for the partial connections which they induce on the spinor bundles $\Pi^*\mathcal{S}_B$ and $\pi^*\mathcal{S}_B$:

$$(\pi_N^*\nabla)^{\pi_N^{!!}\Psi^M, \Pi^*\mathcal{S}_B}_{X_a^N}(\pi_N^* s) = \pi_N^*\left(\nabla^{\Psi^M, \pi^*\mathcal{S}_B}_{(\pi_N)_*(X_a^N)} s\right), \tag{196}$$

for any $a \in \mathfrak{t}^{2k}$ and $s \in \Gamma(\pi^*\mathcal{S}_B)$.

i) We prove that the map τ_1 defined by (195) maps invariant spinors to invariant spinors. For this, we show first that if $\omega \otimes s \in \Gamma(\mathbb{S}_M)$ is $\mathfrak{t}^k$-invariant then $\pi_N^*(\omega \otimes s) \in \Gamma(\mathbb{S}_N)$ is $\mathfrak{t}^{2k}$-invariant. The $\mathfrak{t}^k$-invariance of $\pi_N^*(\omega \otimes s)$ follows from Proposition 67 ii) and $(\pi_N^{!!}\Psi^M)(a) = (\pi_N^!\Psi^M)(a)$ for any $a \in \mathfrak{t}^k$, which implies that $(\pi_N^{!!}\Psi^M)^{\mathbb{S}_N}(a) = (\pi_N^!\Psi^M)^{\mathbb{S}_N}(a)$ for any $a \in \mathfrak{t}^k$, where we recall that $(\pi_N^{!!}\Psi^M)^{\mathbb{S}_N}$ and $(\pi_N^!\Psi^M)^{\mathbb{S}_N}$ are the actions on spinors (sections of $\Gamma(\mathbb{S}_N)$) induced by the $\mathfrak{t}^{2k}$, respectively $\mathfrak{t}^k$-actions $\pi_N^{!!}\Psi^M$ and $\pi_N^!\Psi^M$ on $\pi_N^! E_M$ respectively. In order to prove that $\pi_N^*(\omega \otimes s)$ is $\tilde{\mathfrak{t}}^k$-invariant, we apply the formula

$$(\pi_N^{!!}\Psi^M)^{\mathbb{S}_N}(a)\,(\pi_N^*(\omega\otimes s)) = \mathcal{L}_{X_a^N}(\pi_N^*\omega)\otimes\pi_N^* s + (\pi_N^*\omega)\otimes(\pi_N^*\nabla)^{\pi_N^{!!}\Psi^M,\Pi^*\mathcal{S}_B}_{X_a^N}(\pi_N^* s), \tag{197}$$

see relation (149). If $a\in\tilde{\mathfrak{t}}^k$ then

$$\mathcal{L}_{X_a^N}(\pi_N^*\omega) = \pi_N^*\mathcal{L}_{(\pi_N)_*X_a^N}\omega = 0, \tag{198}$$

since $(\pi_N)_*X_a^N = 0$. From (196), we deduce that $(\pi_N^*\nabla)^{\pi_N^{!!}\Psi^M,\Pi^*\mathcal{S}_B}_{X_a^N}(\pi_N^* s) = 0$ by using again $(\pi_N)_*X_a^N = 0$. It follows that $\pi_N^*(\omega\otimes s)$ is $\tilde{\mathfrak{t}}^k$-invariant. We have proven that $\pi_N^*(\omega\otimes s)$ is $\mathfrak{t}^{2k}$-invariant. From Corollary 62 and the invariance of F_1, we deduce that $(F_1)_{\mathbb{S}}\pi_N^*(\omega\otimes s)$ is $\mathfrak{t}^{2k}$-invariant (in particular, $\tilde{\mathfrak{t}}^k$-invariant) and from Proposition 67 ii) we obtain that $\tau_1(\omega\otimes s) = (\tilde{\pi}_N)_*(F_1)_{\mathbb{S}}\pi_N^*(\omega\otimes s)$ is $\tilde{\mathfrak{t}}^k$-invariant, as needed.

ii) Let $u=\xi+r+X\in\Gamma_{\mathfrak{t}^k}(E_M)$. Then X and ξ are invariant with respect to the standard (by Lie derivatives) action of $\mathfrak{t}^k$ on M and r is ∇^{Ψ^M}-parallel. We obtain

$$\mathcal{L}_{X_a^N}(\pi_N^*\xi) = 0,\ \forall a\in\mathfrak{t}^{2k},\ (\pi_N^*\nabla)^{\pi_N^{!!}\Psi^M}(\pi_N^* r) = 0, \tag{199}$$

where in the second relation we used (181). We claim that there is a unique $\mathfrak{t}^{2k}$-invariant lift $\widehat{X}_0\in\mathfrak{X}_{\mathfrak{t}^{2k}}(N)$ of X with the property that

$$\mathrm{pr}_{T^*N}F_1(\pi_N^*(\xi+r)+\widehat{X}_0) = \tilde{\pi}_N^*(\tilde{\xi}) \tag{200}$$

for an (invariant) 1-form $\tilde{\xi}\in\Omega^1(\tilde{M})$. To prove the claim we assume that the isomorphism F_1 is defined by data (β,K,Φ) as in Section 2.2.1, where $\beta\in\Omega^2(N)$, $K\in\Gamma\,\mathrm{Isom}(\Pi^*\mathcal{G}_B,\Pi^*\tilde{\mathcal{G}}_B)$ and $\Phi\in\Omega^1(N,\Pi^*\tilde{\mathcal{G}}_B)$. Let $\widehat{X}$ be an arbitrary $\mathfrak{t}^{2k}$-invariant lift of X. Then

$$\begin{aligned}
&\mathrm{pr}_{TN}F_1(\pi_N^*(\xi+r)+\widehat{X}) = \widehat{X},\\
&\mathrm{pr}_{\tilde{\pi}_N^*\tilde{\mathcal{G}}}F_1(\pi_N^*(\xi+r)+\widehat{X}) = K(\pi_N^* r)+\Phi(\widehat{X})\\
&\mathrm{pr}_{T^*N}F_1(\pi_N^*(\xi+r)+\widehat{X}) = \pi_N^*(\xi)-2\Phi^*K(\pi_N^* r)+i_{\widehat{X}}\beta-\Phi^*\Phi(\widehat{X}).
\end{aligned} \tag{201}$$

From (199), $\pi_N^*(\xi+r)$ is $\mathfrak{t}^{2k}$-invariant and we obtain that $F_1(\pi_N^*(\xi+r)+\widehat{X})$ is also $\mathfrak{t}^{2k}$-invariant. On the other hand, from the non-degeneracy condition (183), we can choose a $\mathfrak{t}^{2k}$-invariant lift $\widehat{X}_0$ of X such that

$$(\beta-\Phi^*\Phi)(\widehat{X}_0,Y) = \xi((d\pi_N)(Y)) - 2\langle K(\pi_N^* r),\Phi(Y)\rangle,\ \forall Y\in\mathrm{Ker}\,d\tilde{\pi}_N. \tag{202}$$

The third relation (201) together with (202) imply that the (invariant) form $\mathrm{pr}_{T^*N} F_1(\pi_N^*(\xi + r) + \widehat{X}_0)$ is horizontal with respect to $\tilde{\pi}_N$, hence basic. Considering the lift $\widehat{X}_0$ we obtain relation (200).

