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003

HIGHER INTEGRABILITY FOR SOLUTIONS TO A SYSTEM OF CRITICAL ELLIPTIC PDE*

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Abstract. We give new estimates for a critical elliptic system introduced by Rivière-Struwe which generalises PDE solved by (almost) harmonic maps from a Euclidean ball $B_1 \subset \mathbb{R}^n$ into closed Riemannian manifolds $\mathcal{N} \hookrightarrow \mathbb{R}^m$. Solutions $u : B_1 \to \mathbb{R}^m$ take the form

$$-\Delta u^i = \Omega^i_j.\nabla u^j$$

where Ω maps into antisymmetric $m \times m$ matrices with entries in $\mathbb{R}^n$. Here Ω and ∇u belong to a Morrey space which makes the PDE critical from a regularity perspective. We use the Coulomb frame method employed by Rivière-Struwe along with the Hölder regularity already acquired therein, coupled with an extension of a Riesz potential estimate, in order to improve the known regularity and estimates for solutions u. These methods apply when $n = 2$ thereby re-proving the full regularity in this case using Coulomb gauge methods. Moreover they lead to a self contained proof of the local regularity of stationary harmonic maps in high dimension.

Key words. Harmonic maps, regularity for critical elliptic systems, Riesz potential estimates for Morrey-Hardy spaces.

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1. Introduction. The system of PDE under consideration here, introduced by Rivière [Riv07] and Rivière-Struwe [RS08], generalises many PDE present in the study of critical geometric problems. It has 'hidden' regularity properties by virtue of the anti-symmetry of the term 'Ω', and one can both recover and somewhat improve the regularity theory for the aforementioned geometric PDE via a much more general theory. For instance, consider weakly harmonic maps $u \in W^{1,2}(B_1, \mathcal{N})$ from the Euclidean ball $B_1 \subset \mathbb{R}^n$ into a closed Riemannian manifold $\mathcal{N} \hookrightarrow \mathbb{R}^m$. These are defined to be critical points with respect to outer variations of

$$E(u) := \frac{1}{2}\int_{B_1} |\nabla u|^2,$$

meaning that maps $u \in W^{1,2}(B_1, \mathcal{N})$ form the natural admissible function space for this problem. It is known that u weakly solves a PDE of the form

$$-\Delta u = A(u)(\nabla u, \nabla u) \tag{1}$$

where A is the second fundamental form of the embedding $\mathcal{N} \hookrightarrow \mathbb{R}^m$. Since $\nabla u \in L^2$ one can estimate the right hand side of (1) in L^1 - A is a bounded bilinear form. From here the L^1 theory for the Laplacian is not enough to begin any kind of bootstrapping argument unless $n = 1$. For instance, when $n = 2$ standard theory gives estimates on ∇u in the Lorentz space $L^{2,\infty}$, a space slightly larger than L^2. For $n \geq 3$ the analogous estimates give control on ∇u in $L^{\frac{n}{n-1},\infty}$ which is much worse than being in L^2. Indeed the regularity theory for weakly harmonic maps in higher dimensions is doomed to fail since there exist 'nowhere continuous' weakly harmonic maps whenever $n \geq 3$ [Riv92]. In two dimensions however, all weakly harmonic maps are smooth,

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which one can prove using a moving frame technique due to Hélein [Hél91]. The same technique can be used to prove partial regularity in higher dimensions ([Eva91] for $\mathcal{N} = S^{m-1}$ and [Bet93] for general closed $\mathcal{N}$) where one can prove smoothness under the condition that $\|\nabla u\|_{L^2}$ decays sufficiently quickly on small balls - i.e. it lies in a Morrey space. Roughly speaking, these techniques use the fact that under an appropriate frame the 'bad' L^1 terms can be replaced by better $\mathcal{H}^1$ terms. The space $\mathcal{H}^1$ is a Hardy space - functions which are in L^1 but have additional cancellation properties - see below for a definition.

Given $g \in L^1_{loc}(O)$ and $O \subset \mathbb{R}^n$, we say that g lies in the Morrey space $M^{p,\beta}(O)$ if the following function

$$M_\beta[|g|^p](x) := \sup_r r^{-\beta} \int_{B_r(x)\cap O} |g|^p$$

is bounded; with a norm given by $\|g\|^p_{M^{p,\beta}(O)} := \|M_\beta[|g|^p]\|_{L^\infty(O)}$. See appendix A for more details. We have followed [Gia83] in terms of our choice of indices, though we use $M^{p,\beta}$ rather than $L^{p,\beta}$ so as to avoid confusion with Lorentz spaces.

From now on we will denote $M_n[g] = M[g]$ giving the usual maximal function. We now define a more refined version for $g : \mathbb{R}^n \to \mathbb{R}$: let $\phi \in C^\infty_c(B_1)$ with $\|\nabla\phi\|_{L^\infty} \leq 1$ and $\phi_t(x) := t^{-n}\phi(\frac{x}{t})$. We set

$$g_*(x) = \sup_{0<t<\infty} |\phi_t * g(x)|$$

and say that $g \in \mathcal{H}^1(\mathbb{R}^n)$ if $g_* \in L^1(\mathbb{R}^n)$ with norm $\|g\|_{\mathcal{H}^1(\mathbb{R}^n)} = \|g_*\|_{L^1(\mathbb{R}^n)}$. It can be shown that this norm is independent of the choice of ϕ up to a constant - see [Sem94]. The related local Hardy space h^1 is defined similarly; we require that

$$g_{\tilde{*}}(x) = \sup_{0<t<1} |\phi_t * g(x)| \in L^1$$

with the obvious associated norm - we clearly have the continuous embeddings $\mathcal{H}^1(\mathbb{R}^n) \hookrightarrow h^1(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$. Recall that $M[g] \in L^p$ iff $g \in L^p$ when $p > 1$, and that $M[g] \in L^1$ iff $g \equiv 0$. Therefore $g \in \mathcal{H}^1$ iff the more refined version of the maximal function g_*, is integrable. We see $\mathcal{H}^1$ as being a replacement for L^1 in this sense. Most importantly for us the Hardy spaces behave well with respect to singular integral operators and therefore in elliptic regularity theory, again in contrast to the L^1 theory.

As mentioned above, when $n \geq 3$, there exist nowhere regular weakly harmonic maps. If however, we consider weakly stationary harmonic maps - critical points of E with respect to outer *and* inner variations - then we obtain that

$$\rho(r) = r^{2-n} \int_{B_r(x)} |\nabla u|^2$$

is monotone increasing for any $x \in B_1$, $r < 1 - |x|$ see [Sim96]. In particular this gives $\nabla u \in M^{2,n-2}_{loc}(B_1)$. Moreover given any $\epsilon > 0$ there exists a closed set S such that for any $z \notin S$ we can find $r_0 = r_0(z) > 0$ with $\sup_{r<r_0} r^{2-n} \int_{B_r(z)} |\nabla u|^2 \leq \epsilon$, and $H^{n-2}(S) = 0$ where H is the Hausdorff measure - see for instance [Gia83, Chapter IV]. It is an easy exercise to check that by the monotonicity of ρ we can equivalently say that for any $z \notin S$ there is some $r_0(z) > 0$ with $\|\nabla u\|_{M^{2,n-2}(B_{r_0}(z))} \leq \epsilon$.

The ϵ regularity results of Evans-Bethuel for $n \geq 3$ state that there is an $\epsilon > 0$ such that whenever a weakly stationary harmonic map u satisfies $\rho(R) \leq \epsilon$ for some

$x \in B_1$ and $0 < R < 1 - |x|$, then u is smooth on $B_{\frac{R}{2}}(x)$. Thus we can conclude that weakly stationary harmonic maps are smooth away from a singular set S with $H^{n-2}(S) = 0$.

In [Riv07] and [RS08] it is shown that by adding a null term to the harmonic map equation (1) one has that $u : B_1 \to \mathcal{N} \hookrightarrow \mathbb{R}^m$ solves

$$-\Delta u = \Omega^A.\nabla u, \tag{2}$$

for some $\Omega^A : B_1 \to so(m) \otimes \wedge^1\mathbb{R}^n$ and the product $[\Omega^A.\nabla u]^i = \sum_j \langle (\Omega^A)^i_j, \nabla u^j\rangle$ is an inner product of forms coupled with matrix multiplication. As we shall see, the anti-symmetry of Ω^A is crucial for obtaining any higher regularity of this PDE. In this paper we will study this equation under very natural and critical conditions on arbitrary Ω and u solving (2).

In the setting of harmonic maps one also knows that there is some $C = C(\mathcal{N}) > 0$ such that $|\Omega^A(x)| \leq C|\nabla u(x)|$. Therefore the natural conditions for Ω and ∇u are that they lie in $M^{2,n-2}$ as in [RS08], where they prove that, when $\|\Omega\|_{M^{2,n-2}(B_1)}$ is small enough, u is Hölder continuous, Theorem 1.1.

The starting point for us will be to consider arbitrary $u \in W^{1,2}(B_1, \mathbb{R}^m)$ weakly solving

$$-\Delta u = \Omega.\nabla u + f \tag{3}$$

on the unit ball $B_1 \subset \mathbb{R}^n$, for some $\Omega \in M^{2,n-2}(B_1, so(m) \otimes \wedge^1\mathbb{R}^n)$ and $f \in L^p$ under the assumptions that $\nabla u \in M^{2,n-2}(B_1, \mathbb{R}^m \otimes \wedge^1\mathbb{R}^n)$ and $\frac{n}{2} < p < n$. The question which will be addressed is whether or not one has higher integrability of ∇u (beyond L^2) for general systems (3). In [RS08], [Rup08] and [Sch10] the Hölder regularity is obtained by improving decay on $\|\nabla u\|_{L^s(B_r)}$ for $s < 2$. Therefore any higher integrability does not follow directly from such a result. Moreover it is difficult to see how one might boot-strap this information back into the PDE to improve the regularity.

Estimating the right hand side of (3) using Hölder's inequality leaves us with $\Delta u \in M^{1,n-2}$ ($=L^1$ when $n = 2$), and the best we can do using singular integral estimates is to conclude that $\nabla u \in M^{(2,\infty),n-2}$ ($= L^{(2,\infty)}$ when $n = 2$). See appendix A for definitions and results if necessary. These spaces are slightly worse than the spaces we started with, therefore we have lost some information and bootstrapping fails. The antisymmetry condition on Ω is therefore key to unlocking the hidden regularity of this system as first noticed by Rivière [Riv07] - it is known that by dropping the antisymmetry of Ω the regularity theory fails.

