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# THE F. AND M. RIESZ THEOREM FOR NONELLIPTIC VEKUA'S EQUATIONS*

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**Abstract.** We present a class of nonelliptic first order equations in the plane whose solutions satisfy the F. and M. Riesz property.

**Key words.** Vekua's equation, boundary value.

**AMS subject classifications.** Primary 35B60, 35F05; Secondary 30B40, 35H10.

**1. Introduction.** The celebrated F. and M. Riesz Theorem states that if $\mu$ is a measure on the unit circle $\mathcal{T}$ with the property that the Fourier coefficients

$$\hat{\mu}(k) = \int_0^{2\pi} e^{-ik\theta}\, d\mu(\theta) = 0 \quad \text{for } k = -1, -2, \dots \tag{1.1}$$

then $\mu$ is absolutely continuous with respect to the Lebesgue measure $d\theta$. Condition (1.1) is equivalent to the existence of a holomorphic function $f(z)$ defined on the unit disc $\Delta$ whose weak boundary value in the sense of distributions is $\mu$. This theorem has a local version which can be stated as follows: Suppose $f(z)$ is holomorphic on the rectangle $Q = (-a, a) \times (0, b)$ and has a weak boundary value denoted $bf$ which is a distribution. That is, given $\psi(x) \in C_0^\infty(-a, a)$,

$$\lim_{y \to 0^+} \int f(x + iy)\, \psi(x)\, dx = \langle bf, \psi \rangle. \tag{1.2}$$

The local version of the F. and M. Riesz Theorem states that if $bf$ is a measure, then it is absolutely continuous with respect to Lebesgue measure $dx$. That is, $bf \in L^1_{\text{loc}}(-a, a)$. One way of proving this local version uses two facts:

1. A distribution $bf$ which is a weak boundary value of a holomorphic function is microlocally smooth in one direction at each $x \in (-a, a)$, that is, its Fourier transform decays rapidly in one direction - a condition which is akin to the classical situation (1.1).
2. If the Fourier transform $\hat{\mu}(\xi)$ of a measure $\mu$ is rapidly decaying in one direction, then $\mu \in L^1_{\text{loc}}$.

In [BH1] we extended this local version of the F. and M. Riesz Theorem to the solutions of the equation $Lf = 0$ where

$$L = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}$$

is any smooth, complex, and locally integrable vector field. Recall that the vector field $L$ is called locally integrable in an open set $D$ in the plane if for each $p \in D$, there is a neighborhood $U$ which admits a smooth function $Z(x, y)$ which satisfies $LZ = 0$ and its differential $dZ \neq 0$. Examples of locally integrable vector fields include nonzero $(|a(x, y)| + |b(x, y)| \neq 0)$ real analytic vector fields and smooth, locally solvable vector

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fields. We mention that the class of locally integrable vector fields is much larger than these two classes and the reader is referred to [BCH] and [T] for more on these vector fields.

In [BH2], we established a local version of the F. and M. Riesz property for solutions of the classical elliptic Vekua's equation which we recall has the form

$$\frac{\partial u}{\partial \overline{z}} = A(x,t)u + B(x,t)\overline{u} \tag{1.3}$$

where $A$ and $B$ are smooth, complex-valued functions. Here we will use this result to extend the F. and M. Riesz property for solutions of a class of nonelliptic Vekua's equations of the form

$$Lu = A(x,t)u + B(x,t)\overline{u} \tag{1.4}$$

where $L$ is a complex vector field of the form $L = \frac{\partial}{\partial t} + \sqrt{-1}b(x)\frac{\partial}{\partial x}$ and $b(x)$ is a smooth, real-valued function.

Equation (1.3) arises in the theory of elasticity and the geometric problem of the existence of infinitesimal bendings of surfaces of positive curvature (see [V]). In recent years, in several articles, A. Meziani has shown that equation (1.4) is intimately linked with the existence of infinitesimal bendings for certain surfaces of nonnegative curvature (see for example M1 and [M2]).

