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ON LOWER BOUNDS FOR THE SOLUTION, AND ITS SPATIAL DERIVATIVES, OF THE MAGNETOHYDRODYNAMICS EQUATIONS IN LEBESGUE SPACES*

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Abstract. In this paper, the authors establish lower bounds for the usual Lebesgue norms of the maximal solution of the Magnetohydrodynamics Equations and present some criteria for global existence of solution. Thus, we can understand better on the blow-up behavior of this same solution. In addition, it is important to point out that we reach our main results by using standard techniques obtained from Navier-Stokes Equations.

Key words. Magnethohydrodinamics equations, Blow-up criteria, Global existence of solution.

Mathematics Subject Classification. 35Q40, 35Q60, 35Q61, 76W05.

1. Introduction.

In this paper, we consider the three–dimensional Magnetohydrodynamics (MHD) equations for incompressible flows:

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla(p + \frac{1}{2}|\,\mathbf{b}\,|^2) \;=\; \mu\Delta\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{b}, \\ \mathbf{b}_t + \mathbf{u}\cdot\nabla\mathbf{b} \;=\; \nu\,\Delta\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{u}, \\ \nabla\cdot\,\mathbf{u} \;=\; \nabla\cdot\,\mathbf{b} \;=\; 0, \\ \mathbf{u}(\cdot,0) \;=\; \mathbf{u}_0(\cdot),\;\; \mathbf{b}(\cdot,0) \;=\; \mathbf{b}_0(\cdot), \end{cases} \tag{1}$$

where $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \in \mathbb{R}^3$ denotes the incompressible velocity field, $\mathbf{b}(x,t) = (b_1(x,t), b_2(x,t), b_3(x,t)) \in \mathbb{R}^3$ the divergence free magnetic field and $p(x,t) \in \mathbb{R}$ the hydrostatic pressure. As usual, $x \in \mathbb{R}^3$ denotes the space variable and $t \in [0,T]$ the time variable. The positive constants μ and ν are associated with specific properties of the fluid; more precisely, μ is the kinematic viscosity and ν^{-1} is the magnetic Reynolds number. The initial data for the velocity and magnetic fields, given by $\mathbf{u}_0$ and $\mathbf{b}_0$ in (1), are divergence free, i.e., $\nabla\cdot\,\mathbf{u}_0 = \nabla\cdot\,\mathbf{b}_0 = 0$.

The existence and uniqueness of solutions for the system (1) have been extensively studied in [1, 2, 3, 5, 13, 15, 16, 17, 18]. It is important to recall that the existence of smooth solutions for the equations (1) is still an open problem. However, one can ensure the existence of a maximal time $T^* > 0$ for which the MHD system (1) has a classical solution $(\mathbf{u},\mathbf{b})(x,t)$ defined for $(x,t) \in \mathbb{R}^3\times[0,T^*)$. The discussion presented above is related to the incompressible Navier-Stokes problem. In fact, in the absence of a magnetic field, the MHD equations become the classical Navier-Stokes equations.

In this paper, the authors present extensions for most of the results obtained in [12]. More specifically, J. Lorenz and P. R. Zingano [12] show how to establish, by using standard arguments, properties at potential blow-up times; as well as, global existence, for the solution of the Navier-Stokes equations by proving some lower bounds

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considering the usual Lebesgue norms. Our interest is to demonstrate that these techniques can be adapted to the case of the MHD equations.

Now, let us be straight and list our main results whose statements are presented and established as propositions in Section 3 below.

THEOREM 1.1. *Let* $(\mathbf{u},\mathbf{b})(t)$ *be the strong solution for the MHD system* (1) *defined in the maximal interval* $[0,T^*)$. *Let* $0<T<T^*$. *Then, it holds:*

i) If $M_1 := \sup\limits_{0\le t<T}\|(\mathbf{u},\mathbf{b})(t)\|_\infty<\infty$, *we have* $\sum\limits_{|\alpha|=n}\|(D^\alpha\mathbf{u},D^\alpha\mathbf{b})(t)\|_2^2\le K_1$;

ii) Let $\frac{3}{2}<q\le 2$. *If* $M_2:=\sup\limits_{0\le t<T}\|(D\mathbf{u},D\mathbf{b})(t)\|_q<\infty$, *we obtain* $\sup\limits_{0\le t<T}\|(\mathbf{u},\mathbf{b})(t)\|_\infty\le K_2$;

iii) Let $\frac{3}{2}<q\le 2$. *If* $M_3:=\sup\limits_{0\le t<T}\|(D\mathbf{u},D\mathbf{b})(t)\|_q<\infty$, *we conclude* $\sup\limits_{0<t<T} t^{\frac{3}{4}}\|(\mathbf{u},\mathbf{b})(t)\|_\infty\le K_3$;

iv) Let $3<q\le\infty$. *If* $M_4:=\sup\limits_{0\le t<T}\|(\mathbf{u},\mathbf{b})(t)\|_q<\infty$, *we infer* $\sup\limits_{0\le t<T}\|(D\mathbf{u},D\mathbf{b})(t)\|_2\le K_4$,

for all $n\in\mathbb{N}$, *where* $K_1=K_1(n,M_1,T,\mu,\nu,\|(\mathbf{u}_0,\mathbf{b}_0)\|_{H^n})$, $K_2=K_2(\|(\widehat{\mathbf{u}}_0,\widehat{\mathbf{b}}_0)\|_1,M_2,q,T,\ \mu,\nu)$, $K_3=K_3(\|(\mathbf{u}_0,\mathbf{b}_0)\|_2,\ M_3,q,\mu,\nu,T)$ *and* $K_4=K_4(M_4,\|(D\mathbf{u}_0,D\mathbf{b}_0)\|_2,q,T,\mu,\nu)$.

As a result, from Theorem 1.1, we can write the following corollary which establishes sufficient conditions to assure global existence in time for the system (1).

COROLLARY 1.2. *Let* $(\mathbf{u},\mathbf{b})(t)$ *be the strong solution for the MHD system* (1) *defined in the maximal interval* $[0,T^*)$. *Assuming that* $T^*<\infty$, *we obtain:*

i) For $\frac{3}{2}<q\le 2$, *one has* $\sup\limits_{0\le t<T^*}\|(D\mathbf{u},D\mathbf{b})(t)\|_q=\infty$;

ii) For $3<q\le\infty$, *one concludes* $\sup\limits_{0\le t<T^*}\|(\mathbf{u},\mathbf{b})(t)\|_q=\infty$.

We present right below all the lower bounds obtained in this paper for $(\mathbf{u},\mathbf{b})(t)$ and $(D\mathbf{u},D\mathbf{b})(t)$ when one considers that the time of maximal existence for the solution $(\mathbf{u},\mathbf{b})(t)$ of (1), $t=T^*$, is finite.

THEOREM 1.3. *Let* $(\mathbf{u},\mathbf{b})(t)$ *be the strong solution for the MHD system* (1) *defined in the maximal interval* $[0,T^*)$. *Assuming that* $T^*<\infty$ *and* $0\le t<T^*$, *it holds the following statements:*

i) Leray's Inequality: $\|(D\mathbf{u},D\mathbf{b})(t)\|_2\ge C\gamma^{\frac{3}{4}}(T^*-t)^{-\frac{1}{4}}$;

ii) Let $3<q<\infty$. *Then,* $\|(\mathbf{u},\mathbf{b})(t)\|_q\ge K_5(T^*-t)^{-\frac{q-3}{2q}}$;

iii) Let $0<\epsilon<\frac{1}{2}$. *Hence,* $\|(D\mathbf{u},D\mathbf{b})(t)\|_3\ge K_6(T^*-t)^{-\frac{1}{2}+\epsilon}$;

iv) Let $\frac{3}{2}<q<3$. *Thus,* $\|(D\mathbf{u},D\mathbf{b})(t)\|_q\ge K_7(T^*-t)^{-\frac{2q-3}{2q}}$;

v) Let $3<r<\infty$ *and* $\frac{3r}{r+3}\le q\le\infty$. *Therefore,* $\|(D\mathbf{u},D\mathbf{b})(t)\|_q\ge K_8(T^*-t)^{-\frac{(r-3)(5q-6)}{6q(r-2)}}$,

where $\gamma=\min\{\mu,\nu\}$, C *is a positive constant,* $K_5=K_5(q,\mu,\nu)$, $K_6=K_6(\epsilon,\mu,\nu,\|(\mathbf{u}_0,\mathbf{b}_0)\|_2)$, $K_7=K_7(q,\mu,\nu)$ *and* $K_8=K_8(r,q,\mu,\nu,\|(\mathbf{u}_0,\mathbf{b}_0)\|_2)$.

REMARK 1.4. It follows directly from Theorem 1.3 the limits below:

1. $\lim\limits_{t\nearrow T^*}\|(\mathbf{u},\mathbf{b})(t)\|_q=\infty$, for $3<q<\infty$;

2. $\lim_{t \nearrow T^*} \|(D\mathbf{u}, D\mathbf{b})(t)\|_q = \infty$, for $\frac{3}{2} < q \leq \infty$.

Let us enunciate all the lower bounds, in this paper obtained, related to $(D^n\mathbf{u}, D^n\mathbf{b})(t)$, $n \geq 2$, when we consider that the maximal time of existence for the strong solution $(\mathbf{u}, \mathbf{b})(t)$ of (1), $t = T^*$, is finite.

THEOREM 1.5. *Let $(\mathbf{u}, \mathbf{b})(t)$ be the strong solution for the MHD system (1) defined in the maximal interval $[0, T^*)$. Considering that $T^* < \infty$ and $0 \leq t < T^*$, it holds the following statements:*

i) Let $1 < q < \frac{3}{2}$. Then, $\|(D^2\mathbf{u}, D^2\mathbf{b})(t)\|_q \geq K_9(T^* - t)^{-\frac{3q-3}{2q}}$;

ii) Let $n \geq 3$, $3 < q < \infty$ and $1 \leq r \leq \infty$. Hence, $\|(D^n\mathbf{u}, D^n\mathbf{b})(t)\|_r \geq K_{10}(T^* - t)^{-\frac{(q-3)(3r+2nr-6)}{6r(q-2)}}$;

iii) Let $3 < q < \infty$ and $r \geq \frac{3q}{2q+3}$. Thus, $\|(D^2\mathbf{u}, D^2\mathbf{b})(t)\|_r \geq K_{11}(T^* - t)^{-\frac{(q-3)(7r-6)}{6r(q-2)}}$,

where $K_9 = K_9(q, \mu, \nu)$, $K_{10} = K_{10}(q, r, n, \mu, \nu, \|(\mathbf{u}_0, \mathbf{b}_0)\|_2)$ and $K_{11} = K_{11}(q, r, \mu, \nu, \|(\mathbf{u}_0, \mathbf{b}_0)\|_2)$.

REMARK 1.6. It follows from Theorem 1.5 that:

1. $\lim_{t \nearrow T^*} \|(D^2\mathbf{u}, D^2\mathbf{b})(t)\|_q = \infty$, for $1 < q \leq \infty$;
2. $\lim_{t \nearrow T^*} \|(D^n\mathbf{u}, D^n\mathbf{b})(t)\|_q = \infty$, for $1 \leq q \leq \infty$ and $n \geq 3$.

In the next theorem, we expose a way to compare the blow up criteria presented in Theorem 1.3; items *i)* and *ii)*.

THEOREM 1.7. *Let $(\mathbf{u}, \mathbf{b})(t)$ be the strong solution for the MHD system (1) defined in the maximal interval $[0, T^*)$. Assuming that $T^* < \infty$. Then, for all $0 \leq t < T^*$, it holds the statements below:*

i) Let $3 \leq q < r < \infty$. Then, $\dfrac{\|(\mathbf{u}, \mathbf{b})(t)\|_r}{\|(\mathbf{u}, \mathbf{b})(t)\|_q} \geq K_{12}(T^* - t)^{-\frac{r-3}{r-2} \cdot \frac{r-q}{qr}}$;

ii) Let $2 \leq q \leq 6$. Thus, $\dfrac{\|(D\mathbf{u}, D\mathbf{b})(t)\|_2}{\|(\mathbf{u}, \mathbf{b})(t)\|_q} \geq K_{13}(T^* - t)^{-\frac{6-q}{8q}}$,

where $K_{12} = K_{12}(q, r, \mu, \nu, \|(\mathbf{u}_0, \mathbf{b}_0)\|_2)$ and $K_{13} = K_{13}(q, r, \mu, \nu, \|(\mathbf{u}_0, \mathbf{b}_0)\|_2)$.

REMARK 1.8. From Theorem 1.7, we can guarantee that:

1. $\lim_{t \nearrow T^*} \dfrac{\|(\mathbf{u}, \mathbf{b})(t)\|_r}{\|(\mathbf{u}, \mathbf{b})(t)\|_q} = \infty$, for $3 \leq q < r < \infty$;
2. $\lim_{t \nearrow T^*} \dfrac{\|(D\mathbf{u}, D\mathbf{b})(t)\|_2}{\|(\mathbf{u}, \mathbf{b})(t)\|_q} = \infty$, for $2 \leq q \leq 6$.

The result written below establishes two sufficient conditions in order to obtain global existence, in time, for the strong solution of the system (1).

THEOREM 1.9. *Let $(\mathbf{u}, \mathbf{b})(t)$ be the strong solution for the MHD system (1) defined in the maximal interval $[0, T^*)$. Let $3 < q < \infty$. Then $T^* = \infty$, provided that*

i) $\|(\mathbf{u}_0, \mathbf{b}_0)\|_2^{\frac{2q-6}{3q-6}} \|(\mathbf{u}_0, \mathbf{b}_0)\|_q^{\frac{q}{3q-6}} < 2\gamma(qC_qC_q^{'})^{-1}$ or

ii) $\sup_{0 \leq t < T^*} \left\{ \dfrac{\|(\mathbf{u}, \mathbf{b})(t)\|_\infty^q}{\|(\mathbf{u}, \mathbf{b})(t)\|_q^q} \cdot \dfrac{\|(\mathbf{u}, \mathbf{b})(t)\|_3}{\|(\mathbf{u}, \mathbf{b})(t)\|_\infty^2} \right\} < \infty$,

where $\gamma = \min\{\mu, \nu\}$; C_q and $C_q^{'}$ are given in (43) and (45), respectively.

Remark 1.10. It is important to point out that Theorem 1.9 *i)* is going to be proved after we assure that $\|(\mathbf{u},\mathbf{b})(t)\|_q$ $(3<q<\infty)$ is decreasing in $[0,T^*)$. Actually, we can also prove (see Lemma 3.13) that $\|(\mathbf{u},\mathbf{b})(t)\|_3$ is decreasing if it is assumed

$$\|(\mathbf{u}_0,\mathbf{b}_0)\|_3 < \frac{4}{3}\gamma C_3^{-1}, \tag{2}$$

where $\gamma=\min\{\mu,\nu\}$ and C_3 is the constant given in (43).

Remark 1.11. By recalling that the MHD equations (1) admit a unique local solution in time $(\mathbf{u},\mathbf{b})(t)$ for any given data $(\mathbf{u}_0,\mathbf{b}_0)\in H^s(\mathbb{R}^3)$ $(:=\{f\in S'(\mathbb{R}^3): \int_{\mathbb{R}^3}(1+|\xi|^s)^2|\widehat{f}(\xi)|^2\,d\xi<\infty\}$, where $S'(\mathbb{R}^3)$ is the set of tempered distributions) such that $\nabla\cdot\mathbf{u}_0=\nabla\cdot\mathbf{b}_0=0$, with $s\geq 3$ (see [15]), it is important to point out the statements below:

- Y. Zhou and J. Fan [19] proved that if $T>0$, $(\mathbf{u}_0,\mathbf{b}_0)\in H^3(\mathbb{R}^3)$ and $\mathbf{u}(t)$ satisfies one of the following items
  i) $\displaystyle\int_0^T \frac{\|\mathbf{u}(t)\|_r^{\frac{2r}{r-3}}}{1+\ln(e+\|\mathbf{u}(t)\|_\infty)}dt<\infty,$ for some $r>3$;
  ii) $\displaystyle\int_0^T \frac{\|D\mathbf{u}(t)\|_r^{\frac{2r}{2r-3}}}{1+\ln(e+\|D\mathbf{u}(t)\|_\infty)}dt<\infty,$ for some $r>3/2$,
  then this solution can be extended for $T'>T$. Under the same assumptions, if it is considered that $p(t)$ is the pressure associated with $(\mathbf{u},\mathbf{b})(t)$ and satisfies one of the assumptions below:
  iii) $\displaystyle\int_0^T \frac{\|p(t)\|_r^{\frac{2r}{2r-3}}}{1+\ln(e+\|p(t)\|_r)}dt<\infty,$ for some $r>3/2$;
  iv) $\displaystyle\int_0^T \frac{\|Dp(t)\|_r^{\frac{2r}{3r-3}}}{1+\ln(e+\|Dp(t)\|_r)}dt<\infty,$ for some $r>1$,
  then $(\mathbf{u},\mathbf{b})(t)$ is smooth at $t=T$. These authors also proved that the case $r=\infty$ in *iii)* and *iv)* may be replaced by
  iii') $\displaystyle\int_0^T \frac{\|p(t)\|_\infty}{1+\ln(e+\|p(t)\|_q)}dt<\infty,$ for some $1<q<\infty$;
  iv') $\displaystyle\int_0^T \frac{\|Dp(t)\|_\infty^{\frac{2}{3}}}{1+\ln(e+\|Dp(t)\|_q)}dt<\infty,$ for some $1<q<\infty$,
  respectively. The paper [19] also establishes logarithmically improved regularity criteria in the multiplier spaces and the space of the functions of bounded mean oscillation (for more details see [19]);
- X. Jia and Y. Zhou [7] proved that if $(\mathbf{u}_0,\mathbf{b}_0)\in H^s(\mathbb{R}^3)$ such that $\nabla\cdot\mathbf{u}_0=\nabla\cdot\mathbf{b}_0=0$, with $s\geq 3$, and
  i) $u_3,b_i\in L_T^{\alpha,\gamma}(\mathbb{R}^3)(i=1,2,3)$, with $\frac{2}{\alpha}+\frac{3}{\gamma}\leq 1, 3<\gamma\leq\infty$ and $D_3u_1,D_3u_2\in L_T^{\alpha,\gamma}(\mathbb{R}^3)$, with $\frac{2}{\alpha}+\frac{3}{\gamma}\leq 2, \frac{3}{2}<\gamma\leq\infty$ or
  ii) $u_3,b_i\in L_T^{\alpha,\gamma}(\mathbb{R}^3)(i=1,2,3)$, with $\frac{2}{\alpha}+\frac{3}{\gamma}\leq\frac{3}{4}+\frac{1}{2\gamma}, \frac{10}{3}<\gamma\leq\infty$ or
  iii) $\|u_3\|_{L_T^{\infty,\frac{10}{3}}(\mathbb{R}^3)}$ and $\|\mathbf{b}\|_{L_T^{\infty,\frac{10}{3}}(\mathbb{R}^3)}$ are small enough on $[0,T]$ or
  iv) $\|u_3\|_{L_T^{\infty,3}(\mathbb{R}^3)}$, $\|\mathbf{b}\|_{L_T^{\infty,3}(\mathbb{R}^3)}$, $\|D_3u_1\|_{L_T^{\infty,\frac{3}{2}}(\mathbb{R}^3)}$ and $\|D_3u_2\|_{L_T^{\infty,\frac{3}{2}}(\mathbb{R}^3)}$ are small enough on $[0,T]$,
  then $(\mathbf{u},\mathbf{b})(t)$ remains smooth on $[0,T]$. Here $L_T^{p,q}(\mathbb{R}^3)=L^p(0,T;L^q(\mathbb{R}^3)), 1\leq p,q\leq\infty$.