We interpret Ω as connection forms for the trivial bundle $B_1 \times \mathbb{R}^m$, for which $\mathrm{d}u \in \Gamma(\mathrm{T}^*B_1 \otimes \mathbb{R}^m)$ and (3) reads $\mathrm{d}^*_\Omega(\mathrm{d}u) = f$ where d^*_Ω is the induced covariant divergence given by Ω (the formal adjoint of the covariant exterior derivative, $\mathrm{d}^*_\Omega(\mathrm{d}u) = \mathrm{d}^*\mathrm{d}u - *[\Omega \wedge *\mathrm{d}u]$). With this more geometric setting it is possible to talk of changes of frame or gauge, and crucially the antisymmetry of Ω means that any change of gauge lies in the orthogonal group and carries a natural L^∞ bound. A change of gauge here is a purely local affair and consists of a map $P : B_1 \to SO(m)$ in which we can express $\mathrm{d}u$, Ω and therefore our PDE. Under this change of gauge the new connection forms Ω_P look like $\Omega_P = P^{-1}\mathrm{d}P + P^{-1}\Omega P$ and we have that $\mathrm{d}^*_\Omega(\mathrm{d}u) = P(\mathrm{d}^*_{\Omega_P}(P^{-1}\mathrm{d}u))$. Therefore, solutions to (3) are also solutions to

$$\mathrm{d}^*_{\Omega_P}(P^{-1}\mathrm{d}u) = \mathrm{d}^*(P^{-1}\mathrm{d}u) - *[\Omega_P \wedge *P^{-1}\mathrm{d}u] = P^{-1}f$$

for any such P.

1.1. Two-dimensional domains. Given that the problem here is concerned with improving regularity, the game has been to find a gauge that forces this equation to exhibit nice regularity properties. When $n = 2$ it was shown in [Riv07] that we can change the gauge such that the term $\Omega.\nabla u$ is effectively replaced by a Jacobian determinant. Thus we may use Hardy space methods [CLMS93] or Wente-type estimates [Wen69] to improve our situation. It was shown in [Riv07] that the most suitable gauge transform is a small perturbation of the Coulomb gauge (or Uhlenbeck gauge) and in fact it is necessary for the gauge to leave the orthogonal group; moreover these methods allow us to write the PDE as a conservation law.

Solutions with $f \equiv 0$ are shown (in [Riv07]) to describe critical points of conformally invariant elliptic Lagrangians under some natural growth assumptions and for appropriate Ω. In particular when $f \equiv 0$, (2) describes harmonic maps and prescribed mean curvature equations from Riemannian surfaces into closed, C^2 Riemannian manifolds $N \hookrightarrow \mathbb{R}^m$ isometrically embedded in some Euclidean space.

This PDE has subsequently been studied from a regularity and compactness perspective, see for instance [RL13], [LZ09], [MS09], [Sch10], [ShTo13]. In [ShTo13], it is shown that general solutions to (2) (when $n = 2$) are in $W^{2,s}_{loc}$ for all $s < 2$ by means of a Morrey estimate, and we see that Theorem 1.2 and Proposition 2.1 are the analogues of [ShTo13, Theorem 1.1] and [ShTo13, Lemma 7.3] in the higher dimensional setting.

1.2. Higher dimensional domains. For $n \geq 2$ Rivière-Struwe [RS08] showed that we can find a Coulomb gauge in the Morrey space setting (see appendix B), and that this is enough to conclude partial regularity for general solutions. Again this comes down to the appearance of terms that lie in the Hardy space $\mathcal{H}^1$. It is shown that solutions to (2) describe harmonic (and almost harmonic) maps from the Euclidean ball into arbitrary Riemannian manifolds. As outlined in [RS08] it would be difficult to carry out the same techniques when $n \geq 3$ as in the case $n = 2$, however Laura Keller [Kel10] has shown that when Ω and ∇u lie in a (slightly more restrictive) Besov-Morrey space, then the methods as in the two dimensional case apply.

The regularity obtained in [RS08] and [Rup08] is as follows (see also [Sch10]):

Theorem 1.1. *Let u, Ω and f be as in* (3). *There exists $\epsilon = \epsilon(n, m, p)$ such that whenever $\|\Omega\|^2_{M^{2,n-2}(B_1)} \leq \epsilon$ then $u \in C^{0,\gamma}_{loc}$ where $\gamma = 2 - \frac{n}{p} \in (0, 1)$.*

The optimal Hölder regularity was shown in [Rup08] along with an estimate. To see the optimality just consider the case $\Omega \equiv 0$; we have that $u \in W^{2,p}_{loc} \hookrightarrow C^{0,2-\frac{n}{p}}_{loc}$ when $\frac{n}{2} < p < n$ by Calderon-Zygmund theory and Morrey estimates.

As stated in [RS08], this theorem allows us to extend the regularity theory for stationary harmonic maps from the Euclidean ball into closed C^2 Riemannian manifolds immersed in some Euclidean space. More precisely it is possible to show that any weakly stationary harmonic map is smooth away from a singular set S with $H^{n-2}(S) = 0$. This follows from a classical theorem stating that continuous weakly harmonic maps are smooth.

The methods for the higher dimensional theory have also been used in the study of Dirac harmonic maps (e.g. [WaXu09] [CJWZ13]) and weakly harmonic maps into pseudo-Riemannian manifolds [Zhu13].

1.3. Statement of results. In this paper we will show improved regularity along with a new estimate. In order to get this estimate we use the Coulomb gauge obtained in [RS08], Theorem 1.1 and we crucially require an extension of a result of Adams [Ada75] Theorem 1.7.

THEOREM 1.2. *For $n \geq 2$ let $u \in W^{1,2}(B_1, \mathbb{R}^m)$ with $\nabla u \in M^{2,n-2}(B_1, \mathbb{R}^m)$, $\Omega \in M^{2,n-2}(B_1, so(m) \otimes \wedge^1 \mathbb{R}^n)$ and $f \in L^p(B_1)$, for $\frac{n}{2} < p < n$, weakly solve*

$$-\Delta u = \Omega.\nabla u + f.$$

Then for any $U \subset\subset B_1$ there exist $\epsilon = \epsilon(n,m,p) > 0$ and $C = C(n,m,p,U) > 0$ such that whenever $\|\Omega\|_{M^{2,n-2}(B_1)} \leq \epsilon$ we have

$$\|\nabla^2 u\|_{M^{\frac{2p}{n},n-2}(U)} + \|\nabla u\|_{M^{\frac{2p}{n-p},n-2}(U)} \leq C(\|u\|_{L^1(B_1)} + \|f\|_{L^p(B_1)}).$$

We see that this generalises [ShTo13] to higher dimensions, and that if $\nabla u \in M^{\frac{2p}{n-p},n-2}$ then $u \in C^{0,\gamma}$ with γ as in Theorem 1.1. Moreover, with p in the above range we have $\frac{2p}{n} > 1$ and $\frac{2p}{n-p} > 2$ - i.e. we have obtained integrability above the critical level. An interesting question here is whether the integrability of ∇u can be improved further when $n \geq 3$. One might expect that we should have estimates on ∇u in $L^{\frac{np}{n-p}}$ (consider the case $\Omega \equiv 0$). Clearly the case $n = 2$ is no problem as this gives the optimal regularity expected, moreover we have found solutions with $f \equiv 0$ that are not in $W^{2,2}$ or even $W^{2,(2,\infty)}$. We leave it to the reader to check that defining $u : B_{e^{-1}} \to \mathbb{R}^2$, by $u(x,y) := (\log r)^2 \begin{pmatrix} x \\ y \end{pmatrix}$ with $r = (x^2+y^2)^{\frac{1}{2}}$ and

$$\Omega = \frac{2(1+2\log r)}{(r\log r)^2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (-y\mathrm{d}x + x\mathrm{d}y)$$

gives the desired example - see [Sh12, Chapter 4.3]. Thus we cannot expect that $\nabla u \in L^\infty$ or even $\nabla u \in BMO$ in general. This also explains the range of f that we consider.

REMARK 1.3. Since writing this paper there have been a few generalisations and extensions of Theorem 1.2, namely Roger Moser [Mos12] has studied almost harmonic maps: functions $u : B_1 \to \mathcal{N} \hookrightarrow \mathbb{R}^m$ solving

$$-\Delta u = A(u)(\nabla u, \nabla u) + f$$

with $f \in L^p$, $p > 1$ and has obtained an appropriate $W^{2,p} \cap W^{1,2p}$ estimate on such u when $\|\nabla u\|_{M^{2,n-2}}$ is small. One would expect this result to hold for the system under consideration in this paper, under the further condition that $|\Omega| \leq C|\nabla u|$.

Armin Schikorra [Sch12] has proved that an analogue of Theorem 1.2 holds for more general systems involving non-local operators, using different techniques.

Finally, in a joint work with Miaomiao Zhu [ShZhu13], the author has studied the boundary regularity problem for similar systems in order to obtain a full regularity theory in the free boundary problem for Dirac harmonic maps from surfaces.

An easy consequence of Theorem 1.2 is the following

COROLLARY 1.4. *Let u and Ω be as in Theorem 1.2 with $f \equiv 0$. For any $q < \infty$ and $U \subset\subset B_1$, setting $s = \frac{2q}{2+q} < 2$, there exist $\epsilon = \epsilon(q,m,n) > 0$ and $C = C(q,m,n,U)$ such that if $\|\Omega\|_{M^{2,n-2}(B_1)} \leq \epsilon$ then*

$$\|\nabla^2 u\|_{M^{s,n-2}(U)} + \|\nabla u\|_{M^{q,n-2}(U)} \leq C\|u\|_{L^1(B_1)}.$$

REMARK 1.5. In the case that $|\Omega| \leq C|\nabla u|$ (as is the case for Harmonic maps) this automatically gives that when $\|\nabla u\|_{M^{2,n-2}(B_1)}$ is small enough then $u \in W^{2,q}$

for some $q > n$ yielding $u \in C^{1,\gamma}$ for some $\gamma \in (0,1)$. If we knew furthermore that Ω depended on u and ∇u in a smooth way (as is also the case for Harmonic maps) then we could immediately conclude smoothness by a simple bootstrapping argument using Schauder theory. Thus we recover a proof of the regularity of weakly stationary harmonic maps into Riemannian manifolds (away from a singular set S with $H^{n-2}(S) = 0$). We also mention that passing to the smooth local estimates for harmonic maps also easily follows - we leave the details to the reader.