Since $F_1(\pi_N^*(\xi + r) + \widehat{X}_0)$ is $\mathfrak{t}^{2k}$-invariant, its projection to $\Pi^*\tilde{\mathcal{G}}_B$ is parallel with respect to $(\tilde{\pi}_N^*\tilde{\nabla})^{\tilde{\pi}_N^{!!}\tilde{\Psi}^{\tilde{M}}}$ and is therefore the $\tilde{\pi}_N$-pullback of a $\tilde{\nabla}^{\tilde{\Psi}^{\tilde{M}}}$-parallel section $\tilde{r}$ of $\tilde{\pi}^*\tilde{\mathcal{G}}_B$. To summarize,

$$F_1(\pi_N^*(\xi + r) + \widehat{X}_0) = \tilde{\pi}_N^*(\tilde{\xi} + \tilde{r}) + \widehat{X}_0 \tag{203}$$

where $\tilde{\xi} \in \Omega^1(\tilde{M})$ is $\tilde{\mathfrak{t}}^k$-invariant and $\tilde{\nabla}^{\tilde{\Psi}^{\tilde{M}}}\tilde{r} = 0$. We define

$$\rho_1(u) := \tilde{\xi} + \tilde{r} + (\tilde{\pi}_N)_*\widehat{X}_0. \tag{204}$$

Obviously, $\rho_1(u)$ is $\tilde{\mathfrak{t}}^k$-invariant and the resulting map $\rho_1 : \Gamma_{\mathfrak{t}^k}(E_M) \to \Gamma_{\tilde{\mathfrak{t}}^k}(\tilde{E}_{\tilde{M}})$ is $C^\infty(B)$-linear. It remains to prove that (ρ_1, τ_1) satisfy relations (194). In order to prove the first relation (194), let $u := \xi + r + X \in \Gamma_{\mathfrak{t}^k}(E_M)$, $\rho_1(u) = \tilde{\xi} + \tilde{r} + (\tilde{\pi}_N)_*\widehat{X}_0$ constructed as above, $s \in \Gamma_{\mathfrak{t}^k}(\mathbb{S}_M)$ and $\sigma \in \Gamma(\tilde{\mathbb{S}}_N)$. The lifts $\widehat{X}_0$ (for any $X \in \mathfrak{X}(M)$) define a distribution $\mathcal{D}$ on N and we can apply Remark 30 with f replaced by π_N and distribution $\mathcal{D}$. Using (61) and Remark 38,

$$\pi_N^*\gamma_u(s) = \gamma_{\pi_N^*(\xi+r)+\widehat{X}_0}\pi_N^* s,\ (\tilde{\pi}_N)_*\gamma_{\tilde{\pi}_N^*(\tilde{\xi}+\tilde{r})+\widehat{X}_0}\sigma = \gamma_{\rho_1(u)}(\tilde{\pi}_N)_*\sigma, \tag{205}$$

and we write

$$\begin{aligned}
\tau_1\gamma_u(s) &= (\tilde{\pi}_N)_*(F_1)_{\mathbb{S}}(\pi_N)^*\gamma_u(s) = (\tilde{\pi}_N)_*(F_1)_{\mathbb{S}}\gamma_{\pi_N^*(\xi+r)+\widehat{X}_0}\pi_N^* s\\
&= (\tilde{\pi}_N)_*\gamma_{\tilde{\pi}_N^*(\tilde{\xi}+\tilde{r})+\widehat{X}_0}(F_1)_{\mathbb{S}}\pi_N^*(s) = \gamma_{\rho_1(u)}(\tilde{\pi}_N)_*(F_1)_{\mathbb{S}}(\pi_N)^*(s)\\
&= \gamma_{\rho_1(u)}\tau_1(s),
\end{aligned}$$

where in the third equality we used the compatibility of $(F_1)_{\mathbb{S}}$ with F_1 and relation (203). The first relation of (194) is proved. The second relation (194) follows from the next computation, which uses the first relation (194), the defining property

$$\gamma_{[u,v]} = [[\not{d}, \gamma_u], \gamma_v]$$

for Dirac generating operators and $\tau_1 \circ \not{d}_M = \not{d}_{\tilde{M}} \circ \tau_1$ proved in part i) of Theorem 79 (where $\not{d}_M$ and $\not{d}_{\tilde{M}}$ are the Dirac generating operators of E_M

and $\tilde{E}_{\tilde{M}}$ acting on $\Gamma(\mathbb{S}_M)$ and $\Gamma(\mathbb{S}_{\tilde{M}})$ respectively). Namely,

$$\begin{aligned}\gamma_{\rho_1[u,v]_{E_M}}\tau_1(s) &= \tau_1\gamma_{[u,v]_{E_M}}(s) = \tau_1[[\not{d}_M,\gamma_u],\gamma_v](s)\\ &= [[\not{d}_{\tilde{M}},\gamma_{\rho_1(u)}],\gamma_{\rho_1(v)}]\tau_1(s)\\ &= \gamma_{[\rho_1(u),\rho_1(v)]_{\tilde{E}_{\tilde{M}}}}\tau_1(s),\end{aligned}$$

for any $u,v\in\Gamma_{\mathfrak{t}^k}(E_M)$ and $s\in\Gamma_{\mathfrak{t}^k}(\mathbb{S}_M)$. In order to prove the third relation of (194), we remark that for any $u\in\Gamma_{\mathfrak{t}^k}(E_M)$, $\langle u,u\rangle_{E_M}$ is $\mathfrak{t}^k$-invariant and hence is the pullback of a function $g\in C^\infty(B)$. Similarly, $\langle\rho_1(u),\rho_1(u)\rangle_{\tilde{E}_{\tilde{M}}}$ is the pullback of a function $\tilde{g}\in C^\infty(B)$. We need to show that $g=\tilde{g}$. This follows from the next computation which uses the first relation (194): for any $s\in\Gamma(\mathbb{S}_M)$,

$$\begin{aligned}\tilde{\pi}^*(g)\tau_1(s) &= \tau_1(\pi^*(g)s) = \tau_1(\langle u,u\rangle_{E_M}s) = \tau_1\gamma_u^2(s) = \gamma_{\rho_1(u)}^2\tau_1(s)\\ &= \langle\rho_1(u),\rho_1(u)\rangle_{\tilde{E}_{\tilde{M}}}\tau_1(s) = \tilde{\pi}^*(\tilde{g})\tau_1(s).\end{aligned}$$

From the third relation of (194) we obtain that ρ_1 is an isomorphism (of vector spaces and even of $C^\infty(B)$-modules). □

The proof of the theorem is completed.

Corollary 82. *The map*

$$\tau := (\tilde{\pi}_N)_! \circ F_{\mathbb{S}} \circ \pi_N^! : \Gamma_{\mathfrak{t}^k}(\mathbb{S}_E) \to \Gamma_{\tilde{\mathfrak{t}}^k}(\mathbb{S}_{\tilde{E}}) \tag{206}$$

is an isomorphism of $C^\infty(B)$-modules.

Proof. This follows from the irreducibility of the spinor bundles together with the fact that τ is $C^\infty(B)$-linear, is not the zero map and intertwines the Clifford multiplications in the commutative diagram

$$\begin{tikzcd}\Gamma_{\mathfrak{t}^k}(E)\times\Gamma_{\mathfrak{t}^k}(\mathbb{S}_E) \arrow[r] \arrow[d, "\rho\times\tau"'] & \Gamma_{\mathfrak{t}^k}(\mathbb{S}_E) \arrow[d, "\tau"] \\ \Gamma_{\tilde{\mathfrak{t}}^k}(\tilde{E})\times\Gamma_{\tilde{\mathfrak{t}}^k}(\mathbb{S}_{\tilde{E}}) \arrow[r] & \Gamma_{\tilde{\mathfrak{t}}^k}(\mathbb{S}_{\tilde{E}}).\end{tikzcd}$$

□

Remark 83. As in the T-duality for exact or heterotic Courant algebroids, the map ρ constructed in Theorem 79 can be interpreted as an isomorphism between Courant algebroids over B (see [2] and [10]).