By recalling that G. Duvaut and J.-L. Lions [2] proved that (1) admits at least one global weak solution $(\mathbf{u}, \mathbf{b})(t)$ for every divergence free initial data $(\mathbf{u}_0, \mathbf{b}_0) \in L^2(\mathbb{R}^3)$ and it has a (unique) local strong solution, if additionally, $(\mathbf{u}_0, \mathbf{b}_0) \in H^s(\mathbb{R}^3)$, with $s \geq 3$, it is important to emphasize the following statements:

- X. Jia and Y. Zhou [9] proved that the solution $(\mathbf{u}, \mathbf{b})(t)$ is smooth on $(0, T]$ provided that $M \in L_T^{\alpha,\gamma}(\mathbb{R}^3)$ with
  - *i)* $1 \leq \alpha < \infty, \frac{3}{2} < \gamma \leq \infty$, $\frac{2}{\alpha} + \frac{3}{\gamma} = 2$ and $M = B^1_{[i,i]} (i = 1, 2, 3), U^4_{[1,1;1,2;2,1;2,2]}, U^4_{[1,1;1,3;3,1;3,3]}, U^4_{[2,2;2,3;3,2;3,3]}, B^4_{[1,3;2,3;3,1;3,2]}, B^4_{[1,2;1,3;2,1;3,1]}$, or $B^4_{[1,2;2,3;2,1;3,2]}$;
  - *ii)* $1 < \alpha < \infty$, $\frac{3}{2} < \gamma < \infty$, $\frac{2}{\alpha} + \frac{3}{\gamma} = 2$ and $M = U^3_{[1,2;2,1;3,3]}, U^3_{[1,3;3,1;2,2]}$, or $U^3_{[2,3;3,2;1,1]}$;
  - *iii)* $2 \leq \alpha \leq 4, 2 \leq \gamma \leq 3$, $\frac{2}{\alpha} + \frac{3}{\gamma} = 2$ and $M = B^3_{[1,1;1,2;1,3]}, B^3_{[2,1;2,2;2,3]}$, or $B^3_{[3,1;3,2;3,3]}$.

  Here

$$U^4_{[2,2;2,3;3,2;3,3]} = \begin{pmatrix} D_1 b_1 & D_1 b_2 & D_1 b_3 \\ D_2 b_1 & D_2 u_2 & D_2 u_3 \\ D_3 b_1 & D_3 u_2 & D_3 u_3 \end{pmatrix}, \quad U^3_{[1,3;3,1;2,2]} = \begin{pmatrix} D_1 b_1 & D_1 b_2 & D_1 u_3 \\ D_2 b_1 & D_2 u_2 & D_2 b_3 \\ D_3 u_1 & D_3 b_2 & D_3 b_3 \end{pmatrix},$$

$$B^1_{[1,1]} = \begin{pmatrix} D_1 b_1 & D_1 u_2 & D_1 u_3 \\ D_2 u_1 & D_2 u_2 & D_2 u_3 \\ D_3 u_1 & D_3 u_2 & D_3 u_3 \end{pmatrix}, \quad B^4_{[1,2;1,3;2,1;3,1]} = \begin{pmatrix} D_1 u_1 & D_1 b_2 & D_1 b_3 \\ D_2 b_1 & D_2 u_2 & D_2 u_3 \\ D_3 b_1 & D_3 u_2 & D_3 u_3 \end{pmatrix}.$$

  The other matrixes above are defined in a similar way.
- X. Jia and Y. Zhou [8] proved that the solution $(\mathbf{u}, \mathbf{b})(t)$ is smooth on $(0, T]$ provided that
  - *i)* $(D_i u_1, D_j u_2, D_k u_3) \in L_T^{\alpha,\gamma}(\mathbb{R}^3) (i, j, k = 1, 2, 3)$, $\frac{2}{\alpha} + \frac{3}{\gamma} \leq 1 + \frac{1}{\gamma}$ and $2 \leq \gamma \leq \infty$.

  Moreover, these authors demonstrated that any component (respectively components) of $(D_i u_1, D_j u_2, D_k u_3)$ in the criterion determined in *i)* above can be replaced by the corresponding velocity component (respectively components) which is (respectively are) in the space $L_T^{\alpha,\gamma}(\mathbb{R}^3)$, with $\frac{2}{\alpha} + \frac{3}{\gamma} \leq 1$ and $3 < \gamma \leq \infty$. Furthermore, X. Jia and Y. Zhou [8] obtained a Ladyzhenskaya-Prodi-Serrin type regularity condition involving two components of the gradient of pressure (see [8] for more details).

An outline of the paper is as follows: In the next section, we list the notation that will be used throughout the paper and recall lemmas that play an important role in the proofs of our main results. The rest of the paper is concerned with the proof of Theorems 1.1, 1.3, 1.5, 1.7, 1.9 and Corollary 1.2.

2. Preliminaries. Here, we introduce some notations and state the results that we will use in the rest of the paper.

2.1. Notations. First of all, we establish some notations that we use in this paper.

- The Euclidean norm of $\mathbf{a} = (a_1, ..., a_N) \in \mathbb{R}^N$ $(N \in \mathbb{N})$ is denoted by $|\mathbf{a}|^2 := \sum_{i=1}^N a_i^2$. The scalar product between the vectors $\mathbf{a} = (a_1, ..., a_N)$ and $\mathbf{b} = (b_1, ..., b_N) \in \mathbb{R}^N$ is given by $\mathbf{a} \cdot \mathbf{b} := \sum_{i=1}^N a_i b_i$;
- Let $f \in L^p(\mathbb{R}^N)$ (usual Lebesgue space). The L^p-norm of f is given by $\|f\|_p := \left(\int_{\mathbb{R}^3} |f(x)|^p \, dx\right)^{\frac{1}{p}}$ if $1 \leq p < \infty$, and $\|f\|_\infty := \text{esssup}_{x \in \mathbb{R}^N} \{|f(x)|\}$ for $p = \infty$;

- We define the L^2- inner product by $(\mathbf{a},\mathbf{b})_2 := \int_{\mathbb{R}^3} \mathbf{a}(x)\cdot\mathbf{b}(x)\,dx$, where $\mathbf{a},\mathbf{b} : \mathbb{R}^3 \to \mathbb{R}^N$ $(N \in \mathbb{N})$ are mensurable functions;
- For any multi-indices $\boldsymbol{\alpha}$, we denote $|\boldsymbol{\alpha}| := \sum_{i=1}^{3} \alpha_i$ where $\boldsymbol{\alpha} = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{N}^3$. We also denote $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3}$, where the partial derivatives are given by $D_i = \frac{\partial}{\partial x_i}$;
- Define $\|D^\alpha \mathbf{u}\|_2^2 := \sum_{i=1}^{N} \|D^\alpha u_i\|_2^2$ and $\|\mathbf{u}\|_\infty := \max\{\|u_i\|_\infty : 1 \le i \le N\}$, where $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^N$ $(N \in \mathbb{N})$ is a mensurable function that is written by $\mathbf{u} = (u_1, u_2, ..., u_N)$;
- Denote $J_n^2(t) = \sum_{|\alpha|=n} \|(D^\alpha \mathbf{u}, D^\alpha \mathbf{b})(t)\|_2^2$, for all $n \in \mathbb{N} \cup \{0\}$, where
$$\|D^n \mathbf{u}\|_p := \left(\sum_{i=1}^{3} \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} |D_{j_1} \cdots D_{j_n} u_i(x)|^p \, dx \right)^{\frac{1}{p}},$$
for $1 \le p < \infty$, and $\|D^n \mathbf{u}\|_\infty := \text{supess}\,\{|D^\alpha u_i(x)| : x \in \mathbb{R}^3,\ 1 \le i \le 3,\ |\alpha| = n\}$;
- Let $\mathbf{u} = (u_1, u_2, ..., u_N)$ and $\mathbf{v} = (v_1, v_2, ..., v_N)$ $(N \in \mathbb{N})$ be mensurable functions. Assume that $j \in \mathbb{N}$. Let $(\mathbf{u},\mathbf{v})_{H^j} := \sum_{i=1}^{N} \sum_{|\alpha|\le j} (D^\alpha u_i, D^\alpha v_i)_2$ denote the H^j- inner product; and $\|\mathbf{u}\|_{H^j}^2 := \sum_{i=1}^{N} \sum_{|\alpha|\le j} \|D^\alpha u_i\|_2^2$ the norm associated with this product;
- Consider that $\Delta\mathbf{u} = (\Delta u_1, ..., \Delta u_N)$, $\nabla\mathbf{u} = (\nabla u_1, ..., \nabla u_N)$ and $\nabla\cdot\mathbf{v} = \sum_{i=1}^{3} D_i v_i$, where $\mathbf{u} = (u_1, ..., u_N)$ $(N \in \mathbb{N})$, $\mathbf{v} = (v_1, v_2, v_3)$, $\Delta u_j = \sum_{i=1}^{3} D_i^2 u_j$, $\nabla u_j = (D_1 u_j, D_2 u_j, D_3 u_j)$ $(j = 1, ..., N)$;
- Fourier Transform of $\mathbf{u} = (u_1, u_2, ..., u_N)$ $(N \in \mathbb{N})$ is given by $\widehat{\mathbf{u}}(k) := (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-ik\cdot x} \mathbf{u}(x)\,dx$.
- The constants may change line to line; however, they have the same notation. We also write $C_{q,r,s}$ in order to denote the constants that depend, at least, on q, r and s, for instance. Here C is always an absolute positive constant. At last, we sometimes drop the dependence of x and t as, for example, $\|\mathbf{u}\|_2$ or $\|\mathbf{u}(t)\|_2$ mean $\|\mathbf{u}(\cdot,t)\|_2$.

2.2. Preliminary results. In this paper, we will apply some results related to the following Cauchy problem:
$$\begin{cases} \mathbf{u}_t = \sigma\Delta\mathbf{u}, & t > 0,\ x \in \mathbb{R}^N, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{3}$$
Here $\sigma > 0$ and $\mathbf{u}_0 \in L^{p_0}(\mathbb{R}^N)$ (for some $p_0 \in [1,\infty]$ and $N \in \mathbb{N}$).

The next two lemmas will play an important role in the proofs of our main results.

Lemma 2.1. *Let $\mathbf{u}(t)$ be the solution for the problem* (3) *defined for all $t > 0$. Assume that $1 \le r \le q \le \infty$. Then,*
$$\|\mathbf{u}(t)\|_q \le (4\pi t)^{-\lambda} \sigma^{\frac{N}{2}[1-\frac{1}{r}+\frac{1}{q}]} \|\mathbf{u}_0\|_r, \quad \forall\, t > 0,$$
where $\lambda = \frac{N}{2}\left[\frac{1}{r} - \frac{1}{q}\right]$.

Proof. The proof of this result is only a slight adaptation of Theorem 7.1 in [12]. □

Lemma 2.2. *Let $\mathbf{u}(t)$ be the solution for the problem* (3) *defined for all $t > 0$. Assume that $1 \le r \le q \le \infty$. Then,*
$$\|D^\alpha \mathbf{u}(t)\|_q \le C_{\alpha,N} \sigma^{\frac{N}{2}\left[1-\frac{1}{r}+\frac{1}{q}\right]-\frac{|\alpha|}{2}} t^{-\lambda-\frac{|\alpha|}{2}} \|\mathbf{u}_0\|_r, \quad \forall\, t > 0,$$

where $\lambda = \frac{N}{2}\left[\frac{1}{r} - \frac{1}{q}\right]$*; and* $C_{\alpha,N}$ *depends only on* α *and* N.

Proof. Lemma 2.2 follows directly of the proof of Theorem 7.1 in [12]. □

The next two lemmas will also play an important role in this paper and are related to Gronwall's Lemma.

LEMMA 2.3. *Let* $A \geq 0, B > 0, 0 < \omega < 1$ *and* $\phi \in C^0([0,T))$ *satisfy*

$$0 \leq \phi(t) \leq A + B\int_0^t (t-s)^{-\omega}\phi(s)\,ds, \quad \forall\, 0 \leq t < T.$$

Then, $\phi(t) \leq 2A\exp\{2BC_{\omega,B}T\}$*, for all* $0 \leq t < T$*, where* $C_{\omega,B} > 0$ *depends only on* ω, B.

Proof. See Lemma 3.1 in [12]. □

LEMMA 2.4. *Let* $W \in C^1([0,T))$ *be a positive function satisfying the inequality*

$$W'(t) \leq CW(t)^{\omega}, \quad \forall\, 0 \leq t < T,$$

where $\omega > 1$ *and* $C > 0$ *are positive constant. Moreover, if* $T < \infty$ *and* $\sup_{0\leq t<T} W(t) = \infty$*, then*

$$W(t) \geq \frac{1}{[C(\omega-1)]^{\frac{1}{\omega-1}}}(T-t)^{\frac{-1}{\omega-1}}, \quad \forall\, 0 \leq t < T.$$

Proof. See Lemma 3.3 in [12]. □

The last two lemmas written in this section will be applied in the proof of Theorem 1.1 *i)*.

LEMMA 2.5. *Let* $(\mathbf{u},\mathbf{b})(t)$ *be the strong solution for the MHD system* (1) *defined in the maximal interval* $[0,T^*)$*. Assume that* $m \geq j+2$*. Then,*

$$\|(D^j\mathbf{u}, D^j\mathbf{b})(t)\|_\infty \leq C_j(J_m(t) + J_0(t)), \quad \forall\, t \in [0,T^*),$$

where C_j *relies only on* j.

Proof. See inequalities (2.2) and (2.3) in [12]. □

LEMMA 2.6. *Let* $m > n$*. Then, the following inequality holds:*

$$J_n^2(t) \leq C_n(J_m^2(t) + J_0^2(t)), \quad \forall\, t \geq 0,$$

where C_n *relies only on* n.

Proof. See remark after (2.3) in [12] and also Appendix in [6]. □

3. Proof of the main results. In this section, we establish the proofs of our main results. These ones are presented, with more details, into propositions.

3.1. Proof of Theorem 1.1 *i).* Supposing that $\sup_{0\leq t<T}\|(\mathbf{u},\mathbf{b})(t)\|_\infty$ is finite, we are interested in showing the boundedness below:

$$\sum_{|\alpha|=n}\|(D^\alpha\mathbf{u},D^\alpha\mathbf{b})(t)\|_2^2\leq K_n,\quad\forall\, 0\leq t<T.$$

In order to prove the statement above, let us examine the next lemma.

LEMMA 3.1. *Let* $(\mathbf{u},\mathbf{b})(t)$ *be the strong solution for the MHD system* (1) *defined in the maximal interval* $[0,T^*)$*. Then,*

$$\frac{1}{2}\frac{d}{dt}\|(\mathbf{u},\mathbf{b})(t)\|_2^2\leq-\gamma\|(D\mathbf{u},D\mathbf{b})(t)\|_2^2,\quad\forall\ 0\leq t<T^*. \tag{4}$$

In particular, we have

$$\|(\mathbf{u},\mathbf{b})(t)\|_2\leq\|(\mathbf{u}_0,\mathbf{b}_0)\|_2 \text{ and } \int_0^t\|(D\mathbf{u},D\mathbf{b})(s)\|_2^2ds\leq\frac{1}{2\gamma}\|(\mathbf{u}_0,\mathbf{b}_0)\|_2^2,\quad\forall\, 0\leq t<T^*. \tag{5}$$

Here $\gamma=\min\{\mu,\nu\}$*.*

Proof. By applying the inner products $(\mathbf{u},\cdot)_2$ and $(\mathbf{b},\cdot)_2$ in the first and second equations of the system (1), respectively, one obtains

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_2^2+\mu\|D\mathbf{u}\|_2^2=(\mathbf{u},\mathbf{b}\cdot\nabla\mathbf{b})_2 \tag{6}$$

and also

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{b}\|_2^2+\nu\|D\mathbf{b}\|_2^2=(\mathbf{b},\mathbf{b}\cdot\nabla\mathbf{u})_2 \tag{7}$$

since $\nabla\cdot\mathbf{u}=0$. Thus, by adding the equalities (6) and (7), we get

$$\frac{1}{2}\frac{d}{dt}\|(\mathbf{u},\mathbf{b})\|_2^2+\mu\|D\mathbf{u}\|_2^2+\nu\|D\mathbf{b}\|_2^2=(\mathbf{u},\mathbf{b}\cdot\nabla\mathbf{b})_2+(\mathbf{b},\mathbf{b}\cdot\nabla\mathbf{u})_2. \tag{8}$$

By using the fact that $\mathbf{b}$ is divergence free, we conclude that the right hand side of the equality (8) is null. Hence, if we consider that $\gamma=\min\{\mu,\nu\}>0$, one infers

$$\frac{1}{2}\frac{d}{dt}\|(\mathbf{u},\mathbf{b})\|_2^2\leq-\gamma\|(D\mathbf{u},D\mathbf{b})\|_2^2,\quad\forall\ 0\leq t<T^*. \tag{9}$$

The inequality above proves (4). As a result, by integrating over $[0,t]$, with $0\leq t<T^*$, the inequality (9), we establish the two inequalities given in (5). Consequently, the proof of Lemma 3.1 follows. □

By applying Lemma 3.1, we are going to show that the L^2-norm of all the spatial derivatives of the strong solution for the system (1) is bounded provided that the sup norm of this same solution can be bounded as well. More precisely, we present the next result.