Remark 1.6. We remark that Theorem 1.2 and Corollary 1.4 should hold (with some added technicalities) given any smooth metric g on B_1, with u, Ω and f as above weakly solving

$$-\Delta_g u = \langle \Omega, \nabla u \rangle_g + f. \tag{4}$$

Following [Riv07] or [RS08] one can check that a harmonic map $u : (B_1, g) \to \mathcal{N} \hookrightarrow \mathbb{R}^m$ solves (4) with $f \equiv 0$, and therefore the regularity theory for general domains would follow from this.

The method we use to prove Theorem 1.2 is (for the most part) broadly the same as that employed in [ShTo13], the real difference comes in section 4 where we obtain a decay type estimate (5) using both Hardy and Morrey space methods via a slight improvement of a result of Adams [Ada75].

To illustrate the requirement for an improved Adams result consider the case that Ω is divergence free, i.e. that $\Omega = *\mathrm{d}\xi$ for some $\xi \in W_0^{1,2}(B_1, so(m) \otimes \wedge^{n-2}\mathbb{R}^n)$. Using Hölder's inequality and the results in [CLMS93] we have

$$-\Delta u = *(\mathrm{d}\xi \wedge \mathrm{d}u) \in \mathcal{H}^1 \cap M^{1,n-2}$$

after extending ξ by zero and extending u to $W^{1,2}(\mathbb{R}^n)$ appropriately. This follows from an important result of Coiffman et al, [CLMS93] that (in particular) given two one forms $D, E \in L^2(\mathbb{R}^n, \bigwedge^1 \mathbb{R}^n)$ such that $\mathrm{d}E = 0$ and $\mathrm{d}^*D = 0$ weakly, then $\langle E, D\rangle \in \mathcal{H}^1(\mathbb{R}^n)$ with $\|\langle E, D\rangle\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C\|E\|_{L^2(\mathbb{R}^n)}\|D\|_{L^2(\mathbb{R}^n)}$. The key to attaining sub-critical integrability lies in finding better decay for the L^2 norm of ∇u on small balls, thus what we would like to do is estimate $\|\nabla u\|_{L^2}$. When $n = 2$ we have that $\mathcal{H}^1 \hookrightarrow H^{-1} = (W_0^{1,2})^*$, thus getting such an estimate is not a problem. In higher dimensions one certainly does not have $\mathcal{H}^1 \hookrightarrow H^{-1}$, however by slightly improving a result of Adams [Ada75] we are able to prove that $\mathcal{H}^1 \cap M^{1,n-2} \hookrightarrow H^{-1}$ and we can estimate $\|\nabla u\|_{L^2}$. Actually we prove a local version: $h^1 \cap M^{1,n-2} \hookrightarrow H^{-1}(K)$ - Corollary 1.8. By re-writing with respect to a Coulomb gauge, and using the Hölder continuity already obtained in Theorem 1.1, we are able to essentially reduce to the above situation. Indeed, the anti-symmetry of Ω can be thought of as replacing the condition that it is divergence free.

Before we state the result we first introduce some notation. Let $N[g]$ be the Newtonian potential of a function g, that is convolution by the fundamental solution of the Laplacian $\Gamma : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$. We have $\Gamma(x) = C(n)|x|^{2-n}$ for $n \geq 3$, where $C(n)$ is a dimensional constant, and $\Gamma(x) = -\frac{1}{2\pi}\log|x|$ when $n = 2$. Thus $N[g] = \Gamma * g$ and $-\Delta N[g] = g$. We will need to estimate the operation $g \mapsto \nabla N[g]$ which is a convolution operator given by $\nabla\Gamma * g$.

For $0 < \alpha < n$, define A_α to be convolution of a function by a_α where a_α is homogeneous of degree $\alpha - n$ and is smooth as a function on the sphere S^{n-1} with $\|a_\alpha\|_{C^1(S^{n-1})} \leq C(n)$. For instance N is of the form A_2 when $n \geq 3$, moreover the operator defined by taking a derivative of the Newtonian potential $\nabla N[g]$ is of the form A_1 for all $n \geq 2$. The A_α are essentially Riesz potentials.

A few words of warning are necessary here: our notation differs from that used in [Ada75], in particular we define our maximal functions M_β differently and the Riesz potentials we consider are slightly more general.

The next theorem is a replacement of a weak L^q-estimate given by Proposition 3.2 in [Ada75]. We replace the maximal function $M[g]$ by g_* thereby allowing us to estimate functions in the (local) Hardy space.

Theorem 1.7. *Let $0 \leq \beta < n$, $0 < \alpha p < n - \beta$ and $g \in M^{p,\beta}(\mathbb{R}^n)$ we have*

$$|A_\alpha[g](x)| \leq C(\alpha, \beta, n, p)(M_\beta[g^p](x))^{\frac{\alpha}{n-\beta}}(g_*(x))^{\frac{n-\beta-\alpha p}{n-\beta}},$$

with $C(\alpha, \beta, n, p) \leq C(n) \sup \left\{ \frac{1}{1-\left(\frac{1}{2}\right)^\alpha}, \frac{1}{1-\left(\frac{1}{2}\right)^{\alpha - \frac{n-\beta}{p}}} \right\}$.

As alluded to above we have the following

Corollary 1.8. *Let $g \in h^1 \cap M^{1,n-2}(\mathbb{R}^n)$, then for any compact $K \subset \mathbb{R}^n$ there exists some $C = C(K, n)$ such that*

$$\|g\|_{H^{-1}(K)} \leq C \|g\|_{M^{1,n-2}}^{\frac{1}{2}} \|g\|_{h^1}^{\frac{1}{2}}.$$

Theorem 1.7 also recovers the following potential estimates, due to Adams [Ada75]. We provide a proof of the following because we wish to keep track of the constants more closely.

Corollary 1.9. *Let $0 \leq \beta < n$, $0 < \alpha p < n - \beta$ and $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{\alpha}{n-\beta}$.*

1. *When $p > 1$ we have that $A_\alpha : M^{p,\beta}(\mathbb{R}^n) \to M^{\tilde{p},\beta}(\mathbb{R}^n)$ is bounded. Moreover there exists*

$$C(n, p, \alpha, \beta) \leq C(n) \left(\frac{p}{p-1} \right)^{\frac{1}{\tilde{p}}} \sup \left\{ \frac{1}{1 - \left(\frac{1}{2}\right)^\alpha}, \frac{1}{1 - \left(\frac{1}{2}\right)^{\alpha - \frac{n-\beta}{p}}} \right\}$$

such that $\|A_\alpha[g]\|_{M^{\tilde{p},\beta}(\mathbb{R}^n)} \leq C(n, p, \alpha, \beta) \|g\|_{M^{p,\beta}(\mathbb{R}^n)}$.

2. *When $p = 1$ we have that $A_\alpha : M^{1,\beta}(\mathbb{R}^n) \to M^{(\tilde{p},\infty),\beta}(\mathbb{R}^n)$ is bounded. Moreover there exists*

$$C(n, \alpha, \beta) \leq C(n) \sup \left\{ \frac{1}{1 - \left(\frac{1}{2}\right)^\alpha}, \frac{1}{1 - \left(\frac{1}{2}\right)^{\alpha - (n-\beta)}} \right\}$$

such that $\|A_\alpha[g]\|_{M^{(\tilde{p},\infty),\beta}(\mathbb{R}^n)} \leq C(n, \alpha, \beta) \|g\|_{M^{p,\beta}(\mathbb{R}^n)}$.

These reduce to well known estimates when $\beta = 0$.

Remark 1.10. Setting $\alpha = 1$ and recalling that $\nabla N[g]$ is an operator of the form A_1 we see that given a, b with $1 < a \leq p \leq b < n - \beta$ then there is a uniform constant $C = C(a, b)$ such that for $g \in M^{p,\beta}$ then $\|\nabla N[g]\|_{M^{\tilde{p},\beta}} \leq C \|g\|_{M^{p,\beta}}$ for any p and β in this range.

Furthermore, let u, Ω and f be as in (3). Define $g := \Omega . \nabla u$ and extend both f and g by zero to $\mathbb{R}^n$. Then there is some harmonic function $h : B_1 \to \mathbb{R}^m$ with

$u = N[g] + N[f] + h$, since $-\Delta(u - N[g] - N[f]) = 0$ on B_1. Standard estimates (using Corollary 1.9) give

$$\|h\|_{L^1(B_1)} \leq C(\|u\|_{L^1(B_1)} + \|f\|_{L^p(B_1)} + \|\Omega\|_{M^{2,n-2}(B_1)}\|\nabla u\|_{M^{2,n-2}(B_1)})$$

and therefore since h is harmonic, for any $r < 1$

$$\|\nabla h\|_{L^\infty(B_r)} \leq \frac{C(n)}{(1-r)^{n+1}}(\|u\|_{L^1(B_1)} + \|f\|_{L^p(B_1)} + \|\Omega\|_{M^{2,n-2}(B_1)}\|\nabla u\|_{M^{2,n-2}(B_1)}).$$

Thus we gain local control on $\nabla u = \nabla N[g] + \nabla N[f] + \nabla h$ by estimating $\nabla N[g]$ and $\nabla N[f]$.

Remark 1.11. We point out that by combining Theorem 1.7 and Corollary 1.9 we have $A_\alpha : \mathcal{H}^1 \cap M^{1,\beta}(\mathbb{R}^n) \to L^{\tilde{p}} \cap M^{(\tilde{p},\infty),\beta}(\mathbb{R}^n)$ with an estimate - the details are left to the reader. In particular, when $\beta = 0$ we have $A_\alpha : \mathcal{H}^1(\mathbb{R}^n) \to L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$ - see also [Sem94, Theorem 1.77].

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2. Proof of Theorem 1.2. We prove Theorem 1.2 based on the following proposition, analogous to [ShTo13, Lemma 7.3], the proof of which is left to section 3.

Proposition 2.1. *Let u, Ω and f be as in Theorem 1.2. There exits $\epsilon = \epsilon(n, m, p)$ such that whenever $\|\Omega\|^2_{M^{2,n-2}(B_1)} \leq \epsilon$ then $\nabla u \in M^{2,n-2(\frac{n}{p}-1)}_{loc}(B_1, \mathbb{R}^m)$.*

Proof of Theorem 1.2. This proof generalises the ideas needed in the proofs of [ShTo13, Lemmata 7.1 and 7.2] to Morrey spaces.

