Methods and Applications of Analysis.
Vol. 25, No. 3, pp. 205–224, September 2018

INTRODUCTION TO THE STUDY OF ARNOLD DIFFUSION*

CHONG-QING CHENG† AND MIN ZHOU†

Abstract. This article is written for general dynamicists to understand the ideas for the proof of Arnold diffusion conjecture, not in pursuit of rigorousness. Readers need not to be familiar with the variational method based on the Mather theory.

Key words. Dynamical instability, Arnold diffusion.

Mathematics Subject Classification. 37J40, 37D05.

1. Introduction. The problem of Arnold diffusion is raised for nearly integrable Hamiltonian systems to study the dynamical instability

$$H(p,q) = h(p) + \epsilon P(p,q), \qquad (p,q) \in \mathbb{R}^n \times \mathbb{T}^n,\ n \geq 3. \tag{1.1}$$

Because the Hamiltonian H is autonomous, each energy level set is invariant for the Hamiltonian flow, denoted by Φ_H^t. According to the KAM theory, large part of each energy level set is occupied by invariant tori where each orbit of Φ_H^t is quasi-periodic. However, with respect to $(2n-1)$ dimensional energy level set, the complimentary part of the set of n-dimensional KAM tori is open-dense and path connected if $n \geq 3$. After he constructed an example where there are initial conditions leading to a significant drift in the action variables, Arnold foresaw a much more stronger form of instability in which neighborhoods of any two points on a given energy surface would be connected by a drifting orbit. Here is the precise formulation of what has come to be known as Arnold's conjecture for the system above (cf. [A66], or [A94] Section 1.8 p.17):

Arnold's conjecture. For $n \geq 3$, the *general case* for nearly integrable Hamiltonian $H(p,q) = h(p) + \epsilon P(p,q)$ is represented by the situation that for arbitrary pair of neighborhoods of tori $p = p'$ and $p = p''$ in one component of the level set $h(p') = h(p'')$ there exists, for sufficiently small ϵ, an orbit intersects both neighborhoods.

The study of Arnold diffusion conjecture has proved quite daunting, engaging scores of investigators over the last half century. For long time, people used to apply Arnold's idea developed in [A64] to construct diffusion orbit. It relies on the existence of NHIC (normally hyperbolic invariant cylinder). The difficulties encountered in this direction are *the gap problem* and *the transversality problem*, the former puzzled people for forty years while the latter appears to be more substantial (cf. [CY04, T04, DLS06, B08, CY09]). The resolution of these problems opened the way to construct diffusion orbits along the path of single resonance. However, as pointed out by Arnold [A66], *in order to take the final step in the proof of the above conjecture, it is necessary to examine the transition from single to double resonance*, because NHIC may collapse. Since the announcement of Mather in 2001 ([M03]), it becomes known as a notorious difficulty. We don't know the details Mather designed to cross double resonance [M09], his work in [M11] is obviously towards to the goal. Recently, Kaloshin-Zhang [KZ] and Marco [Mar] proposed that a cylinder with hole could be used to cross double resonance, their way in the cohomology space appears the same as Mather's way.

*Received August 29, 2018; accepted for publication January 4, 2019.
†Department of mathematics, Nanjing Univerisity, Nanjing 210093, China (chengcq@nju.edu.cn; minzhou@nju.edu.cn).

We shall discuss some issues in the final section which need to be well understood if one wants to cross double resonance along their way. By turning around instead of passing through the double resonant point, we found a different way to join two pieces of cylinders, the problem of double resonance was therefore solved in [C17b]. Recently, the mechanism of turning around the point was observed numerically in [GSV].

To bring readers to the stage of understanding what is the gap problem, we illustrate Arnold's idea for his celebrated example in the second section. The third section is for the "proof" of diffusion along the path of single resonance (*a priori* unstable case), to explain what are the gap problem and transversality problem and to show how to solve them. In the fourth section, we explain why one has to handle the problem of strong double resonance. The ideas to solve the problem in [CZ16, C17a, C17b] are illustrated in the sections from 5 to 8.