In the next remark we discuss Theorem 79 without the assumption that $F_{\mathbb{S}}$ is globally defined.

Remark 84. i) We claim that the isomorphism $(F_1)_{\mathbb{S}}$ introduced before Remark 78 (hence also $F_{\mathbb{S}}$) is always defined on subsets of N of the form $\Pi^{-1}(V)$, where $V \subset B$ is open and sufficiently small. Indeed, $(F_1)_{\mathbb{S}|_U} \in \mathrm{Isom}(\mathbb{S}_N|_U, \tilde{\mathbb{S}}_N|_U)$ is defined, whenever $U \subset N$ is open and sufficiently small (see Lemma 20). Letting $V := \Pi(U)$, we can find (reducing V if necessary) invariant frames (s_i) and $(\tilde{s}_i)$ of $\mathbb{S}_N$ and $\mathbb{S}_{\tilde{N}}$ on $\Pi^{-1}(V)$ and write

$$(F_1)_{\mathbb{S}|_U}(s_i) = \sum_j C_{ji}\tilde{s}_j \tag{207}$$

for some functions $C_{ji} \in C^\infty(U)$. From the invariance of $(F_1)_{\mathbb{S}|_U}$, we deduce that $C_{ji} = \Pi^*(c_{ji})|_U$ where $c_{ji} \in C^\infty(V)$. Then

$$(F_1)_{\mathbb{S}|_{\Pi^{-1}(V)}}(s_i) := \sum_j \Pi^*(c_{ji})\tilde{s}_j, \tag{208}$$

defines an invariant extension of $(F_1)_{\mathbb{S}|_U}$ to $\Pi^{-1}(V)$ compatible with $F_1|_{\Pi^{-1}(V)}$.

ii) From the above, the map

$$\tau_V := (\tilde{\pi}_N)_! \circ F_{\mathbb{S}} \circ \pi_N^! : \Gamma(\mathbb{S}_E|_{\pi^{-1}(V)}) \to \Gamma(\mathbb{S}_{\tilde{E}}|_{\tilde{\pi}^{-1}(V)})$$

is defined. Theorem 79 still holds, the only difference being that τ is replaced by the locally defined maps τ_V, for any $V \subset B$ open and sufficiently small. (The isomorphism ρ remains globally defined.)

6.3. Existence of a T-dual

Let $\pi : M \to B$ be a principal T^k-bundle and $\mathcal{H}$ a principal connection on π, with connection form $\theta = \sum_{i=1}^k \theta_i e_i \in \Omega^1(M, \mathfrak{t}^k)$, where (e_i) is a basis of $\mathfrak{t}^k$. Let (E, Ψ) be a standard Courant algebroid with an action $\Psi : \mathfrak{t}^k \to \mathrm{Der}(E)$ which lifts the vertical parallellism of π, defined by a quadratic Lie algebra bundle $(\mathcal{G}_B, [\cdot,\cdot]_{\mathcal{G}_B}, \langle\cdot,\cdot\rangle_{\mathcal{G}_B})$ whose adjoint representation $\mathrm{ad} : \mathcal{G}_B \to \mathrm{Der}(\mathcal{G}_B)$ onto the bundle of skew-symmetric derivations is an isomorphism, a connection ∇^B on $\mathcal{G}_B$ which preserves $[\cdot,\cdot]_{\mathcal{G}_B}$ and $\langle\cdot,\cdot\rangle_{\mathcal{G}_B}$, a 3-form $H^B_{(3)}$, 2-forms $H^{i,B}_{(2)}$ and sections $r^B_i \in \Gamma(\mathcal{G}_B)$ $(1 \leq i \leq k)$ as in Example 56 (see also

Corollary 55). Recall that the curvature R^B of ∇^B is of the form $R^B = \mathrm{ad}_{\mathfrak{r}^B}$, where $\mathfrak{r}^B \in \Omega^2(B, \mathcal{G}_B)$. We denote by (e^i) the dual basis of (e_i).

Theorem 85. *Assume that the closed forms $\mathcal{K}_i$ defined by (140) represent integral cohomology classes in $H^2(B, \mathbb{R})$ and let $\tilde{\pi} : \tilde{M} \to B$ be a principal $\tilde{T}^k$-bundle with connection form $\tilde{\theta} = \sum_{i=1}^k \tilde{\theta}_i \tilde{e}^i$, such that $(d\tilde{\theta}_i)^B = \mathcal{K}_i$ for any i. Then E admits a T-dual $\tilde{E}$ defined on $\tilde{M}$ and*

$$\left[\sum_{i=1}^{k} (d\theta_i)^B \wedge (d\tilde{\theta}_i)^B\right] = [\langle \mathfrak{r}^B \wedge \mathfrak{r}^B \rangle_{\mathcal{G}_B}] \in H^4(B, \mathbb{R}). \tag{209}$$

Proof. From the expression (140) of $\mathcal{K}_i = (d\tilde{\theta}_i)^B$, we have

$$(d\tilde{\theta}_i)^B = H^{i,B}_{(2)} + 2\langle \mathfrak{r}^B, r^B_i \rangle_{\mathcal{G}_B} - \langle r^B_i, r^B_j \rangle_{\mathcal{G}_B} (d\theta_j)^B. \tag{210}$$

We consider the data formed by the quadratic Lie algebra bundle

$$(\tilde{\mathcal{G}}_B, [\cdot, \cdot]_{\tilde{\mathcal{G}}_B}, \langle \cdot, \cdot \rangle_{\tilde{\mathcal{G}}_B}) := (\mathcal{G}_B, [\cdot, \cdot]_{\mathcal{G}_B}, \langle \cdot, \cdot \rangle_{\mathcal{G}_B}),$$

connection $\tilde{\nabla}^B := \nabla^B$, sections $\tilde{r}^B_i \in \Gamma(\mathcal{G}_B)$ (arbitrarily chosen), 3-form $\tilde{H}^B_{(3)} := H^B_{(3)}$ and 2-forms

$$\tilde{H}^{i,B}_{(2)} := (d\theta_i)^B - 2\langle \mathfrak{r}^B, \tilde{r}^B_i \rangle_{\mathcal{G}_B} + \langle \tilde{r}^B_i, \tilde{r}^B_j \rangle_{\mathcal{G}_B} (d\tilde{\theta}_j)^B. \tag{211}$$