Using the improved regularity from Proposition 2.1 and the Hölder inequality we see that $\Omega.\nabla u \in M^{1,n(1-\frac{1}{p})}_{loc}$. We also have $f \in M^{1,n(1-\frac{1}{p})}$ - see appendix D if necessary.

By applying Corollary 1.9 (for $\alpha = 1$, $p = 1$ and $\beta = n(1-\frac{1}{p})$) and Remark 1.10 we see that this implies $\nabla u \in M^{(\frac{n}{n-p},\infty),n(1-\frac{1}{p})}_{loc}$, which in turn gives, $\nabla u \in M^{\frac{\theta n}{n-p},n(1-\frac{\theta}{p})}_{loc}$ for any $\theta < 1$ - we leave the details to the reader.

The fact that $\nabla u \in M^{\frac{\theta n}{n-p},n(1-\frac{\theta}{p})}_{loc}$ implies $\Omega.\nabla u \in M^{s,n(1-\frac{s}{p})}_{loc}$ where $\frac{1}{s} = \frac{1}{2} + \frac{n-p}{\theta n}$. We can choose θ so that $s > 1$ but note that we also have $s < \frac{2n}{3n-2p} < \frac{2p}{n}$ for $p \in (\frac{n}{2}, n)$. We make the following claim:

$$\Omega.\nabla u \in M^{s_k,n(1-\frac{s_k}{p})}_{loc}, s_k \in (1, \frac{2p}{n}) \Rightarrow$$
$$\Omega.\nabla u \in M^{s_{k+1},n(1-\frac{s_{k+1}}{p})}_{loc}, s_k < s_{k+1} = \frac{2ns_k}{ns_k + 2(n-p)} \in (1, \frac{2p}{n}).$$

To prove the claim we can easily check that $f \in M^{s_k,n(1-\frac{s_k}{p})}$ with a uniform estimate for any such s_k (see appendix D). Therefore we may apply Corollary 1.9 (for $\alpha = 1$, $p = s_k$ and $\beta = n(1 - \frac{s_k}{p})$) and Remark 1.10 to yield $\nabla u \in M^{s_k\frac{n}{n-p},n(1-\frac{s_k}{p})}_{loc}$. Again by Hölder's inequality we have $\Omega.\nabla u \in M_{loc}^{s_{k+1},n(1-\frac{s_{k+1}}{p})}$ where $\frac{1}{s_{k+1}} = \frac{1}{2} + \frac{n-p}{s_k n}$. We check that

$$\frac{s_k}{s_{k+1}} = \frac{s_k}{2} + \frac{n-p}{n} < \frac{p}{n} + 1 - \frac{p}{n} = 1.$$

If we assume, to get a contradiction, that $s_{k+1} \geq \frac{2p}{n}$ then we have $\frac{2ns_k}{ns_k+2(n-p)} \geq \frac{2p}{n}$. Which implies $2ns_k \geq 2ps_k + 2(n-p)\frac{2p}{n}$ and therefore $s_k \geq \frac{2p}{n}$, a contradiction. Thus the claim holds.

We have the recursive relation $s_{k+1} = \frac{2ns_k}{ns_k+2(n-p)}$, so we have $s_k \uparrow \frac{2p}{n}$ and we have proved that $\Omega.\nabla u \in M_{loc}^{s,n(1-\frac{s}{p})}$ for all $s < \frac{2p}{n}$. Thus $\nabla u \in M_{loc}^{s\frac{n}{n-p},n(1-\frac{s}{p})}$ for all s in this range (again by Corollary 1.9).

For some $B_R(x_0) \subset\subset B_1$ letting $\hat{u}(x) = u(x_0 + Rx)$, $\hat{\Omega}(x) = R\Omega(x_0 + Rx)$ and $\hat{f}(x) = R^2 f(x_0 + Rx)$ we have

$$-\Delta \hat{u} = \hat{\Omega}.\nabla \hat{u} + \hat{f}$$

on B_1. The above argument gives us that $\nabla \hat{u} \in M^{s\frac{n}{n-p},n(1-\frac{s}{p})}(B_1)$ for all $s < \frac{2p}{n}$.

Now let $s \in (\frac{2p+n}{2n}, \frac{2p}{n})$ and denote by t the next value given in the bootstrapping argument (if $s = s_k$ then $t = s_{k+1}$). Notice that within this range, by Remark 1.10 there is a $C = C(n,p)$ independent of s and t such that

$$\begin{aligned}\|\nabla \hat{u}\|_{M^{t\frac{n}{n-p},n(1-\frac{t}{p})}(B_{\frac{1}{2}})} &\leq C(\|\hat{\Omega}\|_{M^{2,n-2}(B_1)}\|\nabla \hat{u}\|_{M^{s\frac{n}{n-p},n(1-\frac{s}{p})}(B_1)} + \|\hat{f}\|_{L^p(B_1)} + \|\hat{u}\|_{L^1(B_1)}) \\ &\leq C(\|\hat{\Omega}\|_{M^{2,n-2}(B_1)}\|\nabla \hat{u}\|_{M^{t\frac{n}{n-p},n(1-\frac{t}{p})}(B_1)} + \|\hat{f}\|_{L^p(B_1)} + \|\hat{u}\|_{L^1(B_1)})\end{aligned}$$

since whenever $1 < s < t < \frac{2p}{n}$ we have the estimate

$$\|\nabla \hat{u}\|_{M^{s\frac{n}{n-p},n(1-\frac{s}{p})}} \leq C\|\nabla \hat{u}\|_{M^{t\frac{n}{n-p},n(1-\frac{t}{p})}}$$

for $C = C(n,p)$. Raising to the power $\mu := t\frac{n}{n-p}$ we see that

$$\|\nabla \hat{u}\|^{\mu}_{M^{\mu,n(1-\frac{t}{p})}(B_{\frac{1}{2}})} \leq C(\|\hat{\Omega}\|^{\mu}_{M^{2,n-2}(B_1)}\|\nabla \hat{u}\|^{\mu}_{M^{\mu,n(1-\frac{t}{p})}(B_1)} + (\|\hat{f}\|_{L^p(B_1)} + \|\hat{u}\|_{L^1(B_1)})^{\mu}),$$

where we can still pick C independent of t since $\mu < \frac{2p}{n-p}$. Undoing the scaling leaves (see appendix D)

$$\begin{aligned}R^{\mu+\frac{nt}{p}}\|\nabla u\|^{\mu}_{M^{\mu,n(1-\frac{t}{p})}(B_{\frac{R}{2}}(x_0))} \leq C(&\|\Omega\|^{\mu}_{M^{2,n-2}(B_1)}R^{\mu+\frac{nt}{p}}\|\nabla u\|^{\mu}_{M^{\mu,n(1-\frac{t}{p})}(B_R(x_0))} + \\ &+(R^{2-\frac{n}{p}}\|f\|_{L^p(B_1)} + R^{-n}\|u\|_{L^1(B_1)})^{\mu}).\end{aligned}$$

Since $R < 1$, $\mu < \frac{2p}{n-p}$ and $t < \frac{2p}{n}$ we have that

$$\begin{aligned}\|\nabla u\|^{\mu}_{M^{\mu,n(1-\frac{t}{p})}(B_{\frac{R}{2}}(x_0))} \leq &C\|\Omega\|^{\mu}_{M^{2,n-2}(B_1)}\|\nabla u\|^{\mu}_{M^{\mu,n(1-\frac{t}{p})}(B_R(x_0))} + \\ &+C(\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)})^{\mu})R^{-((n+1)\frac{2p}{n-p}+2)}.\end{aligned}$$

We are now in a position to apply Lemma C.1 for $\Gamma = C(\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)})^{\mu})$, $\epsilon \leq \left(\frac{\epsilon_0}{C}\right)^{\frac{n-p}{2p}}$ and $\epsilon_0 = \epsilon_0(n,k)$ is found for $k = (n+1)\frac{2p}{n-p} + 2$ to give the estimate

$$\|\nabla u\|_{M^{\mu,n(1-\frac{t}{p})}(B_{\frac{1}{2}})} \leq C(\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)})$$

with C independent of t. We may now pass to the limit $t \uparrow \frac{2p}{n}$ to give

$$\|\nabla u\|_{M^{\frac{2p}{n-p},n-2}(B_{\frac{1}{2}})} \leq C(\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)}).$$

For the second estimate, note that we have $\Omega.\nabla u + f \in M^{\frac{2p}{n},n-2}_{loc}$ by Hölder's inequality, thus by Theorem A.1 and the proceeding remarks we have finished the proof after applying a covering argument. ◻

3. Proof of Proposition 2.1. We begin with a proposition stating the main decay estimate required, the proof of this is left until section 4. This decay estimate is analogous to that of part 2. from [ShTo13, Theorem 1.5], except that here we crucially require the Hölder regularity already obtained in order to prove (5).

PROPOSITION 3.1. *With the set-up as in Theorem 1.2. Let $\delta > 0$, then there exist $\epsilon = \epsilon(n, m, p) > 0$ and $C = C(\delta, m, n)$ such that when $\|\Omega\|_{M^{2,n-2}(B_1)} \leq \epsilon$ we have the following estimate ($\gamma = 2 - \frac{n}{p}$)*

$$\|\nabla u\|^2_{L^2(B_r)} \leq C(\delta)(\|\Omega\|^2_{M^{2,n-2}(B_1)}[u]^2_{C^{0,\gamma}(B_1)} + \|f\|^2_{L^p(B_1)}) + r^n(1+\delta)\|\nabla u\|^2_{L^2(B_1)}. \tag{5}$$

Proof of Proposition 2.1. We follow the argument for the proof of [ShTo13, Lemma 7.3]: Pick $\delta = \delta(n, p)$ sufficiently small so that

$$\lambda := \frac{1+\delta}{2^n} < \frac{1}{2^{n-2+2\gamma}} := \Lambda \in \left(\frac{1}{2^n}, 1\right) \tag{6}$$

since $\gamma = 2 - \frac{n}{p} \in (0, 1)$.

