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On a quasimorphism of Hamiltonian diffeomorphisms and quantization

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In the setting of geometric quantization, we associate to any prequantum bundle automorphism a unitary map of the corresponding quantum space. These maps are controlled in the semiclassical limit by two invariants of symplectic topology: the Calabi-Weinstein morphism and a quasimorphism on the universal cover of the Hamiltonian diffeomorphism group introduced by Entov, Py, Shelukhin.

1. Introduction

Geometric quantization deals with defining a quantum system corresponding to a given classical system, usually given by the Hamiltonian formalism [25]. From its introduction, it has been deeply connected to representation theory and the first application [15] was the orbit method: constructing irreducible representations of a Lie group by quantizing its coadjoint orbits. Later on, geometric quantization has been applied to flat bundle moduli spaces to produce projective representations of mapping class groups [14] [2], which are of fundamental importance in quantum topology.

Besides these achievements, semiclassical methods for geometric quantization have been developed successfully after the seminal work [6], broadening the field of applications to any Hamiltonian of a prequantizable compact symplectic manifold.

Our goal in this paper is to study a natural asymptotic representation for the group of Hamiltonian diffeomorphisms, actually a central extension of this group, defined in the context of geometric quantization. Here the adjective "asymptotic" refers to the fact that our representation satisfies the homomorphism equation up to an error small in the semiclassical limit. This limitation is inherent to the analytical methods we use but is also meaningful. Indeed, the simplicity of the group of Hamiltonian diffeomorphisms of a compact symplectic manifold [3] imposes severe restrictions on the possible representations as was first noticed in [12]. This simplicity explains that quasimorphisms are of big interest in symplectic topology [20]. Actually, our main result says that our asymptotic representation is controlled at first order by a quasimorphism on the universal cover of Hamiltonian diffeomorphism group introduced in [10], [22], [24].

1.1. The asymptotic representation

Let M be a symplectic compact manifold equipped with a prequantum line bundle L. The quantum space will be defined as a subspace of $\mathcal{C}^\infty(M, L)$ depending on some auxiliary data. Typically, this additional data is a holomorphic structure and the quantum space consists of the corresponding holomorphic sections. The group we will work with is the group Preq of prequantum bundle automorphisms of L. It acts naturally on $\mathcal{C}^\infty(M, L)$ by push-forward, but without preserving the quantum space. To remedy this, we will consider parallel transport in the bundle of quantum spaces along some specific paths.

Recall first that L is a Hermitian line bundle over M equipped with a connection ∇ whose curvature is $\frac{1}{i}$ times the symplectic form ω. Any complex structure j of M compatible with ω has a natural lift to a holomorphic structure of L determined by the condition that ∇ becomes the Chern connection. Denote by $\mathcal{Q}(j)$ the corresponding space of holomorphic sections of L. The prequantum bundle automorphisms of L are the vector bundle automorphisms preserving the Hermitian structure and the connection. The push-forward by a prequantum bundle automorphism φ sends $\mathcal{Q}(j)$ to $\mathcal{Q}(\pi(\varphi)_* j)$, where $\pi(\varphi)$ is the diffeomorphism of M lifted by φ. The important observation is that if we were able to identify equivariantly the various $\mathcal{Q}(j)$, we would have a representation of Preq.

One geometrical way to produce such an identification is to consider each $\mathcal{Q}(j)$ as a fiber of a bundle over the space of complex structures and to introduce a flat equivariant connection. To do this, it is more convenient to work with the space $\mathcal{J}$ of almost complex structures of M compatible with ω, because it is a smooth contractible (infinite dimensional) manifold. Following [13], we can still define $\mathcal{Q}(j)$ for any $j \in \mathcal{J}$ as a sum of some eigenspaces of a convenient Laplacian. Simplifying slightly, this defines a vector subbundle $\mathcal{Q} \to \mathcal{J}$ of $\mathcal{J} \times \mathcal{C}^\infty(M, L)$. Then using the natural scalar product of $\mathcal{C}^\infty(M, L)$, we obtain a connection of $\mathcal{Q} \to \mathcal{J}$, sometimes called the L^2-connection. This connection is equivariant with respect to the action of Preq, but unfortunately it is known not to be flat [11].

Still we will use this connection to construct a map

$$\Psi : \mathrm{Preq} \to \mathrm{U}(\mathcal{H}), \tag{1}$$

where $\mathcal{H} = \mathcal{Q}(j_0)$, j_0 being a given base point of $\mathcal{J}$. We will need a particular family of paths of $\mathcal{J}$ which was introduced in [24]. For any two points j_0, j_1 of $\mathcal{J}$, for any $x \in M$, $j_0(x)$ and $j_1(x)$ are linear complex structures of (T_xM, ω_x). The space of linear complex structures of a symplectic vector space has a natural Riemannian metric such that any two points are connected by a unique geodesic, and in particular there is a unique geodesic $j_t(x)$ joining $j_0(x)$ to $j_1(x)$. This defines a path (j_t) of $\mathcal{J}$ that we call abusively the geodesic from j_0 to j_1.

Let us now define the map Ψ. For any $\varphi \in \mathrm{Preq}$, $\Psi(\varphi)$ is the composition of the push-forward $\varphi_* : \mathcal{Q}(j_0) \to \mathcal{Q}(j_1)$, $j_1 = \pi(\varphi)_* j_0$, with the parallel transport along the geodesic joining j_1 to j_0.

The inspiration comes from the geometric construction of the quantum representations of the mapping class group of a surface [14], [2]. The connections used in these papers are projectively flat so that the resulting representations are projective and the choice of paths does not matter. To the contrary, our result will depend essentially on the choice of paths. The idea to use these particular geodesics in the context of geometric quantization is new.

Before we continue, let us introduce the *semiclassical limit*. For any positive integer k, we replace in the previous definitions the bundle L by its kth tensor power L^k, which defines $\mathcal{Q}_k \to \mathcal{J}$, $\mathcal{H}_k := \mathcal{Q}_k(j_0)$ and

$$\Psi_k : \mathrm{Preq} \to \mathrm{U}(\mathcal{H}_k), \tag{2}$$

The semiclassical limit is the large k limit.

Let us emphasize that $\mathcal{Q}_k \to \mathcal{J}$ is not a genuine vector bundle. But this can be solved in the semiclassical limit [11]. Indeed, for any compact submanifold N of $\mathcal{J}$, there exists $k_0(N)$ such that the restriction of $\mathcal{Q}_k$ to N is a vector bundle when $k \geqslant k_0(N)$. Consequently, $\Psi_k(\varphi)$ is well-defined only when $k \geqslant k_0(\varphi)$. However for this introduction, we will keep our simplified version.

1.2. Two invariants of symplectic geometry

Our main result regarding the mappings Ψ_k connects them with two invariants of symplectic topology, the Calabi-Weinstein morphism [28] and a quasimorphism introduced by Shelukhin [24]. The Calabi-Weinstein morphism is an invariant of automorphisms of a prequantum bundle over a compact manifold, which is similar to the invariant introduced by Calabi [7] for Hamiltonian diffeomorphims of an open symplectic manifold.

By the Kostant-Souriau prequantization theory [15] [25], the Lie algebra of prequantum bundle infinitesimal automorphisms is naturally isomorphic with the Poisson algebra of M. Let $\mathcal{P}(\mathrm{Preq})$ be the set of smooth paths of Preq starting from the identity. To any such path $(\gamma_t,\ t \in [0,1])$, we associate a path (H_t) of $\mathcal{C}^\infty(M,\mathbb{R})$ representing the derivative $(\dot{\gamma}_t)$ through the Kostant-Souriau isomorphism, that we call the generating Hamiltonian of γ.

For any path $\gamma \in \mathcal{P}(\mathrm{Preq})$ with generating Hamiltonian (H_t), we set

$$\mathrm{Cal}(\gamma) = \int_0^1 dt \ \int_M H_t(x) d\mu(x) \tag{3}$$

where $\mu = \omega^n/n!$ is the Liouville measure. The map Cal factorizes to a morphism from the universal cover $\widetilde{\mathrm{Preq}}^0$ of the identity component Preq^0 of Preq, into $\mathbb{R}$, already considered in [23].

The definition of the Shelukhin quasimorphism is more involved and will be postponed to Section 4. Let us discuss its main properties. It is a map

$$\mathrm{Sh}_j : \widetilde{\mathrm{Ham}} \to \mathbb{R} \tag{4}$$

defined for any compact symplectic manifold M equipped with an almost complex structure j. Here $\widetilde{\mathrm{Ham}}$ is the universal cover of the Hamiltonian diffeomorphism group of M. Moreover Sh_j is a quasi-morphism, i.e.

$$|\,\mathrm{Sh}_j(\alpha\beta) - \mathrm{Sh}_j(\alpha) - \mathrm{Sh}_j(\beta)| \leqslant C \tag{5}$$

for a constant C not depending on α, β. This condition is meaningful because the group $\widetilde{\mathrm{Ham}}$ being perfect [3], there exists no non trivial morphism from $\widetilde{\mathrm{Ham}}$ to $\mathbb{R}$. Sh_j is non trivial in the sense that its homogeneisation $\overline{\mathrm{Sh}}(\alpha) := \lim_{\ell\to\infty} \mathrm{Sh}_j(\alpha^\ell)/\ell$ is not zero. This homogeneisation $\overline{\mathrm{Sh}}$ is itself a quasimorphism, it does not depend on j and it had been defined before for specific classes of symplectic manifolds in [10] and [22]. As a last comment, the construction of Sh_j is soft in the sense that it does not use pseudo-holomorphic curves. To the contrary, important quasimorphisms of $\widetilde{\mathrm{Ham}}$ can be obtained from Floer theory [20]. For a general introduction to quasimorphisms in symplectic topology, we refer the reader to [20].

1.3. Semiclassical results

Recall the well-known fact that the Hamiltonian diffeomorphisms of M are precisely the diffeomorphisms that can be lifted to a prequantum bundle automorphism isotopic to the identity. Furthermore, M being connected, the lift is unique up to a constant rotation, so we have a central extension

$$1 \to \mathrm{U}(1) \to \mathrm{Preq}^0 \xrightarrow{\pi} \mathrm{Ham} \to 1, \tag{6}$$

where Preq^0 is the identity component of Preq. So any path $\gamma \in \mathcal{P}(\mathrm{Preq})$ defines a path $\pi(\gamma) \in \mathcal{P}(\mathrm{Ham})$ and consequently a class $[\pi(\gamma)]$ in $\widetilde{\mathrm{Ham}}$.