PROPOSITION 3.2. *Let* $(\mathbf{u},\mathbf{b})(t)$ *be the strong solution of the MHD system* (1) *defined in the maximal interval* $[0,T^*)$*. Assume that*

$$M_1:=\sup_{0\leq t<T}\|(\mathbf{u},\mathbf{b})(t)\|_\infty<\infty. \tag{10}$$

Then,

$$J_n(t) \le K_n, \quad \forall\ 0 \le t < T \ e\ n \in \mathbb{N},$$

where K_n *depends only on* n, M_1, T, μ, ν *and* $\|(\mathbf{u}_0, \mathbf{b}_0)\|_{H^n}$.

Proof. Fist of all, notice that

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=n}\|D^\alpha \mathbf{u}\|_2^2 = \sum_{|\alpha|=n}[\mu(D^\alpha \mathbf{u}, D^\alpha \Delta \mathbf{u})_2 - (D^\alpha \mathbf{u}, D^\alpha(\mathbf{u}\cdot\nabla\mathbf{u}))_2 + (D^\alpha \mathbf{u}, D^\alpha(\mathbf{b}\cdot\nabla\mathbf{b}))_2$$
$$- (D^\alpha \mathbf{u}, D^\alpha(\nabla(p + \frac{1}{2}|\mathbf{b}|^2)))_2].$$

Therefore, by using the fact that $\nabla \cdot \mathbf{u} = 0$, one concludes

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=n}\|D^\alpha \mathbf{u}\|_2^2 + \mu\sum_{|\beta|=n+1}\|D^\beta \mathbf{u}\|_2^2$$
$$= \sum_{|\alpha|=n}[-(D^\alpha \mathbf{u}, D^\alpha(\mathbf{u}\cdot\nabla\mathbf{u}))_2 + (D^\alpha \mathbf{u}, D^\alpha(\mathbf{b}\cdot\nabla\mathbf{b}))_2]. \tag{11}$$

Similarly, we can obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=n}\|D^\alpha \mathbf{b}\|_2^2 + \nu\sum_{|\beta|=n+1}\|D^\beta \mathbf{b}\|_2^2$$
$$= \sum_{|\alpha|=n}[-(D^\alpha \mathbf{b}, D^\alpha(\mathbf{u}\cdot\nabla\mathbf{b}))_2 + (D^\alpha \mathbf{b}, D^\alpha(\mathbf{b}\cdot\nabla\mathbf{u}))_2]. \tag{12}$$

By adding (11) and (12), one gets

$$\frac{1}{2}\frac{d}{dt}J_n^2(t) + \gamma J_{n+1}^2(t) \le S_n(t), \tag{13}$$

where $\gamma := \min\{\mu, \nu\}$ and

$$S_n(t) := \sum_{|\alpha|=n}[-(D^\alpha \mathbf{u}, D^\alpha(\mathbf{u}\cdot\nabla\mathbf{u}))_2 + (D^\alpha \mathbf{u}, D^\alpha(\mathbf{b}\cdot\nabla\mathbf{b}))_2$$
$$-(D^\alpha \mathbf{b}, D^\alpha(\mathbf{u}\cdot\nabla\mathbf{b}))_2 + (D^\alpha \mathbf{b}, D^\alpha(\mathbf{b}\cdot\nabla\mathbf{u}))_2].$$

Now we are ready to prove that $J_n(t)$ bounded in $[0, T)$, for all $n \in \mathbb{N}$. The proof of this statement is going to be given by mathematical induction.

At first, we establish that $J_1(t)$ is bounded in $[0, T)$. To this end, the inequality (13) lets us know that it is enough to estimate $S_1(t)$. More specifically, by checking the definition of $S_1(t)$, we can see that the boundedness of $J_1(t)$ follows from the term $(D\mathbf{u}, D(\mathbf{b}\cdot\nabla\mathbf{b}))_2$ since the other ones contribute at a similar way in the next estimates. With this in mind, note that

$$(D\mathbf{u}, D(\mathbf{b}\cdot\nabla\mathbf{b}))_2 = -\sum_{i=1}^{3}((D_i D\mathbf{u})Db_i, \mathbf{b})_2 + \sum_{i=1}^{3}(D\mathbf{u}, b_i DD_i\mathbf{b})_2,$$

because $\nabla \cdot \mathbf{b} = 0$. Thus, it is sufficient to analyse the terms that have the forms $(D\mathbf{b}, \mathbf{b}D^2\mathbf{u})_2$ and $(D\mathbf{u}, \mathbf{b}D^2\mathbf{b})_2$. By using Hölder's inequality and (10), we conclude

$$|(D\mathbf{b}, \mathbf{b}D^2\mathbf{u})_2| \le \|\mathbf{b}\|_\infty \|D\mathbf{b}\|_2 \|D^2\mathbf{u}\|_2 \le M_1 J_1(t) J_2(t).$$

Analogously, one obtains

$$|(D\mathbf{u}, \mathbf{b}D^2\mathbf{b})_2| \le M_1 J_1(t) J_2(t).$$

Hence,

$$|(D\mathbf{u}, D(\mathbf{b}\cdot\nabla\mathbf{b}))_2| \le CM_1 J_1(t) J_2(t).$$

In a similar way, we get

$$|(D\mathbf{u}, D(\mathbf{u}\cdot\nabla\mathbf{u}))_2|, |(D\mathbf{b}, D(\mathbf{u}\cdot\nabla\mathbf{b}))_2|, |(D\mathbf{b}, D(\mathbf{b}\cdot\nabla\mathbf{u}))_2| \le CM_1 J_1(t) J_2(t).$$

Consequently, from (13) and Young's inequality, we infer

$$\frac{1}{2}\frac{d}{dt}J_1^2(t) + \frac{\gamma}{2}J_2^2(t) \le CM_1^2\gamma^{-1}J_1^2(t).$$

As a result, by applying Gronwall's Lemma, we obtain

$$J_1^2(t) \le \exp\{CM_1^2\gamma^{-1}T\}\|(\mathbf{u}_0, \mathbf{b}_0)\|_{H^1}^2, \quad \forall\, 0 \le t < T.$$

Now, let us estimate $J_2(t)$ in the interval $[0, T)$ (this boundedness is important in the process of induction). By (13), one has

$$\frac{1}{2}\frac{d}{dt}J_2^2(t) + \gamma J_3^2(t) \le S_2(t).$$

In order to estimate $S_2(t)$, it is enough to prove that the term $(D^2\mathbf{u}, D^2(\mathbf{b}\cdot\nabla\mathbf{b}))_2$ is bounded (this statement was already discussed above). Thus, we infer

$$\begin{aligned}
&(D^2\mathbf{u}, D^2(\mathbf{b}\cdot\nabla\mathbf{b}))_2 \\
&= \sum_{i=1}^{3}\Big\{\binom{2}{0}(D^2D_i\mathbf{b}, b_iD^2\mathbf{u})_2 - \binom{2}{1}\Big[(D^3\mathbf{u}, b_iDD_i\mathbf{b})_2 + (D^2D_i\mathbf{b}, b_iD^2\mathbf{u})_2\Big] \\
&\quad - \binom{2}{2}(D_iD^2\mathbf{u}, \mathbf{b}D^2b_i)_2\Big\},
\end{aligned}$$

since $\nabla\cdot\mathbf{b} = 0$. Therefore, it is sufficient to analyse the terms that present the forms $(D^3\mathbf{b}, \mathbf{b}D^2\mathbf{u})_2$ and $(D^3\mathbf{u}, \mathbf{b}D^2\mathbf{b})_2$. By using Hölder's inequality and (10), one gets

$$|(D^3\mathbf{b}, \mathbf{b}D^2\mathbf{u})_2| \le M_1 J_2(t) J_3(t) \quad \text{and} \quad |(D^3\mathbf{u}, \mathbf{b}D^2\mathbf{b})_2| \le M_1 J_2(t) J_3(t).$$

So, it is easy to see that

$$|(D^2\mathbf{u}, D^2(\mathbf{b}\cdot\nabla\mathbf{b}))_2| \le CM_1 J_2(t) J_3(t).$$

Analogously, one obtains

$$|(D^2\mathbf{u}, D^2(\mathbf{u}\cdot\nabla\mathbf{u}))_2|, |(D^2\mathbf{b}, D^2(\mathbf{u}\cdot\nabla\mathbf{b}))_2|, |(D^2\mathbf{b}, D^2(\mathbf{b}\cdot\nabla\mathbf{u}))_2| \le CM_1 J_2(t) J_3(t).$$

Then, by Young's inequality and (13), one infers

$$\frac{1}{2}\frac{d}{dt}J_2^2(t) + \frac{1}{2}\gamma J_3^2(t) \le CM_1^2\gamma^{-1}J_2^2(t).$$

By applying Gronwall's Lemma, we have

$$J_2^2(t) \leq \exp\{CM_1^2\gamma^{-1}T\}\|(\mathbf{u}_0, \mathbf{b}_0)\|_{H^2}^2, \quad \forall\, 0 \leq t < T.$$

Now, we consider that $J_n(t)$ is bounded for all $t \in [0, T)$ and $n \geq 3$. Our interest is to verify that $J_{n+1}(t)$ is also bounded in this same time interval. We saw in (13) that

$$\frac{1}{2}\frac{d}{dt}J_{n+1}^2(t) + \gamma J_{n+2}^2(t) \leq S_{n+1}(t). \tag{14}$$

In order to estimate $S_{n+1}(t)$, it is enough to show that the term $(D^{n+1}\mathbf{u}, D^{n+1}(\mathbf{b} \cdot \nabla \mathbf{b}))_2$ is bounded since the other ones follow from a similar argument. Thus, notice that

$$(D^{n+1}\mathbf{u}, D^{n+1}(\mathbf{b} \cdot \nabla \mathbf{b}))_2 = -\sum_{j=0}^{n}\sum_{i=1}^{3}\binom{n}{j}(D^{n+2}\mathbf{u}, D^j b_i D^{n-j} D_i \mathbf{b})_2.$$

Consequently, it is sufficient to analyse the term of the form $(D^{n+2}\mathbf{u}, D^j\mathbf{b}D^{n+1-j}\mathbf{b})_2$, where $0 \leq j \leq n$. Let us examine three cases:

1°Case: At first, assume that $0 \leq j \leq n-2$.

By using Hölder's inequality, one has

$$\begin{aligned} |(D^{n+2}\mathbf{u}, D^j\mathbf{b}D^{n+1-j}\mathbf{b})_2| &\leq \|D^j\mathbf{b}\|_\infty \|D^{n+2}\mathbf{u}\|_2 \|D^{n+1-j}\mathbf{b}\|_2 \\ &\leq K_n[J_{n+1}(t) + J_0(t)]J_{n+2}(t), \end{aligned}$$

where in the last passage we used the induction hypothesis and Lemmas 2.5 and 2.6.

2°Case: Secondly, consider that $j = n-1$.

By applying Hölder's inequality again, Lemma 2.5 and the fact that $J_2(t)$ is bounded, we get

$$\begin{aligned} |(D^{n+2}\mathbf{u}, D^{n-1}\mathbf{b}D^2\mathbf{b})_2| &\leq \|D^{n-1}\mathbf{b}\|_\infty \|D^{n+2}\mathbf{u}\|_2 \|D^2\mathbf{b}\|_2 \\ &\leq C_n[J_{n+1}(t) + J_0(t)]J_{n+2}(t). \end{aligned}$$

3°Case: Assume that $j = n$.

Once again, apply Hölder's inequality in order to obtain

$$|(D^{n+2}\mathbf{u}, D^n\mathbf{b}D\mathbf{b})_2| \leq \|D\mathbf{b}\|_\infty \|D^{n+2}\mathbf{u}\|_2 \|D^n\mathbf{b}\|_2 \leq K_n[J_{n+1}(t) + J_0(t)]J_{n+2}(t),$$

where in the last inequality we have used Lemmas 2.5 and 2.6, and the induction hypothesis.

By observing the three cases above, we conclude that

$$|(D^{n+1}\mathbf{u}, D^{n+1}(\mathbf{b} \cdot \nabla \mathbf{b}))_2| \leq K_n[J_{n+1}(t) + J_0(t)]J_{n+2}(t), \quad \forall\, 0 \leq t < T, \tag{15}$$

where K_n depends on $n, M_1, T, \mu, \nu, \|(\mathbf{u}_0, \mathbf{b}_0)\|_{H^n}$. Analogously, one proves that

$$(D^{n+1}\mathbf{u}, D^{n+1}(\mathbf{u} \cdot \nabla \mathbf{u}))_2, (D^{n+1}\mathbf{b}, D^{n+1}(\mathbf{u} \cdot \nabla \mathbf{b}))_2, (D^{n+1}\mathbf{b}, D^{n+1}(\mathbf{b} \cdot \nabla \mathbf{u}))_2$$

can be bounded by the same limit given in (15).

Thus, by using Young's inequality and Lemma 3.1, (14) becomes

$$\frac{d}{dt}J_{n+1}^2(t) + \gamma J_{n+2}^2(t) \leq K_n^2\gamma^{-1}[J_{n+1}^2(t) + \|(\mathbf{u}_0, \mathbf{b}_0)\|_2^2].$$

At last, by Gronwall's Lemma, we obtain

$$J_{n+1}^2(t) \leq \exp\{K_n^2\gamma^{-1}T\}\left[\|(\mathbf{u}_0, \mathbf{b}_0)\|_{H^{n+1}}^2(1 + K_n^2\gamma^{-1}T)\right], \quad \forall\, 0 \leq t < T.$$

Therefore, Proposition 3.2 follows. □

3.2. Proof of Theorem 1.1 *ii)* and Corollary 1.2 *i)*. Now, let us assume that the solution $(\mathbf{u},\mathbf{b})(t)$ for the system (1) in $[0,T^*)$ presents blow up at $t = T^* < \infty$ in order to prove that

$$\sup_{0\leq t<T^*} \|(D\mathbf{u}, D\mathbf{b})(t)\|_q = \infty, \quad \frac{3}{2} < q \leq 2. \tag{16}$$

It is important to point out that if $\|(\mathbf{u},\mathbf{b})(t)\|_\infty$ is bounded in $[0,T)$, for some T finite, then $(\mathbf{u},\mathbf{b})(t)$ can be continued smoothly beyond T. This statement plays an important role in the proof of (16). More specifically, we assure that $\sup_{0\leq t<T^*}\|(\mathbf{u},\mathbf{b})(t)\|_\infty < \infty$ if (16) does not hold and, thereafter, this result leads us to a contradiction.

PROPOSITION 3.3. *Let $\frac{3}{2} < q \leq 2$. Let $(\mathbf{u},\mathbf{b})(t)$ be the strong solution for the MHD system* (1) *defined in the maximal interval $[0,T^*)$. Assume that*

$$M_2 := \sup_{0\leq t<T} \|(D\mathbf{u}, D\mathbf{b})(t)\|_q < \infty.$$

Then

$$\sup_{0\leq t<T} \|(\mathbf{u},\mathbf{b})(t)\|_\infty \leq K_{2,q},$$

where $K_{2,q}$ depends only on $\|(\widehat{\mathbf{u}}_0, \widehat{\mathbf{b}}_0)\|_1, M_2,\ q, T, \mu, \nu$. In particular, if $T^* < \infty$, we have

$$\sup_{0\leq t<T^*} \|(D\mathbf{u}, D\mathbf{b})(t)\|_q = \infty.$$

Proof. By applying Fourier Transform to the first equation of the system (1), one obtains

$$\widehat{\mathbf{u}}_t = -\mu|k|^2\widehat{\mathbf{u}} + \widehat{\mathbf{Q}}, \tag{17}$$

where $\mathbf{Q} := -\mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{b} - \nabla(p + \frac{1}{2}|\mathbf{b}|^2)$. Notice that,

$$\widehat{\mathbf{u}\cdot\nabla\mathbf{u}} - \widehat{\mathbf{b}\cdot\nabla\mathbf{b}} = -\widehat{\mathbf{Q}} - [(\nabla(p + \frac{1}{2}|\mathbf{b}|^2)]^\wedge$$

is the orthogonal decomposition of the vector $\widehat{\mathbf{u}\cdot\nabla\mathbf{u}} - \widehat{\mathbf{b}\cdot\nabla\mathbf{b}}$ into a vector orthogonal to k and a vector parallel to k, respectively. Thus,

$$|\widehat{\mathbf{Q}}| \leq |\widehat{\mathbf{u}\cdot\nabla\mathbf{u}}| + |\widehat{\mathbf{b}\cdot\nabla\mathbf{b}}|.$$

Consequently, by using the heat semigroup $e^{-\mu|k|^2(t-s)}$ to (17) and integrating over $[0,t]$, we conclude

$$\begin{aligned}|\widehat{\mathbf{u}}| &\leq e^{-\mu|k|^2t}|\widehat{\mathbf{u}}_0| + \int_0^t e^{-\mu|k|^2(t-s)}|\widehat{\mathbf{Q}}|\,ds \\ &\leq |\widehat{\mathbf{u}}_0| + \int_0^t e^{-\mu|k|^2(t-s)}[|\widehat{\mathbf{u}\cdot\nabla\mathbf{u}}| + |\widehat{\mathbf{b}\cdot\nabla\mathbf{b}}|]\,ds.\end{aligned}$$

By integrating over $\mathbb{R}^3$ the inequality above, one infers

$$\begin{aligned}\|\mathbf{u}\|_\infty &\le \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}|\widehat{\mathbf{u}}|\,dk\\ &\le \frac{1}{(2\pi)^{\frac{3}{2}}}\|\widehat{\mathbf{u}}_0\|_1+\frac{1}{(2\pi)^{\frac{3}{2}}}\int_0^t\int_{\mathbb{R}^3}e^{-\mu|k|^2(t-s)}[|\widehat{\mathbf{u}\cdot\nabla\mathbf{u}}|+|\widehat{\mathbf{b}\cdot\nabla\mathbf{b}}|]\,dkds.\end{aligned}$$

Analogously, we get

$$\|\mathbf{b}\|_\infty \le \frac{1}{(2\pi)^{\frac{3}{2}}}\|\widehat{\mathbf{b}}_0\|_1+\frac{1}{(2\pi)^{\frac{3}{2}}}\int_0^t\int_{\mathbb{R}^3}e^{-\nu|k|^2(t-s)}[|\widehat{\mathbf{u}\cdot\nabla\mathbf{b}}|+|\widehat{\mathbf{b}\cdot\nabla\mathbf{u}}|]\,dkds.$$