## 2. Statement and proof of the main result.

THEOREM. *Let $L = \frac{\partial}{\partial t} + \sqrt{-1}b(x)\frac{\partial}{\partial x}$ where $b(x)$ is smooth and real-valued in a neighborhood of zero. Let $U$ be a neighborhood of $(0,0)$ in $\mathbb{R}^2$, $U_+ = \{(x,t) \in U : t > 0\}$, $A, B \in C^\infty(U)$. Suppose $f \in C^1(U_+)$ satisfies*

$$Lf(x,t) = A(x,t)f(x,t) + B(x,t)\overline{f(x,t)} \ \text{ in } U_+ \tag{2.1}$$

*and assume that*

$$|f(x,t)| = O(t^{-N}) \ \text{ as } t \to 0^+. \tag{2.2}$$

*If the weak boundary value $bf$ on $U \cap \mathbb{R}$ is a measure $\mu$, then it is in $L^1_{loc}$.*

REMARK. It was shown in [BH2] that a solution $f$ that satisfies (2.1) and (2.2) has a weak boundary value $bf$.

*Proof.* Since this is a local statement, we only need to prove that $bf$ is integrable in a neighborhood of the origin. We can thus assume that $U_+ = (-\alpha,\alpha) \times (0,\beta)$. If at a point $x_0 \in (-\alpha,\alpha)$, $b(x_0) \neq 0$, then $L$ is elliptic at $(x_0, 0)$ and in this case, by our results in [BH2], $bf$ is integrable in a neighborhood of $x_0$. We therefore only need to show that $bf$ is absolutely continuous with respect to Lebesgue measure in a neighborhood of each point in the set

$$F = \{x \in (-\alpha,\alpha) : b(x) = 0\}.$$

Define the operator $S$ by

$$Su = \frac{\partial u}{\partial t} + \sqrt{-1}b(x)\frac{\partial u}{\partial x} - Au - \overline{B}\overline{u}.$$

We will show that given $\varphi(x) \in C_0^\infty(-\alpha,\alpha)$, for each $k = 1, 2, \dots$, there is a $C^\infty$ function $u_k(x,t)$ such that

1. $u_k(x,0) = \varphi(x)$ and
2. $u_k(x,t) = \sum_{j=0}^{k} \varphi_j(x)\frac{t^j}{j!}$ where for $1 \le j \le k$, the $\varphi_j$ satisfy the estimate

$$|\varphi_j(x)| \le C_j \left( \sum_{m=0}^{j} |b(x)|^m |D^m \varphi(x)| \right), \ \varphi_0 = \varphi$$

   with $C_j$ depending only on the functions $b(x), A(x,t)$ and $B(x,t)$, and

$$\varphi_j(x) = \sum_{m=0}^{j} c_m^j(x) b(x)^m D^m \varphi(x) + \sum_{n=0}^{j-1} d_n^j(x) b(x)^n D^n \overline{\varphi(x)}$$

   where the $c_m^j$ and $d_n^j$ depend only on $b(x), A(x,t)$ and $B(x,t)$.
3. 

$$Su_k(x,t) = \left( \sum_{m=0}^{k+1} p_m^k(x,t) b(x)^m D^m \varphi(x) + \sum_{n=0}^{k} q_n^k(x,t) b(x)^n D^n \overline{\varphi(x)} \right) t^k$$

   where the $p_m^k$ and $q_n^k$ depend only on $b(x), A(x,t)$ and $B(x,t)$.