Theorem 1.1. *Assume j_0 is integrable. Then for any path $\gamma \in \mathcal{P}(\mathrm{Preq})$, the variation $v_k(\gamma)$ of the argument of $t \mapsto \det \Psi_k(\gamma_t)$ is equal to*

$$v_k(\gamma) = -\Big(\frac{k}{2\pi}\Big)^n \Big(\big(k + \lambda'\big) \,\mathrm{Cal}(\gamma) + \tfrac{1}{2}\, \mathrm{Sh}_{j_0}([\pi(\gamma)]) + \mathcal{O}(k^{-1}) \Big)$$

where $2n$ is the dimension of M and $\lambda' = \frac{n}{2}[c_1(M) \cup c_1(L)^{n-1}]/[c_1(L)^n]$

Interestingly, Shelukhin interpreted as well some related quasimorphisms on finite-dimensional Lie groups, as rotation numbers of determinants [24, Section 1.9].

For the proof of Theorem 1.1, we use several remarkable results: on one hand by [11], the curvature of $\mathcal{Q}_k \to \mathcal{J}$ is given at first order by the scalar curvature; on the other hand the definition of Sh_j is based on the action of the group Ham of Hamiltonian diffeomorphisms on $\mathcal{J}$, action which is Hamiltonian with a momentum map given by the scalar curvature by [9].

As a corollary, we will deduce that the lift of Ψ_k to the universal covers

$$\widetilde{\Psi}_k : \widetilde{\mathrm{Preq}}^0 \to \widetilde{\mathrm{U}}(\mathcal{H}_k) \tag{7}$$

is asymptotically a morphism. Introduce the geodesic distance $\tilde{d}$ of $\widetilde{\mathrm{U}}(\mathcal{H}_k)$ corresponding to the operator norm. This distance is controlled at large scale by the lift of the determinant $\widetilde{\det} : \widetilde{\mathrm{U}}(\mathcal{H}) \to \mathbb{R}$. More precisely, if the dimension of $\mathcal{H}$ is N, then for any $\tilde{u}$, $\tilde{v}$ in $\widetilde{\mathrm{U}}(\mathcal{H})$, we have

$$\frac{\left|\widetilde{\det}\,\tilde{u} - \widetilde{\det}\,\tilde{v}\right|}{N} \leqslant \tilde{d}(\tilde{u}, \tilde{v}) \leqslant \frac{\left|\widetilde{\det}\,\tilde{u} - \widetilde{\det}\,\tilde{v}\right|}{N} + 2\pi \tag{8}$$

So the estimate of the argument variation in Theorem 1.1 and the fact that Cal is a mophism and Sh_j a quasimorphism will show the following fact.

Corollary 1.2. *There exists $C > 0$ such that for any $\gamma_1, \gamma_2 \in \widetilde{\mathrm{Preq}}^0$,*

$$\tilde{d}(\widetilde{\Psi}_k(\gamma_1)\widetilde{\Psi}_k(\gamma_2), \widetilde{\Psi}_k(\gamma_1\gamma_2)) \leqslant C + \mathcal{O}(k^{-1}) \tag{9}$$

with $\mathcal{O}(k^{-1})$ depending on γ_1, γ_2.

It is also interesting to compare the map Ψ_k with the quantum propagator defined through Toeplitz quantization. For any $f \in \mathcal{C}^\infty(M, \mathbb{R})$, we let $T_k(f)$ be the endomorphism of $\mathcal{H}_k$ such that $\langle T_k(f)\psi, \psi' \rangle = \langle f\psi, \psi' \rangle$ for any $\psi, \psi' \in \mathcal{H}_k$. Define the map

$$\widetilde{\Phi}_k : \mathcal{P}(\mathrm{Preq}) \to \widetilde{\mathrm{U}}(\mathcal{H}_k) \tag{10}$$

as follows. Let $\gamma \in \mathcal{P}(\mathrm{Preq})$ with generating Hamiltonian (H_t). Solve the Schrödinger equation

$$U_t' = \frac{k}{i} T_k(H_t) U_t, \qquad U_0 = \mathrm{id}_{\mathcal{H}_k}. \tag{11}$$

where $(U_t) \in \mathcal{C}^\infty([0,1], \mathrm{U}(\mathcal{H}_k))$. Lift the path (U_t) to a path $(\widetilde{U}_t)$ of $\widetilde{\mathrm{U}}(\mathcal{H}_k)$ starting at the identity element. And set $\widetilde{\Phi}_k(\gamma) := \widetilde{U}_1$.

If we have two families $(g_k, h_k \in \widetilde{\mathrm{U}}(\mathcal{H}_k),\ k \in \mathbb{N})$, we write $g_k = h_k + \mathcal{O}(r_k)$ to say that $\tilde{d}(g_k, h_k) = \mathcal{O}(r_k)$.

Theorem 1.3. *For any path γ in $\mathcal{P}(\mathrm{Preq})$, $\widetilde{\Phi}_k(\gamma) = \widetilde{\Psi}_k([\gamma]) + \mathcal{O}(1)$ with a $\mathcal{O}$ depending on γ.*

As a consequence if γ, $\gamma' \in \mathcal{P}(\gamma)$ are homotopic with fixed endpoints, then

$$\widetilde{\Phi}_k(\gamma) = \widetilde{\Phi}_k(\gamma') + \mathcal{O}(1). \tag{12}$$

Furthermore by Corollary 1.2,

$$\widetilde{\Phi}_k(\gamma_1\gamma_2) = \widetilde{\Phi}_k(\gamma_1)\widetilde{\Phi}_k(\gamma_2) + \mathcal{O}(1). \tag{13}$$

Actually it could be possible to deduce (12) and (13) directly from the commutator estimate $\big[T_k(f), T_k(g)\big] = (ik)^{-1}T_k(\{f,g\}) + \mathcal{O}(k^{-2})$.

The slight difference between $\widetilde{\Psi}_k$ and $\widetilde{\Phi}_k$ is that the $\mathcal{O}(1)$ in (13) depends on γ_1, γ_2, whereas in (9) only the $\mathcal{O}(k^{-1})$ depends on γ_1, γ_2. Actually, (12) and (13) still hold if we modify $\widetilde{\Phi}_k$ by a $\mathcal{O}(1)$. For instance, we can use any quantization T'_k such that $T'_k = T_k + \mathcal{O}(k^{-1})$. Or we can define $\widetilde{\Phi}_k$ through the L^2-connection by using arbitrary paths in $\mathcal{J}$. On the contrary, we can not modify $\widetilde{\Psi}_k$ arbitrarily by a $\mathcal{O}(1)$ and still having Corollary 1.2. So we can view $\widetilde{\Psi}_k$ as a specific choice amongst all the maps in $\widetilde{\Phi}_k + \mathcal{O}(1)$, such that (9) holds.

As a last remark, observe that by composing the maps $\widetilde{\Phi}_k$, $\widetilde{\Psi}_k$ with the projection $\widetilde{\mathrm{U}}(\mathcal{H}_k) \to \mathrm{U}(\mathcal{H}_k)$, we do not obtain anything interesting because the diameter of $\mathrm{U}(N)$ for the geodesic distance associated to the uniform norm, is bounded independently of N. So any family in $\mathrm{U}(\mathcal{H}_k)$ is in $\mathcal{O}(1)$.

1.4. Structure of the article

Because our results are essentially on the quantization of prequantum bundle automorphisms, Section 2 will be devoted to the group Preq and the universal cover of its identity component. We will prove that

$$\widetilde{\mathrm{Preq}}^0 \simeq \mathbb{R} \times \widetilde{\mathrm{Ham}}$$

the isomorphism being the product of the Calabi-Weinstein morphism and the lift of the projection $\mathrm{Preq}^0 \to \mathrm{Ham}$. We will also show that the central extension (6) is essentially controlled by the Weinstein action morphism $\pi_1(\mathrm{Ham}) \to \mathrm{U}(1)$.

Section 3 is devoted to the geodesic distance of the universal cover of the unitary group induced by the uniform norm. We will prove estimate (8), compute explicitly the distance, and show that the distance between the identity and a point is always achieved by a one-parameter semi-group.

This does not follow from general results in Finsler geometry because the uniform norm is not sufficiently regular. Our proof is actually based on a theorem by Thompson [26], which follows itself from the Horn conjecture.

In Section 4, we give more details on the definition of the maps Ψ_k. Theorem 1.1 is proved in Section 5. Corollary 1.2 and Theorem 1.3 are proved in Section 6.

2. Prequantum bundle automorphisms

We study the geometry of the central extension (6). In a first subsection, we introduce a similar finite dimensional extension with genuine Lie groups, which despite of its simplicity, already has the main features. These extensions are also relevant because they appear in our setting when we restrict the Hamiltonian diffeomorphisms group to the subgroup of isometries for a given metric. The case of projective manifolds and more generally toric manifolds has been studied in the literature, as will be explained in the second subsection. In the third subsection, we prove more specific results on the Calabi-Weinstein morphism and the universal covers of Preq^0 and Ham. In the last subsection, we explain the relation with the usual Calabi morphism.

2.1. A finite-dimensional model

Consider a central extension G of a Lie group H by $\mathrm{U}(1)$. In other words we have an exact sequence of Lie group morphisms

$$1 \to \mathrm{U}(1) \to G \xrightarrow{\pi} H \to 1. \tag{14}$$

such that $\mathrm{U}(1)$ is sent in the center of G. We assume as well that H and G are connected and that the corresponding exact sequence of Lie algebras $0 \to \mathbb{R} \to \mathfrak{g} \to \mathfrak{h} \to 0$ splits. So $\mathfrak{g} \simeq \mathbb{R} \oplus \mathfrak{h}$ with a Lie bracket of the form

$$[(s,\xi),(t,\eta)]_{\mathfrak{g}} = (0, [\xi,\eta]_{\mathfrak{h}}).$$

Assume for a moment that there exists a group morphism $\sigma : H \to G$ integrating the Lie algebra morphism $\mathfrak{h} \to \mathfrak{g}$, $\xi \mapsto (0,\xi)$. Then (14) splits. Indeed, $\sigma \circ \pi = \mathrm{id}_H$ because the derivative of $\sigma \circ \pi$ is the identity of $\mathfrak{h}$ and H is connected. This implies that $\mathrm{U}(1) \times H \to G$, $(\theta, g) \to \theta\sigma(g)$ is an isomorphism with inverse the map sending g into $(g\sigma(\pi(g))^{-1}, \sigma(\pi(g)))$.

In general, by Lie's second Theorem, there exists a unique Lie group morphism $\widetilde{\sigma}$ from the universal cover $\widetilde{H}$ of H to G integrating the Lie algebra morphism $\mathfrak{h} \to \mathfrak{g}$, $\xi \mapsto (0, \xi)$. The derivative of $\widetilde{\sigma} \circ \pi$ being the identity of $\mathfrak{h}$, $\widetilde{\sigma} \circ \pi$ is the projection of $\widetilde{H}$ onto H. So there exists a unique morphism $A : \pi_1(H) \to \mathrm{U}(1)$ such that the following diagram commutes.

