Methods and Applications of Analysis.
Vol. 9, No. 3, pp. 329–344, September 2002

001

A CONVEX DARBOUX THEOREM *

IVAR EKELAND† AND LOUIS NIRENBERG‡

Abstract. We give necessary and sufficient conditions for a smooth, generic, differential one-form ω on $\mathbb{R}^n$ to decompose into a sum $\omega = a^1 du_1 + \ldots + a^k du_k$, where the functions a^ℓ are positive and the u_ℓ convex (or quasi-convex) near the origin.

1. Introduction. We are given a smooth differential one-form in a neighbourhood of the origin in $\mathbb{R}^n$ (the Einstein summation convention is used throughout):

$$\omega = \omega_i dx^i$$

its exterior derivative being:

$$d\omega = \omega_{i,j} dx^j \wedge dx^i$$

with:

$$\omega_{i,j} = \frac{\partial \omega_i}{\partial x^j}$$

Problem 1. Under what conditions can we represent ω (near the origin) in the form:

$$\omega = \sum_{\ell=1}^{k} a^\ell du_\ell \tag{1}$$

where the a^ℓ are positive functions and the u_ℓ are strictly convex functions?

The last requirement will be understood to mean that, for each ℓ, the matrix

$$\frac{\partial^2 u_\ell}{\partial x^i \partial x^j} =: u_{\ell,ij}$$

is positive definite.

An obvious necessary condition is:

$$\omega \wedge (d\omega)^k \equiv 0. \tag{2}$$

Indeed, if (1) holds, then

$$d\omega = da^\ell \wedge du_\ell$$

and

$$(d\omega)^k = \alpha \wedge du_1 \wedge \ldots \wedge du_k$$

for some form α, so that (2) follows.

*Received November 9, 2000; accepted for publication February 15, 2001.
†CEREMADE, Université Paris-Dauphine (ekeland@dauphine.fr).
‡Courant Institute of Mathematical Sciences, New York University (nirenberg@cims.nyu.edu).

We will always assume the generic condition:

$$\omega \wedge (d\omega)^{k-1} \neq 0. \tag{3}$$

Definition 1. We say that a smooth one-form ω satisfies the *k-fold Darboux condition* at the origin if (2) holds on some neighbourhood of the origin.

It is a theorem of Darboux (see [2], theorem II.3.3) that, provided (3) holds, this condition is necessary and sufficient for the representation (1) near the origin, if we do not require the positivity of the a^ℓ nor the convexity of the u_ℓ.

In [3], Chiappori and Ekeland asked what further conditions are needed to have this representation with the a^ℓ positive and the u_ℓ convex. They treated the case when ω is real analytic by using Cartan-Kähler theory, and they found the following to be necessary and sufficient:

Condition 1. *There is some neighbourhood of the origin where the matrix with coefficients $\omega_{i,j}$ is the sum of two matrices, a positive definite one and another one of rank k.*

Shortly afterwards, V.M. Zakalyukin studied the nonanalytic case, when ω is merely smooth. He found a necessary and sufficient condition, theorem 1 of [5], which, surprisingly, is slightly different. He introduces the space $A_2(\omega)$ of all tangent vector fields ξ such that:

$$\langle \omega \mid \xi \rangle = 0$$
$$\langle d\omega, (\xi, \eta) \rangle = 0 \ \ \forall \eta,$$

and in addition to condition 1, he requires the following:

Condition 2. *There is some neighbourhood of the origin where the matrix $\omega_{i,j} + \omega_{j,i}$ is positive definite on $A_2(\omega)$.*

The present paper began as an attempt to understand the proof in [5]. We began by investigating the low-dimensional case, and to our surprise we found counterexamples to the preceeding results, both [3] and [5]. We present two, with $k = 2$, that is, we are trying to write

$$\omega = a^1 du_1 + a^2 du_2. \tag{4}$$

Example 1. Here $n = 4$, the Darboux condition holds and condition 1 as well. Set:

$$\omega = (1 + x^1 + x^4)dx^1 + x^2 dx^2 + (x^3 + x^2)dx^3.$$

Then

$$d\omega = dx^4 \wedge dx^1 + dx^2 \wedge dx^3$$

and clearly (3) holds, as well as the Darboux condition. Further, we have:

$$\omega_{i,j} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{Id} + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The last matrix has rank 2, so condition 1 holds. However, the problem has no solution. Indeed, if relation (4) holds, then u_1 satisfies:

$$du_1 \wedge \omega \wedge d\omega = du_1 \wedge a^2 du_2 \wedge d\omega \equiv 0 \tag{5}$$

On the other hand:

$$\omega \wedge d\omega = (1 + x^1 + x^4) dx^1 \wedge dx^2 \wedge dx^3.$$

Substituting into (5) yields

$$(1 + x^1 + x^4) u_{1,4} \equiv 0.$$

So $u_{1,4} \equiv 0$ and u_1 cannot be strictly convex.

Example 2. Here $n = 5$, the Darboux condition holds, and conditions 1 and 2 as well. Set:

$$\omega = -x^2 dx^1 + x^1 dx^2 + (1 + x^3) dx^3 + (1 + x^4) dx^4 + (1 + x^5) dx^5. \tag{6}$$

Then

$$\begin{aligned} d\omega &= 2dx^1 \wedge dx^2 \\ \omega \wedge d\omega &= 2dx^1 \wedge dx^2 \left[(1 + x^3) dx^3 + (1 + x^4) dx^4 + (1 + x^5) dx^5 \right]. \end{aligned}$$

so that (3) holds, as well as the Darboux condition. We also have:

$$\omega_{i,j} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \mathrm{Id} + \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

so condition 1 holds as well. Finally, at the origin, ξ belonging to $A_2(\omega)$ means that

$$\begin{aligned} \xi^3 + \xi^4 + \xi^5 &= 0 \\ 2(\xi^1 \eta^2 - \xi^2 \eta^1) &= 0 \ \ \forall (\eta^1, \eta^2) \end{aligned}$$

the latter equation meaning that $\xi^1 = \xi^2 = 0$. We see that the matrix $\omega_{i,j} + \omega_{j,i}$ is positive definite on vectors of the form $(0, 0, \xi^3, \xi^4, \xi^5)$. So the condition 2 is satisfied also. However, in Appendix A, we present a proof that for the one-form ω given by (6) the problem has no solution.

