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TURÁN INEQUALITIES AND ZEROS OF ORTHOGONAL POLYNOMIALS*

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Abstract. We use Turán type inequalities to give new non-asymptotic bounds on the extreme zeros of orthogonal polynomials in terms of the coefficients of their three term recurrence. Most of our results deal with symmetric polynomials satisfying the three term recurrence $p_{k+1} = xp_k - c_k p_{k-1}$, with a nondecreasing sequence $\{c_k\}$. As a special case they include a non-asymptotic version of Máté, Nevai and Totik result on the largest zeros of orthogonal polynomials with $c_k = c\, k^{2\delta}(1 + o(k^{-2/3}))$. Our proof is based on new Turán inequalities which are obtained by analogy with higher order Laguerre inequalities.

Key words. orthogonal polynomials, Turán inequalities, three term recurrence

AMS subject classifications. 33C45

1. Introduction.

Let $\mathcal{P} = \{p_i(x)\}_{k=-1}^{\infty}$ be a family of orthogonal polynomials satisfying the three term recurrence relation

$$b_k p_{k+1}(x) = (x - a_k)p_k(x) - c_k p_{k-1}(x), \quad b_k, c_k > 0, \tag{1}$$

with the initial conditions $p_{-1} = 0$, $p_0 = 1$, and let $x_{1k} < ... < x_{kk}$, be the zeros of $p_k(x)$. We are interested in finding uniform bounds on the extreme zeros, that is an interval $I = [A(k), B(k)]$, such that $A(k) < x_{1k} < x_{kk} < B(k)$, in terms of the coefficients of the recurrence. Such a setting arises naturally if one deals with a family depending on parameters, as in the case of classical Jacobi and Laguerre polynomials, and is seeking for bounds uniform in all the parameters involved. The main aim of this paper is to show that the classical Turán inequality

$$T_k(\mathcal{P}, x) = p_k^2(x) - p_{k-1}(x)p_{k+1}(x) \geq 0, \tag{2}$$

and its analogues (abbreviated as TI in the sequel), provide a convenient tool for tackling the problem. It is known that (2) holds for some families of orthogonal polynomials including Laguerre, Jacobi and some other polynomials (see [21] and the references therein).

At present there are two general approaches to the problem, one is based on the chain sequences [8] and another exploiting the Rayleigh quotient to find the extreme eigenvalues of the corresponding Jacobi matrix (see e.g. [6], [7], [15]). The last one yields the following elegant representation for the extreme zeros.

$$x_{1k} = \min \left(\sum_{i=0}^{k-1} a_i x_i^2 - 2 \sum_{i=0}^{k-2} x_i x_{i+1} \sqrt{b_i c_{i+1}} \right), \tag{3}$$

$$x_{kk} = \max \left(\sum_{i=0}^{k-1} a_i x_i^2 + 2 \sum_{i=0}^{k-2} x_i x_{i+1} \sqrt{b_i c_{i+1}} \right), \tag{4}$$

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where the extrema are taken over all (or only over positive) $x_0, x_1, ..., x_{k-1}$, subjected to $\sum_{i=0}^{k-1} x_i^2 = 1$.
For symmetric polynomials (i.e. when $p_k(-x) = (-1)^k p_k(x)$) and the monic normalization, the case we mainly deal with in this paper, the recurrence (1) can be rewritten as

$$p_{k+1}(x) = xp_k(x) - c_k p_{k-1}(x), \tag{5}$$

and (3), (4) become

$$x_{kk} = -x_{1k} = 2\ \max\left(\sum_{i=0}^{k-2} x_i x_{i+1} \sqrt{c_{i+1}}\right). \tag{6}$$

Thus x_{kk} as a function of the vector $(c_1, c_2, ...)$, possesses many nice properties, e.g. it is clearly subadditive, continuous, increasing with k. This makes transparent many otherwise puzzling questions concerning the behaviour of the extreme zeros. The most striking result obtained via (3), (4) is due to A. Máté, P. Nevai and V. Totik [17] and states that if $c, \delta > 0$, are fixed and $c_k = c^2 k^{2\delta}\left(1 + o(k^{-2/3})\right)$, then

$$x_{kk} k^{-\delta} = 2c - c \cdot 3^{-1/3} (2\delta)^{2/3} i_1 k^{-2/3} + o(k^{-2/3}), \tag{7}$$