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(H) & \longrightarrow & \widetilde{H} & \longrightarrow & H & \longrightarrow & 1 \\
 & & \big\downarrow A & & \big\downarrow \widetilde{\sigma} & & \Big\| & & \\
1 & \longrightarrow & \mathrm{U}(1) & \longrightarrow & G & \xrightarrow{\pi} & H & \longrightarrow & 1.
\end{array} \tag{15}
$$

Observe that the previous morphism σ exists if and only if $\widetilde{\sigma}(\pi_1(H)) = \{1_G\}$ if and only if A is trivial. If σ does not exist, introduce the subgroup $K := \widetilde{\sigma}(\widetilde{H})$ of G as a replacement of $\sigma(H)$. Then by an easy diagram chase in (15), we have first that

$$1 \to \ker A \to \widetilde{H} \xrightarrow{\widetilde{\sigma}} K \to 1,$$

so $\widetilde{H}$ is the universal cover of K and $\pi_1(K) \simeq \ker A$. Second,

$$1 \to \operatorname{Im} A \to K \xrightarrow{\pi} H \to 1$$

so K is a central extension of H by $\operatorname{Im} A$.

The typical example is $G = \mathrm{U}(n)$ with its subgroup $\mathrm{U}(1)$ of diagonal matrices so that $H = \mathrm{U}(n)/\mathrm{U}(1) = \mathrm{PU}(n)$. Since the projection $\mathrm{SU}(n) \to \mathrm{PU}(n)$ is the universal cover, we get an identification between the Lie algebras of $\mathrm{PU}(n)$ and $\mathrm{SU}(n)$. We have $\mathfrak{u}(n) = \mathbb{R} \oplus \mathfrak{su}(n)$ as required and the group morphism $\widetilde{\sigma} : \mathrm{SU}(n) \to \mathrm{U}(n)$ is merely the inclusion. So A is the embedding $\mathbb{Z}/n\mathbb{Z} \hookrightarrow \mathrm{U}(1)$ and $K = \mathrm{SU}(n)$.

Another thing that can be done in general is to introduce the isomorphism of the universal covers

$$\widetilde{G} \simeq \mathbb{R} \times \widetilde{H} \tag{16}$$

corresponding to the isomorphism of Lie algebras $\mathfrak{g} \simeq \mathbb{R} \oplus \mathfrak{h}$. Observe that the morphisms $\widetilde{\sigma}$ and A can be recovered from (16). Indeed $\widetilde{\sigma}$ is the composition of $\widetilde{H} \to \widetilde{G}$, $h \to (0, h)$ with the projection $\widetilde{G} \to G$.

2.2. Diffeomorphism group

Let (M, ω) be a connected compact symplectic manifold and Ham be its group of Hamiltonian diffeomorphism. Assume M is equipped with a prequantum bundle $P \to M$ and let Preq be the the group of prequantum bundle automorphisms of P. The precise definition will be given in Section 2.3. For now recall the exact sequence of groups

$$1 \to \mathrm{U}(1) \to \mathrm{Preq}^0 \xrightarrow{\pi} \mathrm{Ham} \to 1, \tag{17}$$

where the projection π sends a prequantum bundle automorphism φ of P to the diffeomorphism of M lifted by φ. Furthermore, it is a well-known fact due to Kostant and Souriau that the Lie algebra of infinitesimal prequantum bundle automorphisms of P is isomorphic with $\mathcal{C}^\infty(M, \mathbb{R})$, the Lie bracket being sent to the Poisson bracket. The Lie algebra exact sequence corresponding to (17) is

$$0 \to \mathbb{R} \to \mathcal{C}^\infty(M, \mathbb{R}) \to \mathcal{C}^\infty(M, \mathbb{R})/\mathbb{R} \to 0.$$

It has a natural splitting

$$\mathcal{C}^\infty(M, \mathbb{R})/\mathbb{R} \simeq \mathcal{C}_0^\infty(M, \mathbb{R})$$

where $\mathcal{C}_0^\infty(M, \mathbb{R})$ is the subalgebra of $\mathcal{C}^\infty(M, \mathbb{R})$ consisting of the functions f having a null average with respect to the Liouville measure.

So we are exactly in the situation described in Section 2.1 except that Preq and Ham are infinite-dimensional Lie groups. These groups do not have all the good properties of Lie groups, notably they may have elements arbitrarily close to the identity and not belonging to any one-parameter subgroup, cf. the remarkable general introduction [19] and [21] for results specific to Ham. However, the constructions presented in Section 2.1 can be extended to our situation. In particular, we have an isomorphism

$$\widetilde{\mathrm{Preq}}^0 \simeq \mathbb{R} \times \widetilde{\mathrm{Ham}}. \tag{18}$$

Here the universal covers will be very concretely defined as quotients of path groups of Preq and Ham, and the isomorphism will be given by integrating the generating vector fields. The details will be given in the next section.

The morphism $\mathrm{Cal} : \widetilde{\mathrm{Preq}}^0 \to \mathbb{R}$ given by the projection on the first factor in (18) will be called the *Calabi-Weinstein morphism* and has already been considered in [23]. Its relation with the usual Calabi morphism will be explained in Section 2.4.

The obstruction $A : \pi_1(\mathrm{Ham}) \to \mathrm{U}(1)$ to the splitting of (17) can be defined from (18) as explained at the end of Section 2.1. This morphism A is actually known in the symplectic topology literature as the *Weinstein action homomorphism* and was introduced in [28].

When M has a complex structure j compatible with ω, so that M is Kähler, we can also introduce the subgroup H of Ham consisting of the holomorphic Hamiltonian diffeomorphisms, and the subgroup G of Preq consisting of the automorphisms lifting an element of H. The groups G and H are genuine Lie groups and satisfy all the assumptions of Section 2.1. The corresponding morphism $A_j : \pi_1(H) \to \mathrm{U}(1)$ is the composition of $A : \pi_1(\mathrm{Ham}) \to \mathrm{U}(1)$ with the map $\pi_1(H) \to \pi_1(\mathrm{Ham})$ induced by the inclusion $H \subset \mathrm{Ham}$.

For instance, if M is the projective space $\mathbb{CP}(n)$ with its standard symplectic, complex and prequantum structures, then we recover the example discussed in Section 2.1 where $G = \mathrm{U}(n+1)$, $H = \mathrm{PU}(n+1)$. We deduce that $\mathbb{Z}/(n+1)\mathbb{Z}$ embeds into $\pi_1(\mathrm{Ham})$, a well-known fact. More generally, the morphism A_j is discussed in [18] for toric manifolds. It is proved that in most cases, A_j is injective and its image is not finite, [17, Corollary 2.4 and Proposition 2.5]. So in all these cases, $\pi_1(H) \subset \pi_1(\mathrm{Ham})$ and the image of A is not finite.

2.3. $\widetilde{\mathbf{Preq}}^{\mathbf{0}}$, $\widetilde{\mathbf{Ham}}$ and the Calabi-Weinstein morphism

Consider a compact connected symplectic manifold (M, ω). Our sign convention for the Hamiltonian vector field X of $f \in \mathcal{C}^\infty(M, \mathbb{R})$ and the Poisson bracket is

$$\omega(X, \cdot) + df = 0, \qquad \{f, g\} = X.g.$$

Let $\mu = \omega^n/n!$ be the Liouville volume form. A Hamiltonian $f \in \mathcal{C}^\infty(M, \mathbb{R})$ is *normalised* if $\int_M f\ \mu = 0$.

Let $\mathcal{P}(\mathrm{Ham})$ be the group of smooth paths of Ham based at the identity element, the law group being the pointwise product. Associating each path (ϕ_t) of Ham to its generating vector field X_t,

$$\frac{d}{dt}\phi_t(x) = X_t(\phi_t(x)), \qquad x \in M,\ t \in [0,1]$$

we obtain a one-to-one correspondence between $\mathcal{P}(\mathrm{Ham})$ and the space of time-dependent Hamiltonians $f \in \mathcal{C}^\infty([0,1] \times M, \mathbb{R})$ which are normalised at each time t. As customary in symplectic topology, the group $\widetilde{\mathrm{Ham}}$ is defined as the quotient of $\mathcal{P}(\mathrm{Ham})$ by the relation of being smoothly homotopic with fixed endpoints, cf. [20] and [4].

Assume now that M is equipped with a prequantum bundle $P \to M$, that is a $\mathrm{U}(1)$-principal bundle over M endowed with a connection form $\alpha \in \Omega^1(P, \mathbb{R})$ such that $d\alpha + \pi^*\omega = 0$. Here we identify the Lie algebra of

U(1) with Herm(1) $= \mathbb{R}$. So if ∂_θ is the infinitesimal generator of the U(1)-action corresponding to 1, we have that $\alpha(\partial_\theta) = 1$ and $\mathcal{L}_{\partial_\theta}\alpha = 0$.

An *infinitesimal automorphism* of P is a vector field of P commuting with the U(1)-action and preserving α. Any such vector field Y has the form

$$Y = X^{\mathrm{hor}} - (\pi^* f)\partial_\theta \tag{19}$$

where $f \in \mathcal{C}^\infty(M, \mathbb{R})$, X is the corresponding Hamiltonian vector field and X^{hor} is the lift of X such that $\alpha(X^{\mathrm{hor}}) = 0$. The map sending Y to f is an isomorphism from the space of prequantum infinitesimal automorphims of P to $\mathcal{C}^\infty(M, \mathbb{R})$. The Lie bracket is sent to the Poisson bracket.

The *prequantum bundle automorphisms* of P are by definition the diffeomorphisms of P preserving α and commuting with the U(1)-action. The identity component Preq^0 is a central extension of Ham by U(1), cf. (17). The embedding $\mathrm{U}(1) \hookrightarrow \mathrm{Preq}$ is given by the action of the structure group of P. The proofs of the previous claims starting from (19) may be found in [5, Section 7.1].

As for the Hamiltonian diffeomorphisms, let $\mathcal{P}(\mathrm{Preq})$ be the group of smooth paths of prequantum bundle automorphisms based at the identity. To any (γ_t) in $\mathcal{P}(\mathrm{Preq})$, we associate its generating vector field (Y_t) and the corresponding time-dependent Hamiltonian (f_t) through (19). This defines a bijection between $\mathcal{P}(\mathrm{Preq})$ and $\mathcal{C}^\infty([0,1] \times M, \mathbb{R})$.

We define $\widetilde{\mathrm{Preq}}^0$ as the quotient of $\mathcal{P}(\mathrm{Preq})$ by the relation of being smoothly homotopic with fixed endpoints. To handle the difference between $\widetilde{\mathrm{Preq}}^0$ and $\widetilde{\mathrm{Ham}}$, we will need the following Lemma.

Lemma 2.1. *Let* $(\gamma_t^s, (t,s) \in [0,1]^2)$ *be a smooth family in* Preq *such that for any* $s \in [0,1]$, γ_0^s *is the identity map of* P *and* γ_1^s *lifts the identity map of* M. *For any* $s \in [0,1]$, *set*

$$C(s) = \int_0^1 \Big(\int_M f_t^s \mu \Big) dt$$

where $(f_t^s, t \in [0,1])$ *is the Hamiltonian generating* $(\gamma_t^s, t \in [0,1])$. *Then* $s \mapsto C(s)$ *is constant if and only if* $s \mapsto \gamma_1^s$ *is constant.*

Proof. Let (g_t^s) be the smooth family in $\mathcal{C}^\infty(M, \mathbb{R})$ such that for any $t \in [0,1]$, $(g_t^s, s \in [0,1])$ is the Hamiltonian generating $(\gamma_t^s, s \in [0,1])$. Then

1) $g_0^s = 0$ because γ_0^s is the identity.