In this paper, we find the necessary and sufficient conditions, always assuming (3), that a smooth ω must satisfy for the problem to have a solution.

First, we introduce in the space of all one-forms α a subset $\mathcal{I}$ defined as follows:

$$\mathcal{I} = \left\{ \alpha \mid \alpha \wedge \omega \wedge (d\omega)^{k-1} \equiv 0 \right\} \tag{8}$$

Condition 3. *There is a k-dimensional subspace V of $\mathcal{I}(0)$, containing $\omega(0)$, and such that on $N = V^{\perp}$, the matrix $\omega_{i,j}(0)$ is symmetric and positive definite.*

In their seminal paper, [1], Browning and Chiappori investigated the consumption behaviour of a two-person household spending jointly one unit of account. This amounts to a particular case of problem 1, with $k = 2$ and:

$$\omega_j(x)x^j = 1.$$

Browning and Chiappori then consider the matrix $S = (s_{ij})$ defined by:

$$s_{ij} = \omega_{i,j} + x^k \omega_{i,k} \omega_j,$$

and they show that $S = \Sigma + A$, where Σ is a symmetric matrix, positive semi-definite, and $A = (a_{ij})$ has rank at most one.

Suppose Σ is in fact positive definite and A has rank exactly one: $a_{ij} = u_i v_j$, where $\alpha = \sum u_i dx^i$ and $\beta = \sum v_i dx^i$ are both non-zero. A simple computation shows that:

$$\omega \wedge d\omega = -\omega \wedge \alpha \wedge \beta$$

so that α and β belong to $\mathcal{I}$. One then checks that S coincides with Σ on the orthogonal of the two-dimensional subspace spanned by ω and α, so the Browning-Chiappori condition amounts to condition 3 in that particular case.

Theorem 1. *Assume ω is a smooth one-form satisfying (3). Problem 1 has a solution if and only if the k-fold Darboux condition is satisfied near the origin, and condition 3 is satisfied at the origin.*

Let us rewrite condition 3 at the origin. Define N as the subspace of vectors ξ, tangent at the origin, such that:

$$\langle \xi \mid \alpha \rangle = 0 \ \ \forall \alpha \in V.$$

Condition 3 requires that:

$$\omega_{i,j}(0)\xi^i \eta^j = \omega_{i,j}(0)\eta^i \xi^j \ \ \forall \xi, \eta \in N,$$
$$\omega_{i,j}(0)\xi^i \xi^j > 0 \ \ \forall 0 \neq \xi \in N.$$

Note that condition 3 by itself is not open (if it holds at some point, it need not hold in a neigbourhood). Theorem 1 implies that it becomes open if we impose in addition the Darboux condition.

Proof of necessity. Assume problem (1) has a solution. We already know that the Darboux condition has to hold. At the origin (and hence nearby) the du_i are linearly independent; for otherwise, we could express $\omega(0)$ as a linear combination of $k-1$ of them, and it would then follow that $\omega \wedge (d\omega)^{k-1} = 0$ at the origin, contradicting (3).

Note next, as we did in example 1 for the case $k = 2$, that relation (1) implies:

$$\omega \wedge (d\omega)^{k-1} = \Theta \wedge du_1 \wedge \ldots \wedge du_k$$

for some $(k-1)$-form Θ, so that

$$du_i \wedge \omega \wedge (d\omega)^{k-1} = 0.$$

Thus du_i lies in $\mathcal{I}$ for all i. Let V be the k-dimensional subspace spanned by $du_1, \ldots, du_k$. By (1), the linear form $\omega(0)$ lies in V. Set $N = V^{\perp}$and take ξ and η in N. Differentiating (1), we find:

$$\omega_{i,j} = a_j^{\ell} u_{\ell,i} + a^{\ell} u_{\ell,ij}. \tag{9}$$

Writing $u_{\ell,i}\xi^i = u_{\ell,i}\eta^i = 0$ in the preceding equation , we get:

$$\omega_{i,j}\xi^i\eta^j = a^{\ell} u_{\ell,ij}\xi^i\eta^j$$

The right-hand side is symmetric in ξ and η, and so therefore is the left. Furthermore, taking $\xi = \eta$, we see that $\omega_{i,j}$ is positive definite, as announced. Necessity is proved. $\square$

To conclude this section, let us state (and solve) a related problem. Recall first that a function u is *quasi-convex* iff the level sets $\{x \mid u(x) \leq h\}$ are convex for every h, and *strictly* quasi-convex if these level sets are stricly convex. A second-order necessary and sufficient condition for quasi-convexity is that the restriction of the quadratic form $u_{ij}\xi^i\xi^j$ to the subspace $u_i\xi^i = 0$ be positive semidefinite (see [4], theorem M.C.4). If it is positive definite, the function is strictly quasi-convex.

Such functions play an important role in economics, since direct utility functions are typically quasi-concave, and the indirect utility functions quasi-convex (see [4], proposition 3.D.3). In this context, the following is a natural question:

Problem 2. Under what conditions can we represent ω (near the origin) in the form:

$$\omega = \sum_{\ell=1}^{k} a^{\ell} du_{\ell} \tag{10}$$

where the a^{ℓ} are positive functions and the u_{ℓ} are strictly quasi-convex functions?