where $i_1 = 3.3721...$, is the smallest zero of the Airy function. This is a far reaching generalization of classical bounds in the Hermite case (see e.g. [20]).
If x_{kk} is represented by a series in decreasing powers of k, it is naturally to distinguish between the first and the second order bounds, e.g. $2c$ and $2c(1 - K\delta^{2/3} k^{-2/3})$ in (7), where to maintain the uniformity in k we will allow a weaker constant K than the exact asymptotic one. Whereas the first order bounds can be obtained rather easily, say, by replacing $2x_i x_{i+1}$ in (6) by $x_i^2 + x_{i+1}^2$, or by similar elementary arguments [2],[15], the second order estimates uniform in all the parameters, which are the main subject of this paper, were found only recently for the case of classical orthogonal polynomials [10] - [14].
It is worth also noticing that it is rather easy to extract bounds on the zero x_{kk}^* corresponding to the perturbed recurrence $p_{k+1}(x) = xp_k(x) - c_k(1 + \epsilon_k)^2 p_{k-1}(x)$, provided one knows the result for $\epsilon = 0$. Indeed, $\sum_{i=0}^{k-2} x_i x_{i+1} \leq 1$, and thus, x_{kk}^* is in the interval $(1 \pm \epsilon)\ x_{kk}$, for $|\epsilon_k| < \epsilon < 1$. For example it would be enough to establish (7) for $c_k = c^2 k^{2\delta}$, the general case with the extra factor $(1 + o(k^{-2/3}))$, follows from the above arguments. In this paper we will not state explicitly such obvious generalizations, but the reader should keep them in mind.
Our approach to the problem is quite different from the above two and based on the following observation [9]. Let $f = f(x)$ and $g = g(x)$, $|\deg(f) - \deg(g)| \leq 1$, be two real polynomials with only real interlacing zeros. Suppose also that we know a form $\sum_{i=0}^m A_i(x) f^{m-i} g^i \geq 0$, which is indefinite in f and g viewed as formal variables. Then one can routinely obtain bounds on the extreme zeros of f and g similar to (7) but, of course, with a weaker constant instead of i_1. The existence of such a form is far from being obvious and in fact the main difficulty is to find an appropriate one. For classical orthogonal polynomials, when a second order differential equation is known, one may choose $f = p_k,\ g = p_k'$, and the quadratic form obtained from the Laguerre inequality $f'^2 - ff'' > 0$, or its higher order generalizations [9, 10, 11, 14]. In the discrete case $f = p_k(x+1), g = p_k(x)$, play a similar role [9, 12, 13].
In this paper we will show that the required forms, in particular these giving second

order bounds, may be obtained directly from TI via the three term recurrence. To this end we will establish two new sets of TI, Theorems 9 and 12 below, yielding second order bounds. The following theorem is one of our main results.

THEOREM 1. *Let p_k be a polynomial satisfying (5) and suppose that c_k are nondecreasing. Then*

$$x_{kk}^2 < 4c_{k-1}, \quad k \geq 2. \tag{8}$$

Moreover if $d_i = \frac{c_i - c_{i-1}}{c_i} \geq 0, \;\; c_0 = 0$, satisfy

$$\frac{d_i}{2(1+d_i)} < d_{i+1} < \frac{d_i(1+2\sqrt{d_i}+2d_i)}{1+d_i}\,, \quad i = 1, 2, \ldots; \tag{9}$$

then for $k \geq 2$,

$$x_{kk}^2 < \tag{10}$$

$$4c_k\left(1 - (\frac{d_{k+1}}{36})^{2/3}\ \left((\sqrt{6d_{k+1}+81}+9)^{1/3} - (\sqrt{6d_{k+1}+81}-9)^{1/3}\right)^2\right) <$$

$$4c_k\left(1 - \frac{d_{k+1}^{2/3}}{(2^{1/3}+d_{k+1}^{1/3}\ )^2}\right)\,.$$

Inequality (8) is not new but holds unconditionally and is given here for completeness. In fact slightly better first order bounds $x_{kk} \leq 2\sqrt{c_{k-1}}\cos\frac{\pi}{k+1}$, are known [8].
As a corollary of Theorem 1 we obtain the following uniform version of (7).

THEOREM 2. *Let $c_k = c^2k^{2\delta}$, where $c > 0, \; \delta \geq 0$, are fixed. Then*

$$x_{kk}k^{-\delta} < 2c\sqrt{1 - \frac{\delta^{2/3}}{((k+\frac{1}{2})^{1/3}+\delta^{1/3})^2}}\,, \quad k \geq 2.$$

It would be important to have lower bounds corresponding to (10). A trivial one $x_{kk} > \sqrt{c_{k-1}}$ readily follows from (6) (see also [2], [14] and Lemma 7 below in this connection). One hardly could expect that (10) is sharp for a faster than polynomial rate of growth of c_k. It would be also interesting to obtain similar results for polynomials orthogonal on $[-1, 1]$ and $[0, \infty)$.
The paper is organized as follows. In the next section we survey some known TI which will be used in section 3 for obtaining bounds on the extreme zeros. In the last section we establish two new sets of TI and give second order bounds for a vast class of orthogonal polynomials satisfying (5) with a nondecreasing sequence c_k.

2. Turán Inequalities. The reason for study TI in the theory of orthogonal polynomials is that they have a few quite important applications. For example, under appropriate restrictions $T_k(\mathcal{P}, x)$ converges uniformly on compact subsets of $(-1, 1)$ to $\frac{2\sqrt{1-x^2}}{\pi\alpha'(x)}$, where $\alpha'(x)$ is the absolutely continuous part of the corresponding orthogonality measure on $[-1, 1]$. The existing technique enables one to obtain similar limiting expressions for higher order analogues of TI as well, e.g. the right hand side

of inequality (18) below tends to $\frac{8(1-x^2)}{\pi^2(\alpha'(x))^2}$, see [5] and the references therein.
Today there are two almost independent theories related to TI, the first dealing with the validity of (2) for different families of orthogonal polynomials and goes back to the pioneer work of Turán [22]. Another one is motivated by the theory of entire functions, in particular by the Riemann hypothesis. In the last case much more precise higher order generalization of (2) are known, but the orthogonal polynomials must have a generating function of a very special type. A survey of this theory and the relevant references can be found in [3], [4], [5].
Progress in the first direction was recently summarized by R. Szwarc [21], who established the following result.