2) $g_1^s =: D(s) \in \mathbb{R}$ because γ_1^s lifts the identity of M

3) $\partial f_t^s/\partial s - \partial g_t^s/\partial t = \left\{f_t^s, g_t^s\right\}$, as a consequence of the differential homotopy formula, cf . [3, Proposition I.1.1.]

Using that $\int_M \{f_t^s, g_t^s\}\mu = 0$, we obtain

$$\begin{aligned} C'(s) &= \int_0^1 \int_M \frac{\partial f_t^s}{\partial s}\mu \, dt = \int_0^1 \int_M \frac{\partial g_t^s}{\partial t}\mu \, dt = \int_0^1 \frac{\partial}{\partial t}\Big(\int_M g_t^s \mu\Big) dt \\ &= \int_M g_1^s \mu - \int_M g_0^s \mu = \mathrm{Vol}(M) D(s) \end{aligned}$$

which concludes the proof. □

Introduce the group morphism

$$R : \mathbb{R} \to \widetilde{\mathrm{Preq}}^0$$

lifting the embedding of $U(1)$ into Preq. More explicitly, $R(\tau)$ is the class of the path $t \in [0, 1] \mapsto e^{it\tau}$. The image of R is contained in the center of $\widetilde{\mathrm{Preq}}^0$.

Define the map Cal from $\mathcal{P}(\mathrm{Preq})$ to $\mathbb{R}$ by

$$\mathrm{Cal}(\gamma_t) = \int_0^1 \Big(\int_M f_t \mu\Big) dt$$

where (f_t) is the Hamiltonian generating (γ_t). As we will see, Cal factorizes to $\widetilde{\mathrm{Preq}}^0$ which defines our *Calabi-Weinstein morphism.*

Proposition 2.2.

1) Cal *is a group morphism, which factorizes to a morphism* Cal *from* Preq^0 *to* $\mathbb{R}$.

2) *for any* τ, $\mathrm{Cal}(R(\tau)) = -\tau \, \mathrm{Vol}(M)$.

3) $\mathrm{Cal}(\gamma) = 0$ *if and only if* γ *has a representative whose generating Hamiltonian* (f_t) *is normalised, that is* $\int_M f_t \mu = 0$, *for every* $t \in [0, 1]$.

Proof. To check that $\mathrm{Cal}(\gamma_t\gamma_t') = \mathrm{Cal}(\gamma_t) + \mathrm{Cal}(\gamma_t')$, one first computes the generating Hamiltonians of $(\gamma_t\gamma_t')$ in terms of the generating Hamiltonian of

(γ_t) and (γ'_t):

$$(f_t) \star (f'_t) = (f_t + f'_t \circ \alpha_t^{-1}) \tag{20}$$

where α_t is the Hamiltonian flow of (f_t). The same proof for the Calabi morphism is presented in [20, Theorem 4.1.1]. The fact that $\mathrm{Cal}(\gamma_t) = \mathrm{Cal}(\gamma'_t)$ when (γ_t) and (γ_t) are smoothly homotopic with fixed endpoints, follows from Lemma 2.1.

By (19), the Hamiltonian generating the path $R(\tau)$ is the constant function $f = -\tau$, so $\mathrm{Cal}(R(\tau)) = -\tau \operatorname{Vol}(M)$.

Let $(\gamma_t) \in \mathcal{P}(\mathrm{Preq})$ with generating Hamiltonian (f_t). Let $\theta : [0,1] \to \mathbb{R}$ be a smooth function. Then the generating Hamiltonian of $(e^{i\theta(t)}\gamma_t)$ is $f_t - \theta'(t)$. This Hamiltonian is normalized if we define θ by

$$\theta(t) = \int_0^t \Big(\int_M f_t \mu \Big) dt.$$

Furthermore, when $\mathrm{Cal}(\gamma_t) = 0$, we have $\theta(0) = \theta(1) = 0$, so that (γ_t) and $(e^{i\theta(t)}\gamma_t)$ are smoothly homotopic with fixed endpoints. □

The projection $\pi : \mathrm{Preq}^0 \to \mathrm{Ham}$ induces a group morphism

$$\pi : \widetilde{\mathrm{Preq}}^0 \to \widetilde{\mathrm{Ham}}.$$

We can also define a left inverse

$$L : \widetilde{\mathrm{Ham}} \to \widetilde{\mathrm{Preq}}^0$$

by sending the class of a path (φ_t) to the class of the path (γ_t) lifting (φ_t) and having a normalized generating Hamiltonian.

Proposition 2.3.

1) *L is well-defined, it is a group morphism,*

2) *$\pi \circ L$ is the identity of $\widetilde{\mathrm{Ham}}$, the image of L is the kernel of the Calabi-Weinstein morphism,*

3) *the kernel of π is the image of R.*

Proof. The path (γ_t) certainly exists because it is the flow of the vector field associated to the normalized generating Hamiltonian of (φ_t). The fact that the class of (γ_t) only depends on the class of (φ_t) follows from Lemma

2.1. L is a group morphism because the product of normalized Hamiltonians corresponding to the product of $\mathcal{P}(\mathrm{Ham})$ is also given by formula (20).

The identity $\pi \circ L = \mathrm{id}_{\widetilde{\mathrm{Ham}}}$ is obvious, the assertion on the image of L is the third assertion of Proposition 2.2.

For the last point, the image of R is certainly contained in the kernel of π. To show the converse, consider $(\gamma_t) \in \mathcal{P}(\mathrm{Preq})$ with generating Hamiltonian (f_t). Write $f_t = \theta(t) + g_t$ where for any t, $\theta(t) \in \mathbb{R}$ and g_t is normalized. Then $[\gamma_t] = R(\tau)L(\varphi_t)$ with $\tau = -\int_0^1 \theta(t)dt$ and (φ_t) the path of Ham generated by (g_t). Now $\pi(\gamma_t) = 0$ implies that $[\varphi_t] = 0$ because $\pi \circ L = \mathrm{id}_{\widetilde{\mathrm{Ham}}}$. □

We deduce that the groups $\widetilde{\mathrm{Preq}}^0$ and $\widetilde{\mathrm{Ham}} \times \mathbb{R}$ are isomorphic.

Corollary 2.4. *The group isomorphisms*

$$\begin{aligned} \widetilde{\mathrm{Preq}}^0 &\to \widetilde{\mathrm{Ham}} \times \mathbb{R} \\ g &\mapsto (\pi(g), \mathrm{Cal}(g)), \end{aligned} \qquad \begin{aligned} \widetilde{\mathrm{Ham}} \times \mathbb{R} &\to \widetilde{\mathrm{Preq}}^0 \\ (h, \tau) &\mapsto L(h)R\big(-\tau \,\mathrm{Vol}(M)\big) \end{aligned}$$

are inverse of each other.

2.4. The Calabi morphism

Let $x \in M$. Introduce the subgroup Ham_x of Ham consisting of the Hamiltonian diffeomorphisms fixing x, and the subgroup Preq_x of Preq consisting of the prequantum bundle automorphisms fixing the fiber P_x. We claim that the morphism $\mathrm{Preq}_x \to \mathrm{Ham}_x$, $\gamma \mapsto \pi(\gamma)$ is an isomorphism. The injectivity follows directly from the exact sequence (17). The surjectivity is a consequence of the connectedness of Ham_x. This can be proved by using the long exact sequence for homotopy groups associated to the fibration $\mathrm{Ham} \to M$, $\phi \mapsto \phi(x)$, and the fact that the morphism $\pi_1(\mathrm{Ham}) \to \pi_1(M)$ is trivial, a folk Theorem according to [17, Footnote 3].

Since Ham_x and Preq_x are isomorphic, the same holds for their universal covers and composing with the Calabi-Weinstein morphism introduced in Proposition 2.2, we obtain a morphism $\widetilde{\mathrm{Ham}}_x \to \mathbb{R}$.

Introduce now the open set $U = M \setminus \{x\}$ and let $\mathrm{Ham}_c(U)$ be the group of compactly supported Hamiltonian diffeomorphisms of U. This group is a subgroup of Ham_x, so we get a morphism $\widetilde{\mathrm{Ham}_c(U)} \to \widetilde{\mathrm{Ham}}_x$, which after composition with the previous morphism gives us $\widetilde{\mathrm{Ham}_c(U)} \to \mathbb{R}$. This morphism is the usual Calabi morphism, defined for instance in [20, Section 4.1]. Indeed, both are defined by the same formula.

3. Geodesic distance induced by operator norm

We start with geodesic distance of the unitary group. The results are certainly standard but we do not know any reference. The second subsection is devoted to the universal cover of the unitary group.

3.1. The unitary group

Let $\mathcal{H}$ be a finite-dimensional Hilbert space. We denote by $\mathrm{U}(\mathcal{H})$ and $\mathrm{Herm}(\mathcal{H})$ the spaces of unitary and Hermitian endomorphisms of $\mathcal{H}$ respectively. We consider $\mathrm{Herm}(\mathcal{H})$ as the Lie algebra of $\mathrm{U}(\mathcal{H})$. Denote by $\|A\|$ the operator norm of any endomorphism A of $\mathcal{H}$. We define for any piecewise $\mathcal{C}^1$ curve $\gamma : [a, b] \to \mathrm{U}(\mathcal{H})$, its length

$$L(\gamma) = \int_a^b \|\gamma'(t)\| \, dt \tag{21}$$

and the corresponding distance in $\mathcal{H}$

$$d(u, v) = \inf\{L(\gamma);\ \gamma \text{ is a piecewise } \mathcal{C}^1 \text{ curve with endpoints } u, v\}.$$

Since $\|\gamma'(t)\| = \|\gamma(t)^{-1}\gamma'(t)\|$, this distance is the geodesic distance of $\mathrm{U}(\mathcal{H})$ for the invariant Finsler metric corresponding to the operator norm. There is a large literature on Finsler geometry, but it does not say anything about d because the operator norm is not regular enough to apply the variational method. So we have to study d from its definition.

We easily see that L is invariant under reparameterization and that d is symmetric and satisfies the triangle inequality. Since $\| \cdot \|$ is left and right-unitarily invariant, we have for any $g \in \mathrm{U}(\mathcal{H})$, $L(\gamma) = L(\gamma g) = L(g\gamma)$. Furthermore $L(\gamma) = L(\gamma^{-1})$. Consequently

$$d(u, v) = d(gu, gv) = d(ug, vg) = d(u^{-1}, v^{-1}).$$

Let us compute explicitly $d(u, v)$, which will prove that d is non-degenerate in the sense that $d(u, v) = 0$ only when $u = v$.