Clearly, (strictly) convex functions are (strictly) quasi-convex, so any solution to problem 1 is a solution to problem 2. So problem 2 would seem to be more general than problem 1. This is not the case: the problems are equivalent. This follows from

Lemma 1. *Assume that the function u is strictly quasi-convex at the origin, in the sense that the restriction of the quadratic form $u_{ij}(0)\xi^i\xi^j$ to the subspace $u_i(0)\xi^i = 0$ is positive definite. Then there is an increasing function $\phi(t)$, defined on a small interval containing $u(0)$, such that $\phi \circ u$ is strictly convex at the origin, in the sense that the quadratic form $(\phi \circ u)_{ij}(0)\xi^i\xi^j$ is positive definite.*

Proof. We may assume $u(0) = 0$ and $du(0) = dx_1$. Take $\phi'(0) = 1$ and $\phi''(0) = K$ We have:

$$(\phi \circ u)_{ij}(0)\xi^i\xi^j = u_{ij}(0)\xi^i\xi^j + K(\xi^1)^2$$

which can be made positive definite by choosing K large enough. $\square$

So any decomposition

$$\omega = \sum_{\ell=1}^{k} a^{\ell} du_{\ell}$$

in terms of functions u_{ℓ} strictly quasi-convex at the origin (and hence strictly quasi-convex near the origin) yields a similar decomposition in terms of strictly convex functions. Indeed, choose for each u_ℓ a function ϕ_ℓ such that $\phi'_\ell(u_\ell) = 1$ at the origin and $v_\ell = \phi_\ell \circ u_\ell$ is strictly convex. Then:

$$\omega = \sum_{\ell=1}^{k} \frac{a^\ell}{\phi'_\ell(u_\ell)} dv_\ell$$

and the coefficients are clearly positive. So theorem 1 provides an answer both to problems 1 and 2.

2. Preliminaries. The following algebraic result will be used repeatedly. It is proposition I.1.6. of [2].

Lemma 2. *Let $\alpha_1, \dots, \alpha_{p+1}$ be linearly independent one-forms and Ω a two-form such that:*

$$\alpha_1 \wedge \dots \wedge \alpha_p \wedge \Omega^q = 0$$

for some positive integers p and q. Then

$$\alpha_1 \wedge \dots \wedge \alpha_{p-1} \wedge \Omega^{q+1} = 0$$

From now on we assume that ω satisfies the conditions of theorem 1.

There are $(n-1)$ one-forms $\alpha_1, \dots, \alpha_{n-1}$, which, together with ω, span all 1-forms. We write:

$$d\omega = \omega \wedge \beta + \Omega \tag{11}$$

where Ω is a two-form involving the α_i only. Then, for any ℓ,

$$\omega \wedge (d\omega)^\ell = \omega \wedge (\ell\omega \wedge \beta \wedge \Omega^{\ell-1} + \Omega^\ell) = \omega \wedge \Omega^\ell. \tag{12}$$

Thus, setting $\ell = k-1$ and k in succession, we find that:

$$\Omega^{k-1} \neq 0, \quad \Omega^k \equiv 0.$$

By another algebraic result, namely theorem I.1.7 in [2], there exist $2k-2$ linearly independent one-forms, $\sigma_1, \dots, \sigma_{k-1}, \sigma'_1, \dots, \sigma'_{k-1}$, in the span of the α_i, and such that:

$$\Omega = \sum_{\ell=1}^{k-1} \sigma_\ell \wedge \sigma'_\ell \tag{13}$$

Inserting this in relation (11), we finally get

$$d\omega = \omega \wedge \beta + \sum_{\ell=1}^{k-1} \sigma_\ell \wedge \sigma'_\ell \tag{14}$$

One last algebraic result will be needed.

Lemma 3. *Let $\alpha_1, \dots, \alpha_{\ell-1}$ be one-forms such that $\alpha_1, \dots, \alpha_{\ell-1}, \omega$ are linearly independent and satisfy:*

$$\alpha_1 \wedge \dots \wedge \alpha_{\ell-1} \wedge \omega \wedge (d\omega)^{k-\ell+1} \equiv 0.$$

Define $\mathcal{J}_\ell$ to be the set of one-forms α such that:

$$\alpha \wedge \alpha_1 \wedge \ldots \wedge \alpha_{\ell-1} \wedge \omega \wedge (d\omega)^{k-\ell} \equiv 0.$$

Then:
(i) $\mathcal{J}_\ell$ is spanned by $2k-\ell$ one-forms $\tau_1, \ldots, \tau_{2k-\ell}$
(ii) If Θ is a two-form satisfying

$$\Theta \wedge \alpha_1 \wedge \ldots \wedge \alpha_{\ell-1} \wedge \omega \wedge (d\omega)^{k-\ell} \equiv 0.$$

then there exist one-forms μ_i such that

$$\Theta = \sum_{i=1}^{2k-\ell} \mu_i \wedge \tau_i$$

The proof is deferred to Appendix B.
We are going to construct the functions $u_1, \ldots, u_k$ by the successive equations:

$$du_1 \wedge \omega \wedge (d\omega)^{k-1} \equiv 0, \quad ie \;\; du_1 \in \mathcal{I} \tag{15.1}$$

$$du_2 \wedge du_1 \wedge \omega \wedge (d\omega)^{k-2} \equiv 0, \tag{15.2}$$

$$du_\ell \wedge \ldots \wedge du_1 \wedge \omega \wedge (d\omega)^{k-\ell} \equiv 0, \tag{15.ℓ}$$

$$du_{k-1} \wedge du_{k-2} \wedge \ldots \wedge du_1 \wedge \omega \wedge d\omega \equiv 0, \tag{15.$k-1$}$$

$$du_k \wedge \ldots \wedge du_1 \wedge \omega \equiv 0. \tag{15.k}$$

At the origin the du_ℓ are to be linearly independent and to lie in V. Furthermore, each u_ℓ will be strictly convex, the a^ℓ will be positive, and there will be conditions on the $du_\ell(0)$. We will do this inductively on ℓ.