As a result, one obtains

$$\begin{aligned}\|(\mathbf{u},\mathbf{b})\|_\infty \le{}& \frac{1}{(2\pi)^{\frac{3}{2}}}\|(\widehat{\mathbf{u}}_0,\widehat{\mathbf{b}}_0)\|_1+\frac{1}{(2\pi)^{\frac{3}{2}}}\int_0^t\int_{\mathbb{R}^3}e^{-\mu|k|^2(t-s)}[|\widehat{\mathbf{u}\cdot\nabla\mathbf{u}}|+|\widehat{\mathbf{b}\cdot\nabla\mathbf{b}}|]\,dkds\\ &+\frac{1}{(2\pi)^{\frac{3}{2}}}\int_0^t\int_{\mathbb{R}^3}e^{-\nu|k|^2(t-s)}[|\widehat{\mathbf{u}\cdot\nabla\mathbf{b}}|+|\widehat{\mathbf{b}\cdot\nabla\mathbf{u}}|]\,dkds.\end{aligned}$$

Let us estimate the integrals on the right hand side of the inequality above. At first, by Hölder's inequality, we reach

$$\begin{aligned}\int_{\mathbb{R}^3}e^{-\nu|k|^2(t-s)}|\widehat{\mathbf{u}\cdot\nabla\mathbf{b}}|\,dk &\le \left(\int_{\mathbb{R}^3}e^{-q\nu|k|^2(t-s)}\,dk\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^3}|\widehat{\mathbf{u}\cdot\nabla\mathbf{b}}|^{q'}\,dk\right)^{\frac{1}{q'}}\\ &= C_q\nu^{-\frac{3}{2q}}(t-s)^{-\frac{3}{2q}}\|\widehat{\mathbf{u}\cdot\nabla\mathbf{b}}\|_{q'},\end{aligned}$$

where $\frac{1}{q}+\frac{1}{q'}=1$. Hence, by Hausdorff-Young's inequality (or Parseval's identity in the case $q=2$), one has

$$\begin{aligned}\int_{\mathbb{R}^3}e^{-\nu|k|^2(t-s)}|\widehat{\mathbf{u}\cdot\nabla\mathbf{b}}|\,dk &\le C_q\nu^{-\frac{3}{2q}}(t-s)^{-\frac{3}{2q}}\|\mathbf{u}\cdot\nabla\mathbf{b}\|_q\\ &\le C_q\nu^{-\frac{3}{2q}}(t-s)^{-\frac{3}{2q}}\|\mathbf{u}\|_\infty\|D\mathbf{b}\|_q.\end{aligned}$$

Similarly, we conclude

$$\int_{\mathbb{R}^3}e^{-\nu|k|^2(t-s)}|\widehat{\mathbf{b}\cdot\nabla\mathbf{u}}|\,dk \le C_q\nu^{-\frac{3}{2q}}(t-s)^{-\frac{3}{2q}}\|\mathbf{b}\|_\infty\|D\mathbf{u}\|_q,$$

$$\int_{\mathbb{R}^3}e^{-\mu|k|^2(t-s)}|\widehat{\mathbf{u}\cdot\nabla\mathbf{u}}|\,dk \le C_q\mu^{-\frac{3}{2q}}(t-s)^{-\frac{3}{2q}}\|\mathbf{u}\|_\infty\|D\mathbf{u}\|_q,$$

$$\int_{\mathbb{R}^3}e^{-\mu|k|^2(t-s)}|\widehat{\mathbf{b}\cdot\nabla\mathbf{b}}|\,dk \le C_q\mu^{-\frac{3}{2q}}(t-s)^{-\frac{3}{2q}}\|\mathbf{b}\|_\infty\|D\mathbf{b}\|_q.$$

Thus,

$$\|(\mathbf{u},\mathbf{b})\|_\infty \le C\|(\widehat{\mathbf{u}}_0,\widehat{\mathbf{b}}_0)\|_1+C_q\gamma^{-\frac{3}{2q}}M_2\int_0^t(t-s)^{-\frac{3}{2q}}\|(\mathbf{u},\mathbf{b})\|_\infty ds,$$

where $\gamma=\min\{\mu,\nu\}$. By applying Lemma 2.3, one gets

$$\|(\mathbf{u},\mathbf{b})(t)\|_\infty \le C\|(\widehat{\mathbf{u}}_0,\widehat{\mathbf{b}}_0)\|_1\exp\{C_q\gamma^{-\frac{3}{2q}}M_2T\},\quad \forall\ 0\le t<T.$$

In particular, if $T^*<\infty$ and $M_2<\infty$, then by using the inequality above, we assure that the solution for the system (1) can be continued beyond T^*. This fact contradicts the maximality of T^*. □

3.3. Proof of Theorem 1.1 *iii).* By adapting the proof of Proposition 3.3, it is possible to guarantee that $t^{\frac{3}{4}}\|(\mathbf{u},\mathbf{b})(t)\|_\infty$ is bounded in $(0,T)$ provided that $\sup\limits_{0\leq t<T}\|(D\mathbf{u},D\mathbf{b})(t)\|_q<\infty$, where $\frac{3}{2}<q\leq 2$.

Proposition 3.4. *Let $\frac{3}{2}<q\leq 2$. Let $(\mathbf{u},\mathbf{b})(t)$ be the strong solution for the MHD system* (1) *defined in the maximal interval $[0,T^*)$. Assume that*

$$M_3:=\sup_{0\leq t<T}\|(D\mathbf{u},D\mathbf{b})(t)\|_q<\infty,$$

then

$$\sup_{0<t<T}t^{\frac{3}{4}}\|(\mathbf{u},\mathbf{b})(t)\|_\infty\leq K_{2,q},$$

where $K_{2,q}$ depends only on $\|(\mathbf{u}_0,\mathbf{b}_0)\|_2, M_3,\ q,\mu,\nu,T.$

Proof. Apply $s^{\frac{3}{4}}e^{-\mu|k|^2(t-s)}$ $(0<s\leq t<T)$ to (17) and, thereafter, integrate the result over $[0,t]$ in order to obtain

$$\begin{aligned}|t^{\frac{3}{4}}\widehat{\mathbf{u}}|&\leq\frac{3}{4}\int_0^t s^{-\frac{1}{4}}e^{-\mu|k|^2(t-s)}|\widehat{\mathbf{u}}|\,ds+\int_0^t s^{\frac{3}{4}}e^{-\mu|k|^2(t-s)}|\widehat{\mathbf{Q}}|\,ds\\&\leq\frac{3}{4}\int_0^t s^{-\frac{1}{4}}e^{-\mu|k|^2(t-s)}|\widehat{\mathbf{u}}|\,ds+\int_0^t s^{\frac{3}{4}}e^{-\mu|k|^2(t-s)}[|\widehat{\mathbf{u}\cdot\nabla\mathbf{u}}|+|\widehat{\mathbf{b}\cdot\nabla\mathbf{b}}|]\,ds.\end{aligned}$$

Now, by integrating over $\mathbb{R}^3$, one reaches

$$\begin{aligned}t^{\frac{3}{4}}\|\mathbf{u}\|_\infty\leq&\frac{3}{4(2\pi)^{\frac{3}{2}}}\int_0^t s^{-\frac{1}{4}}\int_{\mathbb{R}^3}e^{-\mu|k|^2(t-s)}|\widehat{\mathbf{u}}|\,dkds\\&+\frac{1}{(2\pi)^{\frac{3}{2}}}\int_0^t s^{\frac{3}{4}}\int_{\mathbb{R}^3}e^{-\mu|k|^2(t-s)}[|\widehat{\mathbf{u}\cdot\nabla\mathbf{u}}|+|\widehat{\mathbf{b}\cdot\nabla\mathbf{b}}|]\,dkds.\end{aligned}$$

By using Hölder's inequality, we infer

$$\begin{aligned}t^{\frac{3}{4}}\|\mathbf{u}\|_\infty\leq&\frac{3}{4(2\pi)^{\frac{3}{2}}}\int_0^t s^{-\frac{1}{4}}\left(\int_{\mathbb{R}^3}e^{-2\mu|k|^2(t-s)}\,dk\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}|\widehat{\mathbf{u}}|^2\,dk\right)^{\frac{1}{2}}ds\\&+\frac{1}{(2\pi)^{\frac{3}{2}}}\int_0^t s^{\frac{3}{4}}\left(\int_{\mathbb{R}^3}e^{-q\mu|k|^2(t-s)}\,dk\right)^{\frac{1}{q}}\\&\times\left[\left(\int_{\mathbb{R}^3}[|\widehat{\mathbf{u}\cdot\nabla\mathbf{u}}|]^{q'}\,dk\right)^{\frac{1}{q'}}+\left(\int_{\mathbb{R}^3}[|\widehat{\mathbf{b}\cdot\nabla\mathbf{b}}|]^{q'}\,dk\right)^{\frac{1}{q'}}\right]ds,\end{aligned}$$

where $\frac{1}{q}+\frac{1}{q'}=1$. By following a similar argument applied in the proof of Proposition 3.3, one concludes

$$\begin{aligned}t^{\frac{3}{4}}\|\mathbf{u}\|_\infty\leq&\,C\mu^{-\frac{3}{4}}\|(\mathbf{u}_0,\mathbf{b}_0)\|_2\int_0^t s^{-\frac{1}{4}}(t-s)^{-\frac{3}{4}}\,ds\\&+C_qM_3\mu^{-\frac{3}{2q}}\int_0^t(t-s)^{-\frac{3}{2q}}s^{\frac{3}{4}}\|(\mathbf{u},\mathbf{b})\|_\infty\,ds,\end{aligned}$$

where we used Parseval's identity and Lemma 3.1. It is easy to prove that

$$\int_0^t s^{-\frac{1}{4}}(t-s)^{-\frac{3}{4}}\,ds \leq \frac{16}{3}.$$

Then,

$$t^{\frac{3}{4}}\|\mathbf{u}\|_\infty \leq C\mu^{-\frac{3}{4}}\|(\mathbf{u}_0,\mathbf{b}_0)\|_2 + C_q M_3 \mu^{-\frac{3}{2q}} \int_0^t (t-s)^{-\frac{3}{2q}} s^{\frac{3}{4}}\|(\mathbf{u},\mathbf{b})\|_\infty\,ds.$$

By applying a similar argument to the second equation of the system (1), we obtain

$$t^{\frac{3}{4}}\|\mathbf{b}\|_\infty \leq C\nu^{-\frac{3}{4}}\|(\mathbf{u}_0,\mathbf{b}_0)\|_2 + C_q M_3 \nu^{-\frac{3}{2q}} \int_0^t (t-s)^{-\frac{3}{2q}} s^{\frac{3}{4}}\|(\mathbf{u},\mathbf{b})\|_\infty\,ds.$$

Therefore,

$$t^{\frac{3}{4}}\|(\mathbf{u},\mathbf{b})(t)\|_\infty \leq C\gamma^{-\frac{3}{4}}\|(\mathbf{u}_0,\mathbf{b}_0)\|_2 + C_q M_3 \gamma^{-\frac{3}{2q}} \int_0^t (t-s)^{-\frac{3}{2q}} s^{\frac{3}{4}}\|(\mathbf{u},\mathbf{b})(s)\|_\infty\,ds,$$

where $\gamma = \min\{\mu,\nu\}$. At last, by Lemma 2.3, one reaches

$$t^{\frac{3}{4}}\|(\mathbf{u},\mathbf{b})(t)\|_\infty \leq C\gamma^{-\frac{3}{4}}\|(\mathbf{u}_0,\mathbf{b}_0)\|_2 \exp\{C_q M_3 \gamma^{-\frac{3}{2q}} T^{\frac{7}{4}}\}, \quad \forall\, 0 < t < T.$$

The proof of Proposition 3.4 is done. □

3.4. Proof of Theorem 1.3 *i)*. Now, we are going to prove the well known Leray's inequality related to the MHD equations (1), established by Leray [10] who studied the classical Navier-Stokes equations. Considering this aim, we present a result that presents a differential inequality involving $\|(D\mathbf{u}, D\mathbf{b})(t)\|_2$. More precisely, we write the next lemma.

LEMMA 3.5. *Let* $(\mathbf{u},\mathbf{b})(t)$ *be the strong solution for the MHD system* (1) *defined in the maximal interval* $[0,T^*)$. *Then,*

$$\frac{d}{dt}\|(D\mathbf{u}, D\mathbf{b})(t)\|_2^2 \leq C\gamma^{-3}\|(D\mathbf{u}, D\mathbf{b})(t)\|_2^6, \quad \forall\, 0 \leq t < T^*,$$

where C *is a positive constant and* $\gamma = \min\{\mu,\nu\}$.

Proof. First of all, notice that the first equation of (1) allows us to obtain

$$\frac{1}{2}\frac{d}{dt}\|D\mathbf{u}\|_2^2 + \mu\|D^2\mathbf{u}\|_2^2 = -\sum_{j=1}^3 (D_j\mathbf{u}, D_j(\mathbf{u}\cdot\nabla\mathbf{u}))_2 + \sum_{j=1}^3 (D_j\mathbf{u}, D_j(\mathbf{b}\cdot\nabla\mathbf{b}))_2, \quad (18)$$

since $\nabla\cdot\mathbf{u} = 0$. Analogously, by examining the second equation of (1), we conclude

$$\frac{1}{2}\frac{d}{dt}\|D\mathbf{b}\|_2^2 + \nu\|D^2\mathbf{b}\|_2^2 = -\sum_{j=1}^3 (D_j\mathbf{b}, D_j(\mathbf{u}\cdot\nabla\mathbf{b}))_2 + \sum_{j=1}^3 (D_j\mathbf{b}, D_j(\mathbf{b}\cdot\nabla\mathbf{u}))_2. \quad (19)$$

By adding (18) and (19), one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(D\mathbf{u}, D\mathbf{b})\|_2^2 + \mu\|D^2\mathbf{u}\|_2^2 + \nu\|D^2\mathbf{b}\|_2^2 \\
&= \sum_{j=1}^3 [-(D_j\mathbf{u}, D_j(\mathbf{u}\cdot\nabla\mathbf{u}))_2 + (D_j\mathbf{u}, D_j(\mathbf{b}\cdot\nabla\mathbf{b}))_2 \\
&\quad -(D_j\mathbf{b}, D_j(\mathbf{u}\cdot\nabla\mathbf{b}))_2 + (D_j\mathbf{b}, D_j(\mathbf{b}\cdot\nabla\mathbf{u}))_2]. \qquad (20)
\end{aligned}$$

Let us study the terms on the right hand side of the equality above. Thus,

$$\begin{aligned}(D_j\mathbf{u}, D_j(\mathbf{b}\cdot\nabla\mathbf{b}))_2 &= \sum_{i,k=1}^{3}\int_{\mathbb{R}^3} D_ju_kD_jb_iD_ib_k\,dx + \sum_{i,k=1}^{3}\int_{\mathbb{R}^3} D_ju_kb_iD_jD_ib_k\,dx\\ &\leq C\int_{\mathbb{R}^3}|D\mathbf{u}||D\mathbf{b}|^2\,dx + \sum_{i,k=1}^{3}\int_{\mathbb{R}^3} D_ju_kb_iD_jD_ib_k\,dx.\end{aligned}$$

Hence, one infers

$$\begin{aligned}(D_j\mathbf{b}, D_j(\mathbf{b}\cdot\nabla\mathbf{u}))_2 &= \sum_{i,k=1}^{3}\int_{\mathbb{R}^3} D_jb_kD_jb_iD_iu_k\,dx + \sum_{i,k=1}^{3}\int_{\mathbb{R}^3} D_jb_kb_iD_jD_iu_k\,dx\\ &\leq \sum_{i,k=1}^{3}\int_{\mathbb{R}^3} |D_jb_k||D_jb_i||D_iu_k|\,dx - \sum_{i,k=1}^{3}\int_{\mathbb{R}^3} D_i(D_jb_kb_i)D_ju_k\,dx\\ &\leq C\int_{\mathbb{R}^3}|D\mathbf{b}|^2|D\mathbf{u}|\,dx - \sum_{i,k=1}^{3}\int_{\mathbb{R}^3}(D_iD_jb_k)b_iD_ju_k\,dx,\end{aligned}$$

since $\mathbf{b}$ is divergence free. Hence, we reach

$$(D_j\mathbf{u}, D_j(\mathbf{b}\cdot\nabla\mathbf{b}))_2 + (D_j\mathbf{b}, D_j(\mathbf{b}\cdot\nabla\mathbf{u}))_2 \leq C\int_{\mathbb{R}^3}|D\mathbf{u}||D\mathbf{b}|^2\,dx. \tag{21}$$

Consequently,

$$-(D_j\mathbf{b}, D_j(\mathbf{u}\cdot\nabla\mathbf{b}))_2 = -\sum_{i,k=1}^{3}\int_{\mathbb{R}^3} D_jb_kD_ju_iD_ib_k\,dx \leq C\int_{\mathbb{R}^3}|D\mathbf{u}||D\mathbf{b}|^2\,dx, \tag{22}$$

since $\nabla\cdot\mathbf{u} = 0$. Similarly, one has

$$(D_j\mathbf{u}, D_j(\mathbf{u}\cdot\nabla\mathbf{u}))_2 \leq C\int_{\mathbb{R}^3}|D\mathbf{u}|^3\,dx. \tag{23}$$

By replacing (21), (22) and (23) in (20), we obtain

$$\frac{1}{2}\frac{d}{dt}\|(D\mathbf{u}, D\mathbf{b})\|_2^2 + \mu\|D^2\mathbf{u}\|_2^2 + \nu\|D^2\mathbf{b}\|_2^2 \leq C\|(D\mathbf{u}, D\mathbf{b})\|_3^3.$$

By using Gagliardo-Nirenberg's inequality

$$\|v\|_3 \leq C\|v\|_2^{\frac{1}{2}}\|Dv\|_2^{\frac{1}{2}}, \quad \forall\, v\in C_0^\infty(\mathbb{R}^3),$$

one infers

$$\frac{1}{2}\frac{d}{dt}\|(D\mathbf{u}, D\mathbf{b})\|_2^2 + \gamma\|(D^2\mathbf{u}, D^2\mathbf{b})\|_2^2 \leq C\|(D\mathbf{u}, D\mathbf{b})\|_2^{\frac{3}{2}}\|(D^2\mathbf{u}, D^2\mathbf{b})\|_2^{\frac{3}{2}},$$

where $\gamma = \min\{\mu,\nu\}$. The result follows by applying Young's inequality. □

The next proposition shows us how to obtain a lower bound to the L^2-norm of the gradient of the solution for the system (1), in its maximal interval of existence, assuming that this solution presents blow up in finite time. It is important to point out that Proposition 3.6 below refers to Leray [10] in the particular case $\mathbf{b} = 0$.

PROPOSITION 3.6 (Leray's Inequality). *Let $(\mathbf{u}, \mathbf{b})(t)$ be the strong solution for the MHD system (1) defined in the maximal interval $[0, T^*)$. Assume that $T^* < \infty$, then*

$$\|(D\mathbf{u}, D\mathbf{b})(t)\|_2 \geq C\gamma^{\frac{3}{4}}(T^* - t)^{-\frac{1}{4}}, \quad \forall\, 0 \leq t < T^*,$$

where C is a positive constant and $\gamma = \min\{\mu, \nu\}$.