Consider the solution on some small ball $B_R(x_0) \subset B_1$ by re-scaling via $\hat{u}$, see appendix D. Since the hypotheses of Proposition 3.1 are also satisfied for $\hat{u}$ we have (by (5))

$$\|\nabla \hat{u}\|^2_{L^2(B_r)} \leq C(\|\Omega\|^2_{M^{2,n-2}(B_1)}[\hat{u}]^2_{C^{0,\gamma}(B_1)} + \|\hat{f}\|^2_{L^p(B_1)}) + r^n(1+\delta)\|\nabla \hat{u}\|^2_{L^2(B_1)},$$

and setting $r = \frac{1}{2}$ yields

$$\|\nabla \hat{u}\|^2_{L^2(B_{\frac{1}{2}})} \leq \lambda\|\nabla \hat{u}\|^2_{L^2(B_1)} + C(\|\Omega\|^2_{M^{2,n-2}(B_1)}[\hat{u}]^2_{C^{0,\gamma}(B_1)} + \|\hat{f}\|^2_{L^p(B_1)}).$$

Undoing the scaling gives

$$\begin{aligned} &\|\nabla u\|^2_{L^2(B_{\frac{R}{2}}(x_0))} \\ &\leq \lambda\|\nabla u\|^2_{L^2(B_R)} + CR^{n-2+2\gamma}(\|\Omega\|^2_{M^{2,n-2}(B_1)}[u]^2_{C^{0,\gamma}(B_R(x_0))} + \|f\|^2_{L^p(B_R(x_0))}). \end{aligned}$$

Therefore, setting $R = 2^{-k}$, $k \in \mathbb{N}_0$ and $a_k := \|\nabla u\|^2_{L^2(B_{2^{-k}})}$ we have

$$\begin{aligned} a_{k+1} &\leq \lambda a_k + \left(\frac{1}{2}\right)^{k(n-2+2\alpha)} C(\|\Omega\|^2_{M^{2,n-2}(B_1)}[u]^2_{C^{0,\gamma}(B_1)} + \|f\|^2_{L^p(B_1)}) \\ &= \lambda a_k + \Lambda^k C(\|\Omega\|^2_{M^{2,n-2}(B_1)}[u]^2_{C^{0,\gamma}(B_1)} + \|f\|^2_{L^p(B_1)}). \end{aligned}$$

This can be solved to yield (letting $K := C(\|\Omega\|^2_{M^{2,n-2}(B_1)}[u]^2_{C^{0,\gamma}(B_1)} + \|f\|^2_{L^p(B_1)})$)

$$a_{k+1} \leq \lambda^k a_1 + K\Lambda\frac{(\Lambda^k - \lambda^k)}{\Lambda - \lambda},$$

and by (6), this simplifies to

$$\|\nabla u\|^2_{L^2(B_{2^{-k}}(x_0))} =: a_k \leq C\Lambda^k.$$

Therefore, for $r \in (0, 1/2]$

$$\|\nabla u\|^2_{L^2(B_r(x_0))} \leq Cr^{n-2+2\gamma}.$$

We have

$$\|\nabla u\|^2_{M^{2,n-2+2\gamma}(B_{\frac{1}{2}})} \leq C,$$

and a covering argument concludes the proof. □

4. Proof of Proposition 3.1.

Proof of Proposition 3.1. We will use the Coulomb gauge in order to re-write our equation. Set ϵ small enough so that we can apply Lemma B.1. We have (see appendix B for the relevant background on Sobolev forms)

$$\mathrm{d}^*(P^{-1}\mathrm{d}u) = \langle \mathrm{d}^*\eta, P^{-1}\mathrm{d}u\rangle + P^{-1}f$$

and

$$\mathrm{d}(P^{-1}\mathrm{d}u) = (\mathrm{d}P^{-1} \wedge \mathrm{d}u).$$

We can also set ϵ small enough in order to apply Theorem 1.1 so that $u \in C^{0,\gamma}$ where $\gamma = 2 - \frac{n}{p}$. Now we wish to extend the quantities arising above in the appropriate way: First of all we may extend η by zero. We also extend $P - \frac{1}{|B_1|}\int_{B_1} P$ to $\tilde{P} \in W^{1,2} \bigcap L^\infty(\mathbb{R}^n)$ and finally u to $\tilde{u} \in C^{0,\gamma}(\mathbb{R}^n)$ where each has compact support in B_2 (we may assume $u \in C^{0,\gamma}(\overline{B}_1)$).

Note that we have $\|\nabla\tilde{P}\|_{L^2} \leq C\|\nabla P\|_{L^2(B_1)} \leq C\|\Omega\|_{M^{2,n-2}(B_1)}$ by Poincaré's inequality and $\nabla\tilde{P} = \nabla P$ in B_1. We also have $\tilde{u} \in C^{0,\gamma}(\mathbb{R}^n)$ with $\|\tilde{u}\|_{C^{0,\gamma}} \leq C\|u\|_{C^{0,\gamma}}$ and (since we may assume $\int u = 0$) we have $\|\tilde{u}\|_{C^{0,\gamma}} \leq C[u]_{C^{0,\gamma}}$, moreover $\tilde{u} = u$ in B_1. All the constants here come from standard extension operators and are independent of the function, see for instance [GT01].

Now we use Lemma B.3 in order to write $P^{-1}\mathrm{d}u = \mathrm{d}a + \mathrm{d}^*b + h$ with a, b, h as in the Lemma. Notice that we have $\Delta a = \langle \mathrm{d}^*\eta, P^{-1}\mathrm{d}u\rangle + P^{-1}f$ and $\Delta b = \mathrm{d}P^{-1} \wedge \mathrm{d}u$ weakly. We proceed to estimate $\nabla u \in L^2$ by estimating $\|\mathrm{d}a\|_{L^2}$, $\|\mathrm{d}^*b\|_{L^2}$ and using standard properties of harmonic functions in order to deal with $\|h\|_{L^2}$.

We start with $\|\mathrm{d}a\|_{L^2}$; notice that

$$\langle \mathrm{d}^*\eta, P^{-1}\mathrm{d}u\rangle = \langle \mathrm{d}^*\eta, \mathrm{d}(P^{-1}u)\rangle - \langle \mathrm{d}^*\eta, \mathrm{d}P^{-1}\rangle u = I + II.$$

For I, pick $\phi \in C_c^\infty(B_1)$ and check (we use that η has zero boundary values)

$$\begin{aligned}
\int *\langle \mathrm{d}^*\eta, \mathrm{d}(P^{-1}u)\rangle\phi &= (\mathrm{d}^*\eta, \mathrm{d}(P^{-1}u)\phi) \\
&= (\mathrm{d}^*\eta, \mathrm{d}(P^{-1}u\phi)) - (\mathrm{d}^*\eta, (\mathrm{d}\phi)P^{-1}u) \\
&= -(\mathrm{d}^*\eta, (\mathrm{d}\phi)P^{-1}u) \\
&\leq \|\nabla\eta\|_{L^2(B_1)}\|\nabla\phi\|_{L^2(B_1)}\|P^{-1}u\|_{L^\infty(B_1)} \\
&\leq C\|\Omega\|_{M^{2,n-2}(B_1)}\|\nabla\phi\|_{L^2(B_1)}[u]_{C^{0,\gamma}(B_1)}.
\end{aligned}$$

We have $I \in H^{-1}(B_1)$ with

$$\|I\|_{H^{-1}(B_1)} \leq C\|\Omega\|_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}. \tag{7}$$

For II notice that $\langle \mathrm{d}^*\eta, \mathrm{d}\tilde{P}^{-1}\rangle\tilde{u} = \langle \mathrm{d}^*\eta, \mathrm{d}P^{-1}\rangle u$ in B_1. Moreover $\langle \mathrm{d}^*\eta, \mathrm{d}\tilde{P}^{-1}\rangle \in \mathcal{H}^1(\mathbb{R}^n)$ with

$$\|\langle \mathrm{d}^*\eta, \mathrm{d}\tilde{P}^{-1}\rangle\|_{\mathcal{H}^1} \leq C\|\nabla\eta\|_{L^2(B_1)}\|\nabla\tilde{P}\|_{L^2(B_1)} \leq C\|\Omega\|^2_{M^{2,n-2}(B_1)}$$

by the result of Coiffman et al, [CLMS93].

The space $\mathcal{H}^1$ is *not* stable by multiplication of smooth functions since $f \in \mathcal{H}$ implies $\int f = 0$. However for the local Hardy space h^1, as long as the multiplier function is sufficiently regular, we have stability. For instance if $h \in h^1$ and $g \in C^{0,\gamma}$, then $gh \in h^1$ and

$$\|gh\|_{h^1} \leq C(\gamma)\|g\|_{C^{0,\gamma}}\|h\|_{h^1}.$$

Therefore, $\langle \mathrm{d}^*\eta, \mathrm{d}\tilde{P}^{-1}\rangle\tilde{u} \in h^1(\mathbb{R}^n)$ with

$$\|\langle \mathrm{d}^*\eta, \mathrm{d}\tilde{P}^{-1}\rangle\tilde{u}\|_{h^1(\mathbb{R}^n)} \leq C\|\Omega\|^2_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}.$$

We also have $\|M_{n-2}(\langle \mathrm{d}^*\eta, \mathrm{d}\tilde{P}^{-1}\rangle\tilde{u})\|_{L^\infty} \leq C\|\Omega\|^2_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}$ since for $R > 0$

$$\begin{aligned} &R^{2-n}\int_{B_R(x_0)} \langle \mathrm{d}^*\eta, \mathrm{d}\tilde{P}^{-1}\rangle\tilde{u} \\ &= R^{2-n}\int_{B_R(x_0)\bigcap B_1} \langle \mathrm{d}^*\eta, \mathrm{d}\tilde{P}^{-1}\rangle\tilde{u} \leq C\|\Omega\|^2_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)} \end{aligned}$$

(remember η was extended by zero). Now, using Corollarly 1.8 we have $\langle \mathrm{d}^*\eta, \mathrm{d}\tilde{P}^{-1}\rangle\tilde{u} \in H^{-1}(B_1)$, moreover $\langle \mathrm{d}^*\eta, \mathrm{d}\tilde{P}^{-1}\rangle\tilde{u} = \langle \mathrm{d}^*\eta, \mathrm{d}P^{-1}\rangle u$ in B_1 so

$$\|II\|_{H^{-1}(B_1)} \leq C\|\Omega\|^2_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}. \tag{8}$$