Proposition 3.1. *For any $u, v \in \mathrm{U}(\mathcal{H})$,*

$$d(u, v) = \max |\arg \lambda_i| \tag{22}$$

where the λ_i's are the eigenvalues of $u^{-1}v$ and arg *is the inverse of the map from* $]-\pi,\pi]$ *to* $\mathrm{U}(1)$ *sending* θ *to* $e^{i\theta}$. *So we have*

$$\|u-v\| \leqslant d(u,v) \leqslant \frac{\pi}{2}\|u-v\| \tag{23}$$

As a last remark, observe that the diameter of $\mathrm{U}(\mathcal{H})$ is π.

Proof. Since $d(u,v) = d(1,u^{-1}v)$, without loss of generality, we may assume that $u=1$. Working with an orthonormal eigenbasis of v, we construct $\xi \in \mathrm{Herm}(\mathcal{H})$ such that $\exp(i\xi) = v$ and $\|\xi\| \leqslant \pi$. The length of the curve $[0,1] \ni t \mapsto \exp(it\xi)$ is $\|\xi\| = \max|\theta_j|$ with $\theta_j = \arg\lambda_j$. So $d(1,v) \leqslant \max|\theta_j|$.

Conversely, choose a normalised eigenvector X_j of v with eigenvalue $e^{i\theta_j}$. Consider any curve $\gamma : [a,b] \to \mathrm{U}(\mathcal{H})$ from 1 to v. Set $X(t) = \gamma(t)X_j$. Since $X(b) = e^{i\theta_j}X(a)$, the geodesic distance in the unit sphere of $\mathcal{H}$ between $X(b)$ and $X(a)$ is $|\theta_j|$. So

$$|\theta_j| \leqslant \int_a^b \|X'(t)\|\,dt \leqslant \int_a^b \|\gamma'(t)\|\,dt = L(\gamma)$$

So $|\theta_j| \leqslant d(1,v)$ for any j, so $\max|\theta_j| \leqslant d(1,v)$.

To prove (23), we can again assume that $u=1$. Clearly, $\|v-1\| = \max|\lambda_j - 1|$. Then observe that for $\lambda = e^{i\theta}$, $|\lambda - 1| = 2|\sin(\theta/2)|$ so if $\theta \in [-\pi,\pi]$, we get that $|\lambda-1| \leqslant |\theta| \leqslant \frac{\pi}{2}|\lambda - 1|$. □

3.2. Universal cover of $\mathrm{U}(\mathcal{H})$

The universal cover $\widetilde{\mathrm{U}}(\mathcal{H})$ of the unitary group $\mathrm{U}(\mathcal{H})$ of the finite-dimensional Hilbert space $\mathcal{H}$ can be realised as the subgroup of $\mathrm{U}(\mathcal{H}) \times \mathbb{R}$ consisting of the pairs (u,φ) such that $\det u = e^{i\varphi}$. Note that the determinant map from $U(\mathcal{H})$ to $\mathrm{U}(1)$ lifts to the map

$$\widetilde{\det} : \widetilde{\mathrm{U}}(\mathcal{H}) \to \mathbb{R}, \qquad \widetilde{\det}(u,\varphi) = \varphi$$

Endowing $\mathrm{Herm}(\mathcal{H})$ with the operator norm, we define the length of the curves of $\widetilde{\mathrm{U}}(\mathcal{H})$. So writing $\widetilde{\gamma} : [a,b] \to \widetilde{\mathrm{U}}(\mathcal{H})$ in the form $\widetilde{\gamma}(t) = (\gamma(t),\varphi(t)) \in \mathrm{U}(\mathcal{H}) \times \mathbb{R}$, the length of $\widetilde{\gamma}$ is equal to the length (21) of γ. Then for any $\tilde{u},\tilde{v} \in \widetilde{\mathrm{U}}(\mathcal{H})$, we define $\tilde{d}(\tilde{u},\tilde{v})$ as the infimum of the lengths of the curves connecting $\tilde{u}$ and $\tilde{v}$. The situation is exactly the same as for d: we easily prove that $\tilde{d}$ is symmetric, satisfies the triangle inequality, is left and right invariant. But it is a priori not clear that $\tilde{d}$ is non degenerate.

Let $\tilde{u} = (u, \varphi)$ and $\tilde{v} = (v, \psi)$. The curves of $\widetilde{\mathrm{U}}(\mathcal{H})$ connecting $\tilde{u}$ and $\tilde{v}$ can be identified with the curves of $\mathrm{U}(\mathcal{H})$ connecting u and v and such that the angle variation[1] of their determinant if $\psi - \varphi$. So

$$d(u, v) \leqslant \tilde{d}(\tilde{u}, \tilde{v}). \tag{24}$$

Furthermore using that for any curve γ of $\mathrm{U}(\mathcal{H})$, the logarithmic derivative of $\det(\gamma(t))$ is $\mathrm{tr}(\gamma^{-1}(t)\gamma'(t))$ and that for any Hermitian matrix, $|\operatorname{tr}(A)| \leqslant N\|A\|$ where $N = \dim \mathcal{H}$, we get

$$\frac{|\psi - \varphi|}{N} \leqslant \tilde{d}(\tilde{u}, \tilde{v}). \tag{25}$$

The inequalities (24) and (25) imply that $\tilde{d}$ is non-degenerate.

We are now going to compute explicitly the distance from $\tilde{u} = (u, \varphi)$ to $\tilde{v} = (v, \psi)$. Denote by λ_i, $i = 1, \dots, N$ the eigenvalues of vu^{-1}. Let (X_i) be an associated orthonormal eigenbasis. Introduce the $(N-1)$-dimensional affine lattice of $\mathbb{R}^N$

$$R = \{\theta \in \mathbb{R}^N;\ e^{i\theta_j} = \lambda_j,\ \forall j \text{ and } \textstyle\sum_{j=1}^N \theta_j = \psi - \varphi\}$$

For any $\theta \in R$, let $H \in \mathrm{Herm}(\mathcal{H})$ be such that $HX_j = \theta_j X_j$. Then $\exp(iH) = vu^{-1}$ and $\operatorname{tr} H = \psi - \varphi$, so the curve

$$[0, 1] \to \widetilde{\mathrm{U}}(\mathcal{H}), \qquad t \mapsto (e^{itH}u, \varphi + t \operatorname{tr} H)$$

goes from (u, φ) to (v, ψ). Its length is $\|H\| = \max |\theta_i|$. This proves that

$$\tilde{d}((u, \varphi), (v, \psi)) \leqslant m \qquad \text{with } m = \min\Big\{ \max_{i=1,\dots,n} |\theta_i|;\ \theta \in R \Big\} \tag{26}$$

By Proposition 3.3 below, we actually have an equality. So there exists a length minimizing curve $(\gamma(t), \varphi(t))$ connecting (u, φ) to (v, ψ) such that $\gamma^{-1}(t)\gamma'(t)$ is constant.

We first prove the following partial result, which is actually sufficient for our applications.

[1]the angle variation of a curve $z : [a, b] \to \mathrm{U}(1)$ is $-i \int_a^b z'(t)/z(t)\, dt$.

Proposition 3.2. *We have*

$$\tilde{d}(\tilde{u},\tilde{v}) \leqslant \frac{\pi}{2N} \Rightarrow \tilde{d}(\tilde{u},\tilde{v}) = d(u,v) = m$$

with $N = \dim \mathcal{H}$. *Furthermore*

$$\frac{|\psi-\varphi|}{N} \leqslant \tilde{d}(\tilde{u},\tilde{v}) \leqslant \frac{|\psi-\varphi|}{N} + 2\pi$$

Proof. Assume that $\tilde{d}(\tilde{u},\tilde{v}) \leqslant \pi/(2N)$. By Proposition 3.1, $d(u,v) = \max|\theta_j|$ where $\theta_j = \arg\lambda_j$. By (24), $d(u,v) \leqslant \pi/2N$, so $|\theta_j| \leqslant \pi/2N$ and consequently $|\sum\theta_j| \leqslant \pi/2$. By (25), $|\psi-\varphi| \leqslant \pi/2$. Since $\psi - \varphi = \sum\theta_j$ modulo 2π, we deduce that $\psi-\varphi = \sum\theta_j$, that is $(\theta_i) \in R$. Since $|\theta_i| \leqslant \pi/2$ for any i, we have $d(u,v) = \max|\theta_i| = m$, which proves the first part.

The second part will follows from $m \leqslant \frac{|\psi-\varphi|}{N} + 2\pi$. Denote by d_N the distance of $\mathbb{R}^N$ associated to the sup norm, so $m = d_N(0,R)$. Let $\alpha = \frac{\psi-\varphi}{N}(1,\dots,1)$. Then $m \leqslant d_N(0,\alpha) + d_N(\alpha,R) = \frac{|\psi-\varphi|}{N} + d_N(0,R-\alpha)$. Now $R-\alpha$ is an affine lattice of the hyperplane $\{\sum\theta_i = 0\}$ having the form

$$R-\alpha = \beta + \textstyle\bigoplus_{j=1}^{N-1}\mathbb{Z}f_j$$

for some $\beta \in \mathbb{R}^N$, with $f_j = 2\pi(e_j - e_{j+1})$, (e_j) being the canonical basis of $\mathbb{R}^N$. Hence $R-\alpha$ contains a point $\beta' = \sum x_i f_j$ with $|x_j| \leqslant 1/2$. So $d_N(0,R-\alpha) \leqslant d_N(0,\beta') \leqslant 2\pi$. □

Proposition 3.3. *We have* $\tilde{d}((u,\varphi),(v,\psi)) = m$.

The proof is based on the Thompson theorem [26], which is not quite an elementary result because it follows from the Horn conjecture. The idea to apply Thompson theorem comes from the paper [1] where the geodesic distance of the unitary group associated to the Schatten norms is computed.

Proof. We prove the result by induction on k where

$$\tilde{d}((u,\varphi),(v,\psi)) \leqslant k\frac{\pi}{2N}$$

For $k=1$, this was the first part of Proposition 3.2. Without loss of generality, we may assume that $(u,\varphi) = (1,0)$. Assume that

$$k\frac{\pi}{2N} < \tilde{d}((1,0),(v,\psi)) \leqslant (k+1)\frac{\pi}{2N} \tag{27}$$

Let $S = \{g \in \widetilde{\mathrm{U}}(\mathcal{H}) |\ \tilde{d}((1,0),g) = \pi/2N\}$. Endow $\widetilde{\mathrm{U}}(\mathcal{H})$ with the subspace topology of $\mathrm{U}(\mathcal{H}) \times \mathbb{R}$. By (24) and (25), $\tilde{d}$ is continuous and S is compact.