Proof. Apply Proposition 3.3, Lemma 3.5 and Lemma 2.4 in order to obtain

$$\|(D\mathbf{u}, D\mathbf{b})(t)\|_2 \geq \left(\frac{1}{C\gamma^{-3}}\right)^{\frac{1}{4}} (T^* - t)^{-\frac{1}{4}}, \quad \forall\, 0 \leq t < T^*.$$

Therefore, Proposition 3.6 is proved. □

3.5. Proof of Theorem 1.1 *iv)* and Corollary 1.2 *ii)*. Now, we are interested in proving that the solution $(\mathbf{u}, \mathbf{b})(t)$ for (1) has L^q-norm $(3 < q \leq \infty)$ unbounded, if $(\mathbf{u}, \mathbf{b})(t)$ presents blow up in finite time. More precisely, allow us to enunciate the next result.

PROPOSITION 3.7. *Let $(\mathbf{u}, \mathbf{b})(t)$ be the solution for the MHD system (1) defined in the maximal interval $[0, T^*)$. Let $3 < q \leq \infty$. Assume that*

$$M_4 := \sup_{0 \leq t < T} \|(\mathbf{u}, \mathbf{b})(t)\|_q < \infty.$$

Then,

$$\sup_{0 \leq t < T} \|(D\mathbf{u}, D\mathbf{b})(t)\|_2 \leq K_{2,q},$$

where $K_{2,q}$ depends only on $M_4, \|(D\mathbf{u}_0, D\mathbf{b}_0)\|_2, q, T, \mu, \nu$. In particular, if $T^* < \infty$, so

$$\sup_{0 \leq t < T^*} \|(\mathbf{u}, \mathbf{b})(t)\|_q = \infty.$$

Proof. First of all, consider that $\frac{6}{5} < r \leq 2$ is such that $\frac{1}{q} + \frac{1}{2} = \frac{1}{r}$. On the other hand, by using the fact that $\nabla \cdot \mathbf{u} = 0$, we get

$$\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{b} \cdot \nabla \mathbf{b} + \nabla(p + \frac{1}{2}|\mathbf{b}|^2)) = -\nabla \cdot (\mathbf{u}_t - \mu \Delta \mathbf{u}) = 0.$$

Consequently, Helmontz's projector P_H (see Section 7.2 in [12] and references therein) is well defined and

$$P_H(\mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{b} \cdot \nabla \mathbf{b}) = \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{b} \cdot \nabla \mathbf{b} + \nabla(p + \frac{1}{2}|\mathbf{b}|^2).$$

Hence, by applying the heat semigroup $e^{\mu\Delta(t-s)}$ to the first equation of (1), integrating over $[0, t]$ and considering the operator D_j, one obtains

$$D_j \mathbf{u} = D_j e^{\mu \Delta t} \mathbf{u}_0 + \int_0^t D_j e^{\mu \Delta(t-s)} P_H(\mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{b} \cdot \nabla \mathbf{b})\, ds.$$

Thus, by using Lemmas 2.1 and 2.2, we conclude

$$\|D_j\mathbf{u}\|_2 \le \mu^{\frac{3}{2}}\|D_j\mathbf{u}_0\|_2 + C\mu^{\frac{3}{2}[\frac{7}{6}-\frac{1}{r}]}\int_0^t (t-s)^{-\lambda-\frac{1}{2}}\|P_H(\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{b}\cdot\nabla\mathbf{b})\|_r\,ds,$$

where $\lambda = \frac{3}{2}\left(\frac{1}{r}-\frac{1}{2}\right)$. Then, by using Theorem 7.2 in [12], we have

$$\|D_j\mathbf{u}\|_2 \le \mu^{\frac{3}{2}}\|D_j\mathbf{u}_0\|_2 + C_r\mu^{\frac{3}{2}[\frac{7}{6}-\frac{1}{r}]}\int_0^t (t-s)^{-k}\|\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{b}\cdot\nabla\mathbf{b}\|_r\,ds,$$

where $k = \frac{3}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{1}{2}$. Notice that, by Hölder's inequality, one obtains

$$\begin{aligned}\|\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{b}\cdot\nabla\mathbf{b}\|_r^r &\le C_r[\|\mathbf{u}\cdot\nabla\mathbf{u}\|_r^r + \|\mathbf{b}\cdot\nabla\mathbf{b}\|_r^r]\\ &\le C_r\|\mathbf{u}\|_q^r\|D\mathbf{u}\|_2^r + C_r\|\mathbf{b}\|_q^r\|D\mathbf{b}\|_2^r\\ &\le C_r\|(\mathbf{u},\mathbf{b})\|_q^r\|(D\mathbf{u},D\mathbf{b})\|_2^r.\end{aligned}$$

Consequently,

$$\|\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{b}\cdot\nabla\mathbf{b}\|_r \le C_rM_4\|(D\mathbf{u},D\mathbf{b})\|_2.$$

Thus, one reaches

$$\|D\mathbf{u}\|_2 \le \mu^{\frac{3}{2}}\|(D\mathbf{u}_0,D\mathbf{b}_0)\|_2 + C_r\mu^{\frac{3}{2}[\frac{7}{6}-\frac{1}{r}]}M_4\int_0^t (t-s)^{-k}\|(D\mathbf{u},D\mathbf{b})\|_2\,ds. \tag{24}$$

Now, aply the same argument to the second equation of (1) in order to obtain

$$\|D_j\mathbf{b}\|_2 \le \nu^{\frac{3}{2}}\|D_j\mathbf{b}_0\|_2 + C_r\nu^{\frac{3}{2}[\frac{7}{6}-\frac{1}{r}]}\int_0^t (t-s)^{-k}\|\mathbf{u}\cdot\nabla\mathbf{b}-\mathbf{b}\cdot\nabla\mathbf{u}\|_r\,ds.$$

Once again, by Hölder's inequality, one has

$$\|\mathbf{u}\cdot\nabla\mathbf{b}-\mathbf{b}\cdot\nabla\mathbf{u}\|_r^r \le C_r\|\mathbf{u}\|_q^r\|D\mathbf{b}\|_2^r + C_r\|\mathbf{b}\|_q^r\|D\mathbf{u}\|_2^r \le C_r\|(\mathbf{u},\mathbf{b})\|_q^r\|(D\mathbf{u},D\mathbf{b})\|_2^r.$$

By using the hypothesis, we get

$$\|\mathbf{u}\cdot\nabla\mathbf{b}-\mathbf{b}\cdot\nabla\mathbf{u}\|_r \le C_rM_4\|(D\mathbf{u},D\mathbf{b})\|_2.$$

Therefore,

$$\|D\mathbf{b}\|_2 \le \nu^{\frac{3}{2}}\|(D\mathbf{u}_0,D\mathbf{b}_0)\|_2 + C_r\nu^{\frac{3}{2}[\frac{7}{6}-\frac{1}{r}]}M_4\int_0^t (t-s)^{-k}\|(D\mathbf{u},D\mathbf{b})\|_2\,ds. \tag{25}$$

From (24) and (25), we infer

$$\begin{aligned}\|(D\mathbf{u},D\mathbf{b})\|_2 &\le (\mu^{\frac{3}{2}}+\nu^{\frac{3}{2}})\|(D\mathbf{u}_0,D\mathbf{b}_0)\|_2\\ &\quad + C_r(\mu^{\frac{3}{2}[\frac{7}{6}-\frac{1}{r}]}+\nu^{\frac{3}{2}[\frac{7}{6}-\frac{1}{r}]})M_4\int_0^t (t-s)^{-k}\|(D\mathbf{u},D\mathbf{b})\|_2\,ds.\end{aligned}$$

By applying Lemma 2.3, one writes

$$\|(D\mathbf{u},D\mathbf{b})(t)\|_2 \le C(\mu^{\frac{3}{2}}+\nu^{\frac{3}{2}})\|(D\mathbf{u}_0,D\mathbf{b}_0)\|_2\exp\{C_r(\mu^{\frac{3}{2}[\frac{7}{6}-\frac{1}{r}]}+\nu^{\frac{3}{2}[\frac{7}{6}-\frac{1}{r}]})M_4T\},$$

$\forall\, 0 \le t < T$. In particular, if $T^* < \infty$ and $M_4 < \infty$ then the inequality above leads us to a contradiction because of Proposition 3.3. This finishes the proof of Proposition 3.7. □

3.6. Proof of Theorem 1.3 *ii)*. Now, our goal is to study the norm $\|(\mathbf{u},\mathbf{b})(t)\|_q$, $3<q<\infty$, at potential blow up time. To this end, let us introduce a result that establishes an integral inequality which is going to be useful in the search for a lower bound for $\|(\mathbf{u},\mathbf{b})(t)\|_q$.

LEMMA 3.8. *Let $(\mathbf{u},\mathbf{b})(t)$ be the strong solution for the MHD system (1) defined in the maximal interval $[0,T^*)$. Let $3<q<\infty$ and $k=\frac{3}{2q}+\frac{1}{2}<1$. Then,*

$$\|(\mathbf{u},\mathbf{b})(t)\|_q \le (\mu^{\frac{3}{2}}+\nu^{\frac{3}{2}})\|(\mathbf{u},\mathbf{b})(t_0)\|_q + C_q(\mu^{1-\frac{3}{2q}}+\nu^{1-\frac{3}{2q}})\int_{t_0}^{t}(t-s)^{-k}\|(\mathbf{u},\mathbf{b})(s)\|_q^2\,ds, \tag{26}$$

whenever $0\le t_0\le t<T^*$, where C_q depends only on q.

Proof. Assume that $r=\frac{q}{2}$. Hence, $k=\frac{3}{2}\left(\frac{1}{r}-\frac{1}{q}\right)+\frac{1}{2}$. Furthermore, by applying the heat semigroup $e^{\mu\Delta(t-s)}$ to the first equation of (1) and considering the L^q-norm in this same equation, one obtains

$$\begin{aligned}\|\mathbf{u}\|_q &\le \|e^{\mu\Delta(t-t_0)}\mathbf{u}(t_0)\|_q + \int_{t_0}^{t}\|e^{\mu\Delta(t-s)}P_H(\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{b}\cdot\nabla\mathbf{b})\|_q\,ds\\ &\le \mu^{\frac{3}{2}}\|\mathbf{u}(t_0)\|_q + C_q\int_{t_0}^{t}\|e^{\mu\Delta(t-s)}(\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{b}\cdot\nabla\mathbf{b})\|_q\,ds,\end{aligned}$$

where we have used Lemma 2.1 and Theorem 7.2 in [12]. By using the facts that $\nabla\cdot\mathbf{u}=\nabla\cdot\mathbf{b}=0$ and Lema 2.2, we conclude

$$\begin{aligned}\|e^{\mu\Delta(t-s)}(\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{b}\cdot\nabla\mathbf{b})\|_q &\le \sum_{i=1}^{3}\|D_ie^{\mu\Delta(t-s)}(u_i\mathbf{u}-b_i\mathbf{b})\|_q\\ &\le C\mu^{1-\frac{3}{2q}}(t-s)^{-\lambda-\frac{1}{2}}\sum_{i=1}^{3}\|u_i\mathbf{u}-b_i\mathbf{b}\|_r\\ &\le C\mu^{1-\frac{3}{2q}}\|(\mathbf{u},\mathbf{b})\|_q^2(t-s)^{-k},\end{aligned}$$

since $q=2r$. Thus,

$$\|\mathbf{u}\|_q \le \mu^{\frac{3}{2}}\|(\mathbf{u},\mathbf{b})(t_0)\|_q + C_q\mu^{1-\frac{3}{2q}}\int_{t_0}^{t}(t-s)^{-k}\|(\mathbf{u},\mathbf{b})\|_q^2\,ds.$$

Analogously, one reaches

$$\begin{aligned}\|\mathbf{b}\|_q &\le \|e^{\nu\Delta(t-t_0)}\mathbf{b}(t_0)\|_q + \int_{t_0}^{t}\|e^{\nu\Delta(t-s)}(-\mathbf{u}\cdot\nabla\mathbf{b}+\mathbf{b}\cdot\nabla\mathbf{u})\|_q\,ds\\ &\le \nu^{\frac{3}{2}}\|\mathbf{b}(t_0)\|_q + \int_{t_0}^{t}\|e^{\nu\Delta(t-s)}(-\mathbf{u}\cdot\nabla\mathbf{b}+\mathbf{b}\cdot\nabla\mathbf{u})\|_q\,ds.\end{aligned}$$

Once again, by using the equalities $\nabla\cdot\mathbf{u}=\nabla\cdot\mathbf{b}=0$ and Lemma 2.2, we have

$$\begin{aligned}\|e^{\nu\Delta(t-s)}(-\mathbf{u}\cdot\nabla\mathbf{b}+\mathbf{b}\cdot\nabla\mathbf{u})\|_q &\le \sum_{i=1}^{3}\|D_ie^{\nu\Delta(t-s)}(-u_i\mathbf{b}+b_i\mathbf{u})\|_q\\ &\le C\nu^{1-\frac{3}{2q}}(t-s)^{-\lambda-\frac{1}{2}}\sum_{i=1}^{3}\|-u_i\mathbf{b}+b_i\mathbf{u}\|_r,\end{aligned}$$

where $\lambda = \frac{3}{2}\left(\frac{1}{r} - \frac{1}{q}\right)$. Then,

$$\|e^{\nu\Delta(t-s)}(-\mathbf{u}\cdot\nabla\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{u})\|_q \leq C\nu^{1-\frac{3}{2q}}(t-s)^{-k}\|(\mathbf{u},\mathbf{b})\|_q^2.$$

Consequently,

$$\|\mathbf{b}\|_q \leq \nu^{\frac{3}{2}}\|(\mathbf{u},\mathbf{b})(t_0)\|_q + C\nu^{1-\frac{3}{2q}}\int_{t_0}^{t}(t-s)^{-k}\|(\mathbf{u},\mathbf{b})\|_q^2\,ds.$$

At last, for all $0 \leq t < T^*$, we infer

$$\|(\mathbf{u},\mathbf{b})(t)\|_q \leq (\mu^{\frac{3}{2}} + \nu^{\frac{3}{2}})\|(\mathbf{u},\mathbf{b})(t_0)\|_q + C_q(\mu^{1-\frac{3}{2q}} + \nu^{1-\frac{3}{2q}})\int_{t_0}^{t}(t-s)^{-k}\|(\mathbf{u},\mathbf{b})(s)\|_q^2\,ds.$$

□

Proposition 3.9 below shows us a lower bound for $\|(\mathbf{u},\mathbf{b})(t)\|_q$, $3 < q < \infty$, when $(\mathbf{u},\mathbf{b})(t)$ is the solution for the system (1) which is supposed to present blow up in finite time.

PROPOSITION 3.9. *Let $(\mathbf{u},\mathbf{b})(t)$ be the strong solution for the MHD system (1) defined in the maximal interval $[0,T^*)$. Assume that $3 < q < \infty$. If $T^* < \infty$, then*

$$\|(\mathbf{u},\mathbf{b})(t)\|_q \geq \frac{C_q}{B_{\mu,\nu}D_{\mu,\nu,q}}(T^*-t)^{-\frac{q-3}{2q}}, \quad \forall\, 0 \leq t < T^*.$$

where C_q depends only on q. Here $B_{\mu,\nu} = (1 + \mu^{\frac{3}{2}} + \nu^{\frac{3}{2}})^2$ and $D_{\mu,\nu,q} = \mu^{1-\frac{3}{2q}} + \nu^{1-\frac{3}{2q}}$.

Proof. Let $k = \frac{3}{2q} + \frac{1}{2}$ and

$$\tau_* = \min\left\{T^*, t_0 + \left[\frac{(1-k)(\lambda_{\mu,\nu}-1)}{C_{\mu,\nu}\lambda_{\mu,\nu}^2 C_q D_{\mu,\nu,q}\|(\mathbf{u},\mathbf{b})(t_0)\|_q}\right]^{\frac{1}{1-k}}\right\}, \tag{27}$$

where C_q is given in (26), $C_{\mu,\nu} = \mu^{\frac{3}{2}} + \nu^{\frac{3}{2}}$, $D_{\mu,\nu,q} = \mu^{1-\frac{3}{2q}} + \nu^{1-\frac{3}{2q}}$ and $\lambda_{\mu,\nu} = 1 + C_{\mu,\nu}^{-1}$. We assure that

$$\|(\mathbf{u},\mathbf{b})(t)\|_q < \lambda_{\mu,\nu}C_{\mu,\nu}\|(\mathbf{u},\mathbf{b})(t_0)\|_q, \quad \forall\, t_0 \leq t < \tau_*. \tag{28}$$

In fact, suppose, by contradiction, that there exists $t_2 \in [t_0, \tau_*)$ such that

$$\|(\mathbf{u},\mathbf{b})(t_2)\|_q \geq \lambda_{\mu,\nu}C_{\mu,\nu}\|(\mathbf{u},\mathbf{b})(t_0)\|_q.$$

By definition of $\lambda_{\mu,\nu}$, we have

$$\lambda_{\mu,\nu}C_{\mu,\nu}\|(\mathbf{u},\mathbf{b})(t_0)\|_q > \|(\mathbf{u},\mathbf{b})(t_0)\|_q.$$

From the continuity of $\|(\mathbf{u},\mathbf{b})(t)\|_q$, we conclude that there is $t_1 < t_2$ such that

$$\|(\mathbf{u},\mathbf{b})(t_1)\|_q = \lambda_{\mu,\nu}C_{\mu,\nu}\|(\mathbf{u},\mathbf{b})(t_0)\|_q \tag{29}$$

and also

$$\|(\mathbf{u},\mathbf{b})(t)\|_q < \lambda_{\mu,\nu}C_{\mu,\nu}\|(\mathbf{u},\mathbf{b})(t_0)\|_q, \quad \forall\, t_0 \leq t < t_1.$$

By applying Lemma 3.8, one reaches

$$\|(\mathbf{u},\mathbf{b})(t_1)\|_q < C_{\mu,\nu}\|(\mathbf{u},\mathbf{b})(t_0)\|_q + C_q D_{\mu,\nu,q}\lambda_{\mu,\nu}^2 C_{\mu,\nu}^2\|(\mathbf{u},\mathbf{b})(t_0)\|_q^2 \int_{t_0}^{t_1} (t_1-s)^{-k}\, ds.$$

Thus, by (29), we have

$$\lambda_{\mu,\nu}C_{\mu,\nu}\|(\mathbf{u},\mathbf{b})(t_0)\|_q < C_{\mu,\nu}\|(\mathbf{u},\mathbf{b})(t_0)\|_q + C_q D_{\mu,\nu,q}\lambda_{\mu,\nu}^2 C_{\mu,\nu}^2\|(\mathbf{u},\mathbf{b})(t_0)\|_q^2 \frac{(\tau_* - t_0)^{1-k}}{1-k}.$$

This is an absurd because of our choice of τ_* (see (27)). This completes the proof of (28).