Putting (7) and (8) together yields $\langle \mathrm{d}^*\eta, P^{-1}\mathrm{d}u\rangle \in H^{-1}(B_1)$ with (assuming $\epsilon < 1$)

$$\|\langle \mathrm{d}^*\eta, P^{-1}\mathrm{d}u\rangle\|_{H^{-1}(B_1)} \leq C\|\Omega\|_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}.$$

It is easy to check that $P^{-1}f \in H^{-1}(B_1)$ with $\|P^{-1}f\|_{H^{-1}(B_1)} \leq C\|f\|_{L^p(B_1)}$, overall this means that $a \in W^{1,2}_0(B_1)$ weakly solves

$$\Delta a = \langle \mathrm{d}^*\eta, P^{-1}\mathrm{d}u\rangle + P^{-1}f,$$

so we have

$$\|\nabla a\|_{L^2(B_1)} \leq C(\|\Omega\|_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)} + \|f\|_{L^p(B_1)}). \tag{9}$$

Now we need to estimate $\|\mathrm{d}^*b\|_{L^2(B_1)}$. We know that $b \in W^{1,2}_N(B_1, \bigwedge^2\mathbb{R}^n)$ (see appendix B for a definition) has $\mathrm{d}b = 0$ and $\Delta b = (\mathrm{d}P^{-1} \wedge \mathrm{d}u)$. We have

$$\|\mathrm{d}^*b\|_{L^2(B_1)} = \sup_{E\in C^\infty(B_1,\bigwedge^1\mathbb{R}^n)\ \|E\|_{L^2(B_1)}\leq 1} (\mathrm{d}^*b, E).$$

Using a smooth version of Lemma B.3 we can decompose each E by $E = \mathrm{d}e_1 + \mathrm{d}^* e_2 + e_3$ where $e_1 \in C_0^\infty(B_1)$, $e_2 \in C_N^\infty(B_1, \bigwedge^2 \mathbb{R}^n)$ with $\mathrm{d}e_2 = 0$ and $\mathrm{d}e_3 = \mathrm{d}^* e_3 = 0$ (e_3 is a harmonic one form). Notice that $(\mathrm{d}^* b, \mathrm{d}e_1) = 0$ since b has zero normal component and $\mathrm{d}^2 e_1 = 0$. Also we have $(\mathrm{d}^* b, e_3) = 0$ since e_3 is harmonic and b has vanishing normal components. Therefore

$$\begin{aligned}
(\mathrm{d}^* b, E) &= (\mathrm{d}^* b, \mathrm{d}^* e_2) \\
&= (P^{-1}\mathrm{d}u, \mathrm{d}^* e_2) \\
&= (\mathrm{d}(P^{-1}u), \mathrm{d}^* e_2) - ((\mathrm{d}P^{-1})u, \mathrm{d}^* e_2) \\
&= -((\mathrm{d}P^{-1})u, \mathrm{d}^* e_2) \\
&\le C\|\Omega\|_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}\|\mathrm{d}^* e_2\|_{L^2(B_1)} \\
&\le C\|\Omega\|_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}\|E\|_{L^2(B_1)}.
\end{aligned}$$

Therefore

$$\|\mathrm{d}^* b\|_{L^2(B_1)} \le C\|\Omega\|_{M^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}. \tag{10}$$

We note here that by [Mor66, Theorem 7.5.1] and $\mathrm{d}b = 0$ that we in fact have the same estimate for ∇b.

We now use the fact that h is harmonic giving that the quantity $r^{-n}\|h\|^2_{L^2(B_r)}$ is increasing, and Lemma B.3 to give

$$\begin{aligned}
\|h\|^2_{L^2(B_r)} &\le r^n\|h\|^2_{L^2(B_1)} \\
&\le r^n\|P^{-1}\mathrm{d}u\|^2_{L^2(B_1)} \\
&= r^n\|\mathrm{d}u\|^2_{L^2(B_1)}
\end{aligned}$$

where the last line follows because P is orthogonal.

Going back to our original Hodge decomposition we see that (using Young's inequality, the orthogonality of P, (9) and (10))

$$\begin{aligned}
\|\mathrm{d}u\|^2_{L^2(B_r)} &= \|P^{-1}\mathrm{d}u\|^2_{L^2(B_r)} \\
&\le (\|h\|_{L^2(B_r)} + \|\mathrm{d}a\|_{L^2(B_r)} + \|\mathrm{d}^* b\|_{L^2(B_r)})^2 \\
&\le (1+\delta)\|h\|^2_{L^2(B_r)} + C_\delta(\|\mathrm{d}a\|_{L^2(B_r)} + \|\mathrm{d}^* b\|_{L^2(B_r)})^2 \\
&\le (1+\delta)r^n\|\mathrm{d}u\|^2_{L^2(B_1)} + C_\delta(\|\Omega\|^2_{M^{2,n-2}(B_1)}[u]^2_{C^{0,\gamma}(B_1)} + \|f\|^2_{L^p(B_1)}).
\end{aligned}$$

□

5. Proof of Theorem 1.7 and Corollaries 1.8, 1.9.

Proof of Theorem 1.7. First assume the theorem holds for $p = 1$. If $q > 1$ and $g \in M^{q,\beta}$ then by Hölder's inequality we have $g \in M^{1,\hat\beta}$ with

$$M_{\hat\beta}[g] \le C(n)M_\beta[g^q]^{\frac{1}{q}}$$

where $n - \hat\beta = \frac{1}{q}(n-\beta)$. Thus if $\alpha q < n - \beta$ (giving $\alpha < n - \hat\beta$), by applying the theorem for $p = 1$ we have

$$\begin{aligned}
|A_\alpha[g](x)| &\le C(\alpha,\hat\beta,n)(M_{\hat\beta}[g](x))^{\frac{\alpha}{n-\hat\beta}}(g_*(x))^{\frac{n-\hat\beta-\alpha}{n-\hat\beta}} \\
&\le C(\alpha,\beta,n,q)(M_\beta[g^q](x))^{\frac{\alpha}{n-\beta}}(g_*(x))^{\frac{n-\beta-\alpha q}{n-\beta}}.
\end{aligned}$$

Where we know that

$$C(\alpha,\beta,n,q) \leq C(n) \sup\left\{ \frac{1}{1-\left(\frac{1}{2}\right)^{\alpha}}, \frac{1}{1-\left(\frac{1}{2}\right)^{\alpha-\frac{n-\beta}{q}}} \right\}.$$

It remains to prove Theorem 1.7 for $p = 1$.

We split $A_\alpha(g)$ up into its near and far parts using a partition of unity subordinate to dyadic annuli of a chosen modulus δ. More precisely, let $\theta(x) \in C_c^\infty(B_4 \backslash B_{\frac{1}{2}})$ with $\theta(x) > 0$ for $1 \leq |x| \leq 2$. Similarly as is done in Semmes [Sem94, Theorem 1.77] we can arrange so that

$$\sum_{j\in\mathbb{Z}} \theta(\delta^{-1}2^{-j}x) = 1$$

for all $x \in \mathbb{R}^n \backslash \{0\}$. Moreover we want for our choice of a_α that $C\|\nabla[\theta(4\cdot)a_\alpha(\cdot)]\|_{L^\infty(B_1)} \leq 1$ for some constant $C(n)$ (for reasons that will become apparent below). We can always find such a C since we have assumed a uniform C^1 bound on a_α when restricted to the sphere. Notice that $\theta(4\cdot)a(\cdot) \in C_c^\infty(B_1)$ also.

Now define $\eta^j(x) := \delta^{-1}2^{-j}\theta(\delta^{-1}2^{-j}x)a_\alpha(x)$. Notice that $\delta 2^j \eta^j(x)$ is the piece of a_α around $\delta 2^{j-1} \leq |x| \leq \delta 2^{j+2}$, so that

$$A_\alpha(g) = \sum_{j\in\mathbb{Z}} \delta 2^j \eta^j * g = \sum_{j\leq 0} \delta 2^j \eta^j * g + \sum_{j\geq 1} \delta 2^j \eta^j * g = I_{inner} + I_{outer}.$$

The intuition here is that we use the decay estimate we have on g in order to deal with I_{outer} and we use the Hardy space qualities of g in order to deal with I_{inner}.

With that in mind we start with estimating I_{inner}. We use the following claim:

$$|\eta^j * g(x)| \leq C(\delta^{-1}2^{-j})^{1-\alpha} g_*(x).$$

This is easy enough to see, first of all we remark that in our definition of g_* we choose to use the function $\psi(x) := C\theta(4x)a_\alpha(x)$, therefore $g_*(x) := \sup_{t>0} |\psi_t * g(x)|$.

$$\begin{aligned}
&|\eta^j * g(x)| \\
&= \left| \int_{B_{\delta 2^{j+2}}} \delta^{-1}2^{-j}\theta(\delta^{-1}2^{-j}(x-y))a_\alpha(x-y)g(y)dy \right| \\
&= \left| \int_{B_{\delta 2^{j+2}}} \delta^{-1}2^{-j}\theta(\delta^{-1}2^{-j}(x-y))a_\alpha(\delta^{-1}2^{-(j+2)}(x-y))g(y)(\delta^{-1}2^{-(j+2)})^{n-\alpha} dy \right| \\
&= C(\delta^{-1}2^{-j})^{1-\alpha} |\psi_{\delta 2^{j+2}} * g(x)| \leq C(\delta^{-1}2^{-j})^{1-\alpha} g_*(x).
\end{aligned}$$

We estimate

$$\begin{aligned}
|I_{inner}| &\leq \sum_{j\leq 0} \delta 2^j |\eta^j * g(x)| \\
&\leq C \sum_{j\leq 0} \delta 2^j (\delta^{-1}2^{-j})^{1-\alpha} g_*(x) \\
&\leq C \frac{1}{1-\left(\frac{1}{2}\right)^{\alpha}} \delta^\alpha g_*(x).
\end{aligned}$$

Now we estimate I_{outer}

$$
\begin{aligned}
|I_{outer}| &\leq C \sum_{j\geq 1} \int_{\delta 2^{j-1}\leq |x-y|\leq \delta 2^{j+2}} |\theta(\delta^{-1}2^{-j}(x-y))||a_\alpha(x-y)||g(y)|dy \\
&\leq C \sum_{j\geq 1} \int_{\delta 2^{j-1}\leq |x-y|\leq \delta 2^{j+2}} |x-y|^{\alpha-n}|g(y)|dy \\
&\leq C \sum_{j\geq 1} (\delta 2^{j-1})^{\alpha-n} \int_{|x-y|\leq \delta 2^{j+2}} |g(y)|dy \\
&\leq C \sum_{j\geq 1} (\delta 2^{j-1})^{\alpha-n} (\delta 2^{j+2})^{\beta} M_\beta(g)(x) \\
&\leq C \frac{1}{1-\left(\frac{1}{2}\right)^{\alpha-(n-\beta)}} \delta^{\alpha-(n-\beta)} M_\beta(g)(x).
\end{aligned}
$$