THEOREM 3. *(i) Let $\mathcal{P}$ be a family of symmetric polynomials orthogonal on* $[-1,1]$, *where three term recurrence (1) is normalized by*

$$b_k + c_k = 1,\ c_0 = 0,\ c_{k+1} > 0,\ b_k > 0, \tag{11}$$

that is by $p_k(1) = 1,\ p_k(-1) = (-1)^k$. *Then*

$$T_k(\mathcal{P}, x) \geq 0, \quad |x| \leq 1, \tag{12}$$

provided one of the following conditions holds
(i_a) c_k *is nondecreasing and* $c_k \leq \frac{1}{2}$;
(i_b) c_k *is nonincreasing and* $c_k \geq \frac{1}{2}$.
(ii) Let $\mathcal{P}$ be a family of polynomials orthogonal on $[0,\infty)$, *which are normalized by* $p_k(0) = 1$, *with the three term recurrence*

$$xp_k(x) = -b_k p_{k+1}(x) + (b_k + c_k)p_k(x) - c_k p_{k-1}(x), \tag{13}$$

where $b_0 = 1, c_0 = 0, b_k > 0,\ c_{k+1} > 0$. *Suppose* b_k *and* c_k *are nondecreasing, then*

$$T_k(\mathcal{P}, x) \geq 0, \quad x \geq 0, \tag{14}$$

provided one of the following conditions holds
(ii_a) $c_k \leq b_k,\ \ c_k - c_{k-1} \geq b_k - b_{k-1}$;
(ii_b) $c_k \geq b_k,\ \ c_k - c_{k-1} \leq b_k - b_{k-1}$.

R. Szwarc [21] also obtained similar yet rather technical conditions which guarantee the validity of (2) for a general nonsymmetric polynomial with the orthogonality measure supported on $[-1,1]$. For the sake of simplicity we did not state them here. It seems nothing is known in the case of nonsymmetric polynomials orthogonal on the whole real axis. On the other hand the case of symmetric polynomials orthogonal on $(-\infty,\infty)$ is almost trivial and was given in [1].

THEOREM 4. *Let $\mathcal{P}$ be a family of orthogonal polynomials satisfying (5) with a nondecreasing sequence* $\{c_k\}_{k=1}^{\infty}$. *Then* $T_k(\mathcal{P}, x) \geq 0$.

Proof. The result follows by $T_0 = 1$, and the following easy to check identity

$$T_{k+1}(\mathcal{P}, x) = c_k T_k(\mathcal{P}, x) + (c_{k+1} - c_k)p_k^2(x). \tag{15}$$

□

A few higher order generalization of (2) are known. To state them we shall consider the Laguerre-Pólya class of functions which consists of real polynomials with only real zeros and real entire functions

$$F(z) = ce^{-\alpha z^2 + \beta z} z^r \prod_i (1 - z/z_i) e^{z/z_i},$$

where $\alpha \geq 0,\ c, \beta$, are real, r is a nonnegative integer and $\sum_i z_i^{-2}$ is convergent. Suppose now that a family $\mathcal{U}$ of real functions $u_k = u_k(x),\ k = 0, 1, ...,$ has for some values of x a generating function of the Laguerre-Pólya class,

$$\sum_{k=0}^{\infty} u_k \frac{z^k}{k!} = F(z). \tag{16}$$

An instructive example is provided by the binary Krawtchouk polynomials $u_k = K_k^n(x)$, n is a positive integer, having the generating function

$$\sum_{i=0}^{\infty} K_i^n(x) z^i = (1-z)^x (1+z)^{n-x},$$

which satisfy (16) for $x = 0, 1, ..., n$. Another examples are given by ultraspherical $C_k^{(\lambda)}$, Hermite H_k, and Laguerre $L_k^{(\alpha)}$ polynomials [5], with

$$u_k = \frac{C_k^{(\lambda)}(x)}{C_k^{(\lambda)}(1)}, \ \ \lambda > -\frac{1}{2}, \ \ -1 \leq x \leq 1;$$

$$u_k(x) = H_k(x), \ \ -\infty < x < \infty;$$

$$u_k = \frac{L_k^{(\alpha)}(x)}{L_k^{(\alpha)}(0)}, \ \ \alpha > -1, \ \ x \geq 0.$$

In the following theorem the first part belongs to M. Patrick [19] and the second one to J. Maříc [16] (the extension of it to the whole Laguerre-Pólya class is due to D.K. Dimitrov [5]).

Theorem 5. *For those values of x for which (16) holds*

$$T_k^{(m)}(\mathcal{U}, x) = \frac{1}{2} \sum_{j=0}^{2m} (-1)^{j+m} \binom{2m}{j} u_{k-m+j} u_{k+m-j} \geq 0, \ \ m = 0, 1, \ldots, \tag{17}$$

and

$$S_k(\mathcal{U}, x) = 4(u_k^2 - u_{k-1}u_{k+1})(u_{k+1}^2 - u_k u_{k+2}) - (u_k u_{k+1} - u_{k-1} u_{k+2})^2 \geq 0. \tag{18}$$

Notice that $T_k^{(1)}(\mathcal{U}, x)$ is just $T_k(\mathcal{U}, x)$. The inequality (18) can be viewed as a refinement of (2) and is intimately connected with so-called Newton inequalities [18]. To apply (17),(18) to orthogonal polynomials it would be important to to restate the condition for their validity in terms of the coefficients of (1). For $T_k^{(2)}$ and recurrence (5) this will be accomplished in the last section. The corresponding question for S_k remains open.