Note that Proposition 3.7 informs us that $\|(\mathbf{u},\mathbf{b})(t)\|_q$ is unbounded in $[0,T^*)$. On the other hand, by continuity, this same application is bounded in $[0,t_0]$. Therefore, $\|(\mathbf{u},\mathbf{b})(t)\|_q$ is unbounded in $[t_0,T^*)$. Hence, by (28), $\|(\mathbf{u},\mathbf{b})(t)\|_q$ is bounded in $[t_0,\tau_*)$ with $\tau_* < T^*$. This means that

$$\tau_* = t_0 + \left[\frac{(1-k)(\lambda_{\mu,\nu}-1)}{C_{\mu,\nu}\lambda_{\mu,\nu}^2 C_q D_{\mu,\nu,q}\|(\mathbf{u},\mathbf{b})(t_0)\|_q}\right]^{\frac{1}{1-k}},$$

see (27). Thus,

$$\|(\mathbf{u},\mathbf{b})(t_0)\|_q \geq \frac{(1-k)(\lambda_{\mu,\nu}-1)}{C_{\mu,\nu}\lambda_{\mu,\nu}^2 C_q D_{\mu,\nu,q}}(T^*-t_0)^{-(1-k)}, \quad \forall\, 0 \leq t_0 < T^*.$$

It completes the proof of Proposition 3.9. □

3.7. Proof of Theorem 1.3 *iii), iv)* and *v).* The result below establishes lower bounds for $\|(D\mathbf{u},D\mathbf{b})(t)\|_q$ ($\frac{3}{2} < q \leq 3$).

Proposition 3.10. *Let $(\mathbf{u},\mathbf{b})(t)$ be the strong solution for the MHD system (1) defined in the maximal interval $[0,T^*)$. Assume that $0<\epsilon<\frac{1}{2}$. If $T^* < \infty$, then*

$$\|(D\mathbf{u},D\mathbf{b})(t)\|_3 \geq C_\epsilon C_{\epsilon,\mu,\nu}(T^*-t)^{-\frac{1}{2}+\epsilon}, \quad \forall\, 0 \leq t < T^*, \tag{30}$$

where C_ϵ depends only on ϵ and $\|(\mathbf{u}_0,\mathbf{b}_0)\|_2$; $C_{\epsilon,\mu,\nu} = [(1+\mu^{\frac{3}{2}}+\nu^{\frac{3}{2}})^2(\mu^{\frac{1+\epsilon}{1+4\epsilon}} + \nu^{\frac{1+\epsilon}{1+4\epsilon}})]^{-(1+4\epsilon)}$. Moreover, for $\frac{3}{2}<q<3$, one has

$$\|(D\mathbf{u},D\mathbf{b})(t)\|_q \geq C_q C_{q,\mu,\nu}(T^*-t)^{-\frac{2q-3}{2q}}, \quad \forall\, 0 \leq t < T^*, \tag{31}$$

where C_q relies only on q; and $C_{q,\mu,\nu} = [(1+\mu^{\frac{3}{2}}+\nu^{\frac{3}{2}})^2(\mu^{\frac{3q-3}{2q}}+\nu^{\frac{3q-3}{2q}})]^{-1}$.

Proof. Let $0<\epsilon<\frac{1}{2}$. Consider that $r = \frac{1+4\epsilon}{2\epsilon}$ ($3<r<\infty$). In order to prove (30), it is sufficient to use Gagliardo-Nirenberg's inequality

$$\|v\|_r \leq C_r\|v\|_2^{\frac{2}{r}}\|Dv\|_3^{1-\frac{2}{r}}, \quad \forall\, v \in C_0^\infty(\mathbb{R}^3),$$

Lema 3.1 and Proposition 3.9.

On the other hand, the inequality (31) follows from an immediate application of Proposition 3.9 and Sobolev's inequality below:

$$\|v\|_{\frac{3q}{3-q}} \leq C_q\|Dv\|_q, \quad \forall\, v \in C_0^\infty(\mathbb{R}^3). \qquad \square$$

Another result that assures lower bounds for $\|(D\mathbf{u}, D\mathbf{b})(t)\|_q$ is the following:

PROPOSITION 3.11. *Let* $(\mathbf{u}, \mathbf{b})(t)$ *be the strong solution for the MHD system* (1) *defined in the maximal interval* $[0, T^*)$. *Assume that* $3 < r < \infty$ *and* $T^* < \infty$. *Then,*

$$\|(D\mathbf{u}, D\mathbf{b})(t)\|_q \geq C_{r,q} C_{r,q,\mu,\nu} (T^* - t)^{-\frac{(r-3)(5q-6)}{6q(r-2)}}, \quad \forall\, 0 \leq t < T^*, \tag{32}$$

provided that $\frac{3r}{r+3} \leq q \leq \infty$, *where* $C_{r,q}$ *depends only on* r, q *and* $\|(\mathbf{u}_0, \mathbf{b}_0)\|_2$; *and* $C_{r,q,\mu,\nu} = [(1 + \mu^{\frac{3}{2}} + \nu^{\frac{3}{2}})^2 (\mu^{1-\frac{3}{2r}} + \nu^{1-\frac{3}{2r}})]^{-\frac{2r(5q-6)}{6q(r-2)}}$.

Proof. Notice that (32) follows directly from Lema 3.1, Proposio 3.9 and Gagliardo-Nirenberg's inequality below:

$$\|v\|_r \leq C_{r,q} \|v\|_2^{1-\theta} \|Dv\|_q^{\theta}, \quad \forall\, v \in C_0^\infty(\mathbb{R}^3),$$

where $\theta = \frac{6q(r-2)}{2r(5q-6)}$. □

3.8. Proof of Theorem 1.9 *i).* Now, we are interested in proving a sufficient condition in order to guarantee that the solution for the system (1) exists globally in time. Let us start considering a differential inequality that will paly an important role in this subject.

LEMMA 3.12. *Let* $(\mathbf{u}, \mathbf{b})(t)$ *be the strong solution for the system* (1) *defined in the maximal interval* $[0, T^*)$. *Assume that* $3 < q < \infty$. *Then,*

$$\frac{d}{dt} \|(\mathbf{u}, \mathbf{b})(t)\|_q^q \leq C_q \gamma^{-\frac{q+3}{q-3}} \|(\mathbf{u}, \mathbf{b})(t)\|_q^{\frac{q(q-1)}{q-3}}, \quad \forall\, 0 \leq t < T^*,$$

where C_q *depends only on* q; *and* $\gamma = \min\{\mu, \nu\}$.

Proof. Notice that, by considering the divergent operator in the first equation of (1), we can write

$$-\Delta(p + \frac{1}{2}|\mathbf{b}|^2) = \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{b} \cdot \nabla \mathbf{b}),$$

since $\mathbf{u}$ is divergence free. Then,

$$-\Delta(p + \frac{1}{2}|\mathbf{b}|^2) = \sum_{i,j=1}^{3} D_i D_j (u_i u_j - b_i b_j),$$

since $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{b} = 0$. By applying Calderon-Zygmund's theory to the Poisson equation above (see [4, 11] and references therein), one has

$$\|p + \frac{1}{2}|\mathbf{b}|^2\|_r \leq C_r \Big\| \sum_{i,j=1}^{3} [u_i u_j - b_i b_j] \Big\|_r \leq C_r \|(\mathbf{u}, \mathbf{b})\|_{2r}^2, \tag{33}$$

where $1 < r < \infty$. Given $\delta > 0$, let $L_\delta(\cdot)$ be a regularized sign function (see [6] for more details) and $\Phi_\delta(\cdot) := L_\delta(\cdot)^q$. By multiplying the ith line of the first equation of the system (1) by $\Phi'_\delta(u_i(t))$ and integrating over $\mathbb{R}^3$, one obtains

$$\int_{\mathbb{R}^3} \Phi'_\delta(u_i) u_{it}\, dx - \mu \int_{\mathbb{R}^3} \Phi'_\delta(u_i) \Delta u_i\, dx + \int_{\mathbb{R}^3} \Phi'_\delta(u_i)(\mathbf{u} \cdot \nabla) u_i\, dx$$
$$- \int_{\mathbb{R}^3} \Phi'_\delta(u_i)(\mathbf{b} \cdot \nabla) b_i\, dx + \int_{\mathbb{R}^3} \Phi'_\delta(u_i) D_i (p + \frac{1}{2}|\mathbf{b}|^2)\, dx = 0. \tag{34}$$

Analysing each integral on the left hand side of the equality above. Thus, passing to the limit, as $\delta \to 0$, and using Dominated Convergence Theorem, one reaches

$$\lim_{\delta\to 0}\int_{\mathbb{R}^3}\Phi'_\delta(u_i)u_{it}\,dx = \frac{d}{dt}\|u_i(\cdot,t)\|_q^q. \tag{35}$$

Now, notice that

$$\int_{\mathbb{R}^3}\Phi'_\delta(u_i)\Delta u_i\,dx = -\int_{\mathbb{R}^3}\Phi''_\delta(u_i)|\nabla u_i|^2\,dx.$$

Passing to the limit, as $\delta \to 0$, and using, once again, Dominated Convergence Theorem, one concludes

$$\lim_{\delta\to 0}\int_{\mathbb{R}^3}\Phi'_\delta(u_i)\Delta u_i\,dx = -q(q-1)\int_{\mathbb{R}^3}|u_i|^{q-2}|\nabla u_i|^2\,dx. \tag{36}$$

It is easy to check that

$$\int_{\mathbb{R}^3}\Phi'_\delta(u_i)(\mathbf{u}\cdot\nabla)u_i\,dx = 0, \tag{37}$$

since $\nabla\cdot\mathbf{u} = 0$. Moreover,

$$\int_{\mathbb{R}^3}\Phi'_\delta(u_i)(\mathbf{b}\cdot\nabla)b_i\,dx = -\sum_{j=1}^{3}\int_{\mathbb{R}^3}\Phi''_\delta(u_i)D_ju_ib_jb_i\,dx,$$

since $\mathbf{b}$ is divergence free. Passing to the limit, as $\delta \to 0$, and applying Dominated Convergence Theorem, we have

$$\lim_{\delta\to 0}\int_{\mathbb{R}^3}\Phi'_\delta(u_i)(\mathbf{b}\cdot\nabla)b_i\,dx = -q(q-1)\sum_{j=1}^{3}\int_{\mathbb{R}^3}|u_i|^{q-2}(D_ju_i)b_jb_i\,dx.$$

From Cauchy-Schwarz's inequality, we get

$$-q(q-1)\sum_{j=1}^{3}\int_{\mathbb{R}^3}|u_i|^{q-2}(D_ju_i)b_jb_i\,dx \le q(q-1)\int_{\mathbb{R}^3}|u_i|^{q-2}|\nabla u_i||\mathbf{b}||b_i|\,dx.$$

Furthermore, by Hölder's inequality, one infers

$$\int_{\mathbb{R}^3}|u_i|^{q-2}|\nabla u_i||\mathbf{b}||b_i|\,dx \le \|(\mathbf{u},\mathbf{b})\|_{q+2}^2\|u_i\|_{q+2}^{\frac{q-2}{2}}\Big(\int_{\mathbb{R}^3}|u_i|^{q-2}|\nabla u_i|^2\,dx\Big)^{\frac{1}{2}}.$$

Therefore, passing to the limit, as $\delta \to 0$, we obtain

$$\begin{aligned}&\lim_{\delta\to 0}\int_{\mathbb{R}^3}\Phi'_\delta(u_i)(\mathbf{b}\cdot\nabla)b_i\,dx\\ &\le q(q-1)\|(\mathbf{u},\mathbf{b})\|_{q+2}^2\|u_i\|_{q+2}^{\frac{q-2}{2}}\Big(\int_{\mathbb{R}^3}|u_i|^{q-2}|\nabla u_i|^2\,dx\Big)^{\frac{1}{2}}.\end{aligned} \tag{38}$$

The last integral to be studied in (34) informs us that

$$\int_{\mathbb{R}^3}\Phi'_\delta(u_i)D_i(p+\frac{1}{2}|\mathbf{b}|^2)\,dx = -\int_{\mathbb{R}^3}\Phi''_\delta(u_i)(D_iu_i)(p+\frac{1}{2}|\mathbf{b}|^2)\,dx.$$

Once again, by taking the limit, as $\delta \to 0$, and using Dominated Convergence Theorem, one has

$$\lim_{\delta\to 0}\int_{\mathbb{R}^3} \Phi'_\delta(u_i)D_i(p+\frac{1}{2}|\mathbf{b}|^2)\,dx = -q(q-1)\int_{\mathbb{R}^3} |u_i|^{q-2}(D_iu_i)(p+\frac{1}{2}|\mathbf{b}|^2)\,dx.$$

From Hölder's inequality, we obtain

$$-\int_{\mathbb{R}^3} |u_i|^{q-2}(D_iu_i)(p+\frac{1}{2}|\mathbf{b}|^2)\,dx \le \|p+\frac{1}{2}|\mathbf{b}|^2\|_{\frac{q+2}{2}}\|u_i\|_{q+2}^{\frac{q-2}{2}}\Big(\int_{\mathbb{R}^3} |u_i|^{q-2}|\nabla u_i|^2\,dx\Big)^{\frac{1}{2}}.$$

By using (33), we reach

$$\begin{aligned}&\lim_{\delta\to 0}\int_{\mathbb{R}^3} \Phi'_\delta(u_i)D_i(p+\frac{1}{2}|\mathbf{b}|^2)\,dx\\ &\le q(q-1)C_q\|(\mathbf{u},\mathbf{b})\|_{q+2}^2\|u_i\|_{q+2}^{\frac{q-2}{2}}\Big(\int_{\mathbb{R}^3} |u_i|^{q-2}|\nabla u_i|^2\,dx\Big)^{\frac{1}{2}}.\end{aligned} \tag{39}$$

By replacing (35)-(39) in (34), one has

$$\begin{aligned}&\frac{d}{dt}\|u_i\|_q^q + q(q-1)\mu\int_{\mathbb{R}^3} |u_i|^{q-2}|\nabla u_i|^2\,dx\\ &\le q(q-1)C_q\|(\mathbf{u},\mathbf{b})\|_{q+2}^2\|u_i\|_{q+2}^{\frac{q-2}{2}}\Big(\int_{\mathbb{R}^3} |u_i|^{q-2}|\nabla u_i|^2\,dx\Big)^{\frac{1}{2}}.\end{aligned} \tag{40}$$

Now, let us apply the same argument to the second equation of (1). Thus, multiplying the ith line of this equation by $\Phi'_\delta(b_i(t))$ and integrating over $\mathbb{R}^3$, one obtains

$$\begin{aligned}&\int_{\mathbb{R}^3} \Phi'_\delta(b_i)b_{it}\,dx - \nu\int_{\mathbb{R}^3} \Phi'_\delta(b_i)\Delta b_i\,dx\\ &+\int_{\mathbb{R}^3} \Phi'_\delta(b_i)(\mathbf{u}\cdot\nabla)b_i\,dx - \int_{\mathbb{R}^3} \Phi'_\delta(b_i)(\mathbf{b}\cdot\nabla)u_i\,dx = 0.\end{aligned} \tag{41}$$

By (35) and (36), we conclude

$$\lim_{\delta\to 0}\int_{\mathbb{R}^3} \Phi'_\delta(b_i)b_{it}\,dx = \frac{d}{dt}\|b_i\|_q^q$$

and also

$$\lim_{\delta\to 0}\int_{\mathbb{R}^3} \Phi'_\delta(b_i)\Delta b_i\,dx = -q(q-1)\int_{\mathbb{R}^3} |b_i|^{q-2}|\nabla b_i|^2\,dx.$$

We are going to analyse the last two integrals on the left hand side of (41). Note that,

$$\int_{\mathbb{R}^3} \Phi'_\delta(b_i)(\mathbf{u}\cdot\nabla)b_i\,dx = -\sum_{j=1}^3\int_{\mathbb{R}^3} \Phi''_\delta(b_i)(D_jb_i)u_jb_i\,dx,$$

since $\nabla\cdot\mathbf{u} = 0$. Taking the limit, as $\delta \to 0$, and using Dominated Convergence Theorem, we have

$$\lim_{\delta\to 0}\int_{\mathbb{R}^3} \Phi'_\delta(b_i)(\mathbf{u}\cdot\nabla)b_i\,dx = -q(q-1)\sum_{j=1}^3\int_{\mathbb{R}^3} |b_i|^{q-2}(D_jb_i)u_jb_i\,dx.$$

From Cauchy-Scharwz's inequality, one reaches

$$-q(q-1)\sum_{j=1}^{3}\int_{\mathbb{R}^3}|b_i|^{q-2}(D_jb_i)u_jb_i\,dx \le q(q-1)\int_{\mathbb{R}^3}|b_i|^{q-2}|\nabla b_i||(\mathbf{u},\mathbf{b})|^2\,dx.$$

By using Hölder's inequality, we obtain

$$\begin{aligned}&q(q-1)\int_{\mathbb{R}^3}|b_i|^{q-2}|\nabla b_i||(\mathbf{u},\mathbf{b})|^2\,dx\\ &\le q(q-1)\|(\mathbf{u},\mathbf{b})\|_{q+2}^2\|b_i\|_{q+2}^{\frac{q-2}{2}}\left(\int_{\mathbb{R}^3}|b_i|^{q-2}|\nabla b_i|^2\,dx\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore,

$$\lim_{\delta\to 0}\int_{\mathbb{R}^3}\Phi_\delta'(b_i)(\mathbf{u}\cdot\nabla)b_i\,dx \le q(q-1)\|(\mathbf{u},\mathbf{b})\|_{q+2}^2\|b_i\|_{q+2}^{\frac{q-2}{2}}\left(\int_{\mathbb{R}^3}|b_i|^{q-2}|\nabla b_i|^2\,dx\right)^{\frac{1}{2}}.$$

In order to estimate the last integral in (41), we write

$$\int_{\mathbb{R}^3}\Phi_\delta'(b_i)(\mathbf{b}\cdot\nabla)u_i\,dx = -\sum_{j=1}^{3}\int_{\mathbb{R}^3}\Phi_\delta''(b_i)(D_jb_i)b_ju_i\,dx,$$

since $\nabla\cdot\mathbf{b}=0$. Passing to the limit, as $\delta\to 0$, and using Dominated Convergence Theorem, we get

$$\lim_{\delta\to 0}\int_{\mathbb{R}^3}\Phi_\delta'(b_i)(\mathbf{b}\cdot\nabla)u_i\,dx = -q(q-1)\sum_{j=1}^{3}\int_{\mathbb{R}^3}|b_i|^{q-2}(D_jb_i)b_ju_i\,dx.$$