So there exists (w, ξ) in S such that $\tilde{d}((w,\xi),(v,\psi)) = \tilde{d}(S,(v,\psi))$. We claim that

$$\tilde{d}((1,0),(v,\psi)) = \frac{\pi}{2N} + \tilde{d}((w,\xi),(v,\psi)) \tag{28}$$

Indeed, the left-hand side is smaller than the right-hand side by triangle inequality. Conversely, if $g(t)$ is any curve from $(1,0)$ to (v,ψ), by continuity of the function $t \mapsto \tilde{d}((1,0), g(t))$, g meets S at a point $g(t_0)$. So the length of γ is larger than $\tilde{d}((1,0), g(t_0)) + \tilde{d}(g(t_0),(v,\psi)) \geqslant \frac{\pi}{2N} + \tilde{d}(S,(v,\psi))$, which concludes the proof of (28).

By (27) and (28), we have that $\tilde{d}((w,\xi),(v,\psi)) \leqslant k\pi/2N$. Assume that the result is already proved for (w,ξ), (v,ψ). So there exists $H \in \mathrm{Herm}(\mathcal{H})$ such that

$$\|H\| = \tilde{d}((w,\xi),(v,\psi)), \quad e^{iH} = vw^{-1}, \quad \operatorname{tr} H = \psi - \xi.$$

By the first part of Proposition 3.2, there exists $H' \in \mathrm{Herm}(\mathcal{H})$ such that

$$\|H'\| = \frac{\pi}{2N}, \quad e^{iH'} = w, \quad \operatorname{tr} H' = \xi.$$

So $v = e^{iH}e^{iH'}$. By the Thompson Theorem [26], there exists $K \in \mathrm{Herm}(\mathcal{H})$ such that $e^{iK} = v$ and $K = UHU^* + VH'V^*$ for two unitary endomorphisms U, V of $\mathcal{H}$. Hence $\operatorname{tr} K = \operatorname{tr} H + \operatorname{tr} H' = \psi$. This implies by (26) that $\tilde{d}((1,0),(v,\psi)) \leqslant \|K\|$. On the other hand, $\|K\| \leqslant \|H\| + \|H'\| = \tilde{d}((1,0),(v,\psi))$ by (28). Hence

$$\tilde{d}((1,0),(v,\psi)) = \|K\|.$$

So the curve $[0,1] \ni t \mapsto e^{tK}$ is a length minimizing curve from $(1,0)$ to (v,ψ). □

4. The map $\Psi_k : \mathrm{Preq} \to \mathrm{U}(\mathcal{H}_k)$

Space of complex structures

Let E be a finite dimensional symplectic vector space. Let $\mathcal{J}(E)$ be the space of linear complex structures j of E which are compatible with the symplectic form ω of E in the sense that $\omega(jX, jY) = \omega(X,Y)$ for any X, Y in E and $\omega(X, jX) > 0$ for any $X \in E \setminus \{0\}$. The space $\mathcal{J}(E)$ is isomorphic to the Siegel upper half-space $\mathrm{Sp}(2n)/\mathrm{U}(n)$. It has a natural Kähler metric defined as follows. First $\mathcal{J}(E)$ is a submanifold of the vector space $\mathrm{End}(E)$ and

$$T_j\mathcal{J}(E) = \{a \in \mathrm{End}(E) |\; ja + aj = 0,\; \omega(a\cdot, j\cdot) + \omega(j\cdot, a\cdot) = 0\}$$

The symplectic form of $T_j\mathcal{J}(E)$ is given by $\sigma_j(a,b) = \frac{1}{4}\operatorname{tr}(jab)$. The complex structure of $T_j\mathcal{J}(E)$ is the map sending a to ja. The corresponding Riemannian metric of $\mathcal{J}(E)$ has the property that any two points are connected by a unique geodesic.

Consider now a compact symplectic manifold M and $\mathcal{J}$ be the space of (almost) complex structures of M compatible with the symplectic form. Then $\mathcal{J}$ may be considered as an infinite dimensional manifold, whose tangent space at j consists of the sections a of $\mathrm{End}\, TM$ such that at any point x of M, $a(x)$ belongs to $T_{j(x)}\mathcal{J}(T_xM)$. Define the symplectic product

$$(29) \qquad \sigma_j(a,b) = \int_M \sigma_{j(x)}(a(x), b(x))\mu(x), \qquad \forall a, b \in T_j\mathcal{J}.$$

For any $j \in \mathcal{J}$, introduce the Hermitian scalar curvature $S(j) \in \mathcal{C}^\infty(M, \mathbb{R})$ defined as follow. The canonical bundle $\wedge_j^{n,0}T^*M$ has a natural connection induced by j and ω. Then $S(j)\omega^n = n\rho\omega^{n-1}$, where ρ is $-i$ times the curvature of $\wedge_j^{n,0}T^*M$. As was observed by Donaldson in [9], the action of $\mathrm{Ham}\, M$ on $\mathcal{J}$ is Hamiltonian with momentum $-S(j)$. The sign convention for the momentum is different from ours in [9].

Shelukhin quasi-morphism

Fix $j_0 \in \mathcal{J}(M)$. For any $[\varphi_t] \in \widetilde{\mathrm{Ham}}$ with generating Hamiltonian (H_t), let $j_t = (\varphi_t)_* j_0$. Let $(g_t, t \in [0,1])$ be the curve of $\mathcal{J}$ joining j_1 to j_0 such that for any $x \in M$, $s \mapsto g_s(x)$ is the geodesic joining $j_1(x)$ to $j_0(x)$. In the sequel we call (g_t) the *geodesic* joining j_1 to j_0. Let D be any disc of $\mathcal{J}$ bounded by the concatenation (ℓ_t) of (j_t) and (g_t). Set

$$(30) \qquad \mathrm{Sh}_{j_0}([\varphi_t]) = \int_D \sigma + \int_0^1 \Big(\int_M S(J_t) H_t \mu \Big) dt$$

More precisely, viewing D as a smooth family in discs $D_x \subset \mathcal{J}(T_xM)$, the first integral is $\int_M \big(\int_{D_x} \sigma_x \big) \mu(x)$ where σ_x is the symplectic form of $\mathcal{J}(T_xM)$. Observe also that $\mathcal{J}(T_xM)$ being contractible, $\int_{D_x} \sigma_x$ does not depend on the choice of D_x. A possible choice for D_x is the map sending $r\exp(2\pi i t)$ to the point with coordinate r of the geodesic joining $j_0(x)$ to $\ell_t(x)$.

The map introduced in [24] is $n!\,\mathrm{Sh}_{j_0}$, and the scalar curvature is defined with the opposite sign. By this paper, Sh_{j_0} is well-defined, that is the right-hand side of (30) only depends on the homotopy class of (φ_t), and Sh_{j_0} is a quasimorphism (5).

A quantum bundle on $\mathcal{J}$

Assume now that M is endowed with a prequantum line bundle $L \to M$, so the connection ∇ of L has curvature $\frac{1}{i}\omega$. Let $j \in \mathcal{J}$ and $k \in \mathbb{N}$. Then j and ω induce a metric on T^*M, so that we can define the adjoint of

$$\nabla^{L^k} : \mathcal{C}^\infty(M, L^k) \to \Omega^1(M, L^k).$$

Let $\Delta_k(j) = (\nabla^{L^k})^{*j}\nabla^{L^k}$ be the Laplacian acting on $\mathcal{C}^\infty(M, L^k)$. Define $\mathcal{Q}_k(j)$ as the subspace of $\mathcal{C}^\infty(M, L^k)$ spanned by the eigenvectors of $\Delta_k(j)$ whose eigenvalue is smaller than $nk + \sqrt{k}$. We would like to think of $\mathcal{Q}_k(j)$ as the fiber at j of a vector bundle $\mathcal{Q}_k \to M$. But $\mathcal{Q}_k$ is not a genuine vector bundle: for instance the dimension of $\mathcal{Q}_k(j)$ depends on j.

Nevertheless, as shown in [11], [16], $\Delta_k(j)$ has the following spectral gap: there exists positive constants C_1, C_2 independent of k such that the spectrum of $\Delta_k - nk$ is contained in $(-C_1, C_1) \cup (kC_2, \infty)$. Here, C_1 and C_2 remain bounded when j runs over a bounded subset of $\mathcal{J}$ in $\mathcal{C}^3$-topology. Furthermore, the dimension of the subspace spanned by the eigenvectors with eigenvalue in $(-C_1, C_1)$ is equal to $\int_M \exp(k\omega/(2\pi))\,\mathrm{Todd}\,M$ when k is sufficiently large.

Now, choose any compact submanifold S of $\mathcal{J}$, that is a smooth family in $\mathcal{J}$ indexed by a (finite-dimensional) compact manifold (possibly with boundary). Then by the above spectral gap, there exists a constant $k(S)$, such that when k is larger than $k(S)$, the restriction of $\mathcal{Q}_k$ to S is a vector bundle. Furthermore $Q_k \to S$ being a subbundle of the trivial vector bundle $S \times \mathcal{C}^\infty(M, L^k)$, it has a natural connection $\nabla^{\mathcal{Q}_k} = \Pi_k \circ d$ where d is the usual derivative and $\Pi_k(j)$ is the orthogonal projection from $\mathcal{C}^\infty(M, L^k)$ onto $\mathcal{Q}_k(j)$. The main result of [11] (Theorem 2.1) is that the curvature R_k of $\nabla^{\mathcal{Q}_k}$ has the form

$$R_k(a, b) = -\frac{1}{2}\Pi_k \sigma_j(a, b) + \mathcal{O}(k^{-1}), \qquad a, b \in T_j\mathcal{J} \tag{31}$$

where $\sigma_j(a, b)$ is the symplectic product defined in (29), the $\mathcal{O}$ depending on j, a and b.

The map $\Psi_k : \mathrm{Preq} \to \mathrm{U}(\mathcal{H}_k)$ and its lift

Consider the group Preq of prequantum bundle automorphism of L acting on $\mathcal{C}^\infty(M, L^k)$ by push-forward. If $\varphi \in \mathrm{Preq}$ and $j \in \mathcal{J}$, then φ_* intertwines the Laplacians $\Delta_k(j)$ and $\Delta_k(\pi(\varphi)_*(j))$ where $\pi(\varphi)$ is the symplectomorphism of M lifted by φ. So φ_* restricts to a unitary map from $\mathcal{Q}_k(j)$ to $\mathcal{Q}_k(\pi(\varphi)_*(j))$.

Let j_0 be a fixed complex integrable structure. For any k, set $\mathcal{H}_k := \mathcal{Q}_k(j_0)$. Let $\varphi \in \mathrm{Preq}$, $j_1 = \pi(\varphi)_* j_0$ and g be the geodesic segment joining j_0 and j_1. Assume that k is larger than $k(g)$ and let $\mathcal{T}_k : \mathcal{Q}_k(j_1) \to \mathcal{Q}_k(j_0)$ be the parallel transport from j_1 to j_0 along g. When k is larger than $k(g)$, we set $\Psi_k(\varphi) := \mathcal{T}_k \circ \varphi_* \in \mathrm{U}(\mathcal{H}_k)$.