3. Extreme zeros. In this section we give some first order bounds on the extreme zeros which can be deduced from Theorems 3 and 4. We also show that $S_k(\mathcal{U}, x)$ (which gives a fourth degree form), yields second order bounds for Hermite polynomials. The inequality $T_k^{(2)} \geq 0$, will be considered in more details in the next section. First, we describe simple geometric arguments which enable one to deduce bounds on the extreme zeros from a given form. Let $p = p(x),\ q = q(x)$ be two real polynomials, $\deg(p) = k \geq 2,\ \deg(q) = k-1$, with only real interlacing zeros $x_1, ..., x_k$, and $y_1, ..., y_{k-1}$, respectively, $x_i < y_i < x_{i+1}$. Suppose that for $x \in (M, N),\ M < x_1 < x_k < N$, where M and N can be finite or infinite, there exists a nonnegative form

$$\sum_{i=0}^{m} A_i(x) q^i p^{m-i} \geq 0, \quad m \geq 2, \quad even, \tag{19}$$

where $A_i(x)$ are certain functions defined in all the points of (M, N). Introducing the function $t = t(x) = q/p$, we rewrite it as

$$Q(t, x) = \sum_{i=0}^{m} A_i(x) t^i \geq 0.$$

Since $\lim_{x \to \pm\infty} t(x) = 0$, and the zeros of p and q are interlacing then $t(x)$ consists of two hyperbolic B_0, B_k and $k-1$ cotangent-shaped decreasing branches $B_1, ..., B_{k-1}$, where B_i is defined for $x_i < x < x_{i+1},\ x_0 = -\infty,\ x_{k+1} = \infty$. A function $t_0 = t_0(x)$ will be called an $(i, j) - transversal$ if

(i) t_0 is continuous on $[M, N]$,

(ii) t_0 intersects each of the branches B_l, of t for $i \leq l \leq j$.

(iii) there is an open interval $I \subset [M, N]$ such that $Q(t_0, x) \leq 0$ iff $x \in [M, N] \backslash I$.

Obviously, if an $(i, j)-$transversal exists then $[x_{i+1}, x_j] \subset I$, and to get bounds on x_{i+1}, x_j one needs just to find the extreme roots of the equation $Q(t_0, x) = 0$, on $[M, N]$. Note that any continuous function intersects all the cotangent-shaped branches $B_1, ..., B_{k-1}$, and is, if (iii) holds, a $(1, k-1)-$transversal, thus giving bounds on x_2 and x_{k-1}. For example, $t_0 = cx$, is a $(0, k)-$transversal for $c > 0$, and a $(1, k-1)-$transversal for $c \leq 0$, provided it satisfies (iii). As we will see, in many cases the condition (iii) is automatically fulfilled and moreover, a naive choice of t_0 as a solution of $\frac{\partial Q(t,x)}{\partial t} = 0$, that is as the function providing the minimum to $Q(t, x)$, does work. The situation is especially simple for quadratic forms, the case we mainly exploit here. Then $Q(t, x) = A_0 + A_1 t + A_2 t^2$, and one may try $t_0 = -\frac{A_1}{2A_2}$, with $Q(t_0, x) = \frac{4A_0A_2 - A_1^2}{4A_2}$.

In the rest of the paper we will use $t = p_{k-1}/p_k$, and with one explicitly stated exception, t_0 will be chosen as a solution of $\frac{\partial Q(t,x)}{\partial t} = 0$.

The simplest way to obtain the required quadratic form $Q(t, x)$ for orthogonal polynomials is to express p_{i-1}, p_i, and p_{i+1} in $T_i,\ |k - i| \leq 1$, via p_{k-1}, p_k by the three term recurrence. In this case one gets three (slightly different) bounds on the zeros, we present just one of them in the theorem below. But already for $T_{k\pm2}$, the above expression for t_0 may have singularities and our arguments are not applicable without certain restrictions on the coefficients (see Lemma 8 below). Using Theorems 3 and 4 to guarantee the corresponding TI we get the following first order bounds.

THEOREM 6. (*i*) *Let p_k be a symmetric polynomial satisfying (5) and suppose that c_k are nondecreasing. Then*

$$x_{kk} < 2\sqrt{c_{k-1}}, \quad k \geq 2.$$

(ii) Let p_k be a symmetric polynomial orthogonal on $[-1,1]$ satisfying (1) and (11). Then

$$|x_{ik}| < 2\sqrt{b_{k-1}c_{k-1}},$$

where $i = 1, ..., k$, if c_k is nondecreasing and $c_k \le \frac{1}{2}$; and $i = 2, ..., k-1$, if c_k is nonincreasing and $c_k \ge \frac{1}{2}$.
(iii) Let p_k be a polynomial orthogonal on $[0,\infty)$ satisfying (13). If b_k and c_k are nondecreasing, then

$$(\sqrt{b_k} - \sqrt{c_k}\,)^2 < x_{2,k} < x_{k,k} < (\sqrt{b_k} + \sqrt{c_k}\,)^2,$$

provided $c_k \le b_k$, $c_k - c_{k-1} \ge b_k - b_{k-1}$; and

$$(\sqrt{b_k} - \sqrt{c_k}\,)^2 < x_{1,k} < x_{k,k} < (\sqrt{b_k} + \sqrt{c_k}\,)^2,$$

provided $c_k \ge b_k$, $c_k - c_{k-1} \le b_k - b_{k-1}$.