Observe first that the set $\mathcal{I}$ of one-forms, defined by (8), is a linear space of dimension $(2k-1)$. Indeed, using (12) and (13), we find that α belongs to $\mathcal{I}$ if and only if:

$$\begin{aligned} 0 &\equiv \alpha \wedge \omega \wedge \Omega^{k-1} \\ &\equiv c\alpha \wedge \omega \;\wedge \sigma_1 \wedge \sigma_1' \ldots \wedge \sigma_{k-1} \wedge \sigma_{k-1}' \end{aligned}$$

for some $c \neq 0$. This means that

$$\mathcal{I} = Span \left\{ \omega, \sigma_1, \sigma_1', \ldots, \sigma_{k-1}, \sigma_{k-1}' \right\}.$$

We claim that $\mathcal{I}$ generates a differential ideal. This is equivalent to the Frobenius condition: if $\alpha_1, \dots, \alpha_{2k-1}$ span $\mathcal{I}$, then there are one-forms μ_{ij} such that, for $1 \le i \le 2k-1$:

$$d\alpha_i = \sum_{i=1}^{2k-1} \mu_{ij} \wedge \alpha_j. \tag{16}$$

To verify (16), we take α in $\mathcal{I}$ and apply d to (8):

$$d\alpha \wedge \omega \wedge (d\omega)^{k-1} = \alpha \wedge (d\omega)^k$$

which vanishes by lemma 2. Relation (16) then follows by lemma 3 (ii)

Assume $u_1, \dots, u_{\ell-1}$ have been obtained, satisfying equations (15.1) to $(15.\ell-1)$, with ω and the differentials du_i linearly independent.We then define a set $\mathcal{I}_\ell$ of one-forms α by:

$$\mathcal{I}_\ell = \left\{\alpha \ \mid \ \alpha \wedge du_{\ell-1} \wedge \dots \wedge du_1 \wedge \omega \wedge (d\omega)^{k-\ell} \equiv 0\right\}$$

It follows from lemma 2 that:

$$\mathcal{I}_\ell \subset \mathcal{I}_{\ell-1} \subset \dots \subset \mathcal{I}_1 \tag{17}$$

and the assumption on the $u_i, 1 \le i \le \ell$, means that:

$$u_i \in \mathcal{I}_i \ \ \forall i \le \ell \tag{18}$$

For $\ell = 1$, we get $\mathcal{I}_1 = \mathcal{I}$.

For $\ell = k$, we get:

$$\begin{aligned}\mathcal{I}_k &= \{\alpha \ \mid \ \alpha \wedge du_{k-1} \wedge \dots \wedge du_1 \wedge \omega \equiv 0\} \\ &= Span\,\{\omega, du_1, \dots, du_{k-1}\}\end{aligned}$$

which is a k-dimensional linear space.

We claim that $\mathcal{I}_k$ generates a differential ideal. Indeed, if $\alpha \in \mathcal{I}_k$, we apply d to the system for α, and we get:

$$\begin{aligned}d\alpha \wedge du_{k-1} \wedge \dots \wedge du_1 \wedge \omega &\equiv \pm\alpha \wedge du_{k-1} \wedge \dots \wedge du_1 \wedge d\omega \\ &\equiv 0\end{aligned}$$

because α is a linear combination of $\omega, du_1, \dots, du_{k-1}$, and u_{k-1} has been assumed to satify equation $(15.k-1)$. As above, using lemma 3 (ii), we find that the forms $\omega, du_1, \dots, du_{k-1}$ satisfy the Frobenius condition, so that $\mathcal{I}_k$ generates a differential ideal.

For $1 < \ell < k$, by lemma 3 (i), $\mathcal{I}_\ell$ is at each point a linear space of dimension $2k-\ell$. We claim that $\mathcal{I}_\ell$ generates a differential ideal. For, if $\alpha \in \mathcal{I}_\ell$, then, applying d to the system defining α, we obtain:

$$d\alpha \wedge du_{\ell-1} \wedge \dots \wedge du_1 \wedge \omega \wedge (d\omega)^{k-\ell} \equiv \pm\alpha \wedge du_{\ell-1} \wedge \dots \wedge du_1 \wedge (d\omega)^{k-\ell+1}$$

and the right-hand side vanishes by lemma 2. Using lemma 3 (ii) we conclude that $\mathcal{I}_\ell$ is spanned by one-forms satisfying the Frobenius condition, so that it is a differential ideal.

3. Proof of sufficiency. Without loss of generality, we may suppose that at the origin $\omega = dx^1$, and that V is spanned by $dx^1, \dots, dx^k$. Thus N consists of all vectors ξ, tangent at the origin, such that:

$$\xi^1 = \ldots = \xi^k = 0.$$

and the assumed symmetry of $\omega_{i,j}(0)$ on N means simply that:

$$\omega_{i,j}(0) = \omega_{j,i}(0) \;\; \forall i, j > k.$$

Thus:

$$d\omega(0) = dx^1 \wedge \beta_1 + \tau \;\text{ with }\; \tau = dx^2 \wedge \beta_2 + \ldots + dx^k \wedge \beta_k$$

where each β_i involves only the dx^j with $j > i$. And so at the origin:

$$\omega \wedge (d\omega)^{k-\ell} = \omega \wedge (\tau)^{k-\ell}, \;\text{ with }\; \tau^k = 0.$$

We will need:

Lemma 4. *At the origin, if $\alpha_1, \dots, \alpha_\ell$ are any ℓ linear forms in V, then:*

$$\alpha_1 \wedge \ldots \wedge \alpha_\ell \wedge \omega \wedge (d\omega)^{k-\ell} = 0.$$