By applying Cauchy-Scharwz's inequality, we have

$$-q(q-1)\sum_{j=1}^{3}\int_{\mathbb{R}^3}|b_i|^{q-2}(D_jb_i)b_ju_i\,dx \le q(q-1)\int_{\mathbb{R}^3}|b_i|^{q-2}|\nabla b_i||(\mathbf{u},\mathbf{b})|^2\,dx.$$

From Hölder's inequality, we infer

$$\lim_{\delta\to 0}\int_{\mathbb{R}^3}\Phi_\delta'(b_i)(\mathbf{b}\cdot\nabla)u_i\,dx \le q(q-1)\|(\mathbf{u},\mathbf{b})\|_{q+2}^2\|b_i\|_{q+2}^{\frac{q-2}{2}}\left(\int_{\mathbb{R}^3}|b_i|^{q-2}|\nabla b_i|^2dx\right)^{\frac{1}{2}}.$$

Thus,

$$\begin{aligned}&\frac{d}{dt}\|b_i\|_q^q + q(q-1)\nu\int_{\mathbb{R}^3}|b_i|^{q-2}|\nabla b_i|^2\,dx\\ &\le 2q(q-1)\|(\mathbf{u},\mathbf{b})\|_{q+2}^2\|b_i\|_{q+2}^{\frac{q-2}{2}}\left(\int_{\mathbb{R}^3}|b_i|^{q-2}|\nabla b_i|^2\,dx\right)^{\frac{1}{2}}.\end{aligned}\tag{42}$$

By adding (40) and (42), one obtains

$$\begin{aligned}&\frac{d}{dt}\|(u_i,b_i)\|_q^q + q(q-1)\mu\int_{\mathbb{R}^3}|u_i|^{q-2}|\nabla u_i|^2\,dx + q(q-1)\nu\int_{\mathbb{R}^3}|b_i|^{q-2}|\nabla b_i|^2\,dx\\ &\le q(q-1)C_q\|(\mathbf{u},\mathbf{b})\|_{q+2}^2\|u_i\|_{q+2}^{\frac{q-2}{2}}\left(\int_{\mathbb{R}^3}|u_i|^{q-2}|\nabla u_i|^2\,dx\right)^{\frac{1}{2}}\\ &\quad + 2q(q-1)\|(\mathbf{u},\mathbf{b})\|_{q+2}^2\|b_i\|_{q+2}^{\frac{q-2}{2}}\left(\int_{\mathbb{R}^3}|b_i|^{q-2}|\nabla b_i|^2\,dx\right)^{\frac{1}{2}}.\end{aligned}$$

Let $\mathbf{v}(t) = (v_1(t), v_2(t), v_3(t))$ and $\mathbf{w}(t) = (w_1(t), w_2(t), w_3(t))$ be given by

$$v_i(t) := |u_i(t)|^{\frac{q}{2}} \quad \text{and} \quad w_i(t) := |b_i(t)|^{\frac{q}{2}}, \quad 1 \le i \le 3.$$

Consequently,

$$\frac{d}{dt}\|(v_i, w_i)\|_2^2 + 4\mu\left(1 - \frac{1}{q}\right)\|\nabla v_i\|_2^2 + 4\nu\left(1 - \frac{1}{q}\right)\|\nabla w_i\|_2^2$$
$$\le 2q\left(1 - \frac{1}{q}\right)C_q\|(\mathbf{u}, \mathbf{b})\|_{q+2}^2\|u_i\|_{q+2}^{\frac{q-2}{2}}\|\nabla v_i\|_2 + 4q\left(1 - \frac{1}{q}\right)\|(\mathbf{u}, \mathbf{b})\|_{q+2}^2\|b_i\|_{q+2}^{\frac{q-2}{2}}\|\nabla w_i\|_2.$$

Then,

$$\frac{d}{dt}\|(\mathbf{v}, \mathbf{w})\|_2^2 + 4\gamma\left(1 - \frac{1}{q}\right)\|(\nabla \mathbf{v}, \nabla \mathbf{w})\|_2^2 \le q\left(1 - \frac{1}{q}\right)C_q\|(\mathbf{v}, \mathbf{w})\|_\beta^{\frac{q+2}{q}}\|(\nabla \mathbf{v}, \nabla \mathbf{w})\|_2,$$

where $\beta := 2 + \frac{4}{q}$ and $\gamma = \min\{\mu, \nu\}$. By using Gagliardo-Nirenberg's inequality

$$\|v\|_\beta \le C_\beta\|v\|_2^{\frac{q-1}{q+2}}\|\nabla v\|_2^{\frac{3}{q+2}}, \quad \forall\, v \in C_0^\infty(\mathbb{R}^3),$$

where C_β depends only on β, we have

$$\frac{d}{dt}\|(\mathbf{v}, \mathbf{w})\|_2^2 + 4\gamma\left(1 - \frac{1}{q}\right)\|(\nabla \mathbf{v}, \nabla \mathbf{w})\|_2^2$$
$$\le q\left(1 - \frac{1}{q}\right)C_q\|(\mathbf{v}, \mathbf{w})\|_2^{\frac{q-1}{q}}\|(\nabla \mathbf{v}, \nabla \mathbf{w})\|_2^{\frac{3+q}{q}}. \tag{43}$$

(Note that the proof of (43) given above is valid for $2 < q < \infty$). The proof of Lemma 3.12 follows directly from Young's inequality. □

The next result shows us that if the initial data for the solution $(\mathbf{u}, \mathbf{b})(t)$ of the system (1) has L^3-norm appropriately small then $\|(\mathbf{u}, \mathbf{b})(t)\|_3$ is strictly decreasing in its interval of existence.

LEMMA 3.13. *Let* $(\mathbf{u}, \mathbf{b})(t)$ *be the strong solution for the MHD system* (1) *defined in the maximal interval* $[0, T^*)$. *Assume that*

$$\|(\mathbf{u}_0, \mathbf{b}_0)\|_3 < \frac{4}{3}\gamma C_3^{-1},$$

where $\gamma = \min\{\mu, \nu\}$ *and* C_3 *is the constant given in* (43). *Then,* $\|(\mathbf{u}, \mathbf{b})(t)\|_3$ *is decreasing in* $[0, T^*)$.

Proof. Considering $q = 3$ in (43), we have

$$\frac{d}{dt}\|(\mathbf{v}, \mathbf{w})\|_2^2 + \frac{8}{3}\gamma\|(\nabla \mathbf{v}, \nabla \mathbf{w})\|_2^2 \le 2C_3\|(\mathbf{v}, \mathbf{w})\|_2^{\frac{2}{3}}\|(\nabla \mathbf{v}, \nabla \mathbf{w})\|_2^2, \quad \forall\, 0 \le t < T^*. \tag{44}$$

By hypothesis, we get

$$\|(\mathbf{v}, \mathbf{w})(0)\|_2 < \left(\frac{4}{3}\gamma C_3^{-1}\right)^{\frac{3}{2}}.$$

By applying a continuity argument, there exists $t_1 \in (0, T^*)$ such that

$$\|(\mathbf{v}, \mathbf{w})(t)\|_2 < \left(\frac{4}{3}\gamma C_3^{-1}\right)^{\frac{3}{2}}, \quad \forall\, 0 \le t \le t_1.$$

Replace the result above in (44) in order to obtain

$$\frac{d}{dt}\|(\mathbf{v},\mathbf{w})(t)\|_2^2 < 0, \quad \forall\, 0 \le t \le t_1.$$

Then, $\|(\mathbf{v},\mathbf{w})(t)\|_2$ is decreasing in $[0,t_1]$. Therefore,

$$\|(\mathbf{v},\mathbf{w})(t_1)\|_2 < \|(\mathbf{v},\mathbf{w})(0)\|_2 < \left(\frac{4}{3}\gamma C_3^{-1}\right)^{\frac{3}{2}}.$$

Once again, by continuity, there is $t_2 \in (t_1, T^*)$ such that

$$\|(\mathbf{v},\mathbf{w})(t)\|_2 < \left(\frac{4}{3}\gamma C_3^{-1}\right)^{\frac{3}{2}}, \quad \forall\, t_1 \le t \le t_2.$$

By replacing the estimate above in (44), one has

$$\frac{d}{dt}\|(\mathbf{v},\mathbf{w})(t)\|_2^2 < 0, \quad \forall\, t_1 \le t \le t_2.$$

Thus, $\|(\mathbf{v},\mathbf{w})(t)\|_2$ is decreasing in $[0,t_2]$. Following this process, we can prove that $\|(\mathbf{v},\mathbf{w})(t)\|_2$ is decreasing in $[0,T^*)$. This means that $\|(\mathbf{u},\mathbf{b})(t)\|_3$ is decreasing $[0,T^*)$. □

The result below allows us to establish another sufficient condition in order to obtain global existence in time for the solution of (1). To this end, we will apply Gagliardo-Nirenberg's inequality

$$\|v\|_2 \le C_q' \|v\|_{\frac{4}{q}}^{1-\frac{3q-6}{3q-2}} \|Dv\|_2^{\frac{3q-6}{3q-2}}, \quad \forall\, v \in C_0^\infty(\mathbb{R}^3). \tag{45}$$

(Here $2 \le q < \infty$).

Proposition 3.14. *Let $(\mathbf{u},\mathbf{b})(t)$ be the strong solution for the system* (1) *defined in the maximal interval $[0,T^*)$. Assume that $3 < q < \infty$ and*

$$\|(\mathbf{u}_0,\mathbf{b}_0)\|_2^{\frac{2q-6}{3q-6}} \|(\mathbf{u}_0,\mathbf{b}_0)\|_q^{\frac{q}{3q-6}} < 2\gamma(qC_qC_q')^{-1}, \tag{46}$$

where $\gamma = \min\{\mu,\nu\}$, C_q and C_q' are given (43) *and* (45)*, respectively. Then, $\|(\mathbf{u},\mathbf{b})(t)\|_q$ is decreasing in $[0,T^*)$. In particular, $T^* = \infty$ if* (46) *holds.*

Proof. Notice that, by (45), one has

$$\|(\mathbf{v},\mathbf{w})\|_2^{\frac{q-1}{q}} \|(D\mathbf{v},D\mathbf{w})\|_2^{\frac{q+3}{q}} \le C_q' \|(\mathbf{v},\mathbf{w})\|_2^{\frac{2}{3q-6}} \|(\mathbf{v},\mathbf{w})\|_{\frac{4}{q}}^{\frac{4}{q}\cdot\frac{q-3}{3q-6}} \|(D\mathbf{v},D\mathbf{w})\|_2^2.$$

By applying (43) and Lemma 3.1, we get

$$\begin{aligned}
&\frac{d}{dt}\|(\mathbf{v},\mathbf{w})\|_2^2 + 4\gamma\Big(1-\frac{1}{q}\Big)\|(D\mathbf{v},D\mathbf{w})\|_2^2 \\
&\le q\Big(1-\frac{1}{q}\Big)C_qC_q'\|(\mathbf{v},\mathbf{w})\|_2^{\frac{2}{3q-6}} \|(\mathbf{v},\mathbf{w})\|_{\frac{4}{q}}^{\frac{4}{q}\cdot\frac{q-3}{3q-6}} \|(D\mathbf{v},D\mathbf{w})\|_2^2 \\
&\le q\Big(1-\frac{1}{q}\Big)C_qC_q'\|(\mathbf{u}_0,\mathbf{b}_0)\|_2^{\frac{2q-6}{3q-6}} \|(\mathbf{u},\mathbf{b})\|_q^{\frac{q}{3q-6}} \|(D\mathbf{v},D\mathbf{w})\|_2^2.
\end{aligned}$$

For the same reason exposed in the proof of Lemma 3.13, one concludes that $\|(\mathbf{u},\mathbf{b})(t)\|_q$ is decreasing in $[0,T^*)$ provided that

$$\|(\mathbf{u}_0,\mathbf{b}_0)\|_2^{\frac{2q-6}{3q-6}}\|(\mathbf{u}_0,\mathbf{b}_0)\|_q^{\frac{q}{3q-6}} < 2\gamma(qC_qC_q^{'})^{-1}.$$

At last, suppose, by contradiction, that $T^*<\infty$ and (46) holds. Then, we obtain

$$\|(\mathbf{u}_0,\mathbf{b}_0)\|_q > \|(\mathbf{u},\mathbf{b})(t)\|_q, \quad \forall\, 0<t<T^*.$$

It is an absurd because of Remark 1.4. □

3.9. Proof of Theorem 1.5. The result below establishes a lower bound for $\|(D^2\mathbf{u},D^2\mathbf{b})(t)\|_q$ $(1<q<\frac{3}{2})$.

PROPOSITION 3.15. *Let* $(\mathbf{u},\mathbf{b})(t)$ *be the strong solution for the MHD system* (1) *defined in the maximal interval* $[0,T^*)$. *Assume that* $1<q<\frac{3}{2}$ *and* $T^*<\infty$. *Then,*

$$\|(D^2\mathbf{u},D^2\mathbf{b})(t)\|_q \geq C_qC_{q,\mu,\nu}(T^*-t)^{-\frac{3q-3}{2q}}, \quad \forall\, 0\leq t<T^*,$$

where C_q *depends only on* q*; and* $C_{q,\mu,\nu}=[(1+\mu^{\frac{3}{2}}+\nu^{\frac{3}{2}})^2(\mu^{\frac{4q-3}{2q}}+\nu^{\frac{4q-3}{2q}})]^{-1}$.

Proof. This proposition follows directly from the inequality

$$\|(\mathbf{u},\mathbf{b})\|_{\frac{3q}{3-2q}} \leq C_q\|(D^2\mathbf{u},D^2\mathbf{b})\|_q$$

and Proposition 3.9. □

We will show, as follows, lower bounds for $\|(D^n\mathbf{u},D^n\mathbf{b})(t)\|_r$, where $n\geq 2$. More precisely, let us enunciate the next proposition.

PROPOSITION 3.16. *Let* $(\mathbf{u},\mathbf{b})(t)$ *be the strong solution for the MHD system* (1) *defined in the maximal interval* $[0,T^*)$. *Assume that* $T^*<\infty$. *Then,*

$$\|(D^n\mathbf{u},D^n\mathbf{b})(t)\|_r \geq C_{q,r,n}C_{q,r,n,\mu,\nu}(T^*-t)^{-\frac{(q-3)(3r+2nr-6)}{6r(q-2)}}, \quad \forall\, 0\leq t<T^*,$$

where $n\geq 3$, $3<q<\infty$, $1\leq r\leq\infty$ *and* $C_{q,r,n}$ *depends only on* q,r,n *and* $\|(\mathbf{u}_0,\mathbf{b}_0)\|_2$*; and* $C_{q,r,n,\mu,\nu}=[(1+\mu^{\frac{3}{2}}+\nu^{\frac{3}{2}})^2(\mu^{1-\frac{3}{2q}}+\nu^{1-\frac{3}{2q}})]^{-\frac{q(3r+2rn-6)}{3r(q-2)}}$. *It also holds the following lower bound:*

$$\|(D^2\mathbf{u},D^2\mathbf{b})(t)\|_r \geq C_{q,r}C_{q,r,\mu,\nu}(T^*-t)^{-\frac{(q-3)(7r-6)}{6r(q-2)}}, \quad \forall\, 0\leq t<T^*, \tag{47}$$

where $3<q<\infty$, $r\geq\frac{3q}{2q+3}$, $C_{q,r}$ *depends only on* q,r *and* $\|(\mathbf{u}_0,\mathbf{b}_0)\|_2$*; and* $C_{q,r,\mu,\nu}=[(1+\mu^{\frac{3}{2}}+\nu^{\frac{3}{2}})^2(\mu^{1-\frac{3}{2q}}+\nu^{1-\frac{3}{2q}})]^{-\frac{q(7r-6)}{3r(q-2)}}$.

Proof. Recall Gagliardo's inequality:

$$\|v\|_q \leq C_{q,r}\|v\|_2^{1-\theta}\|D^nv\|_r^{\theta}, \quad \forall\, v\in C_0^\infty(\mathbb{R}^3),$$

where $\theta=\frac{\frac{1}{2}-\frac{1}{q}}{\frac{1}{2}+\frac{n}{3}-\frac{1}{r}}$, $r\geq\max\left\{1,\frac{3q}{nq+3}\right\}$, $n\geq 2$ and $3\leq q\leq\infty$, if $(n,q,r)\neq\left(2,\infty,\frac{3}{2}\right)$ and $(n,q,r)\neq(3,\infty,1)$.

By Lemma 3.1, we have

$$\|(\mathbf{u},\mathbf{b})\|_q \leq C_{q,r}\|(\mathbf{u}_0,\mathbf{b}_0)\|_2^{1-\theta}\|(D^n\mathbf{u},D^n\mathbf{b})\|_r^{\theta}.$$

Thus,

$$\|(D^n\mathbf{u}, D^n\mathbf{b})\|_r \geq C_{q,r,n}\|(\mathbf{u},\mathbf{b})\|_q^{\frac{1}{\theta}}\|(\mathbf{u}_0,\mathbf{b}_0)\|_2^{\frac{\theta-1}{\theta}}, \quad \forall\, r \geq \max\left\{1, \frac{3q}{nq+3}\right\}. \tag{48}$$

If $n \geq 3$, we obtain

$$\|(D^n\mathbf{u}, D^n\mathbf{b})\|_r \geq C_{q,r,n}\|(\mathbf{u}_0,\mathbf{b}_0)\|_2^{\frac{\theta-1}{\theta}}\|(\mathbf{u},\mathbf{b})\|_q^{\frac{1}{\theta}}, \quad \forall\, r \geq 1.$$

If $n = 2$, then, by (48), we get

$$\|(D^2\mathbf{u}, D^2\mathbf{b})\|_r \geq C_{q,r}\|(\mathbf{u},\mathbf{b})\|_q^{\frac{q(7r-6)}{3r(q-2)}}\|(\mathbf{u}_0,\mathbf{b}_0)\|_2^{\frac{-4rq-6r+6q}{3r(q-2)}}, \quad \forall\, r \geq \frac{3q}{2q+3}.$$

Therefore, (47) follows from Proposition 3.9. □

3.10. Proof of Theorem 1.7. Now, we are interested in proving the relationship between the blow up rates presented by $\|(\mathbf{u},\mathbf{b})(t)\|_r$ and $\|(\mathbf{u},\mathbf{b})(t)\|_q$ provided that $3 \leq q < r < \infty$.