From (211), the 2-form

$$\tilde{\mathcal{K}}_i := \tilde{H}^{i,B}_{(2)} + 2\langle \mathfrak{r}^B, \tilde{r}^B_i \rangle_{\mathcal{G}_B} - \langle \tilde{r}^B_i, \tilde{r}^B_j \rangle_{\mathcal{G}_B} (d\tilde{\theta}_j)^B = (d\theta_i)^B$$

is closed. Since

$$d\tilde{H}_{(3)} = dH_{(3)} = \langle \mathfrak{r}^B \wedge \mathfrak{r}^B \rangle_{\mathcal{G}_B} - \mathcal{K}_i \wedge (d\theta_i)^B = \langle \mathfrak{r}^B \wedge \mathfrak{r}^B \rangle_{\mathcal{G}_B} - \tilde{\mathcal{K}}_i \wedge (d\tilde{\theta}_i)^B,$$

we obtain from Example 56 a standard Courant algebroid $\tilde{E}$ with an action $\tilde{\Psi}$ which lifts the vertical parallellism of $\tilde{\pi}$, such that

$$\begin{aligned} \tilde{H}^{pqs,B}_{(0)} &:= -\frac{1}{3} \langle [\tilde{r}^B_p, \tilde{r}^B_q]_{\mathcal{G}_B}, \tilde{r}^B_s \rangle_{\mathcal{G}_B} \\ \tilde{H}^{ij,B}_{(1)} &:= \frac{1}{2} \left(\langle \nabla^B \tilde{r}^B_i, \tilde{r}^B_j \rangle_{\mathcal{G}_B} - \langle \nabla^B \tilde{r}^B_j, \tilde{r}^B_i \rangle_{\mathcal{G}_B} \right) \end{aligned}$$

and the $\mathcal{G}_B$-valued forms $\tilde{R}^{ij,B}_{(0)}$, $\tilde{R}^{i,B}_{(1)}$ and $\tilde{R}^B_{(2)}$ given by the tilde analogue of (135) (i.e. (135) with r^B_i replaced by $\tilde{r}^B_i$ and θ_i replaced by $\tilde{\theta}_i$). The quadratic

Lie algebra bundles of E and $\tilde{E}$ are the pullbacks of $(\mathcal{G}_B, [\cdot,\cdot]_{\mathcal{G}_B}, \langle\cdot,\cdot\rangle_{\mathcal{G}_B})$ and, as vector bundles with scalar products,

$$\pi_N^! E = \tilde{\pi}_N^! \tilde{E} = T^*N \oplus \Pi^*\mathcal{G}_B \oplus TN,$$

where the scalar products are given by (192) and $\Pi = \pi \circ \pi_N = \tilde{\pi} \circ \tilde{\pi}_N$.

We claim that E and $\tilde{E}$ are T-dual, i.e. not only that $\pi_N^! E$ and $\tilde{\pi}_N^! \tilde{E}$ are isomorphic as Courant algebroids but that one can choose the isomorphism to be invariant and such that the non-degeneracy condition (183) is satisfied. Such an isomorphism $F : \pi_N^! E \to \tilde{\pi}_N^! \tilde{E}$ (if it exists) is given by a triple (β, K, Φ), where $\beta \in \Omega^2(N)$, $\Phi \in \Omega^1(N, \Pi^*\mathcal{G}_B)$ and $K = \Pi^* K_B$ where $K_B \in \Gamma\,\mathrm{Aut}(\mathcal{G}_B)$ is a quadratic Lie algebra bundle automorphism (see the proof of Lemma 76). Let

$$\begin{aligned} \nabla^{\theta} &:= \nabla + \theta_i \otimes \mathrm{ad}_{r_i} = \pi^*\nabla^B \\ \nabla^{\tilde{\theta}} &:= \tilde{\nabla} + \tilde{\theta}_i \otimes \mathrm{ad}_{\tilde{r}_i} = \tilde{\pi}^*\nabla^B \end{aligned}$$

be the connections on E and $\tilde{E}$ defined before Lemma 48, where $r_i := \pi^*(r_i^B)$, $\tilde{r}_i := \tilde{\pi}^*(\tilde{r}_i^B)$ and to simplify notation we continue to omit the summation sign and we denote by the same symbol 'ad' the adjoint action in the Lie algebra bundles $\mathcal{G}_B$, $\mathcal{G}$, $\tilde{\mathcal{G}}$ or their pullbacks to N. Then

$$\begin{aligned} \pi_N^*\nabla &= \Pi^*\nabla^B - (\pi_N^*\theta_i) \otimes \Pi^*(\mathrm{ad}_{r_i^B}) \\ \tilde{\pi}_N^*\tilde{\nabla} &= \Pi^*\nabla^B - (\tilde{\pi}_N^*\tilde{\theta}_i) \otimes \Pi^*(\mathrm{ad}_{\tilde{r}_i^B}). \end{aligned} \tag{212}$$

With these preliminary remarks, we now consider separately the relations from Lemma 75 and we look for $(\beta, K = \Pi^* K_B, \Phi)$ such that these relations are satisfied. Relation (187) can be written in the equivalent way

$$\begin{aligned} \Pi^*\left(K_B(\nabla^B)K_B^{-1} - \nabla^B\right) = (\pi_N^*\theta_i) &\otimes \Pi^*(\mathrm{ad}_{K_B(r_i^B)}) \\ &- (\tilde{\pi}_N^*\tilde{\theta}_i) \otimes (\Pi^*\mathrm{ad}_{\tilde{r}_i^B}) + \mathrm{ad} \circ \Phi. \end{aligned} \tag{213}$$

Letting

$$K_B := \mathrm{Id}_{\mathcal{G}_B},\ \Phi := (\tilde{\pi}_N^*\tilde{\theta}_i) \otimes \Pi^*(\tilde{r}_i^B) - (\pi_N^*\theta_i) \otimes \Pi^*(r_i^B), \tag{214}$$

relation (213) is obviously satisfied. Relation (188) is satisfied from Lemma 5 and our hypothesis that the adjoint representation $\mathrm{ad} : \mathcal{G}_B \to \mathrm{Der}(\mathcal{G}_B)$ onto the bundle of skew-symmetric derivations is an isomorphism. It remains to find an invariant 2-form $\beta \in \Omega^2(N)$ such that relation (189) is satisfied. Now, a straightforward computation which uses the definition of Φ shows that the 3-form

$$c_3(X,Y,Z) := \langle \Phi(X), [\Phi(Y),\Phi(Z)]_{\Pi^*\mathcal{G}_B}\rangle_{\Pi^*\mathcal{G}_B},\ \forall X,Y,Z\in\mathfrak{X}(N)$$

is given by

$$\begin{aligned} c_3 &= \frac{1}{6}\left(\langle [\tilde r^B_s, \tilde r^B_i]_{\mathcal{G}_B}, \tilde r^B_j\rangle \tilde\theta_s\wedge\tilde\theta_i\wedge\tilde\theta_j - \langle [r^B_s, r^B_i]_{\mathcal{G}_B}, r^B_j\rangle \theta_s\wedge\theta_i\wedge\theta_j\right)\\ &+\frac{1}{2}\left(\langle [r^B_i, r^B_j]_{\mathcal{G}_B}, \tilde r^B_s\rangle \tilde\theta_s\wedge\theta_i\wedge\theta_j - \langle [\tilde r^B_j, \tilde r^B_s]_{\mathcal{G}_B}, r^B_i\rangle \theta_i\wedge\tilde\theta_j\wedge\tilde\theta_s\right), \end{aligned} \tag{215}$$

where we identify forms on M, $\tilde M$ or B with their pullback to N (we omit the pullback signs) and we denote $\langle\cdot,\cdot\rangle_{\mathcal{G}_B}$ by $\langle\cdot,\cdot\rangle$ for simplicity. On the other hand,

$$\begin{aligned} \pi_N^* R &= R^B_{(2)} + \theta_i\wedge R^{i,B}_{(1)} + R^{ij,B}_{(0)}\otimes(\theta_i\wedge\theta_j),\\ \tilde\pi_N^*\tilde R &= \tilde R^B_{(2)} + \tilde\theta_i\wedge \tilde R^{i,B}_{(1)} + \tilde R^{ij,B}_{(0)}\otimes(\tilde\theta_i\wedge\tilde\theta_j), \end{aligned} \tag{216}$$