Proof. Write, for $i = 1, \dots, \ell$:

$$\alpha_i = \sum_{j=1}^{k} \alpha_{ij} dx^j = \alpha_{i1} dx^1 + \alpha_i'.$$

Then:

$$\begin{aligned} \alpha_1 \wedge \ldots \wedge \alpha_\ell \wedge \omega \wedge (d\omega)^{k-\ell} &= \alpha_1' \wedge \ldots \wedge \alpha_\ell' \wedge \omega \wedge (d\omega)^{k-\ell} \\ &= \alpha_1' \wedge \ldots \wedge \alpha_\ell' \wedge \omega \wedge (\tau)^{k-\ell}, \end{aligned}$$

and the right-hand side vanishes for every term involves k products of the $(k-1)$ one-forms $dx^2, \dots, dx^k$. ☐

3.1. Construction of u_1. Since $\mathcal{I}$ has dimension $2k-1$ and satisfies (16), it follows from the Frobenius theorem (see [2], theorem II.1.1) that there exists $2k-1$ functions $v_1, \dots, v_{2k-1}$, the differentials of which span $\mathcal{I}$. We may choose $v_1, \dots, v_k$ such that, at the origin:

$$\begin{aligned} dv_i(0) &= dx^i, \;\; \textit{for} \;\; i = 1, \dots, k \\ v_j(0) &= 0 \;\; \forall j. \end{aligned}$$

Since ω belongs to $\mathcal{I}$, we may write:

$$\omega = \sum_{\ell=1}^{2k-1} f^\ell \, dv_\ell,$$

with $f^1(0) = 1$ and $f^\ell(0) = 0$ for $\ell > 1$. So:

$$\begin{aligned} \omega_i &= f^\ell v_{\ell,i} \\ \omega_{i,j}(0) &= v_{1,ij}(0) + f^\ell_j(0) v_{\ell,i}(0). \end{aligned}$$

Now use the fact that $\omega_{i,j}(0)$ is positive definite on N, and hence on $\mathcal{I}^{\perp}(0)$, which is a smaller space. For $\xi \in \mathcal{I}^{\perp}(0)$, we have:

$$v_{\ell,i}(0)\xi^i = 0 \quad \ell = 1, \ldots, 2k-1 \tag{19}$$

and:

$$c\|\xi\|^2 \leq \omega_{i,j}(0)\xi^i\xi^j = v_{1,ij}(0)\xi^i\xi^j, \quad c > 0. \tag{20}$$

Set

$$u_1 = v_1 + \epsilon_1 v_2 + K\sum_1^{2k-1}(v_\ell)^2 \tag{21}$$

for $\epsilon_1 > 0$ small and K large. Then u_1 is a solution of (15.1), that is, $du_1 \in \mathcal{I}$. We claim that u_1 is strictly convex at the origin. Indeed, at the origin:

$$u_{1,ij}(0)\xi^i\xi^j = v_{1,ij}(0)\xi^i\xi^j + \epsilon_1 v_{2,ij}(0)\xi^i\xi^j + 2K\sum_\ell (v_{\ell,i}(0)\xi^i)^2.$$

Thus, for $\xi \in \mathcal{I}^{\perp}(0)$, we have, according to (19) and (20):

$$\begin{aligned} u_{1,ij}(0)\xi^i\xi^j &= \omega_{i,j}(0)\xi^i\xi^j + \epsilon_1 v_{2,ij}(0)\xi^i\xi^j \\ &\geq c/2\|\xi\|^2 \ \text{ for } \ \epsilon_1 \ \text{ small} \end{aligned}$$

When ξ belongs to a complementary subspace of $\mathcal{I}^{\perp}(0)$, the last term in (21) takes precedence, so that, for K large enough:

$$u_{1,ij}(0)\xi^i\xi^j \geq \frac{c}{2}\|\xi\|^2, \quad c > 0.$$

for all vectors ξ tangent at the origin. This finishes the construction of u_1. Note that:

$$du_1(0) = dx^1 + \epsilon_1 dx^2 \tag{22}$$

3.2. Construction of the u_ℓ, for $\ell \leq k-1$. We now argue by induction. Suppose we have constructed functions:

$$u_1, \ldots, u_{\ell-1},$$

for $\ell \leq k-1$, and positive numbers

$$\epsilon_1, \ldots, \epsilon_{\ell-1}$$

satisfying recursively (15.1), …, (15.$\ell-1$), the matrix $u_{i,rs}(0)$ being positive definite, and

$$du_i(0) = dx^1 - \epsilon_i dx^i + \epsilon_i dx^{i+1} \tag{23}$$

for $2 \leq i \leq \ell - 1$, while (22) holds for $i = 1$. We now construct u_ℓ with similar properties.

Since $\mathcal{I}_\ell$ generates a differential ideal, and has dimension $2k-\ell$, using Frobenius again we find that there exist $2k-\ell$ functions $v_1, \ldots, v_{2k-\ell}$ spanning $\mathcal{I}_\ell$.