To see this we proceed by induction and define

$$u_1(x,t) = \varphi_0(x) + \varphi_1(x)t = \varphi(x) + \varphi_1(x)t,$$

where

$$\varphi_1(x) = -\sqrt{-1}b(x)\varphi'(x) + A(x,0)\varphi(x) + \overline{B(x,0)\varphi(x)}.$$

Write

$$A(x,t) = A(x,0) + A_1(x,t)t, \quad B(x,t) = B(x,0) + B_1(x,t)t$$

We have:

$$\begin{aligned}
&Su_1(x,t)\\
&= \varphi_1(x) + \sqrt{-1}b(x)(\varphi'(x) + \varphi_1'(x)t) - A(x,t)\varphi(x) - A(x,t)\varphi_1(x)t\\
&- \overline{B(x,t)\varphi(x)} - \overline{B(x,t)\varphi_1(x)}t\\
&= \left(\sqrt{-1}b(x)\varphi_1'(x) - A_1(x,t)\varphi(x) - A(x,t)\varphi_1(x) - \overline{B_1(x,t)\varphi(x)} - \overline{B(x,t)\varphi_1(x)}\right) t\\
&= (b(x)^2 D^2\varphi(x) + (b'(x) + \sqrt{-1}A(x,0))b(x)D\varphi(x) + \sqrt{-1}\,\overline{B(x,0)}b(x)\overline{D\varphi(x)}\\
&+ \sqrt{-1}b(x)A_x(x,0)\varphi(x) + \sqrt{-1}b(x)\overline{B_x(x,0)\varphi(x)} - A_1(x,t)\varphi(x)\\
&+ \sqrt{-1}A(x,t)b(x)D\varphi(x) - A(x,t)A(x,0)\varphi(x) - A(x,t)\overline{B(x,0)\varphi(x)}\\
&- \overline{B_1(x,t)\varphi(x)} - \sqrt{-1}b(x)\overline{B(x,t)D\varphi(x)} - \overline{B(x,t)A(x,0)\varphi(x)} - \overline{B(x,t)}B(x,0)\varphi(x))t\\
&= (b(x)^2 D^2\varphi(x) + (b'(x) + \sqrt{-1}A(x,0) + \sqrt{-1}A(x,t))b(x)D\varphi(x)\\
&+ (\sqrt{-1}\,\overline{B(x,0)} - \sqrt{-1}\,\overline{B(x,t)})b(x)\overline{D\varphi(x)}\\
&+ (\sqrt{-1}b(x)A_x(x,0) - A_1(x,t) - A(x,t)A(x,0) - \overline{B(x,t)}B(x,0))\varphi(x)\\
&+ (\sqrt{-1}b(x)\overline{B_x(x,0)} - A(x,t)\overline{B(x,0)} - \overline{B_1(x,t)} - \overline{B(x,t)A(x,0)})\overline{\varphi(x)})t.
\end{aligned}$$

Thus (1), (2), and (3) hold for $u_1$. Suppose $u_1, \dots, u_k$ have been defined so that they satisfy (1), (2), and (3). Let

$$Su_k(x,t) = \left( \sum_{m=0}^{k+1} p_m^k(x,t) b(x)^m D^m \varphi(x) + \sum_{n=0}^{k} q_n^k(x,t) b(x)^n D^n \overline{\varphi(x)} \right) t^k$$

where the $p_m^k$ and $q_n^k$ depend only on $b(x), A(x,t)$ and $B(x,t)$. Define

$$\varphi_{k+1}(x) = -k! \left( \sum_{m=0}^{k+1} p_m^k(x,0) b(x)^m D^m \varphi(x) + \sum_{n=0}^{k} q_n^k(x,0) b(x)^n D^n \overline{\varphi(x)} \right). \tag{2.3}$$

Let

$$u_{k+1}(x,t) = u_k(x,t) + \varphi_{k+1}(x) \frac{t^{k+1}}{(k+1)!}.$$

Then

$$\begin{aligned} Su_{k+1}(x,t) =& Su_k(x,t) + \varphi_{k+1}(x)\frac{t^k}{k!} + \sqrt{-1} b(x) \left( \varphi'_{k+1}(x) \frac{t^{k+1}}{(k+1)!} \right) \\ & - A(x,t)\varphi_{k+1}(x)\frac{t^{k+1}}{(k+1)!} - \overline{B(x,t)} \frac{\overline{\varphi_{k+1}(x)}}{(k+1)!} t^{k+1}. \end{aligned}$$