where we recall (from the proof of Corollary 55) that

$$R^{ij,B}_{(0)} = \frac{1}{2}[r^B_i, r^B_j]_{\mathcal{G}_B},\ R^{i,B}_{(1)} = \nabla^B r^B_i,\ R^B_{(2)} = \mathfrak{r}^B - d\theta_i\otimes r^B_i \tag{217}$$

and similarly for $\tilde R^{ij,B}_{(0)}$, $\tilde R^{i,B}_{(1)}$ and $\tilde R^B_{(2)}$, with r^B_i replaced by $\tilde r^B_i$ and θ_i replaced by $\tilde\theta_i$. From (216) and (217) we obtain that

$$\begin{aligned} \langle (\pi_N^* R + \tilde\pi_N^*\tilde R)\wedge\Phi\rangle_{\Pi^*\mathcal{G}_B} &= -\theta_i\wedge\tilde\theta_j\wedge d\langle r^B_i, \tilde r^B_j\rangle\\ &+ \langle (2\mathfrak{r}^B - (d\theta_i)\otimes r^B_i - (d\tilde\theta_i)\otimes\tilde r^B_i)\wedge \tilde r_j\rangle\wedge\tilde\theta_j\\ &- \langle (2\mathfrak{r}^B - (d\theta_i)\otimes r^B_i - (d\tilde\theta_i)\otimes\tilde r^B_i)\wedge r_j\rangle\wedge\theta_j\\ &+\frac{1}{2}\left(\langle [r^B_i, r^B_j], \tilde r^B_p\rangle\theta_i\wedge\theta_j\wedge\tilde\theta_p - \langle [\tilde r^B_i, \tilde r^B_j], r^B_p\rangle\tilde\theta_i\wedge\tilde\theta_j\wedge\theta_p\right)\\ &+\frac{1}{2}\left(\langle [\tilde r^B_i, \tilde r^B_j], \tilde r^B_p\rangle\tilde\theta_i\wedge\tilde\theta_j\wedge\tilde\theta_p - \langle [r^B_i, r^B_j], r^B_p\rangle\theta_i\wedge\theta_j\wedge\theta_p\right)\\ &+ \langle \nabla^B r^B_i, r^B_j\rangle\wedge\theta_i\wedge\theta_j - \langle\nabla^B\tilde r^B_i, \tilde r^B_j\rangle\wedge\tilde\theta_i\wedge\tilde\theta_j. \end{aligned} \tag{218}$$

We write the 2-form β as

$$\beta = \beta_{(2)} + \theta_i\wedge\beta^i_{(1)} + \tilde\theta_i\wedge\tilde\beta^i_{(1)} + f_{ij}\theta_i\wedge\tilde\theta_j$$

where $\beta_{(2)}$, $\beta^i_{(1)}$, $\tilde{\beta}^i_{(1)}$ and f_{ij} are defined on B, so that

$$
\begin{aligned}
d\beta &= d\beta_{(2)} + d\theta_i \wedge \beta^i_{(1)} + d\tilde{\theta}_i \wedge \tilde{\beta}^i_{(1)} - \theta_i \wedge (d\beta^i_{(1)} + f_{ij} d\tilde{\theta}_j)\\
&\quad + \tilde{\theta}_i \wedge (-d\tilde{\beta}^i_{(1)} + f_{ji} d\theta_j) - df_{ij} \wedge \tilde{\theta}_j \wedge \theta_i.
\end{aligned} \tag{219}
$$

Finally, we write, as in Section 5.1.1,

$$
\begin{aligned}
\pi_N^* H &= H^B_{(3)} + \theta_i \wedge H^{i,B}_{(2)} + \theta_i \wedge \theta_j \wedge H^{ij,B}_{(1)} + H^{ijs,B}_{(0)} \theta_i \wedge \theta_j \wedge \theta_s\\
\tilde{\pi}_N^* \tilde{H} &= \tilde{H}^B_{(3)} + \tilde{\theta}_i \wedge \tilde{H}^i_{(2)} + \tilde{\theta}_i \wedge \tilde{\theta}_j \wedge \tilde{H}^{ij,B}_{(1)} + \tilde{H}^{ijs,B}_{(0)} \tilde{\theta}_i \wedge \tilde{\theta}_j \wedge \tilde{\theta}_s.
\end{aligned} \tag{220}
$$

Using the expressions of $H^{ijs,B}_{(0)}$ and $\tilde{H}^{ijs,B}_{(0)}$, and (215), (218), (219) and (220), we obtain that relation (189) reduces to the following relations:

$$
\begin{aligned}
&H^B_{(3)} - \tilde{H}^B_{(3)} - d\beta_{(2)} - (d\theta_i)^B \wedge \beta^i_{(1)} - (d\tilde{\theta}_i)^B \wedge \tilde{\beta}^i_{(1)} = 0\\
&d\beta^i_{(1)} + f_{ij}(d\tilde{\theta}_j)^B = -H^{i,B}_{(2)} + \langle r^B_i, \tilde{r}^B_j\rangle (d\tilde{\theta}_j)^B - 2\langle \mathfrak{r}^B, r^B_i\rangle + \langle r^B_i, r^B_j\rangle (d\theta_j)^B\\
&d\tilde{\beta}^i_{(1)} - f_{ji}(d\theta_j)^B = \tilde{H}^{i,B}_{(2)} - \langle \tilde{r}^B_i, \tilde{r}^B_j\rangle (d\tilde{\theta}_j)^B + 2\langle \mathfrak{r}^B, \tilde{r}^B_i\rangle - \langle \tilde{r}^B_i, r^B_j\rangle (d\theta_j)^B\\
&df_{ij} = d\langle r^B_i, \tilde{r}^B_j\rangle.
\end{aligned} \tag{221}
$$

Recall now that $H^B_{(3)} = \tilde{H}^B_{(3)}$ and

$$
\begin{aligned}
(d\tilde{\theta}_i)^B &= H^{i,B}_{(2)} + 2\langle \mathfrak{r}^B, r^B_i\rangle - \langle r^B_i, r^B_j\rangle (d\theta_j)^B\\
(d\theta_i)^B &= \tilde{H}^{i,B}_{(2)} + 2\langle \mathfrak{r}^B, \tilde{r}^B_i\rangle - \langle \tilde{r}^B_i, \tilde{r}^B_j\rangle (d\tilde{\theta}_j)^B.
\end{aligned}
$$

It follows that $\beta_{(2)} := 0$, $\beta^i_{(1)} := 0$, $\tilde{\beta}^i_{(1)} := 0$ and $f_{ij} := \langle r^B_i, \tilde{r}^B_j\rangle - \delta_{ij}$ satisfy relations (221), We obtain that the 2-form

$$
\beta := (\langle r^B_i, \tilde{r}^B_j\rangle - \delta_{ij}) \theta_i \wedge \tilde{\theta}_j
$$

satisfies (189). The existence of F is proved. It is clear that F is invariant. The non-degeneracy condition (183) is satisfied, since

$$
\beta(X_{\tilde{a}}, X_b) = -\langle \tilde{r}^B_a, r^B_b\rangle + \delta_{ab},\ (\Phi^*\Phi)(X_{\tilde{a}}, X_b) = -\langle \tilde{r}_a, r_b\rangle,
$$

for any $\tilde{a} \in \tilde{\mathfrak{t}}^k$ and $b \in \mathfrak{t}^k$. $\square$

6.4. Examples of T-duality

In this section we apply Theorem 85 to various classes of transitive Courant algebroids. In particular, we recover, in our setting, the T-duality for exact Courant algebroids [10] and the T-duality for heterotic Courant algebroids [2].