Proof. (i) By (5) we get

$$Q(t,x) = c_{k-1}p_k^{-2}T_{k-1}(\mathcal{P},x) = 1 - xt + c_{k-1}t^2 \ge 0.$$

In our case $M = N = \infty$, and $t_0 = \frac{x}{2c_{k-1}}$, is clearly a $(0,k)-$transversal. Finally $4c_{k-1}^2 Q(t_0,x) = 4c_{k-1} - x^2$, and the result follows.
(ii) Observe that p_k does not depend on b_k, c_k. Hence we may assume that $b_k = b_{k-1}$, $c_k = c_{k-1}$. Substituting p_{k+1} from (1) (in our case $a_k = 0$), we have for $[M,N] = [-1,1]$,

$$Q(t,x) = b_k p_k^{-2} T_k(\mathcal{P},x) = b_k - xt + c_k t^2 \ge 0,$$

with $t_0 = \frac{x}{2c_k}$, and $4b_kc_kQ(t_0,x) = 4b_kc_k - x^2$. Obviously, t_0 is a $(1,k-1)-$transversal. Finally, using the normalization $p_i(1) = 1$, $p_i(-1) = (-1)^i$, and $b_k + c_k = 1$, one can check that $t_0(-1) \le t(-1)$, and $t_0(1) \ge t(1)$, only if $c_k \le \frac{1}{2}$. Thus, for $c_k \le \frac{1}{2}$, t_0 intersects all the branches of t and hence is a $(0,k)-$transversal.
(iii) Substituting p_{k+1} from (13) we obtain with $[M,N] = [0,\infty]$,

$$b_k p_k^{-2} T_k(\mathcal{P},x) = b_k - (b_k + c_k - x)t + c_k t^2 \ge 0.$$

This yields $t_0 = \frac{b_k + c_k - x}{2c_k}$, which is at least a $(1,k-1)-$transversal, and

$$4b_kc_k\Delta(x) = \Big(x - (\sqrt{b_k} - \sqrt{c_k}\,)^2\Big)\Big((\sqrt{b_k} + \sqrt{c_k}\,)^2 - x\Big) \ge 0. \tag{20}$$

Notice that the polynomials here are normalized so that the sign of the leading coefficient of p_k is $(-1)^k$, and so the branches of $t(x)$ are upside-down in comparison with the previous cases. Thus, to guarantee the intersection of t_0 with B_k one should check

$$\lim_{x->\infty} t(x) = -1 > \lim_{x->\infty} t_0(x) = -\infty,$$

hence t_0 is a $(1,k)-$transversal. On the other hand to be a $(0,k)-$transversal it should satisfy $t_0(0) = \frac{b_k+c_k}{2c_k} \le p_k(0) = 1$. This is the case only if $b_k \le c_k$. This completes the proof. ∎

Note that similar bounds can be obtained for nonsymmetric polynomials orthogonal on a finite interval. The corresponding TI are given in [21]. The restrictions we have to impose in cases (ii) and (iii) to obtain bounds on all the zeros reflect the real situation. An easy example is provided by ultraspherical and Laguerre polynomials with small parameters. Roughly speaking, the reason why the obtained bounds exclude one or both extreme zeros is that these zeros are too close to the ends of the interval of orthogonality and the used inequalities do not have enough precision to distinguish between them.

The following Lemma was suggested by one of the referees and together with (8) provides an answer to the following question arising naturally in connection with the above theorem.

Suppose that p_k satisfies (5), what is the maximal rate of growth of c_k such that

$$\lim_{k\to\infty} \frac{x_{kk}}{2\sqrt{c_{k-1}}} = 1 \ ? \tag{21}$$

LEMMA 7. *Suppose that $\frac{c_{k+1}}{c_k} \to 1$, $c_k \to \infty$ and c_k form a nondecreasind sequence, then (21) holds.*

Proof. For any $n = n(k)$, $0 \le n \le k-2$, one has by (6),

$$\frac{x_{kk}}{2} \ge \max\left(\sum_{i=n}^{k-2} x_i x_{i+1}\sqrt{c_{i+1}}\right) \ge \sqrt{c_{n+1}} \max\left(\sum_{i=n}^{k-2} x_i x_{i+1}\right) =$$

$$\sqrt{c_{n+1}} \cos\frac{\pi}{k-n+1} \, ,$$

where the maximum is taken over all sequences such that $\sum |x_i|^2 \le 1$.
On the other hand it is possible to find a sequence $n(k)$ such that $k - n(k) \to \infty$ and $\frac{c_k}{c_{n(k)}} \to 1$. Then

$$\frac{x_{kk}}{2\sqrt{c_{k-1}}} \to 1.$$

□

The following lemma provides some additional information in this direction and may be of independent interest.