Write

$$p_m^k(x,t) = p_m^k(x,0) + p_{m1}^k(x,t)t \ \text{ for } 0 \le m \le k+1,$$

and

$$q_n^k(x,t) = q_n^k(x,0) + q_{n1}^k(x,t)t \ \text{ for } 0 \le n \le k.$$

Using the expression for $Su_k(x,t)$, we get:

$$\begin{aligned} Su_{k+1}(x,t) = & \left( \sum_{m=0}^{k+1} p_{m1}^k(x,t) b(x)^m D^m \varphi(x) + \sum_{n=0}^{k} q_{n1}^{k+1}(x,t) b(x)^n D^n \overline{\varphi(x)} \right) t^{k+1} \\ & + \sqrt{-1} b(x) \left( \varphi'_{k+1}(x) \frac{t^{k+1}}{(k+1)!} \right) \\ & - A(x,t)\varphi_{k+1}(x)\frac{t^{k+1}}{(k+1)!} - \overline{B(x,t)} \frac{\overline{\varphi_{k+1}(x)}}{(k+1)!} t^{k+1}. \end{aligned} \tag{2.4}$$

From (2.3) and (2.4) we see that for some smooth functions $p_m^{k+1}(x,t)$ and $q_n^{k+1}(x,t)$, $0 \le m \le k+1$, $0 \le n \le k$ depending only on $b(x), A(x,t)$, and $B(x,t)$,

$$Su_{k+1}(x,t) = \left( \sum_{m=0}^{k+2} p_m^{k+1}(x,t) b(x)^m D^m \varphi(x) + \sum_{n=0}^{k+1} q_n^{k+1}(x,t) b(x)^n D^n \overline{\varphi(x)} \right) t^{k+1}. \tag{2.5}$$

Thus the function $u_{k+1}(x,t)$ satisfies (1), (2) and (3).

The vector field $L = \frac{\partial}{\partial t} + \sqrt{-1} b(x) \frac{\partial}{\partial x}$ is locally solvable and so (see [T]) there is a $C^\infty$ function $Z(x,t)$ that satisfies

$$LZ(x,t) = 0, \ Z_x(x,t) \ne 0 \text{ near } (0,0).$$

Let $M = \frac{1}{Z_x(x,t)}\frac{\partial}{\partial x}$. If $h(x,t)$ is a $C^1$ function, its differential

$$dh = (Mh)\,dZ + (Lh)\,dt. \tag{2.6}$$

If $w(x,t)$ is a $C^1$ function, since $Lf = Af + B\overline{f}$, for $\epsilon > 0$, and $Q_\epsilon = (-\alpha_1, \alpha_1)\times(\epsilon, \beta_1)$, by Stokes theorem we have:

$$\begin{aligned}
&\int_{\partial Q_\epsilon} f(x,t)w(x,t)\,dZ \\
&= \iint_{Q_\epsilon} d\,(fw\,dZ) \\
&= \iint_{Q_\epsilon} L(fw)\,dt\wedge dZ \\
&= \iint_{Q_\epsilon} (Af + B\overline{f})w\,dt\wedge dZ + \iint_{Q_\epsilon} f(Lw)\,dt\wedge dZ \\
&= \iint_{Q_\epsilon} (Af + B\overline{f})wZ_x(x,t)\,dt\wedge dx + \iint_{Q_\epsilon} f(Lw)Z_x(x,t)\,dt\wedge dx.
\end{aligned} \tag{2.7}$$

If the $x-$ support of $w(x,t)$ is a compact set in $(-\alpha_1, \alpha_1)$,

$$\begin{aligned}
&\int_{\partial Q_\epsilon} f(x,t)w(x,t)\,dZ(x,t) \\
&= \int_{-\alpha_1}^{\alpha_1} f(x,\beta_1)w(x,\beta_1)Z_x(x,\beta_1)\,dx - \int_{-\alpha_1}^{\alpha_1} f(x,\epsilon)w(x,\epsilon)Z_x(x,\epsilon)\,dx
\end{aligned} \tag{2.8}$$

and so

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\partial Q_\epsilon} f(x,t)w(x,t)\,dZ(x,t) \\
&= \int_{-\alpha_1}^{\alpha_1} f(x,\beta_1)w(x,\beta_1)Z_x(x,\beta_1)\,dx - \langle bf, Z_x(x,0)w(x,0)\rangle.
\end{aligned} \tag{2.9}$$