6.4.1. T-duality for exact Courant algebroids. Let $E = T^*M \oplus TM$ be an exact Courant algebroid over the total space of a principal T^k-bundle $\pi : M \to B$, with Dorfman bracket $[\cdot, \cdot]_H$ twisted by an invariant, closed, 3-form $H \in \Omega^3(M)$, that is,

$$[\xi + X, \eta + Y]_H := \mathcal{L}_X \eta - i_Y d\xi + i_Y i_X H + \mathcal{L}_X Y \tag{222}$$

for any $X, Y \in \mathfrak{X}(M)$, $\xi, \eta \in \Omega^1(M)$, scalar product

$$\langle \xi + X, \eta + Y \rangle := \frac{1}{2}(\xi(Y) + \eta(X))$$

and anchor the natural projection from E to TM. The action of T^k on M lifts naturally to an action on E. Assuming that $(i_X H)|_{\Lambda^2(\mathrm{Ker}\,\pi)} = 0$ for any $X \in TM$, we obtain a Courant algebroid of the type described in Example 56. Choose a connection $\mathcal{H}$ on π, with connection form $\theta = \sum_{i=1}^k \theta_i e_i$ (where (e_i) is a basis of $\mathfrak{t}^k$) and write

$$H = H_{(3)} + \sum_{i=1}^{k} \theta_i \wedge H^i_{(2)},$$

where $H_{(3)}$ and $H^i_{(2)}$ are basic. If $[H] \in H^3(M, \mathbb{R})$ is an integral cohomology class, then so is $[H^{i,B}_{(2)}] \in H^2(B, \mathbb{R})$ (for any i) and Theorem 85 can be applied. We recover the existence of a T-dual for exact Courant algebroids, which was proved in [9] (see also Proposition 2.1 of [10]).

6.4.2. Heterotic T-duality. Let G be a compact semi-simple Lie group, with a fixed invariant scalar product of neutral signature $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\sigma : P \to M$ be a principal G-bundle and $\mathcal{H}$ a connection on σ. By definition, the heterotic Courant algebroid defined by the principal G-bundle $\sigma : P \to M$, connection $\mathcal{H}$ and 3-form $H \in \Omega^3(M)$ is the standard Courant algebroid $E = T^*M \oplus \mathcal{G} \oplus TM$ with the following properties:

1) $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}}, \langle \cdot, \cdot \rangle_{\mathcal{G}})$, as a quadratic Lie algebra bundle, is given by the adjoint bundle $\mathfrak{g}_P := P \times_{\mathrm{Ad}} \mathfrak{g}$. Recall that sections $r \in \Gamma(\mathfrak{g}_P)$ are invariant vertical vector fields on P and can be identified with functions

$f : P \to \mathfrak{g}$ which satisfy the equivariance condition $f(pg) = \mathrm{Ad}_{g^{-1}} f(p)$ for any $p \in P$ and $g \in G$. We shall use the notation $r \equiv f$ to denote this identification. Since the Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$ and scalar product $\langle \cdot,\cdot \rangle_{\mathfrak{g}}$ of $\mathfrak{g}$ are Ad-invariant, they induce a Lie bracket $[\cdot,\cdot]_{\mathfrak{g}_P}$ and a scalar product $\langle \cdot,\cdot \rangle_{\mathfrak{g}_P}$ on $\mathfrak{g}_P$, which make $\mathfrak{g}_P$ a quadratic Lie algebra bundle. The Lie bracket of $\mathfrak{g}_P$ so defined coincides with the usual Lie bracket of invariant, vertical vector fields on P.

2) The connection ∇ which is part of the data (∇, R, H) which defines the standard Courant algebroid E is induced by $\mathcal{H}$, $R = R^{\mathcal{H}} \in \Omega^2(M, \mathfrak{g}_P)$ is the curvature of $\mathcal{H}$ and the 3-form H satisfies $dH = \langle R^{\mathcal{H}} \wedge R^{\mathcal{H}} \rangle_{\mathfrak{g}_P}$.

The following proposition describes all invariant scalar products on compact semi-simple Lie algebras. A similar description can be given for arbitrary reductive Lie algebras. Recall that a semi-simple Lie algebra is called compact it it is the Lie algebra of a compact group.

Proposition 86. *Let $\mathfrak{g} = \bigoplus_{i=1}^{s} k_i \mathfrak{g}_i$ be the decomposition of a real semi-simple Lie algebra into its simple ideals $\mathfrak{g}_i$, of multiplicity $k_i \geq 1$. Assume that $\mathfrak{g}$ is compact (or, more generally, that none of the $\mathfrak{g}_i$ has an invariant complex structure). Then every invariant scalar product on $\mathfrak{g}$ is of the form*

$$\sum B_i \otimes b_i, \tag{223}$$

where B_i is the Killing form of $\mathfrak{g}_i$ and b_i is a scalar product on $\mathbb{R}^{k_i}$. The scalar product (223) is of neutral signature if and only if $\sum (\dim \mathfrak{g}_i) p_i = \sum (\dim \mathfrak{g}_i) q_i$, where (p_i, q_i) is the signature of b_i.

Proof. We compute the space of invariant symmetric bilinear forms on $\mathfrak{g}$ as

$$(\mathrm{Sym}^2 \mathfrak{g}^*)^{\mathfrak{g}} = \bigoplus (\mathrm{Sym}^2 \mathfrak{g_i}^*)^{\mathfrak{g}_i} \otimes \mathrm{Sym}^2(\mathbb{R}^{k_i})^* \oplus \bigoplus (\Lambda^2 \mathfrak{g_i}^*)^{\mathfrak{g}_i} \otimes \Lambda^2(\mathbb{R}^{k_i})^*.$$

Since $\mathfrak{g}_i$ is simple and not complex, every invariant bilinear form on $\mathfrak{g}_i$ is a multiple of B_i and the right-hand side reduces to $\bigoplus B_i \otimes \mathrm{Sym}^2(\mathbb{R}^{k_i})^*$. This implies the first claim. The second claim follows by observing that the signature (p, q) of (223) is given by $p = -\sum (\dim \mathfrak{g}_i) p_i$, $q = -\sum (\dim \mathfrak{g}_i) q_i$. $\square$

Assume that M is the total space of a principal T^k-bundle $\pi : M \to B$ and that $\sigma : P \to M$ is the pullback of a principal G-bundle $\sigma_0 : P_0 \to B$. Then, for any $m \in M$ and $g \in T^k$ there is a natural identification between the fibers $P_m := \sigma^{-1}(m)$, $P_{mg} := \sigma^{-1}(mg)$ and $(P_0)_{\pi(m)} := \sigma_0^{-1}(\pi(m))$. The

T^k-action on M lifts naturally to an action on P (such that g acts as the identity map between P_m and P_{mg} in the above identification). We deduce that on any heterotic Courant algebroid $E = T^*M \oplus \mathfrak{g}_P \oplus TM$ there is an induced action

$$\Psi : \mathfrak{t}^k \to \mathrm{Der}(E), \Psi(a)(\xi + r + X) := \mathcal{L}_{X_a^M}\xi + \mathcal{L}_{X_a^P} r + \mathcal{L}_{X_a^M} X \tag{224}$$

where X_a^M and X_a^P denote the fundamental vector fields of the T^k-action on M and P defined by $a \in \mathfrak{t}^k$ and $r \in \Gamma(\mathfrak{g}_P)$ is viewed as an invariant vertical vector field on P. If $r \equiv f$, then $\mathcal{L}_{X_a^P} r \equiv X_a^P(f)$.