LEMMA 8. *Let p_k be a symmetric polynomial satisfying (5) and suppose that $\frac{3}{4}c_k < c_{k-1} \le c_k$. Then*

$$x_{kk} < 2\sqrt{c_{k-2}}, \quad k \ge 3.$$

Proof. We consider

$$Q(t,x) = c_{k-2}c_{k-1}^2 p_k^{-2} T_{k-2}(\mathcal{P}, x) =$$

$$(c_{k-1}^2 - (c_{k-1} - c_{k-2})x^2)t^2 - x(2c_{k-2} - c_{k-1})t + c_{k-2} \ge 0.$$

In view of Theorem 6, (i) we can choose $[M,N]=[-2\sqrt{c_{k-1}},2\sqrt{c_{k-1}}\,]$. Then

$$t_0=\frac{x(2c_{k-2}-c_{k-1})}{2(c_{k-1}^2-(c_{k-1}-c_{k-2})x^2)},$$

By the assumption $\frac{3}{4}c_k<c_{k-1}\le c_k$, therefore t_0 is a continuous function on $[M,N]$ and thus a $(0,k)-$transversal. Finally

$$Q(t_0,x)=\frac{(4c_{k-2}-x^2)c_{k-1}^2}{4(c_{k-1}^2-(c_{k-1}-c_{k-2})x^2)},$$

and the result follows. □

Similar but more involved calculations with T_{k-3} instead of T_{k-2} yield $x_{kk}<2\sqrt{c_{k-3}}$, provided $\frac{5+\sqrt{5}}{8}c_k<c_{k-1}\le c_k$. We omit the details.
Now we will show that using (18), which yields a fourth degree form, one can obtain much sharper second order bounds. We will consider the simplest case of monic Hermite polynomials H_k defined by (5) with $c_k=k/2$. The corresponding asymptotic for x_{kk} given by (7) is

$$x_{kk}=\sqrt{2k}-2^{-1/2}3^{-1/3}i_1k^{-1/6}\approx\sqrt{2k}-1.65\cdot k^{-1/6}.$$

Putting $u_k=H_k,\ t=H_{k-1}/H_k$, in (18), we get

$$Q(t,x)=4S_k(\mathcal{P},x)u_k^{-4}=k^2(2k-x^2)t^4-2kx(1+4k-2x^2)t^3+$$

$$(4(k+x^2)(2k+1-x^2)-1)t^2-4x(4k+3-2x^2)t+4(2k+2-x^2)\ge 0.$$

Choosing the same $t_0=\frac{x}{k}$, as for the case of the quadratic form given by T_k and calculating $Q(t_0,x)$ we get that all the zeros of p_k satisfy

$$8k^2(k+1)-(6k+1)(2k+1)x^2+(6k+2)x^4-x^6\ge 0.$$

This equation has only one positive root x which gives the required bound,

$$x=\frac{(m^2-1)^2\sqrt{m^4+4m^2+1}}{3\sqrt{3}m^3}=\sqrt{2k}-2^{-7/6}k^{-1/6}+O(k^{-5/6}),$$

where $m=2^{-1/6}(\sqrt{27k+2}+\sqrt{27k}\ \ )^{1/3}$. Thus we get $2^{-7/6}\approx 4/9$, instead of 1.65. Notice that the result can be slightly improved by solving the system $Q(t,x)=0,\ \frac{\partial Q(t,x)}{\partial t}=0$, exactly. This yields $\sqrt{\frac{4k-3k^{1/3}+1}{2}}\approx\sqrt{2k}-0.53k^{-1/6}$, we omit the details.

4. Bounds from higher order Turán inequalities. In this section we will consider only the symmetric case (5). For convenience we put $c_0=0$. First, we will establish sufficient conditions for the validity of the inequality

$$T_k^{(2)}=T_k^{(2)}(\mathcal{P},x)=3p_k^2-4p_{k-1}p_{k+1}+p_{k-2}p_{k+2}\ge 0,$$

in terms of the recurrence (5) and derive the corresponding bounds on the extreme zeros. Next, we will show how to modify $T_k=T_k^{(1)}$, to obtain second order bounds for a vast class of nondecreasing sequences c_k. In particularly, we will prove Theorems 1 and 2.

THEOREM 9. *Let* $\{c_k\}_{k=1}^{\infty}$, *be a nondecreasing positive sequence such that*

$$c_{k-1} - 3c_k + 3c_{k+1} - c_{k+2} \geq 0. \tag{22}$$

Then for $k \geq 2$,

$$T_k^{(2)}(\mathcal{P}, x) = 3p_k^2 - 4p_{k-1}p_{k+1} + p_{k-2}p_{k+2} \geq 0. \tag{23}$$

Proof. We have the following directly checked identity

$$T_{k+1}^{(2)} = c_{k-1}T_k^{(2)} + (c_{k+2} + 3c_k - 4c_{k-1})T_k + (c_{k-1} - 3c_k + 3c_{k+1} - c_{k+2})p_k^2. \tag{24}$$

Now the result follows by the induction on k and
$T_2^{(2)} = (c_0 - 3c_1 + 3c_2 - c_3)x^2 + 3c_1^2 + c_1c_3 > 0.$ □

REMARK 1. *If we set* $c_k = \sum_{i=1}^k \delta_i$, *then the conditions (22) can be rewritten as* $\delta_i \geq 0$, $\delta_{i-1} - 2\delta_i + \delta_{i+1} < 0$, *i.e.* δ_i *should be a nonnegative concave function of* i.