PROPOSITION 3.17. *Let* $(\mathbf{u},\mathbf{b})(t)$ *be the strong solution for the MHD system* (1) *defined in the maximal interval* $[0,T^*)$. *Assume that* $T^* < \infty$ *and* $3 \leq q < r < \infty$. *Then,*

$$\frac{\|(\mathbf{u},\mathbf{b})(t)\|_r}{\|(\mathbf{u},\mathbf{b})(t)\|_q} \geq C_{q,r}C_{q,r,\mu,\nu}(T^*-t)^{-\frac{r-3}{r-2}\cdot\frac{r-q}{qr}}, \quad \forall\, 0 \leq t < T^*,$$

where $C_{q,r,\mu,\nu}$ *depends only on* q, r, μ, ν *and* $\|(\mathbf{u}_0,\mathbf{b}_0)\|_2$; *and* $C_{q,r,\mu,\nu} = [(1+\mu^{\frac{3}{2}}+\nu^{\frac{3}{2}})^2(\mu^{1-\frac{3}{2r}}+\nu^{1-\frac{3}{2r}})]^{-\frac{2(r-q)}{q(r-2)}}$.

Proof. By using interpolation inequality and Lemma 3.1, we have

$$\|(\mathbf{u},\mathbf{b})\|_r^{\lambda} \leq \|(\mathbf{u}_0,\mathbf{b}_0)\|_2^{\lambda}\frac{\|(\mathbf{u},\mathbf{b})\|_r}{\|(\mathbf{u},\mathbf{b})\|_q}, \tag{49}$$

where $\lambda = \frac{\frac{1}{q}-\frac{1}{r}}{\frac{1}{2}-\frac{1}{r}}$ (the inequality above holds for $r = \infty$). Therefore, by Proposition 3.9, we complete the proof of Proposition 3.17. □

Now, we are going to show a relationship between the blow up rates of $\|(D\mathbf{u}, D\mathbf{b})(t)\|_2$ and $\|(\mathbf{u},\mathbf{b})(t)\|_q$ $(2 \leq q \leq 6)$.

PROPOSITION 3.18. *Let* $(\mathbf{u},\mathbf{b})(t)$ *be the strong solution for the MHD system* (1) *defined in the maximal interval* $[0,T^*)$. *Assume that* $T^* < \infty$ *and* $2 \leq q \leq 6$. *Then,*

$$\frac{\|(D\mathbf{u}, D\mathbf{b})(t)\|_2}{\|(\mathbf{u},\mathbf{b})(t)\|_q} \geq C_q\gamma^{\frac{18-3q}{8q}}(T^*-t)^{-\frac{6-q}{8q}}, \quad \forall\ 0 \leq t < T^*,$$

where C_q *depends only on* q *and* $\|(\mathbf{u}_0,\mathbf{b}_0)\|_2$; *and* $\gamma = \min\{\mu,\nu\}$.

Proof. Consider Gagliardo-Nirenberg's inequality

$$\|v\|_q \leq C_q\|v\|_2^{1-\theta}\|Dv\|_2^{\theta}, \quad \forall\, v \in C_0^{\infty}(\mathbb{R}^3),$$

where $\theta = \frac{3(q-2)}{2q}$ and $2 \leq q \leq 6$. By using this inequality, Lemma 3.1 and Proposition 3.6, we obtain

$$\frac{\|(D\mathbf{u}, D\mathbf{b})(t)\|_2}{\|(\mathbf{u},\mathbf{b})(t)\|_q} \geq \frac{C_q}{\|(\mathbf{u}_0,\mathbf{b}_0)\|_2^{1-\theta}}\gamma^{\frac{3}{4}(1-\theta)}(T^*-t)^{-\frac{1}{4}(1-\theta)}, \quad \forall\ 0 \leq t < T^*.$$

Therefore, Proposition 3.18 follows. □

3.11. Proof of Theorem 1.9 *ii)*. The next result informs that the solution $(\mathbf{u},\mathbf{b})(t)$ for the system (1) is global in time if

$$\sup_{0\le t<T^*}\left\{\frac{\|(\mathbf{u},\mathbf{b})(t)\|_\infty^q}{\|(\mathbf{u},\mathbf{b})(t)\|_q^q}\cdot\frac{\|(\mathbf{u},\mathbf{b})(t)\|_3}{\|(\mathbf{u},\mathbf{b})(t)\|_\infty^2}\right\}<\infty,$$

where $3<q<\infty$.

PROPOSITION 3.19. *Let $(\mathbf{u},\mathbf{b})(t)$ be the strong solution for the MHD system (1) defined in the maximal interval $[0,T^*)$. Assume that $T^*<\infty$ and $3<q<\infty$. Then,*

$$M_5:=\sup_{0\le t<T^*}\left\{\frac{\|(\mathbf{u},\mathbf{b})(t)\|_\infty^q}{\|(\mathbf{u},\mathbf{b})(t)\|_q^q}\cdot\frac{\|(\mathbf{u},\mathbf{b})(t)\|_3}{\|(\mathbf{u},\mathbf{b})(t)\|_\infty^2}\right\}=\infty. \tag{50}$$

Proof. By the first equation of the system (1), we have

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\|\mathbf{u}\|_q^q=\mu\int_{\mathbb{R}^3}|\mathbf{u}|^{q-2}\mathbf{u}\cdot\Delta\mathbf{u}\,dx-\int_{\mathbb{R}^3}|\mathbf{u}|^{q-2}\mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\mathbf{u}\,dx\\+\int_{\mathbb{R}^3}|\mathbf{u}|^{q-2}\mathbf{u}\cdot(\mathbf{b}\cdot\nabla)\mathbf{b}\,dx-\int_{\mathbb{R}^3}|\mathbf{u}|^{q-2}\mathbf{u}\cdot\nabla(p+\frac{1}{2}|\mathbf{b}|^2)\,dx.\end{aligned}$$

Let us examine the integrals on the right hand side of the equality above. Thus,

$$\begin{aligned}\int_{\mathbb{R}^3}|\mathbf{u}|^{q-2}\mathbf{u}\cdot\Delta\mathbf{u}\,dx=-(q-2)\sum_{j=1}^3\int_{\mathbb{R}^3}|\mathbf{u}|^{q-4}(\mathbf{u}\cdot D_j\mathbf{u})^2\,dx\\-\int_{\mathbb{R}^3}|\mathbf{u}|^{q-2}|D\mathbf{u}|^2\,dx\le 0.\end{aligned} \tag{51}$$

Moreover, we can write

$$\begin{aligned}-\int_{\mathbb{R}^3}|\mathbf{u}|^{q-2}\mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\mathbf{u}\,dx=\sum_{i,j=1}^3\int_{\mathbb{R}^3}D_j(|\mathbf{u}|^{q-2})u_i^2u_j\,dx\\+\sum_{i,j=1}^3\int_{\mathbb{R}^3}|\mathbf{u}|^{q-2}D_j(u_iu_j)u_i\,dx.\end{aligned}$$

Then, by using the fact that $\nabla\cdot\mathbf{u}=0$, one obtains

$$\int_{\mathbb{R}^3}|\mathbf{u}|^{q-2}\mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\mathbf{u}dx=0.$$

Notice also that,

$$\begin{aligned}\int_{\mathbb{R}^3}|\mathbf{u}|^{q-2}\mathbf{u}\cdot(\mathbf{b}\cdot\nabla)\mathbf{b}\,dx=-(q-2)\sum_{j,k=1}^3\int_{\mathbb{R}^3}|\mathbf{u}|^{q-4}u_k(D_ju_k)(\mathbf{u}\cdot\mathbf{b})b_j\,dx\\-\sum_{i,j=1}^3\int_{\mathbb{R}^3}|\mathbf{u}|^{q-2}(D_ju_i)b_jb_i\,dx,\end{aligned}$$

where we have applied the equality $\nabla\cdot\mathbf{b}=0$. Consequently, by Cauchy-Scharwz and Hölder's inequalities, we get

$$\int_{\mathbb{R}^3}|\mathbf{u}|^{q-2}\mathbf{u}\cdot(\mathbf{b}\cdot\nabla)\mathbf{b}\,dx\le C_q\|\mathbf{u}\|_\infty^{q-2}\|D\mathbf{u}\|_2\|\mathbf{b}\|_4^2.$$

Hence, by Gagliardo-Nirenberg's inequality

$$\|v\|_4 \le C\|v\|_3^{\frac{1}{2}}\|Dv\|_2^{\frac{1}{2}}, \quad \forall\, v \in C_0^\infty(\mathbb{R}^3), \tag{52}$$

we infer

$$\int_{\mathbb{R}^3} |\mathbf{u}|^{q-2}\mathbf{u}\cdot(\mathbf{b}\cdot\nabla)\mathbf{b}\,dx \le C_q\|(\mathbf{u},\mathbf{b})\|_\infty^{q-2}\|(\mathbf{u},\mathbf{b})\|_3\|(D\mathbf{u},D\mathbf{b})\|_2^2.$$

Once again, by using the fact that $\mathbf{u}$ is divergence free, we conclude

$$-\int_{\mathbb{R}^3} |\mathbf{u}|^{q-2}\mathbf{u}\cdot\nabla(p+\frac{1}{2}|\mathbf{b}|^2)\,dx = (q-2)\sum_{j=1}^{3}\int_{\mathbb{R}^3}|\mathbf{u}|^{q-4}(\mathbf{u}\cdot D_j\mathbf{u})u_j(p+\frac{1}{2}|\mathbf{b}|^2)\,dx.$$

By applying Cauchy-Scharwz and Hölder's inequalities, (33) and (52), one obtains

$$\begin{aligned}-\int_{\mathbb{R}^3} |\mathbf{u}|^{q-2}\mathbf{u}\cdot\nabla(p+\frac{1}{2}|\mathbf{b}|^2)\,dx &\le C_q\|(\mathbf{u},\mathbf{b})\|_\infty^{q-2}\|(D\mathbf{u},D\mathbf{b})\|_2\|(\mathbf{u},\mathbf{b})\|_4^2\\ &\le C_q\|(\mathbf{u},\mathbf{b})\|_\infty^{q-2}\|(\mathbf{u},\mathbf{b})\|_3\|(D\mathbf{u},D\mathbf{b})\|_2^2.\end{aligned}$$

Consequently, one has

$$\frac{1}{q}\frac{d}{dt}\|\mathbf{u}\|_q^q \le C_q\|(\mathbf{u},\mathbf{b})\|_\infty^{q-2}\|(\mathbf{u},\mathbf{b})\|_3\|(D\mathbf{u},D\mathbf{b})\|_2^2. \tag{53}$$

Checking the second equation of the system (1), we can write

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\|\mathbf{b}\|_q^q = \nu\int_{\mathbb{R}^3}|\mathbf{b}|^{q-2}\mathbf{b}\cdot\Delta\mathbf{b}\,dx - \int_{\mathbb{R}^3}|\mathbf{b}|^{q-2}\mathbf{b}\cdot(\mathbf{u}\cdot\nabla)\mathbf{b}\,dx\\ +\int_{\mathbb{R}^3}|\mathbf{b}|^{q-2}\mathbf{b}\cdot(\mathbf{b}\cdot\nabla)\mathbf{u}\,dx.\end{aligned} \tag{54}$$

Let us to analyse all the integrals on the right hand side of (54). By (51), we have

$$\nu\int_{\mathbb{R}^3}|\mathbf{b}|^{q-2}\mathbf{b}\cdot\Delta\mathbf{b}dx \le 0.$$

Notice that,

$$\begin{aligned}-\int_{\mathbb{R}^3}|\mathbf{b}|^{q-2}\mathbf{b}\cdot(\mathbf{u}\cdot\nabla)\mathbf{b}\,dx = (q-2)\sum_{j=1}^{3}\int_{\mathbb{R}^3}|\mathbf{b}|^{q-4}(\mathbf{b}\cdot D_j\mathbf{b})u_j|\mathbf{b}|^2\,dx\\ +\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}|\mathbf{b}|^{q-2}(D_jb_i)u_jb_i\,dx.\end{aligned}$$

By using Cauchy-Scharwz and Hölder's inequality, we conclude

$$-\int_{\mathbb{R}^3}|\mathbf{b}|^{q-2}\mathbf{b}\cdot(\mathbf{u}\cdot\nabla)\mathbf{b}dx \le C_q\|\mathbf{b}\|_\infty^{q-2}\|D\mathbf{b}\|_2\|(\mathbf{u},\mathbf{b})\|_4^2.$$

Thus, by applying Gagliardo-Nirenberg's inequality (52), we get

$$-\int_{\mathbb{R}^3}|\mathbf{b}|^{q-2}\mathbf{b}\cdot(\mathbf{u}\cdot\nabla)\mathbf{b}\,dx \le C_q\|(\mathbf{u},\mathbf{b})\|_\infty^{q-2}\|(\mathbf{u},\mathbf{b})\|_3\|(D\mathbf{u},D\mathbf{b})\|_2^2.$$

Analogously, one proves that

$$\int_{\mathbb{R}^3} |\mathbf{b}|^{q-2}\mathbf{b}\cdot(\mathbf{b}\cdot\nabla)\mathbf{u}\,dx \leq C_q\|(\mathbf{u},\mathbf{b})\|_\infty^{q-2}\|(\mathbf{u},\mathbf{b})\|_3\|(D\mathbf{u},D\mathbf{b})\|_2^2.$$

Hence, we obtain

$$\frac{1}{q}\frac{d}{dt}\|\mathbf{b}\|_q^q \leq C_q\|(\mathbf{u},\mathbf{b})\|_\infty^{q-2}\|(\mathbf{u},\mathbf{b})\|_3\|(D\mathbf{u},D\mathbf{b})\|_2^2. \tag{55}$$

Then, adding (53) and (55), one obtains

$$\frac{d}{dt}\|(\mathbf{u},\mathbf{b})\|_q^q \leq C_q\frac{\|(\mathbf{u},\mathbf{b})\|_\infty^q}{\|(\mathbf{u},\mathbf{b})\|_q^q}\frac{\|(\mathbf{u},\mathbf{b})\|_3}{\|(\mathbf{u},\mathbf{b})\|_\infty^2}\|(\mathbf{u},\mathbf{b})\|_q^q\|(D\mathbf{u},D\mathbf{b})\|_2^2.$$

Suppose, by contradiction, that $M_5 < \infty$. Thus,

$$\frac{d}{dt}\|(\mathbf{u},\mathbf{b})\|_q^q \leq C_qM_5\|(\mathbf{u},\mathbf{b})\|_q^q\|(D\mathbf{u},D\mathbf{b})\|_2^2.$$

At last, by using Gronwall's Lemma and Lemma 3.1, we conclude

$$\|(\mathbf{u},\mathbf{b})(t)\|_q \leq \|(\mathbf{u}_0,\mathbf{b}_0)\|_q\exp\left\{C_qM_5\gamma^{-1}\|(\mathbf{u}_0,\mathbf{b}_0)\|_2^2\right\}, \quad \forall\, 0\leq t<T^*,$$

where $3<q<\infty$. It is an absurd because of Proposition 3.9. Therefore, (50) holds. □

4. Conclusions. It is important to point out that the results presented in this paper can be extended if we assume that the limit

$$\lim_{t\nearrow T^*}\|(\mathbf{u},\mathbf{b})(t)\|_3 = \infty, \quad T^*<\infty, \tag{56}$$

is valid and Lemma 3.8 holds in the case $q=\infty$. However, it has not been proved in the literature for the knowledge of the authors. Recall that these statements were established in [14] and [10], respectively, by considering the case of the Navier-Stokes equations.

More precisely, we could add the cases $q=3$ in Theorem 1.9 *i)* (see Remark 1.10 and (56)) and $q=\infty$ in Theorem 1.3 *ii)* (see Lemma 3.8 with $q=\infty$) and Remark 1.4, item 1. In order to verify that Theorem 1.3 *ii)* also holds for $q=3$, it is enough to use a contradiction argument and apply (44) and (56). Moreover, Theorem 1.3 *iv)* is valid for $q=3/2$ by using the inequality $\|\phi\|_3\leq C\|D\phi\|_{\frac{3}{2}}$, for all $\phi\in C_0^\infty(\mathbb{R}^3)$, and Theorem 1.3 *ii)* (with $q=3$). In addition, Theorem 1.3 *iii)* holds for $\varepsilon=1/2$ by using the inequality $\|\phi\|_3\leq C\|\phi\|_2^{\frac{2}{3}}\|D\phi\|_3^{1-\frac{2}{3}}$, for all $\phi\in C_0^\infty(\mathbb{R}^3)$, Theorem 1.3 *ii)* (with $q=3$) and Lemma 3.1. Theorem 1.5 *i)* can be extended to the case $q=1$ through the inequality $\|\phi\|_3\leq C\|D^2\phi\|_1$, for all $\phi\in C_0^\infty(\mathbb{R}^3)$, and Theorem 1.3 *ii)* (where $q=3$). This previous inequality and (56) assure that Remark 1.6, item 1, is valid for $q=1$ as well. By analyzing Theorem 1.7 *i)* and applying (49), we guarantee that this result also holds for $r=\infty$ (see Theorem 1.3 *ii)* with $q=\infty$); as a result, Remark 1.8, item 1, is valid for $r=\infty$.

Some results can be also reached by admitting the veracity of (56) and Lemma 3.8 in the case $q=\infty$. In fact, from the inequality

$$\|\phi\|_\infty\leq C_q\|\phi\|_2^{1-\theta}\|D\phi\|_q^\theta, \quad \forall\phi\in C_0^\infty(\mathbb{R}^3),$$

where $\theta = \frac{3q}{5q-6}, 3 < q \leq \infty$, Theorem 1.3 *ii)* (with $q = \infty$) and Lemma 3.1, one concludes

$$\|(\mathbf{u}, \mathbf{b})(t)\|_q \geq C \|(\mathbf{u}_0, \mathbf{b}_0)\|_2^{-\frac{2q-6}{3q}} (T^* - t)^{-\frac{5q-6}{6q}}, \quad 0 \leq t < T^*,$$

$T^* < \infty$ and $3 < q \leq \infty$. Moreover, by using (49), with $r = \infty$, and (56), one infers

$$\lim_{t \nearrow T^*} \frac{\|(\mathbf{u}, \mathbf{b})(t)\|_\infty^q \|(\mathbf{u}, \mathbf{b})(t)\|_3}{\|(\mathbf{u}, \mathbf{b})(t)\|_q^q \|(\mathbf{u}, \mathbf{b})(t)\|_\infty^2} = \infty,$$

where $T^* < \infty$ and $3 < q < \infty$.

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