Following [2], we shall be interested in heterotic Courant algebroids defined by principal bundles $\sigma = \pi^*\sigma_0$ given by pullback as above and a particular class of connections $\mathcal{H} := \mathcal{H}^\sigma$ on σ. More precisely, we consider a connection $\mathcal{H}^\pi$ on the principal T^k-bundle $\pi : M \to B$, with connection form $\theta = \sum_{i=1}^k \theta_i e_i \in \Omega^1(M, \mathfrak{t}^k)$ (where (e_i) is a basis of $\mathfrak{t}^k$), a connection $\mathcal{H}^{\sigma_0}$ on the principal G-bundle $\sigma_0 : P_0 \to B$, with connection form $A_0 \in \Omega^1(P_0, \mathfrak{g})$ and a $G \times T^k$-equivariant function $\hat{v} : P \to (\mathfrak{t}^k)^* \otimes \mathfrak{g}$. They define a connection $\mathcal{H}^\sigma$ on σ, with connection form

$$A := \pi_0^* A_0 - \langle \sigma^*\theta, \hat{v}\rangle = \pi_0^* A_0 - \sum_{i=1}^k \sigma^*\theta_i \otimes \hat{v}_i, \tag{225}$$

where $\pi_0 : P \to P_0$ is the natural projection, $\langle \cdot, \cdot \rangle$ denotes the natural contraction between $\mathfrak{t}^k$ and $(\mathfrak{t}^k)^*$, and $\hat{v}_i = \langle \hat{v}, e_i\rangle : P \to \mathfrak{g}$. From the equivariance of $\hat{v}$, the functions $\hat{v}_i$ define sections of $\mathfrak{g}_P = \pi^*\mathfrak{g}_{P_0}$ which are pullback of sections of $\mathfrak{g}_{P_0}$, i.e. $\hat{v}_i = \pi^*\hat{v}_i^B$ for $\hat{v}_i^B \in \Gamma(\mathfrak{g}_{P_0})$ (we use the same notation for the functions $\hat{v}_i$, $\hat{v}_i^B$ and the corresponding sections of $\mathfrak{g}_P$ and $\mathfrak{g}_{P_0}$ respectively). We shall denote by $\widetilde{X}^{A_0} \in \mathfrak{X}(P_0)$, $\widetilde{Y}^{\pi_0^* A_0} \in \mathfrak{X}(P)$ and $\widetilde{Y}^A \in \mathfrak{X}(P)$ the horizontal lifts of $X \in \mathfrak{X}(B)$ and $Y \in \mathfrak{X}(M)$ with respect to $\mathcal{H}^{\sigma_0}$, $\pi_0^*\mathcal{H}^{\sigma_0}$ and $\mathcal{H}^\sigma$ respectively. Here $\pi_0^*\mathcal{H}^{\sigma_0} \subset TP$ denotes the G-invariant horizontal distribution in $\sigma : P \to M$ defined by $(\pi_0^*\mathcal{H}^{\sigma_0})_p = (d_p\pi_0)^{-1}\mathcal{H}^{\sigma_0}_{\pi_0(p)}$, $p \in P$. It coincides with the kernel of the connection form $\pi_0^* A_0$.

Lemma 87. *Let $(E = T^*M \oplus \mathfrak{g}_P \oplus TM, \Psi)$ be the heterotic Courant algebroid defined by $\sigma : P \to M$, the connection $\mathcal{H}^\sigma$ with connection form (225) and a 3-form $H \in \Omega^3(M)$ such that $dH = \langle R^{\mathcal{H}^\sigma} \wedge R^{\mathcal{H}^\sigma}\rangle_{\mathfrak{g}_P}$, together with the $\mathfrak{t}^k$-action (224). Then the connection ∇^θ on $\mathfrak{g}_P = \pi^*(\mathfrak{g}_{P_0})$ defined in (114) is the pullback of the connection ∇^{A_0} on $\mathfrak{g}_{P_0}$ induced by $\mathcal{H}^{\sigma_0}$ (equivalently, in the notation of Corollary 55, $\mathcal{G}_B = \mathfrak{g}_{P_0}$ and $\nabla^{\theta,B} = \nabla^{A_0}$).*

Proof. We claim that the horizontal lift $\widetilde{X_a^M}^{\pi_0^* A_0} \in \mathfrak{X}(P)$ of the π-vertical vector field $X_a^M \in \mathfrak{X}(M)$ determined by $a \in \mathfrak{t}^k$ coincides with the fundamental vector field X_a^P of the T^k-action on P, i.e.

$$\widetilde{X_a^M}^{\pi_0^* A_0} = X_a^P,\ \forall a \in \mathfrak{t}^k. \tag{226}$$

In order to prove (226), let $U \subset B$ be open and sufficiently small such that, over U, σ_0 is the trivial G-bundle and

$$\pi_0 : P|_{\pi^{-1}(U)} = \pi^{-1}(U) \times G \to P_0|_U = U \times G,\ \pi_0(p, g) = (\pi(p), g).$$

For any $X \in \mathfrak{X}(\pi^{-1}(U))$,

$$\widetilde{X}^{\pi_0^* A_0} = X - \langle (\pi_0^* A_0)(X), f_i^* \rangle X_{f_i}^P, \tag{227}$$

where (f_i) is a basis of $\mathfrak{g}$ with dual basis (f_i^*) and for $f \in \mathfrak{g}$, X_f^P is the left invariant vector field on G determined by f (viewed as a vector field on $P|_{\pi^{-1}(U)} = \pi^{-1}(U) \times G$). On the other hand, $X_a^M \in \mathfrak{X}(\pi^{-1}(U))$, viewed as a vector field on $P|_{\pi^{-1}(U)}$, satisfies $(\pi_0)_* X_a^M = 0$, since $\pi_* X_a^M = 0$ and $\pi_0 = \pi \times \mathrm{Id}$ in our trivializations. Applying (227) to $X := X_a^M$ and using $(\pi_0)_* X_a^M = 0$ we obtain $\widetilde{X_a^M}^{\pi_0^* A_0} = X_a^M$. On the other hand, the action of T^k on $P = \pi^{-1}(U) \times G$ is given by $R_g(m, \tilde{g}) = (mg, \tilde{g})$ which implies that $X_a^P = X_a^M$. Relation (226) follows.