We lift Ψ_k to a map

$$\widetilde{\Psi}_k : \mathcal{P}(\mathrm{Preq}) \to \widetilde{\mathrm{U}}(\mathcal{H}_k) \tag{32}$$

in such a way that for any path $\gamma = (\gamma_t)$, $\widetilde{\Psi}_k(\gamma)$ is the endpoint of the lift of $t \mapsto \Psi_k(\gamma_t)$. More precisely, let $j_t = \pi(\gamma_t)_* j_0$ and let S be a surface of $\mathcal{J}(M)$ containing the geodesic segments joining j_0 to j_t for any $t \in [0,1]$. Then when k is larger than $k(S)$, the parallel transport in Q_k along these geodesic segments is well-defined and the resulting map $\Psi_k(\gamma_t)$ depends continuously on $t \in [0,1]$. So it can be lifted to $\widetilde{\mathrm{U}}(\mathcal{H}_k)$.

The map $\widetilde{\Psi}_k$ almost factorizes to a map from $\widetilde{\mathrm{Preq}}$ to $\widetilde{\mathrm{U}}(\mathcal{H}_k)$. Indeed, for any two paths γ, γ' which are homotopic with fixed endpoints, there exists $k(\gamma, \gamma')$ such that for $k \geqslant k(\gamma, \gamma')$, $\widetilde{\Psi}_k(\gamma) = \widetilde{\Psi}_k(\gamma')$.

5. Proof of Theorem 1.1

We will prove that for any $\gamma \in \mathcal{P}(\mathrm{Preq})$, we have

$$\widetilde{\det}\big(\widetilde{\Psi}_k(\gamma)\big) = -\Big(\frac{k}{2\pi}\Big)^n \Big(\big(k + \lambda'\big)\,\mathrm{Cal}([\gamma]) + \tfrac{1}{2}\,\mathrm{Sh}_{j_0}([\pi(\gamma)]) + \mathcal{O}(k^{-1})\Big) \tag{33}$$

where the constant λ' is

$$\lambda' = \frac{n}{2} \frac{\int_M \rho \omega^{n-1}}{\int_M \omega^n} = \frac{n}{2} \frac{\big[c_1(T^{1,0}M) \cup c_1(L)^{n-1}\big]}{\big[c_1(L)^n\big]} \tag{34}$$

For any $t \in [0,1]$, let $j_t = \pi(\gamma_t)_* j_0$ and let us introduce three unitary maps

- $U_t = (\gamma_t)_* : \mathcal{Q}_k(j_0) \to \mathcal{Q}_k(j_t)$

- $\mathcal{P}_t : \mathcal{Q}_k(j_0) \to \mathcal{Q}_k(j_t)$ is the parallel transport along the path $[0,t] \ni s \mapsto j_s$
- $\mathcal{T}_t : \mathcal{Q}_k(j_t) \to \mathcal{Q}_k(j_0)$ is the parallel transport along the geodesic joining j_t to j_0.

Then $\Psi_k(\gamma_t) = \mathcal{T}_t \circ U_t = \eta_t \circ \xi_t$ where (η_t) and (ξ_t) are the smooth paths of $\mathrm{U}(\mathcal{H}_k)$ given by

$$\eta_t = \mathcal{T}_t \circ \mathcal{P}_t, \qquad \xi_t = \mathcal{P}_t^{-1} \circ U_t.$$

We denote by $(\tilde{\eta}_t)$ and $(\tilde{\xi}_t)$ their lifts to $\widetilde{\mathrm{U}}(\mathcal{H}_k)$ starting from the identity. Of course, $\widetilde{\Psi}_k(\gamma) = \tilde{\eta}_1\tilde{\xi}_1$.

Define the map $S : [0,1]^2 \to \mathcal{J}, (s,t) \mapsto j_t^s$ such that for any t, $s \mapsto j_t^s$ is the geodesic joining j_0 to j_t. In the sequel we assume that k is larger than $k(S)$.

Proposition 5.1. *We have*

$$\widetilde{\det}(\tilde{\eta}_1) = -\frac{1}{2}\Big(\frac{k}{2\pi}\Big)^n \int_S \sigma + \mathcal{O}(k^{n-1})$$

Proof. Note that η_1 is the parallel transport along the boundary of S. So

$$\widetilde{\det}(\tilde{\eta}_1) = \int_{[0,1]^2} \operatorname{tr} R_k(a_t^s, b_t^s)\; ds \wedge dt$$

where $a_t^s = \partial j_t^s/\partial s$ and $b_t^s = \partial j_t^s / \partial t$. By (31), we have

$$R_k(a_t^s, b_t^s) = -\tfrac{1}{2}\Pi_k \sigma_{j_t^s}(a_t^s, b_t^s) + \mathcal{O}(k^{-1})$$

Furthermore, the $\mathcal{O}$ is uniform with respect to s and t. In particular $\Pi_k\sigma_{j_t^s}(a_t^s, b_t^s)$ is uniformly bounded with respect to s, t and k. To conclude we use that

$$\operatorname{tr} \Pi_k f \Pi_k = \Big(\frac{k}{2\pi}\Big)^n \int_M f\mu + \mathcal{O}(k^{n-1}),$$

the $\mathcal{O}$ being uniform if $\sup|f|$ remains bounded. □

Introduce now the Kostant-Souriau operators: for any $f \in \mathcal{C}^\infty(M)$,

$$K_k(f) = f + \frac{1}{ik}\nabla_X^{L^k} : \mathcal{C}^\infty(M, L^k) \to \mathcal{C}^\infty(M, L^k)$$

where X is the Hamiltonian vector field of f. In the next proposition, we prove that ξ_t^{-1} is the solution of a Schrödinger equation.

Proposition 5.2. *Let (H_t) be the Hamiltonian generating (γ_t) and $\phi_t = \pi(\gamma_t)$. We have*

$$\frac{i}{k}\frac{d}{dt}(\xi_t^{-1}) = -\Pi_k(j_0)K_k(H_t \circ \phi_t)\, \xi_t^{-1}.$$

This result has been proved in [11, Proposition 4.3], with a mistake however: the Hamiltonian $K_k(H_t \circ \phi_t)$ is replaced by $K_k(H_t)$.

In the case $H_t = H$ is time-independent, we have $H \circ \phi_t = H$ and ξ_t commutes with $\Pi_k(j_0)K_k(H)$, we deduce that

$$\frac{i}{k}\frac{d}{dt}\xi_t = \Pi_k(j_0)K_k(H)\xi_t.$$

But this does not hold for a general time dependent (H_t). However, observe that $-H_t \circ \phi_t$ is the generating Hamiltonian of γ_t^{-1}.

Proof. Let $V_t : \mathcal{C}^\infty(M, L^k) \to \mathcal{C}^\infty(M, L^k)$ be the push-forward by γ_t. It is part of Kostant-Souriau theory that

$$\frac{i}{k}\dot{V}_t = K_k(H_t)V_t. \tag{35}$$

For any $\varphi \in \mathrm{Preq}$ and $f \in \mathcal{C}^\infty(M)$, we have $(\varphi_*)^{-1}K_k(f)\varphi_* = K_k(f \circ \pi(\varphi))$ where we denote by $\varphi_* : \mathcal{C}^\infty(M, L^k) \to \mathcal{C}^\infty(M, L^k)$ the push-forward by φ and by $\pi(\phi)$ the symplectomorphism of M lifted by φ. In particular

$$V_t^{-1}K_k(H_t) = K_k(H_t \circ \phi_t)V_t^{-1}. \tag{36}$$

Since U_t is the restriction of V_t to $\mathcal{Q}_k(j_0)$, we have $\xi_t^{-1} = U_t^{-1}\mathcal{P}_t = V_t^{-1}\mathcal{P}_t$. Derivating, we get

$$\frac{d}{dt}\xi_t^{-1} = -V_t^{-1}\dot{V}_tV_t^{-1}\mathcal{P}_t + V_t^{-1}\dot{\mathcal{P}}_t. \tag{37}$$

Here, to give a meaning to $\dot{\mathcal{P}}_t$, we consider that $\mathcal{P}_t$ takes its value in $\mathcal{C}^\infty(M, L^k)$. Now $t \mapsto \mathcal{P}_t$ being the parallel transport along the path $t \mapsto j_t$, we have $\Pi(j_t)\dot{\mathcal{P}}_t = 0$. Since $\Pi(j_t)V_t = V_t\Pi(j_0)$, we have $V_t^{-1}\Pi(j_t) = \Pi(j_0)V_t^{-1}$ and consequently $\Pi(j_0)V_t^{-1}\dot{\mathcal{P}}_t = V_t^{-1}\Pi(j_t)\dot{\mathcal{P}}_t = 0$. So by (37),

$$\begin{aligned}\frac{i}{k}\frac{d}{dt}\xi_t^{-1} &= -\frac{i}{k}\Pi(j_0)V_t^{-1}\dot{V}_tV_t^{-1}\mathcal{P}_t = -\Pi(j_0)V_t^{-1}K_k(H_t)\mathcal{P}_t \\ &= -\Pi(j_0)K_k(H_t \circ \phi_t)V_t^{-1}\mathcal{P}_t = -\Pi(j_0)K_k(H_t \circ \phi_t)\xi_t^{-1}\end{aligned}$$

where we have used (35) and (36). □

Lemma 5.3. *For any* $f \in \mathcal{C}^\infty(M, \mathbb{R})$,

$$\begin{aligned} (38)\quad & \operatorname{tr}(\Pi_k(j_0)K_k(f)\Pi_k(j_0)) \\ &= \Big(\frac{k}{2\pi}\Big)^n \Big(\Big(1+\frac{\lambda'}{k}\Big)\int_M f\mu + \frac{1}{2k}\int_M \overline{f} S(j_0)\mu\Big) + \mathcal{O}(k^{n-2}) \end{aligned}$$

where λ' *is the constant* (34) *and* $\overline{f} = f - \int_M f\mu / \int_M \mu$.