Because of (22) and (23), the $du_i(0), 1 \leq i \leq \ell - 1$, all belong to V. Comparing lemma 4 with the definition of $\mathcal{I}_\ell$, we find that $V \subset \mathcal{I}_\ell(0)$, so that we may assume that:

$$dv_i(0) = dx^i \ \ for \ \ i = 1, \dots, k.$$

Again, we may represent:

$$\omega_i = f^\ell v_{\ell,i},$$

using functions f^ℓ with $f^1(0) = 1$ and $f^\ell(0) = 0$ for $\ell > 1$. Thus, at the origin:

$$\omega_{i,j}(0) = v_{1,ij} + f_j^\ell v_{\ell,i}$$

Since $\mathcal{I}_\ell(0) \supset V$, we have $\mathcal{I}_\ell(0)^\perp \subset N$. For $\xi \in \mathcal{I}_\ell(0)^\perp$, we get:

$$\omega_{i,j}(0)\xi^i\xi^j = v_{1,ij}(0)\xi^i\xi^j$$

By our assumption on V, it follows that, for some $c > 0$, we have:

$$v_{1,ij}(0)\xi^i\xi^j \geq c\|\xi\|^2, \ \ \forall \xi \in \mathcal{I}_\ell(0)^\perp$$

We now define:

$$u_\ell = v_1 - \epsilon_\ell v_\ell + \epsilon_\ell v_{\ell+1} + K \sum_{s=1}^{2k-\ell} v_\ell^2$$

Just as before, we find that

$$u_{\ell,ij}(0)\xi^i\xi^j \geq \frac{c}{2}\|\xi\|^2,$$

provided ϵ_ℓ is small and K large. We have $u_\ell \in \mathcal{I}_\ell$, and at the origin:

$$du_\ell(0) = dx^1 - \epsilon_\ell dx^\ell + \epsilon_\ell dx^{\ell+1}$$

3.3. Construction of u_k and conclusion. We have thus constructed $u_1, \dots, u_{k-1}$ with the desired properties, and we finally construct u_k. Again, $\mathcal{I}_k$ has dimension k and generates a differential ideal, so that it is spanned by the differentials of k functions $w_1, \dots, w_k$. As above, $\mathcal{I}_k(0) = V$, so we may choose

$$dw_i(0) = dx_i(0), \ \ i = 1, \dots, k$$

We now set:

$$u_k = w_1 - \epsilon_k w_k + K \sum_1^k w_\ell^2$$

As before, using the fact that $\omega_{i,j}(0)$ is positive definite on $N = V^\perp$, we find that for ϵ_k small and K large, we have

$$u_{k,ij}(0)\xi^i\xi^j \geq \frac{c}{2}\|\xi\|^2$$

while:

$$du_k(0) = dx^1 - \epsilon_k dx^k$$

To complete the proof of the theorem, we must show that in the representation

$$\omega = \sum_{\ell=1}^{k} a^\ell du_\ell$$

all the a^ℓ are positive at the origin. Well, at the origin, the $du_i(0)$ are independent, so the $a_\ell(0)$ are unique. But at the origin;

$$\begin{aligned}
\omega(0) &= dx^1 \\
du_1(0) &= dx^1 + \epsilon_1 dx^2 \\
du_2(0) &= dx^1 - \epsilon_2 dx^2 + \epsilon_2 dx^3 \\
&\dots \\
du_{k-1}(0) &= dx^1 - \epsilon_{k-1} dx^{k-1} + \epsilon_{k-1} dx^k \\
du_k(0) &= dx^1 - \epsilon_k dx^k.
\end{aligned}$$

Then:

$$\begin{aligned}
\frac{1}{\epsilon_1} du_1(0) &= \frac{1}{\epsilon_1} dx^1 + dx^2 \\
\frac{1}{\epsilon_2} du_2(0) &= \frac{1}{\epsilon_2} dx^1 - dx^2 + dx^3 \\
&\dots \\
\frac{1}{\epsilon_{k-1}} du_{k-1}(0) &= \frac{1}{\epsilon_{k-1}} dx^1 - dx^{k-1} + dx^k \\
\frac{1}{\epsilon_k} du_k(0) &= \frac{1}{\epsilon_k} dx^1 - dx^k.
\end{aligned}$$

Summing up, we get:

$$\sum_i \frac{1}{\epsilon_i} du_i(0) = (\sum_i \frac{1}{\epsilon_i}) dx^1$$

which gives the desired decomposition $\omega(0) = \sum_\ell a^\ell(0) du_\ell(0)$, with

$$a^\ell(0) = [\epsilon_\ell \sum (1/\epsilon_i)]^{-1} > 0$$

.

This concludes the proof.

Appendix A. Proof of Example 2. We prove that for the one-form ω given by (6) the problem has no solution. Assume otherwise; for the sake of convenience, write $u_1 = u$, $u_2 = v$, $a^1 = a$, and $a^2 = b$, so that

$$\omega = adu + bdv$$

In particular, on the plane $x^3 = x^4 = x^5 = 0$, we have:

$$au_1 + bv_1 = -x^2 \tag{24}$$

$$au_2 + bv_2 = x^1. \tag{25}$$

We may assume $u(0) = v(0) = 0$. Expanding u and v near the origin in the plane (x^1, x^2), we have:

$$u = c_1x^1 + c_2x^2 + Q_1(x^1, x^2) + o(\|x\|^2)$$
$$v = d_1x^1 + d_2x^2 + Q_2(x^1, x^2) + o(\|x\|^2)$$

where Q_1 and Q_2 are positive definite quadratic forms.

From (24) and (25), we have:

$$a(u_2v_1 - u_1v_2) = x^1v_1 + x^2v_2 = d_1x^1 + d_2x^2 + 2Q_2(x^1, x^2) + o(\|x\|^2) \qquad (26)$$
$$b(u_1v_2 - u_2v_1) = x^1u_1 + x^2u_2 = c_1x^1 + c_2x^2 + 2Q_1(x^1, x^2) + o(\|x\|^2). \qquad (27)$$

At the origin, the right-hand sides vanish , and since a and b are positive, we must have $u_2v_1 - u_1v_2 = 0$. This implies that the vectors (c_1, c_2) and (d_1, d_2) are parallel. One or both may vanish, but in any case we can choose $(x_1, x_2) \neq 0$ near the origin so that

$$c_1x^1 + c_2x^2 = 0 = d_1x^1 + d_2x^2.$$

For such a choice of (x_1, x_2), the right-hand sides of (26) and (27) are positive. But the left-hand sides have opposite signs. This is a contradiction.