Let ∇ be the connection on $\mathfrak{g}_P$ induced by $\mathcal{H}^\sigma$. It is given by

$$\nabla_X r \equiv \widetilde{X}^A(f) = \widetilde{X}^{\pi_0^* A_0}(f) - \theta_i(X) \mathrm{ad}_{\hat{v}_i} \circ f,\ X \in \mathfrak{X}(M), \tag{228}$$

where $r \in \Gamma(\mathfrak{g}_P)$ and $r \equiv f$. Here we have used relation (225), which implies that

$$\widetilde{X}^A = \widetilde{X}^{\pi_0^* A_0} + \theta_i(X) X_{\hat{v}_i}^P \tag{229}$$

and the G-equivariance of f, which implies $X_v^P(f) = -\mathrm{ad}_v \circ f$ for all $v \in \mathfrak{g}$. (In relation (229) the vector field $X_{\hat{v}_i}^P$ at $p \in P$ is equal to $(X_{\hat{v}_i(p)}^P)_p$). Applying relation (228) to $X := X_a^M$ and using (226) we obtain

$$\nabla_{X_a^M} r \equiv X_a^P(f) - \langle a, e_i^* \rangle \mathrm{ad}_{\hat{v}_i} \circ f,\ \forall a \in \mathfrak{t}^k, \tag{230}$$

which implies that the skew-symmetric derivation A_a of $\mathfrak{g}_P$, from Lemma 43, is given by

$$A_a(r) = (\mathcal{L}_{X_a^P} - \nabla_{X_a^M}) r \equiv \mathrm{ad}_{\langle \hat{v}, a \rangle} \circ f,\ \forall a \in \mathfrak{t}^k, \tag{231}$$

where in the equality we used the definition of the endomorphism A_a. From its definition (114) and relations (228), (231), ∇^θ is given by

$$\nabla^\theta_X r = \nabla_X r + \sum_{i=1}^{k} \theta_i(X) A_i(r) \equiv \widetilde{X}^{\pi_0^* A_0}(f), \tag{232}$$

which implies that $\nabla^\theta = \pi^* \nabla^{A_0}$ as needed. □

Since ∇^{A_0} preserves the Lie bracket and scalar product of $\mathfrak{g}_{P_0}$, its curvature takes values in the bundle of skew-symmetric derivations of $\mathfrak{g}_{P_0}$ and is of the form $\mathrm{ad}_{\mathfrak{r}^{A_0}}$ where $\mathfrak{r}^{A_0} \in \Omega^2(B, \mathfrak{g}_{P_0})$ (since $\mathfrak{g}$ is semi-simple). Like in (118), we decompose $H \in \Omega^3(M)$ using the connection θ.

Proposition 88. *In the above setting, assume that*

$$\begin{aligned} H^{ijs,B}_{(0)} &= -\frac{1}{3}\langle [\hat{v}^B_i, \hat{v}^B_j]_{\mathfrak{g}}, \hat{v}^B_s\rangle_{\mathfrak{g}} \\ H^{ij,B}_{(1)} &= \frac{1}{2}\left(\langle \nabla^{A_0}\hat{v}^B_i, \hat{v}^B_j\rangle_{\mathfrak{g}} - \langle \nabla^{A_0}\hat{v}^B_j, \hat{v}^B_i\rangle_{\mathfrak{g}}\right) \end{aligned} \tag{233}$$

and that the (closed) forms

$$H^{i,B}_{(2)} + 2\langle \mathfrak{r}^{A_0}, \hat{v}^B_i\rangle_{\mathfrak{g}} - \langle \hat{v}^B_i, \hat{v}^B_j\rangle_{\mathfrak{g}} (d\theta_j)^B \tag{234}$$

represent integral cohomology classes. Then (E, Ψ) admits a T-dual which is a heterotic Courant algebroid.

Proof. The conditions (233) mean that (E, Ψ) belongs to the class of standard Courant algebroids with $\mathfrak{t}^k$-action described in Example 56 (in the notation of that example, $r_i = \hat{v}_i$ and $r^B_i = \hat{v}^B_i$). Let $(\tilde{E}, \tilde{\Psi})$ be a T-dual of (E, Ψ), provided by Theorem 85. Then $(\tilde{E}, \tilde{\Psi})$ is defined on the total space of a principal T^k-bundle $\tilde{\pi} : \tilde{M} \to B$, with connection form $\tilde{\theta} = \sum_{i=1}^k \tilde{\theta}_i e^i$, in terms of arbitrarily chosen sections $\tilde{r}^B_i \in \Gamma(\mathfrak{g}_{P_0})$. Let $\tilde{\sigma} : \tilde{P} \to \tilde{M}$ be the pullback of the principal bundle $\sigma_0 : P_0 \to B$ by the map $\tilde{\pi}$. The arguments from Theorem 85 and the above lemma show that $\tilde{E}$ is the heterotic Courant algebroid defined by the principal G-bundle $\tilde{\sigma}$, connection $\mathcal{H}^{\tilde{\sigma}}$ with connection form

$$\tilde{A} = \tilde{\pi}_0^* A_0 - \sum_{i=1}^{k} \tilde{\sigma}^* \tilde{\theta}_i \otimes \tilde{r}_i \tag{235}$$

where $\tilde{\pi}_0 : \tilde{P} \to P_0$ is the natural projection, $\tilde{r}_i = \tilde{\pi}^*(\tilde{r}_i^B) \in \Gamma(\mathfrak{g}_{\tilde{P}})$ and 3-form $\tilde{H}$ is constructed as in Theorem 85 (in particular, $\tilde{H}_{(0)}^{ijs,B}$ and $\tilde{H}_{(1)}^{ij,B}$ are given by (233) with $\hat{v}_i^B$ replaced by $\tilde{r}_i^B$). $\square$

Remark 89. The above treatment provides an alternative view-point for the heterotic T-duality developed in [2]. Heterotic Courant algebroids can be obtained from exact Courant algebroids by a reduction procedure described in [2] and the heterotic T-duality from [2] was obtained as a reduction of the T-duality for exact Courant algebroids [10]. Our approach is more direct and makes no reference to exact Courant algebroids.

In our setting it is natural to relax the definition of a heterotic Courant algebroid [2] by allowing as structure groups of the principal bundle not only compact semi-simple Lie groups but any connected Lie group G such that

P_1) $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}})_0$ is a covering for some invariant scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g} = \mathrm{Lie}\, G$. (Equivalently, $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}})$ is an isomorphism onto the Lie algebra of skew-symmetric derivations, cf. Remark 54.) As before, we restrict to scalar products of neutral signature.

The resulting Courant algebroids are transitive and the corresponding bundles of quadratic Lie algebras $\mathcal{G}$ have the property

P_2) $\mathrm{ad} : \mathcal{G} \to \mathrm{Der}(\mathcal{G})$ is an isomorphism.

Proposition 90. *The class of transitive Courant algebroids $E \to M$ over simply connected manifolds for which the bundle of quadratic Lie algebras $\mathcal{G}$ has the property P_2 coincides with the above (relaxed) class of heterotic Courant algebroids.*

Proof. We first remark that the fibers $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})|_p$, $p \in M$, are all isomorphic to a fixed quadratic Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}})$. In fact, for a transitive Courant algebroid E, any two fibers of $\mathcal{G}$ are related by parallel transport, which preserves the tensor fields $[\cdot,\cdot]_{\mathcal{G}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{G}}$. Note that $\mathcal{G}$ satisfies P_2 if and only if $\mathfrak{g}$ satisfies P_1.

Let us fix a basis a basis in $\mathfrak{g}$. The connection ∇ in the bundle $\mathcal{G}$ induces a connection in the bundle P of standard frames of $\mathcal{G}$. A frame is called standard if its structure constants and the Gram matrix of the scalar product coincide with those of the underlying quadratic Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}})$ with respect to the fixed basis in $\mathfrak{g}$. The structure group of P is $\mathrm{Aut}(\mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}})$ and can be always reduced to the connected group

$\mathrm{Aut}(\mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}})_0$ by holonomy reduction if M is simply connected. The property P_2 implies $\mathrm{Lie}\,\mathrm{Aut}(\mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}})_0 \cong \mathfrak{g}$ and then $G := \mathrm{Aut}(\mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}})_0$ satisfies P_1. In that case we can rewrite the bundle $\mathcal{G}$ as the adjoint bundle with connection induced from the connection in the principal G-bundle P. This shows that E belongs to the (relaxed) class of heterotic Courant algebroids. $\square$

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