Proof. Recall that $\operatorname{tr}(T_k(f)) = \int_M f B_k(x)\mu$ where B_k is the restriction to the diagonal of the Schwartz kernel of $\Pi_k(j_0)$. Recall the well-known asymptotics ([8] for a very short proof and many references):

$$B_k = \Big(\frac{k}{2\pi}\Big)^n \Big(1 + \frac{S(j_0)}{2k} + \mathcal{O}(k^{-2})\Big),$$

By Tuynman formula [27] $\operatorname{tr}(T_k(f)) = \operatorname{tr}(\Pi_k(j_0)K_k(f)\Pi_k(j_0))$, we obtain

$$\operatorname{tr}(\Pi_k(j_0)K_k(f)\Pi_k(j_0)) = \Big(\frac{k}{2\pi}\Big)^n \int_M f\Big(1 + \frac{S(j_0)}{2k}\Big)\,\mu + \mathcal{O}(k^{n-2})$$

which rewritten with the normalised Hamiltonian $\overline{f}$ gives (38). □

Proposition 5.4. *We have*

$$\widetilde{\det}(\tilde{\xi}_1) = -\Big(\frac{k}{2\pi}\Big)^n \Big((k+\lambda')\operatorname{Cal}(H_t) + \frac{1}{2}\int_0^1 \int_M \overline{H}_t S(j_t)\mu\, dt\Big) + \mathcal{O}(k^{n-1})$$

Proof. By Proposition 5.2, we have $ik^{-1}\dot{\xi}_t = \xi_t \Pi_k(j_0)K_k(H_t\circ\phi_t)$. So

$$\begin{aligned} \frac{i}{k}\frac{d}{dt}\ln(\det(\xi_t)) &= \frac{i}{k}\operatorname{tr}(\xi_t^{-1}\dot{\xi}_t) = \operatorname{tr}\Pi_k(j_0)K_k(H_t\circ\phi_t)\Pi_k(j_0) \\ &= \Big(\frac{k}{2\pi}\Big)^n \Big((1+\frac{\lambda'}{k})\int_M H_t\mu + \frac{1}{2k}\int_M \overline{H}_t S(j_t)\mu\Big) + \mathcal{O}(k^{n-2}) \end{aligned}$$

by (38) where we have used that μ is preserved by ϕ_t and $S(j_0)\circ\phi_t^{-1} = S(j_t)$. To conclude, we use $\widetilde{\det}(\tilde{\xi}_1) = \frac{1}{i}\int_0^1 \frac{d}{dt}\ln(\det(\xi_t))dt$. □

Now (33) is a consequence of the definition (30) of Sh_{j_0}, Proposition 5.4, Proposition 5.1 and the fact that $\widetilde{\Psi}_k(\gamma) = \tilde{\eta}_1\tilde{\xi}_1$.

6. Proof of Corollary 1.2 and Theorem 1.3

The first consequence of (33) is Corollary 1.2: there exists $C > 0$ such that for any $\alpha, \beta \in \mathcal{P}(\mathrm{Preq})$,

$$\tilde{d}(\widetilde{\Psi}_k(\alpha)\widetilde{\Psi}_k(\beta), \widetilde{\Psi}_k(\alpha\beta)) \leqslant C + \mathcal{O}(k^{-1}) \tag{39}$$

with a $\mathcal{O}$ depending on α, β.

Proof. By Proposition 3.2, since $\widetilde{\det}$ is a morphism, we have

$$\tilde{d}(\widetilde{\Psi}_k(\alpha)\widetilde{\Psi}_k(\beta), \widetilde{\Psi}_k(\alpha\beta)) \leqslant \frac{\left|\widetilde{\det}\big(\widetilde{\Psi}_k(\alpha)\big) + \widetilde{\det}\big(\widetilde{\Psi}_k(\beta)\big) - \widetilde{\det}\big(\widetilde{\Psi}_k(\alpha\beta)\big)\right|}{\dim \mathcal{H}_k} + 2\pi$$

The result follows from (33) by using that Cal is a morphism, Sh_{j_0} a quasi-morphism (5) and $\dim \mathcal{H}_k = \big(k/2\pi\big)^n(\mathrm{Vol}(M,\omega) + \mathcal{O}(k^{-1}))$. □

Introduce the map $\widetilde{\Phi}_k^{\mathrm{KS}} : \mathcal{P}(\mathrm{Preq}) \to \widetilde{\mathrm{U}}(\mathcal{H}_k)$ defined as $\widetilde{\Phi}_k$ in Section 1.3 except that we use the Kostant-Souriau operators instead of the usual Toeplitz operator. So $\widetilde{\Phi}_k^{\mathrm{KS}}(\gamma_t) = \widetilde{W}_1$ where $(\widetilde{W}_t)$ is the lift of the solution (W_t) of the Schrödinger equation

$$W_t' = \frac{k}{i}\Pi_k(j_0)K_k(H_t)W_t, \qquad W_0 = \mathrm{id}_{\mathcal{H}_k}$$

(H_t) being the generating Hamiltonian of (γ_t).

Theorem 6.1. *For any path γ in $\mathcal{P}(\mathrm{Preq})$, $\widetilde{\Phi}_k^{\mathrm{KS}}(\gamma) = \widetilde{\Psi}_k(\gamma) + \mathcal{O}(1)$.*

Proof. As in Section 5, write $\widetilde{\Psi}_k(\gamma) = \tilde{\eta}_1\tilde{\xi}_1$. By Proposition 3.2, we have

$$\tilde{d}(\tilde{\eta}_1, \mathrm{id}) \leqslant \frac{|\widetilde{\det}(\tilde{\eta}_1)|}{\dim \mathcal{H}_k} + 2\pi.$$

By Proposition 5.1, $\widetilde{\det}(\tilde{\eta}_1) = \mathcal{O}(k^n)$. Furthermore, $\dim \mathcal{H}_k = \big(\frac{k}{2\pi}\big)^n$ $(\mathrm{Vol}(M,\omega) + \mathcal{O}(k^{-1}))$. So

$$\tilde{d}(\tilde{\eta}_1, \mathrm{id}) = \mathcal{O}(1).$$

Consequently, by the right-invariance of $\tilde{d}$,

$$\tilde{d}(\widetilde{\Psi}_k(\gamma), \tilde{\xi}_1) = \tilde{d}(\widetilde{\Psi}_k(\gamma)\tilde{\xi}_1^{-1}, \mathrm{id}) = \tilde{d}(\tilde{\eta}_1, \mathrm{id}) = \mathcal{O}(1).$$

By Proposition 5.2, $\tilde{\xi}_1^{-1} = \widetilde{\Phi}_k^{\mathrm{KS}}(\gamma^{-1})$, so we have proved that

$$\tilde{d}(\widetilde{\Psi}_k(\gamma), \mathrm{I}_k(\gamma)) = \mathcal{O}(1), \tag{40}$$

where $\mathrm{I}_k(\gamma) := \widetilde{\Phi}_k^{\mathrm{KS}}(\gamma^{-1})^{-1}$. For any path $\alpha, \beta \in \mathcal{P}(\mathrm{Preq})$, we have

$$\begin{aligned} \tilde{d}(I_k(\alpha)I_k(\beta), I_k(\alpha\beta)) &\leqslant \tilde{d}(I_k(\alpha)I_k(\beta), \widetilde{\Psi}_k(\alpha)\widetilde{\Psi}_k(\beta)) \\ &\quad + \tilde{d}(\widetilde{\Psi}_k(\alpha)\widetilde{\Psi}_k(\beta), \widetilde{\Psi}_k(\alpha\beta)) + \tilde{d}(\widetilde{\Psi}_k(\alpha\beta), I_k(\alpha\beta)) \\ &= \tilde{d}(I_k(\alpha)I_k(\beta), \widetilde{\Psi}_k(\alpha)\widetilde{\Psi}_k(\beta)) + \mathcal{O}(1) \end{aligned} \tag{41}$$

by (39) and (40). Using left and right invariance, we have $\tilde{d}(ab, a'b') = \tilde{d}(a, a'b'b^{-1}) \leqslant \tilde{d}(a, a') + \tilde{d}(a', a'b'b^{-1}) = \tilde{d}(a, a') + \tilde{d}(b, b')$. So by (41) and (40), we obtain

$$\tilde{d}(\mathrm{I}_k(\alpha)\,\mathrm{I}_k(\beta), \mathrm{I}_k(\alpha\beta)) = \mathcal{O}(1) \tag{42}$$

Consequently

$$\begin{aligned} \tilde{d}(\mathrm{I}_k(\gamma), \widetilde{\Phi}_k^{\mathrm{KS}}(\gamma)) &= \tilde{d}(\mathrm{I}_k(\gamma), \mathrm{I}_k(\gamma^{-1})^{-1}) \\ &= \tilde{d}(\mathrm{I}_k(\gamma)\,\mathrm{I}_k(\gamma^{-1}), \mathrm{id}) \qquad \text{by right-invariance} \\ &= \tilde{d}(\mathrm{I}_k(\mathrm{id}), \mathrm{id}) + \mathcal{O}(1) \qquad \text{by (42)} \\ &= \mathcal{O}(1) \end{aligned}$$

Using (40) one last time, it follows that $\tilde{d}(\widetilde{\Psi}_k(\gamma), \widetilde{\Phi}_k^{\mathrm{KS}}(\gamma)) = \mathcal{O}(1)$. □

We can now prove Theorem 1.3, that is

$$\widetilde{\Phi}_k(\gamma) = \widetilde{\Psi}_k(\gamma) + \mathcal{O}(1).$$

Proof. By Tuynman formula [27], $\Pi_k(j_0)K_k(f) = T_k(f) + \mathcal{O}(k^{-1})$. This will imply

$$\tilde{d}(\Phi_k^{\mathrm{KS}}(\gamma), \Phi_k(\gamma)) = \mathcal{O}(1). \tag{43}$$

Then we will conclude with Theorem 6.1. To prove (43), consider two continuous paths $t \mapsto \hat{H}(t)$, $t \mapsto \delta(t)$ of $\mathrm{Herm}(\mathcal{H}_k)$. For any $\epsilon \in [0,1]$, let $U_\epsilon(t)$ be the solution of Schrödinger equation

$$U_\epsilon'(t) = \tfrac{k}{i}(\hat{H}(t) + \epsilon k^{-1}\delta(t))U_\epsilon(t), \qquad U_\epsilon(0) = \mathrm{id}_{\mathcal{H}_k},$$

and let $\varphi_\epsilon(t)$ be the continuous determination of $\det U_\epsilon(t) = \exp(i\varphi_\epsilon(t))$ with $\varphi_\epsilon(0) = 0$. The distance $\tilde{d}((U_0(1), \varphi_0(1)), (U_1(1), \varphi_1(1)))$ is smaller than the length of the path $\epsilon \mapsto \gamma(\epsilon) = (U_\epsilon(1), \varphi_\epsilon(1))$, which is equal to $\int_0^1 \|V_\epsilon(1)\|\, d\epsilon$ where $V_\epsilon(t) = \partial U_\epsilon(t)/\partial\epsilon$. This derivative satisfies the equation

$$V_\epsilon'(t) = \tfrac{k}{i}(\hat{H}(t) + \epsilon k^{-1}\delta(t))V_\epsilon(t) + b(t), \qquad \text{with} \quad b(t) = \tfrac{1}{i}\delta(t)U_\epsilon(t).$$

Since $V_\epsilon(0) = 0$, by Duhamel's principle,

$$V_\epsilon(t) = \frac{1}{i}\int_0^t U_\epsilon(t)U_\epsilon(s)^{-1}\delta(s)U_\epsilon(s)\, ds.$$

Using that $U_\epsilon(t)$ is unitary, so that $\|U_\epsilon(t)\| = 1$, we obtain

$$\tilde{d}((U_0(1), \varphi_0(1)), (U_1(1), \varphi_1(1))) \leqslant \int_0^1 \|\delta(t)\|\, dt$$

and (43) follows. □

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