Appendix B. Proof of lemma 3. We begin by an algebraic result:

Lemma 5. *Consider a $2m$-dimensional space of one-forms spanned by*

$$\sigma_1, \sigma'_1, \ldots, \sigma_m, \sigma'_m,$$

and set:

$$\Omega = \sigma_1 \wedge \sigma'_1 + \ldots + \sigma_m \wedge \sigma'_m$$

so that $\Omega^m \neq 0$. Let $\gamma_1, \ldots \gamma_{s-1}$ be linearly independent one-forms such that:

$$\gamma_1 \wedge \ldots \wedge \gamma_{s-1} \wedge \Omega^{m-s+2} \equiv 0.$$

Then :

(i) There is a set of one-forms $\beta_1, \ldots, \beta_m, \gamma'_s, \ldots, \gamma'_m$ such that:

$$\Omega = \sum_{i=1}^{s-1} \gamma_i \wedge \beta_i + \sum_{j=s}^{m} \gamma'_j \wedge \beta_j. \qquad (28)$$

(ii) The space of one-forms

$$\mathcal{M}_s = \{\alpha \mid \alpha \wedge \gamma_1 \wedge \ldots \wedge \gamma_{s-1} \wedge \Omega^{m-s+1} \equiv 0\}$$

is spanned by the $\gamma_i, \gamma'_j, \beta_\ell$, for $1 \leq i < s \leq j \leq m$, $1 \leq \ell \leq m$, and so has dimension $2m - s + 1$.

Proof. We first assume (i) to prove (ii).

If (28) holds, since $\Omega^m \neq 0$, the β_i, γ_i and γ'_i must be linearly independent. The system defining $\mathcal{J}_s$ then becomes:

$$0 = \alpha \wedge \gamma_1 \wedge \ldots \wedge \gamma_{s-1} \wedge \left(\sum_{i=s}^{m} \gamma'_i \wedge \beta_i\right)^{m-s+1}$$
$$= (m-s+1)!\, \alpha \wedge \gamma_1 \wedge \ldots \wedge \gamma_{s-1} \wedge \gamma'_s \wedge \beta_s \wedge \ldots \wedge \gamma'_m \wedge \beta_m$$

We end up with:

$$\mathcal{M}_s = Span\left\{\gamma_i, \gamma'_j, \beta_j \mid 1 \le i \le s-1,\ s \le j \le m\right\}$$

so that (ii) is proved.

Now to prove (i). This will be done by induction on s. For $s = 1$, we can take $\beta_i = \sigma_i$ and $\gamma'_i = \sigma'_i$, so the lemma holds in this case.

Suppose (i) holds up to $s = r$; we wish to establish it for $s = r + 1$. Let $\gamma_1, \dots, \gamma_r$ be linearly independent one-forms satisfying

$$\gamma_1 \wedge \dots \wedge \gamma_r \wedge \Omega^{m-r+1} \equiv 0.$$

This means that γ_r belongs to $\mathcal{M}_r$. Since (28) holds for $s = r$, we have seen that γ_r then belongs to the linear span of the $\gamma_i, \gamma'_j, \beta_j$, for $1 \le i < r \le j \le m$:

$$\gamma_r = \sum_{i=1}^{r-1} c_i \gamma_i + \sum_{j=r}^{m} b_j \beta_j + \sum_{j=r}^{m} a_j \gamma'_j$$

At least one of the b_j or a_j must be non-zero. Suppose it is a_r; the case where all the a_j vanish and $b_r \neq 0$ would be treated in the same way. We may assume $a_r = 1$. Then:

$$\gamma'_r = \gamma_r - \sum_{i=1}^{r-1} c_i \gamma_i - \sum_{j=r}^{m} b_j \beta_j - \sum_{j=r+1}^{m} a_j \gamma'_j \tag{29}$$

Because (28) holds up for $s = r$, we have:

$$\Omega = \sum_{i=1}^{r-1} \gamma_i \wedge \beta_i + \sum_{j=r}^{m} \gamma'_j \wedge \beta_j.$$

Replacing γ'_r in this expression by its value, taken from (29), we obtain:

$$\Omega = \sum_{i=1}^{r-1} \gamma_i \wedge (\beta_i - c_i \beta_r) + \gamma_r \wedge \beta_r + \sum_{j=r+1}^{m} (\gamma'_i + b_j \beta_r) \wedge (\beta_j - a_j \beta_r) \tag{30}$$

Setting:

$$\begin{aligned}
&\overline{\beta}_i = \beta_i - c_i \beta_r \ \ for \ \ 1 \le i \le r-1 \\
&\overline{\beta}_r = \beta_r \\
&\overline{\gamma}'_j = \gamma'_j + b_j \beta_r \ \ for \ \ r+1 \le j \le m \\
&\overline{\beta}_j = \beta_j - a_j \beta_r \ \ for \ \ r+1 \le j \le m,
\end{aligned}$$

we rewrite (30) as follows:

$$\Omega = \sum_{i=1}^{r} \gamma_i \wedge \overline{\beta}_i + \sum_{j=r+1}^{m} \overline{\gamma}'_j \wedge \overline{\beta}_j$$

so (28) holds for $s = r + 1$, and the lemma is proved. □

Proof of lemma 3(i). We show first that:

$$\mathcal{J}_\ell = \left\{\alpha \mid \alpha \wedge \alpha_{\ell-1} \wedge \ldots \wedge \alpha_1 \wedge \omega \wedge (d\omega)^{k-\ell} \equiv 0\right\}, \tag{31}$$

has dimension $2k - \ell$.