For orthogonal polynomials Theorem 9 yields

THEOREM 10. *Let* c_k *satisfy the conditions (22) of Theorem 9. Suppose also that the following equation*

$$F(x) = x^6 - 2(4c_{k-1} + c_k + c_{k+1})x^4 + (16c_{k-1}^2 + (c_k + c_{k+1})^2 + \tag{25}$$

$$4c_{k-1}(5c_k + 2c_{k+1}))x^2 - 16c_kc_{k-1}(3c_{k-1} + c_{k+1}) = 0,$$

has only two real roots. Then all the zeros of p_k *are confined between them.*

Proof. Rewriting $T_k^{(2)}$ in terms of t we have

$$Q(t, x) = c_{k-1}p_k^{-2}T_k^{(2)} =$$

$$3c_{k-1} + c_{k+1} - x^2 - x(4c_{k-1} - c_k + c_{k+1} - x^2)t + c_k(4c_{k-1} - x^2)t^2 \geq 0.$$

Clearly, $x^2 < 4c_{k-1}$, and we can choose $[M, N] = [-2\sqrt{c_{k-1}}, 2\sqrt{c_{k-1}}\,]$. Now

$$t_0 = \frac{x(4c_{k-1} - c_k + c_{k+1} - x^2)}{2c_k(4c_{k-1} - x^2)},$$

is a $(0, k)-$ transversal and any zero x satisfies

$$Q(t_0, x) = -\frac{F(x)}{4c_k(4c_{k-1} - x^2)} > 0,$$

yielding the required result. □

To show that (25) indeed gives second order bounds we again consider the monic Hermite polynomials $H_k(x)$. The conditions of Lemma 9 are fulfilled as $c_k = k/2$. Solving (25) we get

$$x_{kk} < \sqrt{2k - \frac{(1 + (\sqrt{k} + \sqrt{k-1}\,)^{2/3})^2}{2(\sqrt{k} + \sqrt{k-1}\,)^{2/3}}} = \sqrt{2k-1} - 2^{-5/3}(2k-1)^{-1/6} + O(k^{-5/6}).$$

Now we will establish a new TI which is valid for a vast class of sequences c_k. Its form is motivated by analogy with the Hermite-Poulain theorem. Namely, one can strengthen, say, Laguerre inequality $f'^2 - ff'' \geq 0$, which holds for real polynomials with only real zeros, by the substitution $f + \lambda f'$ instead of f for an appropriate value of λ. This does not affect the inequality as $f + \lambda f'$ is again a polynomial with only real zeros. Similar method may be applied in the discrete case as well [12]. In our case we use the following transformation.

Given a family $\mathcal{P} = \{p_k\}_{k=-1}^{\infty}$, $p_{-1} = 0$, $p_0 = 1$, of orthogonal polynomials satisfying (5), define $\Delta\mathcal{P} = \{q_k\}_{k=0}^{\infty}$, by $q_k(x) = p_{k+1}(x) - c_k p_{k-1}(x)$. We have the following explicit form

$$p_k^{-2} T_k(\Delta\mathcal{P}, x) = c_k(4c_k - x^2)t^2 - x(2c_{k+1} + 2c_k - x^2)t + 4c_{k+1} - x^2.$$

The following identity can be checked directly.

Lemma 11.

$$T_{k+1}(\Delta\mathcal{P}, x) = c_k T_k(\Delta\mathcal{P}, x) + 2c_k\mu_k T_k(\mathcal{P}, x) + G,$$

where

$$p_k^{-2} G = 2c_k^2(2c_{k+2} - 2c_k - \mu_k)t^2 - 2xc_k(3c_{k+2} - 2c_{k+1} - c_k - \mu_k)t +$$

$$x^2(2c_{k+2} - 3c_{k+1} + c_k) + 4c_{k+1}(c_{k+1} - c_k) - 2c_k\mu_k,$$

$\mu_k = 2(c_{k+1} - c_k) + \frac{1}{2}\left(\sqrt{c_{k+1} - c_k} - \sqrt{2c_{k+2} - 3c_{k+1} + c_k}\,\right)^2.$

Theorem 12. *Let $\{c_k\}_{k=1}^{\infty}$ be a nondecreasing sequence satisfying for $k = 1, 2, ...,$ the following conditions*

$$2c_{k+2} - 3c_{k+1} + c_k \geq 0, \tag{26}$$

$$(c_{k+1} - c_k)\left(\sqrt{c_{k+1} - c_k} + \sqrt{2c_{k+2} - 3c_{k+1} + c_k}\,\right) \geq \sqrt{c_k}\,|c_{k+2} - 2c_{k+1} + c_k|. \tag{27}$$

Then

$$T_k(\Delta\mathcal{P}, x) \geq 0. \tag{28}$$

Proof. As $T_k(\mathcal{P}, x) \geq 0$, by $c_{i+1} \geq c_i$, and $T_1(\Delta\mathcal{P}, x) = (2c_2 - 3c_1)x^2 + 4c_1^2 > 0$, by (26) it is left to show that $G \geq 0$. For, consider

$$H = p_k^{-2} G = 2c_k^2(2c_{k+2} - 2c_k - \mu_k)t^2 - 2xc_k(3c_{k+2} - 2c_{k+1} - c_k - \mu_k)t +$$

$$x^2(2c_{k+2} - 3c_{k+1} + c_k) + 4c_{k+1}(c_{k+1} - c_k) - 2c_k\mu_k,$$

The coefficient at t^2 is positive, hence it is left to check that the discriminant of this quadratic in t is nonpositive, what yields