Putting together these threads gives us

$$|A_\alpha(g)| \leq C(\alpha,\beta,n)(\delta^\alpha g_*(x) + \delta^{\alpha-(n-\beta)}M_\beta(g)(x)).$$

Setting $\delta = \left(\frac{M_\beta(g)(x)}{g_*(x)}\right)^{\frac{1}{n-\beta}}$ gives

$$|A_\alpha(g)| \leq C(\alpha,\beta,n)(M_\beta(g)(x))^{\frac{\alpha}{n-\beta}}(g_*(x))^{\frac{n-\beta-\alpha}{n-\beta}}.$$

$\square$

Proof of Corollary 1.8. By [Sem94, Theorem 1.92] we know that $g \in h^1$ if and only if for any $\psi \in C_c^\infty$ with $\int \psi \neq 0$ there is a constant λ such that $\psi(g-\lambda) \in \mathcal{H}^1(\mathbb{R}^n)$, with

$$\|\psi(g-\lambda)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C\|g\|_{h^1(\mathbb{R}^n)},$$

where $C = C(\psi)$ and λ is chosen such that $\int \psi(g-\lambda) = 0$ i.e. $\lambda := \frac{\int g\psi}{\int \psi}$.

Therefore we let $\tilde{g} = \psi(g-\lambda) + \psi\lambda$ where $\psi \in C_c^\infty$, $\psi \equiv 1$ on K and $0 \leq \psi \leq 1$.

We have $\tilde{g} = g$ in K, $\psi(g-\lambda) \in \mathcal{H}^1$, and

$$\|\lambda\psi\|_{L^\infty(K)} \leq C(K)\|g\|_{L^1(K)} \leq C(K)\|g\|^{\frac{1}{2}}_{M^{1,n-2}}\|g\|^{\frac{1}{2}}_{h^1}.$$

Clearly then we have

$$\|A_1[\lambda\psi]\|_{L^2(K)} \leq C(n,K)\|g\|^{\frac{1}{2}}_{M^{1,n-2}}\|g\|^{\frac{1}{2}}_{h^1}.$$

Moreover by applying Theorem 1.7

$$
\begin{aligned}
\|A_1[\psi(g-\lambda)]\|_{L^2(K)} &\leq C(n)\|\psi(g-\lambda)\|^{\frac{1}{2}}_{M^{1,n-2}}\|\psi(g-\lambda)\|^{\frac{1}{2}}_{\mathcal{H}^1} \\
&\leq C(n,K)\|g\|^{\frac{1}{2}}_{M^{1,n-2}}\|g\|^{\frac{1}{2}}_{h^1}.
\end{aligned}
$$

Therefore $A_1[\tilde{g}] \in L^2(K)$ with

$$\|A_1[\tilde{g}]\|_{L^2(K)} \leq C\|g\|^{\frac{1}{2}}_{M^{1,n-2}}\|g\|^{\frac{1}{2}}_{h^1}.$$

Now set $w = N[\tilde{g}] = \Gamma * \tilde{g}$ where N is the Newtonian potential, we have that $\nabla_i w = \nabla_i N[\tilde{g}] = \nabla_i \Gamma * \tilde{g}$ is an operator of the form $A_1[g]$ for $a_1(x) = C(n)\frac{x_i}{|x|^n}$. Therefore for $\phi \in C_c^\infty(K)$ we can test

$$\int_K g\phi = \int_K (\psi(g-\lambda) + \lambda\psi)\phi = -\int_K \Delta w \phi = \int_K \nabla w . \nabla \phi \leq C\|\nabla w\|_{L^2(K)}\|\phi\|_{W^{1,2}(K)}.$$

Thus $\|g\|_{H^{-1}(K)} \leq C\|g\|_{M^{1,n-2}}^{\frac{1}{2}}\|g\|_{h^1}^{\frac{1}{2}}$. □

Proof of Corollary 1.9. We essentially follow [Ada75] here, except that we keep track of the constants. Let $g \in M^{p,\beta}$, $p \geq 1$ and $0 < \alpha p < n - \beta$. We will show that

$$\begin{cases} \|A_\alpha[g]\|_{L^{\tilde{p}}(B_r(x))}^{\tilde{p}} \leq C\|g\|_{M^{p,\beta}}^{\tilde{p}} r^\beta & \text{if } p > 1 \\ |\{z \in B_r(x) : |A_\alpha[g](z)| > s\}| \leq C\|g\|_{M^{1,\beta}}^{\tilde{p}} s^{-\tilde{p}} r^\beta & \text{if } p = 1. \end{cases}$$

To that end write $g_r = g\chi_{B_{2r}(x)}$ and $g^r = g - g_r$. For g_r we know that $\|g_r\|_{L^p} \leq Cr^{\frac{\beta}{p}}\|g\|_{M^{p,\beta}}$. Thus by standard estimates for the maximal function (see [Ste70, page 5] for example) we have

$$\begin{cases} \|M[g_r]\|_{L^p} \leq C(n)\left(\frac{p}{p-1}\right)^{\frac{1}{p}} r^{\frac{\beta}{p}}\|g\|_{M^{p,\beta}} & \text{if } p > 1 \\ |\{x : M[g_r](x) > s\}| \leq C(n)s^{-1}r^\beta\|g\|_{M^{1,\beta}} & \text{if } p = 1. \end{cases}$$

Now we can directly apply the estimate from Theorem 1.7 to conclude (using the trivial estimate $g_*(x) \leq C(n)M[g](x)$)

$$\begin{cases} \|A_\alpha[g_r]\|_{L^{\tilde{p}}(B_r(x))}^{\tilde{p}} \leq C(n,\alpha,\beta,p)^{\tilde{p}}\frac{p}{p-1}\|g\|_{M^{p,\beta}}^{\tilde{p}} r^\beta & \text{if } p > 1 \\ |\{z \in B_r(x) : |A_\alpha[g_r](z)| > s\}| \leq C(n,\alpha,\beta)^{\tilde{p}}\|g\|_{M^{1,\beta}}^{\tilde{p}} s^{-\tilde{p}} r^\beta & \text{if } p = 1. \end{cases}$$

Where $C(n,\alpha,\beta)$ is the constant appearing in Proposition 1.7.

If $z \in B_r(x)$ then

$$\begin{aligned} |A_\alpha[g^r](z)| &\leq C\int_{\mathbb{R}^n \setminus B_r(z)} \frac{1}{|z-y|^{n-\alpha}}|g(y)|\,\mathrm{d}y \\ &\leq C\sum_{j\geq 1}\int_{B_{2^j r}(z)\setminus B_{2^{j-1}r}(z)} \frac{1}{|z-y|^{n-\alpha}}|g(y)|\,\mathrm{d}y \\ &\leq C\sum (2^j r)^{\alpha-n}\|g\|_{M^{p,\beta}} r^{n-\frac{n-\beta}{p}} \\ &= C\frac{1}{1-\left(\frac{1}{2}\right)^{\alpha-\frac{n-\beta}{p}}}\|g\|_{M^{p,\beta}} r^{\alpha-\frac{n-\beta}{p}} \\ &\leq C(n,\alpha,\beta,p)\|g\|_{M^{p,\beta}} r^{\alpha-\frac{n-\beta}{p}}. \end{aligned}$$

Therefore for $p \geq 1$, $\|A_\alpha[g^r]\|_{L^{\tilde{p}}(B_r(x))}^{\tilde{p}} \leq C(n,\alpha,\beta,p)^{\tilde{p}}\|g\|_{M^{p,\beta}}^{\tilde{p}} r^\beta$. □

Appendix A. Morrey, Campanato and Hölder spaces. Here we state some important facts concerning Morrey spaces. Clearly $M^{p,0} = L^p$ and $M^{p,n} = L^\infty$, also we see that $M_n[\cdot] = M[\cdot]$ is the usual maximal function up to a constant. Also note that if we allow $\beta > n$ then $M^{p,\beta} = \{0\}$.

The related Campanato spaces $\mathcal{C}^{p,\beta}$ are variations on BMO, thus we try to capture an integral measure of oscillation similar to that for the Morrey spaces. For $g \in L^1(E)$ let $g_{r,x} = \fint_{B_r(x)\cap E} g$ and we say that $g \in \mathcal{C}^{p,\beta}(E)$ if $g \in L^p(E)$ and

$$[g]_{\mathcal{C}^{p,\beta}(E)} := \sup_{x\in E,\ r>0} \left(r^{-\beta} \int_{B_r(x)\cap E} |g - g_{r,x}|^p \right)^{\frac{1}{p}} < \infty$$

with norm (making $\mathcal{C}^{p,\beta}$ Banach spaces)

$$\|g\|_{\mathcal{C}^{p,\beta}(E)} = [g]_{\mathcal{C}^{p,\beta}(E)} + \|g\|_{L^p(E)}.$$

For Lipschitz domains we have $M^{p,\beta} = \mathcal{C}^{p,\beta}$, when $0 \le \beta < n$ [Gia83, Chapter III, Proposition 1.2]. However (modulo constants) $\mathcal{C}^{p,n} = BMO$ for all p as opposed to $M^{p,n} = L^\infty$. We actually have that $M^{p,\beta} \subset \mathcal{C}^{p,\beta}$ with a uniform estimate (in n, p and β). The reverse inclusion holds with an estimate whose constant blows up as β approaches n.

Moreover $\mathcal{C}^{p,\beta}$ makes sense for $\beta > n$ and when $n < \beta \le n+p$ we have $\mathcal{C}^{p,\beta} = C^{0,\gamma}$ with $\gamma = \frac{\beta-n}{p}$ [Gia83, Chapter III, Theorem 1.2]. If $\beta > n+p$ then $\mathcal{C}^{p,\beta}$ are the constant functions.