Recall that the $\alpha_1, \ldots, \alpha_{\ell-1}$ and ω are linearly independent, and satisfy

$$\alpha_1 \wedge \ldots \wedge \alpha_\ell \wedge \omega \wedge (d\omega)^{k-\ell+1} \equiv 0. \tag{32}$$

By lemma 2 , $\mathcal{J}_\ell$ and the α_i belong to $\mathcal{I}$, and so is α_i is a linear combination of ω and the σ_j, σ'_j. Thus we may write:

$$\alpha_i = b_i\omega + \gamma_i \ \ for \ \ 1 \leq i \leq \ell - 1 \tag{33}$$

where γ_i is a linear combination of the σ_j, σ'_j. Inserting (33) in (32), we obtain:

$$\gamma_1 \wedge \ldots \wedge \gamma_{\ell-1} \wedge \omega \wedge (d\omega)^{k-\ell+1} \equiv 0. \tag{34}$$

.

On the other hand, by relation (14), we have:

$$d\omega = \omega \wedge \beta + \Omega, \quad \text{with } \ \Omega = \sum_{i=1}^{k-1} \sigma_\ell \wedge \sigma'_\ell$$

so equation (34) becomes:

$$\gamma_1 \wedge \ldots \wedge \gamma_{\ell-1} \wedge \omega \wedge \Omega^{k-\ell+1} \equiv 0. \tag{35}$$

This implies:

$$\gamma_{\ell-1} \wedge \ldots \wedge \gamma_1 \wedge \Omega^{k-\ell+1} \equiv 0 \tag{36}$$

Indeed, the left-hand side is a k-form involving only the σ_j, σ'_j, and if it did not vanish, taking the wedge product with ω, which is linearly independent from the σ_j, σ'_j would contradict (35).

Consider any α in $\mathcal{J}_\ell$. We may write $\alpha = b\omega + \hat{\alpha}$, where $\hat{\alpha}$ is a linear combination of the σ_i, σ'_i, and equation (31) implies:

$$\hat{\alpha} \wedge \gamma_1 \wedge \ldots \wedge \gamma_{\ell-1} \wedge \Omega^{k-\ell} = 0$$

By lemma 5, with $m = k-1$, we see that the set of such $\hat{\alpha}$ has dimension $2k-\ell-1$, so $\mathcal{J}_\ell$, which is spanned by these $\hat{\alpha}$ and by ω, has dimension $2k - \ell$. □

Proof of lemma 3(ii). Suppose the two-form Θ satisfies the prescribed condition. Then:

$$\Theta \wedge \alpha_1 \wedge \ldots \wedge \alpha_{\ell-1} \wedge \omega \wedge \Omega^{k-\ell} \equiv 0.$$

Using (33), we get:

$$\Theta \wedge \gamma_1 \wedge \ldots \wedge \gamma_{\ell-1} \wedge \omega \wedge \Omega^{k-\ell} \equiv 0.$$

and the $\gamma_1, \ldots, \gamma_{\ell-1}$ satisfy (36)

By lemma 5 (i), with $m = k-1$, there is a set of one-forms $\beta_1, \ldots, \beta_{k-1}, \gamma'_\ell, \ldots, \gamma'_{k-1}$ such that:

$$\Omega = \sum_{i=1}^{\ell-1} \gamma_i \wedge \beta_i + \sum_{j=\ell}^{k-1} \gamma'_j \wedge \beta_j,$$

and hence:

$$\Theta \wedge \gamma_1 \wedge \ldots \wedge \gamma_{\ell-1} \wedge \omega \wedge \gamma'_\ell \wedge \beta_\ell \wedge \ldots \wedge \gamma'_{k-1} \wedge \beta_{k-1} \equiv 0.$$

Set $\tau_1 = \omega$, $\tau_2 = \gamma_1$, $\tau_\ell = \gamma_{\ell-1}$, $\tau_{\ell+1} = \gamma'_\ell$, $\ldots$, $\ldots$, $\tau_k = \gamma'_{k-1}$, $\tau_{k+1} = \beta_\ell$, $\ldots$, $\tau_{2k-\ell} = \beta_{k-1}$. By lemma 5 (ii), they are all independent and span $\mathcal{J}_\ell$. Then the preceding equation takes the form:

$$\Theta \wedge \tau_1 \wedge \ldots \wedge \tau_{2k-\ell} \equiv 0. \tag{37}$$

The τ_i, together with $n-(2k-\ell)$ forms $\tau'_{2k-\ell+1}, \ldots, \tau'_n$, span all one-forms. We may therefore write:

$$\Theta = \sum_{i=1}^{2k-\ell} \mu_i \wedge \tau_i + \sum_{2k-\ell<i<j}^{n} f^{ij} \tau'_i \wedge \tau'_j. \tag{38}$$

It then follows from (37) that:

$$\tau_1 \wedge \ldots \wedge \tau_{2k-\ell} \wedge \sum_{2k-\ell<i<j}^{n} f^{ij} \tau'_i \wedge \tau'_j. \equiv 0$$

which means that the f^{ij} all are identically zero. Equation (38) then has the desired form. □

REFERENCES

[1] M. Browning and P.A. Chiappori, *Efficient intra-household allocations: a general characterization and empirical tests*, Econometrica, 66 (1998), pp. 1241–1278.

[2] R. Bryant, S. Chern, R. Gardner, H. Goldschmidt, P. Griffiths, *Exterior Differential Systems*, Springer-Verlag, 1991.

[3] P.A. Chiappori and I. Ekeland, *A convex Darboux theorem*, Annali della Scuola Normale di Pisa, Cl. Sci (4), XXV (1997), pp. 287–297.

[4] A. Mas-Colell, M. Whinston, J. Green, *Microeconomic theory*, Oxford University Press, 1995.

[5] V. Zakalyukin, *Concave Darboux theorem*, CRAS Paris 327, série 1, (1998), pp.633–638.