$$c_k(c_{k+2} - 2c_{k+1} + c_k)^2 - (c_{k+1} - c_k)^2\left(\sqrt{c_{k+1} - c_k} + \sqrt{2c_{k+2} - 3c_{k+1} + c_k}\,\right)^2 \leq 0,$$

and (26), (27) follow. □

Practically the conditions of the above theorem are much less restrictive than those of Theorem 9. Yet formally (22) does not follow from (26),(27), as the example $c_k = k^2 + c$, shows.
The conditions (26) and (27) are rather complicated but can be simplified by the substitution $c_k/c_{k-1} = 1 + d_k,\ d_k > 0$, giving respectively

$$d_{k+1} \geq \frac{d_k}{2(1+d_k)}, \tag{29}$$

$$d_k(\sqrt{d_k} + \sqrt{2d_k d_{k+1} + 2d_{k+1} - d_k}\ ) \geq |d_k d_{k+1} + d_{k+1} - d_k|. \tag{30}$$

More practical criteria are given in the following Lemma.

Lemma 13. *The conditions of Theorem 12 hold if* $d_k > 0$, *and*

$$\frac{d_k}{2(1+d_k)} < d_{k+1} < \frac{d_k(1 + 2\sqrt{d_k} + 2d_k)}{1+d_k}, \tag{31}$$

Proof. Putting in (29),(30) $d_{k+1} = \frac{d_k(1+2y+2y^2)}{1+d_k},\ y \geq -1/2$, gives

$$1 + 2y + 2y^2 \geq \frac{1}{2}, \quad 4d_k^2(1+y)^2(d_k - y^2) > 0,$$

and the result follows. □

Now we are in the position to prove (10) and thus Theorem 1. This is accomplished in the following two lemmas.

Lemma 14. *Suppose that* $d_i = \frac{c_i - c_{i-1}}{c_i} \geq 0$, *satisfy (29),(30). Then*

$$x_{kk}^2 < 4c_k \left(1 - 6^{-4/3} d_{k+1}^{2/3}\ \left((v+9)^{1/3} - (v-9)^{1/3}\right)^2\right)$$

where $v = \sqrt{6d_{k+1} + 81}$.

Proof. Let $[M, N] = [-2\sqrt{c_k}, 2\sqrt{c_k}\ ]$, and consider $Q(t,x) = p_k^{-2} T_k(\Delta\mathcal{P}, x)$ given by (28). Then we find

$$t_0 = \frac{x(2c_{k+1} + 2c_k - x^2)}{2c_k(4c_k - x^2)},$$

and t_0 is a $(0,k)-$transversal. Calculating $Q(t_0, x)$ we conclude that all the zeros of p_k satisfy

$$x^6 - 4(c_{k+1} + 2c_k)x^4 + 4(c_{k+1} + c_k)(c_{k+1} + 5c_k)x^2 - 64c_k^2 c_{k+1} < 0 \tag{32}$$

The corresponding equation has only two real roots giving the required bounds, namely

$$x^2 = 4c_k \left(1 - 6^{-4/3} d_{k+1}^{2/3}\ \left((v+9)^{1/3} - (v-9)^{1/3}\right)^2\right).$$

To show that there are no other roots we calculate the discriminant of (32) which is

$$-2^{28}c_{k+1}c_k^4(c_{k+1}-c_k)^8(2c_{k+1}+25c_k)^2.$$

As it does not change the sign, provided $c_{k+1} > c_k > 0$, the number of real zeros is the same for any such a choice of c_k, c_{k+1}. Choosing $c_k = 1, c_{k+1} = 2$, we obtain the test equation $x^6 - 16x^4 + 84x^2 - 128 = 0$, having only two real roots. □

LEMMA 15. *With the conditions of Lemma 14*

$$x_{kk}^2 < 4c_k\left(1 - \frac{d_{k+1}^{2/3}}{(2^{1/3}+d_{k+1}^{1/3})^2}\right). \tag{33}$$

Proof. It is enough to show that

$$(v+9)^{1/3} - (v-9)^{1/3} > \frac{6^{2/3}}{2^{1/3}+d_{k+1}^{1/3}}.$$

This inequality is transformed into an obvious one, $y > \frac{y}{1+y-y^3}$, by the substitution $d_{k+1} = \frac{2(1-y^3)^3}{y^3}$, $0 < y \leq 1$. □

REMARK 2. *The result of Theorem 1 can be strengthened if one expresses $T_k(\Delta\mathcal{P}, x)$ in terms of $t^* = p_k/p_{k+1}$ instead of t as it has been done in Lemma 8. This yields a bound similar to (10) with c_{k-1} and d_k instead of c_k and d_{k+1} respectively. We don't give the details as the resulting expression is more complicated than (10) and gives only a marginal improvement for c_k with less than exponential rate of growth.*

Finally, Theorem 2 follows from (33) with $d_{k+1} = (1+\frac{1}{k})^{2\delta} - 1$. For we observe that $\frac{d_{k+1}^{1/3}}{2^{1/3}+d_{k+1}^{1/3}}$ is an increasing function in d_{k+1}. Now the result follows by applying the elementary inequality

$$(1+\frac{1}{k})^{2\delta} - 1 \geq \frac{2\delta}{k+\frac{1}{2}}, \quad \delta \geq 0.$$

Moreover, $k + \frac{1}{2}$ may be replaced by k for $\delta \geq \frac{1}{2}$.

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