We say that $g \in M_k^{p,\beta}$ if $g, \nabla^k g \in M^{p,\beta}$. Using the Poincaré inequality we see that if $g \in M_1^{p,\beta}$ for some $0 \le \beta \le n$ then $g \in \mathcal{C}^{p,p+\beta}$. Therefore if $n-p < \beta \le n$ then $g \in C^{0,\frac{\beta+p-n}{p}}$. Also the borderline case ($\beta = n-p$) gives $g \in BMO$. These last facts yield another proof of the Morrey embedding theorem: suppose $g \in W^{1,q}$ for $q > n$. Then $g \in M_1^{n,n-\frac{n^2}{q}} \hookrightarrow C^{0,1-\frac{n}{q}}$.

We also introduce here the related weak Morrey spaces $M^{(p,\infty),\beta}(E)$, consisting of functions g in the Lorentz space $L^{(p,\infty)}(E)$ or 'weak L^p' with

$$\|g\|_{M^{(p,\infty),\beta}(E)} := \sup_{x,\ r>0} r^{-\frac{\beta}{p}} \|g\|_{L^{(p,\infty)}(B_r(x)\cap E)} < \infty.$$

This condition is equivalent to

$$|\{x \in B_r(x_0) \cap E : |g|(x) > s\}| \le C\|g\|^p_{M^{(p,\infty),\beta}(E)} s^{-p} r^\beta$$

with C independent of x_0 and r.

Even though the Campanato spaces do not interpolate the L^p spaces we still have good estimates on singular integrals. The following result of Peetre [Pee66] generalises both Calderon-Zygmund and Schauder estimates. We consider the singular integral operator given by the operation $g \mapsto \nabla^2 N[g]$. In words it is the operator that maps a function to second order derivatives of its Newtonian potential.

THEOREM A.1. *For $1 < p < \infty$ and $0 \le \beta < n+p$, $\nabla^2 N : \mathcal{C}^{p,\beta} \to \mathcal{C}^{p,\beta}$ is a bounded operator.*

Therefore we also have for $1 < p < \infty$ and $0 \le \beta < n$ that $\nabla^2 N : M^{p,\beta} \to M^{p,\beta}$ is bounded.

Appendix B. Hodge Decompositions and Coulomb gauge. We require the following from [RS08], which shows that we can still find the appropriate Coulomb gauge even in the Morrey space setting.

Lemma B.1. *Let* $\Omega \in M^{2,n-2}(B_1, so(m) \otimes \wedge^1\mathbb{R}^n)$. *Then there exists* $\epsilon > 0$ *such that whenever* $\|\Omega\|^2_{M^{2,n-2}(B_1)} \leq \epsilon$ *there exist* $P \in W^{1,2}(B_1, SO(m))$ *and* $\eta \in W^{1,2}_0(B_1, so(m) \otimes \wedge^2\mathbb{R}^n)$ *such that* $\mathrm{d}\eta = 0$ *on* B_1 *and*

$$P^{-1}\mathrm{d}P + P^{-1}\Omega P = \mathrm{d}^*\eta.$$

Moreover $\nabla P, \nabla\eta \in M^{2,n-2}(B_1)$ *with*

$$\|\nabla P\|^2_{M^{2,n-2}(B_1)} + \|\nabla\eta\|^2_{M^{2,n-2}(B_1)} \leq C\|\Omega\|^2_{M^{2,n-2}(B_1)} \leq C\epsilon.$$

Remark B.2. This appears different to the lemma appearing in [RS08], however upon replacing η with $(-1)^{3n+1} * \xi$ as it appears in [RS08] we see they are the same. The notation here should be explained, d is the exterior derivative, $\mathrm{d}^* = (-1)^{k(n-k)} * \mathrm{d}*$ is the divergence operator on k-forms (formal adjoint of exterior derivative) and for any form ω, $\nabla\omega$ refers to the collection of all first order derivatives, as opposed to $\mathrm{d}\omega$ or $\mathrm{d}^*\omega$ which refers to new forms comprised of certain combinations of first order derivatives of ω. Of course $*$ is the hodge star operator.

We recall here that there is a natural point-wise inner product for k-forms given by $\langle\omega^1, \omega^2\rangle = *(\omega^1 \wedge *\omega^2)$ and an L^2-inner product given by $(\omega^1, \omega^2) = \int *\langle\omega^1, \omega^2\rangle$.

Our main reference here is [Mor66, Chapter 7] where we can find all of the results stated below, in particular we require the following.

Lemma B.3. *Suppose* $\omega \in L^2(B_1, \bigwedge^1 \mathbb{R}^n)$ *then there are unique* $a \in W^{1,2}_0(B_1)$, $b \in W^{1,2}_N(B_1, \bigwedge^2 \mathbb{R}^n)$ *and a harmonic one form* $h \in L^2(B_1, \bigwedge^1 \mathbb{R}^n)$ *such that*

$$\omega = \mathrm{d}a + \mathrm{d}^*b + h.$$

Moreover $\mathrm{d}b = 0$ *with*

$$\|a\|_{W^{1,2}(B_1)} + \|b\|_{W^{1,2}(B_1)} + \|h\|_{L^2(B_1)} \leq C\|\omega\|_{L^2(B_1)}$$

and

$$\|\mathrm{d}a\|^2_{L^2(B_1)} + \|\mathrm{d}^*b\|^2_{L^2(B_1)} + \|h\|^2_{L^2(B_1)} = \|\omega\|^2_{L^2(B_1)}.$$

We note here that $W^{1,2}_N(B_1, \bigwedge^k \mathbb{R}^n)$ is the space of forms whose normal boundary part vanishes, which we may define in a trace sense or equivalently for any smooth $k-1$ form ν we have $(\omega, \mathrm{d}\nu) = (\mathrm{d}^*\omega, \nu)$ when $\omega \in W^{1,2}_N(B_1, \bigwedge^k \mathbb{R}^n)$.

Otherwise we have the more general formula for smooth k and $k-1$ forms respectively

$$(\omega, \mathrm{d}\nu) = (\mathrm{d}^*\omega, \nu) + \int_\partial \nu_T \wedge *\omega_N$$

where $_T$ and $_N$ denote the tangential and normal components. (The latter holds for any appropriate Sobolev forms by approximation). Note we could easily define $W^{1,2}_T(B_1, \bigwedge^k \mathbb{R}^n)$ in a weak sense also. We use the following fact: For $a \in W^{1,2}(B_1, \bigwedge^{k-1} \mathbb{R}^n)$, $b \in W^{1,2}(B_1, \bigwedge^k \mathbb{R}^n)$ we have $(\mathrm{d}a, \mathrm{d}^*b) = 0$ if either $a_T = 0$ or $b_N = 0$.

Note that we have $\Delta a = \mathrm{d}^*\omega$ and $\Delta b = \mathrm{d}\omega$ in a weak sense since $\mathrm{d}h = \mathrm{d}^*h = 0$, $\mathrm{d}b = 0$ and since a is a function ($\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$).

Appendix C. Absorption lemma. We have changed the hypotheses of the following lemma compared to how it appears in [Sim96] and [ShTo13], however upon inspection of the proof (which can be found in [Sim96]) it can be checked that the lemma as it is stated here is also proved.

LEMMA C.1. *(Leon Simon [Sim96, §2.8, Lemma 2].) Let $B_\rho(y) \subset \mathbb{R}^n$ be any ball, $k \in \mathbb{R}$, $\Gamma > 0$, and let φ be any $[0,\infty)$-valued convex sub additive function on the collection of balls in $B_\rho(y)$; thus $\varphi(A) \leq \sum_{j=1}^N \varphi(A_j)$ whenever $A, A_1, A_2, ..., A_N$ are balls in $B_\rho(y)$ with $A \subset \bigcup_{j=1}^N A_j$ and $A \bigcap A_j \neq \emptyset$ for any j. There is an $\epsilon_0 = \epsilon_0(k,n) > 0$ such that if*

$$\sigma^k \varphi(B_{\sigma/2}(z)) \leq \epsilon_0 \sigma^k \varphi(B_\sigma(z)) + \Gamma$$

whenever $B_{2\sigma}(z) \subset B_\rho(y)$, then there exists some $C = C(k,n) < \infty$ such that

$$\rho^k \varphi(B_{\rho/2}(y)) \leq C\Gamma.$$

In particular we can apply this lemma when $\varphi(A) = \|k\|^p_{M^{p,\beta}(A)}$.

Appendix D. Scaling. We will need to consider u, Ω and f solving

$$-\Delta u = \Omega.\nabla u + f$$

on some small ball $B_R(x_0) \subset B_1$. In order to do so we re-scale $\hat{u}(x) := u(x_0 + Rx)$, $\hat{\Omega}(x) := R\Omega(x_0 + Rx)$ and $\hat{f} := R^2 f(x_0 + Rx)$. First of all we see that

$$-\Delta \hat{u} = \hat{\Omega}.\nabla \hat{u} + \hat{f}$$

on B_1 and we list the scaling properties of the related norms as follows.

- $\|\hat{\Omega}\|_{M^{2,n-2}(B_1)} = \|\Omega\|_{M^{2,n-2}(B_R(x_0))}$.
- $[\hat{u}]_{C^{0,\gamma}(B_1)} = R^\gamma [u]_{C^{0,\gamma}(B_R(x_0))}$.
- $\|\hat{u}\|_{L^1(B_1)} = R^{-n}\|u\|_{L^1(B_R(x_0))}$.
- $\|\nabla \hat{u}\|_{M^{l,\nu}(B_1)} = R^{\frac{l-(n-\nu)}{l}} \|\nabla u\|_{M^{l,\nu}(B_R(x_0))}$ and setting $\nu = 0$ gives $\|\nabla \hat{u}\|_{L^l(B_1)} = R^{1-\frac{n}{l}} \|\nabla u\|_{L^l(B_R(x_0))}$.
- We also have that the Lorentz spaces $L^{(l,\infty)}$ or 'weak'-L^l scale in the same fashion as the usual L^l spaces, $\|\nabla \hat{u}\|_{L^{(l,\infty)}(B_1)} = R^{1-\frac{n}{l}} \|\nabla u\|_{L^{(l,\infty)}(B_R(x_0))}$.
- $\|\hat{f}\|_{L^p(B_1)} = R^{2-\frac{n}{p}} \|f\|_{L^p(B_R(x_0))}$.
- For $f \in L^p(B_1)$ and $1 \leq s \leq p$ we have $\|f\|_{M^{s,n(1-\frac{s}{p})}(B_1)} \leq C\|f\|_{L^p(B_1)}$ for $C = C(n,p)$.

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