If in addition, $w$ satisfies the equation

$$Lw = -Aw + \left(\frac{\overline{BZ_x}}{Z_x}\right)\overline{w} + E(x,t) \text{ on } [-\alpha_1, \alpha_1]\times[0, \beta_1] \tag{2.10}$$

with $|E(x,t)| \leq Ct^N$ for some $C > 0$, then $f(x,t)E(x,t) \in L^1([-\alpha_1, \alpha_1]\times[0, \beta_1])$ and so

$$\begin{aligned}
&\iint_{Q_\epsilon} (Af + B\overline{f})wZ_x(x,t)\,dt\wedge dx + \iint_{Q_\epsilon} f(Lw)Z_x\,dt\wedge dx \\
&= 2\Re\left(\iint_{Q_\epsilon} B\overline{f}w\,dZ\right) + \iint_{Q_\epsilon} f(x,t)E(x,t)\,dt\wedge dZ.
\end{aligned} \tag{2.11}$$

Since $f(x,t)E(x,t) \in L^1([-\alpha_1, \alpha_1]\times[0, \beta_1])$,

$$\lim_{\epsilon\to 0}\iint_{Q_\epsilon} f(x,t)E(x,t)\,dt\wedge dZ = \int_{-\alpha_1}^{\alpha_1}\int_0^{\beta_1} f(x,t)E(x,t)\,dt\wedge dZ. \tag{2.12}$$

Thus if the $x-$ support of $w$ is compact in $(-\alpha_1, \alpha_1)$, and it satisfies equation (2.10), from (2.8) to (2.12), we conclude that

$$\begin{aligned}(2.13)\qquad &\int_{-\alpha_1}^{\alpha_1} f(x,\beta_1)w(x,\beta_1)Z_x(x,\beta_1)\,dx - \langle bf, Z_x(x,0)w(x,0)\rangle \\ &= \int_{-\alpha_1}^{\alpha_1}\int_0^{\beta_1} f(x,t)E(x,t)\,dt\wedge dZ + 2\lim_{\epsilon\to 0}\Re\left(\iint_{Q_\epsilon} B\overline{f}w\,dZ\right).\end{aligned}$$

Recall now that $F = \{x : b(x) = 0\}$. Let $K \subset F$ be a compact set with Lebesgue measure $|K| = 0$. Let $\{\varphi_\epsilon(x)\}_{\epsilon>0}$ be functions in $C_0^\infty(-\alpha_1, \alpha_1)$ satisfying $0 \le \varphi_\epsilon \le 1$, $\varphi_\epsilon(x) = 1$ on $K$, $\varphi_\epsilon(x) = 0$ when $d(x, K) > \epsilon$, and $|D^\alpha\varphi_\epsilon(x)| \le C_\alpha\epsilon^{-\alpha}$. Let $\psi(x) \in C_0^\infty(-\alpha_1, \alpha_1)$. Set

$$w_N^\epsilon(x,t) = \sum_{j=0}^{N}\varphi_j^\epsilon(x)\frac{t^j}{j!},\ \ \varphi_0^\epsilon = \varphi_\epsilon\psi$$

where the $\varphi_j^\epsilon$ satisfy estimate (2) and

$$\frac{\partial w_N^\epsilon}{\partial t} + \sqrt{-1}b(x)\frac{\partial w_N^\epsilon}{\partial x} = -Aw_N^\epsilon + \left(\frac{\overline{BZ_x}}{Z_x}\right)w_N^\epsilon + E_N^\epsilon(x,t),$$

with

$$E_N^\epsilon(x,t) = \left(\sum_{m=0}^{N+1} p_m(x,t)b(x)^m D^m\varphi_0^\epsilon(x) + \sum_{n=0}^{N} q_n(x,t)b(x)^n D^n\overline{\varphi_0^\epsilon(x)}\right)t^N.$$

Plugging $w = w_N^\epsilon$ in (2.13) leads to

$$\begin{aligned}(2.14)\qquad &\int_{-\alpha_1}^{\alpha_1} f(x,\beta_1)w_N^\epsilon(x,\beta_1)\,dZ - \langle bf, Z_x(x,0)\varphi_\epsilon\psi\rangle \\ &= 2\Re\left(\int_{-\alpha_1}^{\alpha_1}\int_0^{\beta_1} B\overline{f}w_N^\epsilon\,dt\wedge dZ\right) + \int_{-\alpha_1}^{\alpha_1}\int_0^{\beta_1} f(x,t)E_N^\epsilon(x,t)\,dt\wedge dZ.\end{aligned}$$

We consider next each term in (2.14). Observe that

$$\begin{aligned}\left|\int_{-\alpha_1}^{\alpha_1} f(x,\beta_1)w_N^\epsilon(x,\beta_1)\,dZ\right| &\le C\sum_{m=0}^{N}\int_{-\alpha_1}^{\alpha_1}|b(x)|^m|D^m(\varphi_\epsilon(x)\psi(x))|\,dx \\ (2.15)\qquad &\le C'\sum_{m=0}^{N}\int_{-\alpha_1}^{\alpha_1} d(x,F)^m\ |D^m(\varphi_\epsilon(x)\psi(x))|\ dx.\end{aligned}$$

Here $d(x, F)$ =dist$(x, F)$. For each $0 \le m \le N$,

$$d(x,F)^m\,|D^m(\varphi_\epsilon(x)\psi(x))| \le C_1\frac{d(x,F)^m}{\epsilon^m} \le C_1\frac{d(x,F)^m}{d(x,K)^m} \le C_1.$$

Moreover, as $\epsilon \to 0$, $d(x, F)^m D^m(\varphi_\epsilon(x)\psi(x)) \to 0$ pointwise. It follows that

$$(2.16)\qquad \lim_{\epsilon\to 0}\int_{-\alpha_1}^{\alpha_1} f(x,\beta_1)w_N^\epsilon(x,\beta_1)\,dZ = 0.$$

The functions $f(x,t)E_N^\epsilon(x,t)$ also satisfy a similar estimate independent of $\epsilon$ which implies that

$$\lim_{\epsilon\to 0}\int_{-\alpha_1}^{\alpha_1}\int_0^{\beta_1} f(x,t)E_N^\epsilon(x,t)\,dt\wedge dZ = 0. \tag{2.17}$$

From (2.14), (2.16), and (2.17), since $\varphi_\epsilon(x)$ converges to the characteristic function of $K$, and $bf = d\mu$ is a measure, we get:

$$\int_K \psi Z_x d\mu = 2\lim_{\epsilon\to 0}\Re\left(\int_{-\alpha_1}^{\alpha_1}\int_0^{\beta_1} B(x,t)\overline{f(x,t)}w_N^\epsilon(x,t)\,dt\wedge dZ\right).$$

That is, for any $\psi\in C_0^\infty(-\alpha_1,\alpha_1)$, the integral $\int_K \psi\,d\mu$ is real-valued, which implies that $\int_K \psi\,d\mu = 0$. The latter in turn implies that $\int_K g\,d\mu = 0$ for every continuous function $g$ of compact support $K$ in $F$. Hence, $|\mu|(E) = 0$ whenever $E\subset F$ is a Borel set with Lebesgue measure $|E| = 0$. This proves that $\mu$ is absolutely continuous with respect to Lebesgue measure. By the Rado-Nikodym theorem, it follows that the measure $\mu$ is a locally integrable function. ∎

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