2. What is the Arnold's mechanism. Arnold's model of instability is the following Hamiltonian system with two and half degrees of freedom [A64]:

$$H(p,q,t)=\frac{1}{2}(p_1^2+p_2^2)-\epsilon(1-\cos q_1)(1+\mu(\cos q_2+\cos t)). \tag{2.1}$$

If $\mu=0$, it is a decoupled rotator-pendulum system, the rotator $\frac{1}{2}p_2^2$ and the pendulum $\frac{1}{2}p_1^2-\epsilon(1-\cos q_1)$. The hyperbolic fixed point of the pendulum $(p_1,q_1)=0$ has its stable and unstable manifolds which merge into one separatrix

$$\Gamma_0=\{(p_1,q_1):p_1=\pm\sqrt{2\epsilon(1-\cos q_1)}\}.$$

In the product phase space $\mathbb{R}^2\times\mathbb{T}^2$, the fixed point is suspended into a cylinder

$$\Pi_0=\{(p,q)\in\mathbb{R}^2\times\mathbb{T}^2:(p_1,q_1)=0\},$$

which is invariant and normally hyperbolic for the time-2π-map $\Phi_H=\Phi_H^t|_{t=2\pi}$. The cylinder is the phase space of the rotator, admitting a foliation of invariant circles. Each circle has its stable and unstable manifold with 2-dimension which is the product of the separatrix with the circle.

For $\mu\neq 0$, the perturbation is chosen is so special that the dynamics on the cylinder Π_0 remains unchanged. However, the stable and unstable manifold of each circle in the cylinder split and intersect transversally. Consequently, for any two numbers $A<B$ there is a sequence $\{p_{2,i}:i=0,1,\cdots,k\}$ so that $p_{2,0}<A<B<p_{2,k}$, the unstable manifold of the circle where $p_2=p_{2,i}$ transversally intersects the stable manifold of the circle where $p_2=p_{2,i+1}$, denoted by $W^u_{p_{2,i}}\pitchfork W^s_{p_{2,i+1}}$. Arnold call it *transition chain*. In this case, a long orbit with $p_2(t_1)<A$ and $p_2(t_2)>B$ can be constructed by λ-lemma, see the left of Figure 1. It is so-called Arnold's mechanism for the construction of connecting orbits.

Let us look at it from variational point of view. Indeed, it was observed in [Bs] that some homoclinic orbit minimizes the Lagrange action. The orbits homoclinic to the circle $\Gamma_{p_2^*}=\{(p,q):p_2=p_2^*,(p_1,q_1)=0\}$ are obtained by searching for a sequence of minimal curves

$$\liminf_{k\to\infty}\min_{\substack{\gamma(-2k\pi)=\gamma(2k\pi)\in\pi\Pi_0\\ [\gamma]=(1,0)}}\int_{-2k\pi}^{2k\pi}\Big(\frac{1}{2}(\dot\gamma_1^2(t)+(\dot\gamma_2^2-p_2^*)^2)+V(\gamma(t),t)\Big)dt$$

where $\gamma=(\gamma_1,\gamma_2)$, $[\gamma]\in H_1(\mathbb{T}^2,\mathbb{Z})$, π denotes the projection along the cotangent fiber $T^*\mathbb{T}^2\to\mathbb{T}^2$ and $V(q,t)=\epsilon(1-\cos q_1)(1+\mu(\cos q_2+\cos t))$. The observation hints the possibility to apply Mather theory to the problem of Arnold diffusion.

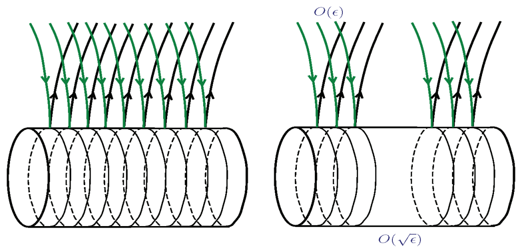


FIG. 1. *The left: no gap, the right: there exist gaps.*

Indeed, because we work in the symplectic space, both stable and unstable manifold are Lagrangian sub-manifold. Restricted on the place where it keeps horizontal (being graph over the configuration manifold), each Lagrangian has its own generating function, as the symplectic form $dp\wedge dq$ vanishes when it is restricted on a Lagrangian. Let $u^{u,s}_{p_2}$ be the generating function for $W^{u,s}_{p_2}$ respectively, it holds at the intersection point that $du^u_{p_2} = du^s_{p_2}$, namely, each critical point of the barrier function

$$B_{p_2}(q) = u^u_{p_2}(q) - u^s_{p_2}(q) \tag{2.2}$$

corresponds to a homoclinic orbit. Since the Hamiltonian (1.1) is positive definite, the barrier function has its minimal points not lying on the circle $\pi\Pi_0$. Passing through each minimal point there is a minimal configuration $q(t)$ producing a homoclinic orbit $(p(t), q(t))$. The transversal intersection implies that the orbit is disconnected to other homoclinic orbits.

3. Diffusion along cylinders in generic case. The perturbation introduced in Arnold's example is so artificial that the dynamics on the cylinder is still integrable, which guarantees the existence of transition chain. Under generic perturbation, although the NHIC still exists with slight deformation, the dynamics on the cylinder is no longer integrable. The restriction of the symplectic map on the cylinder is an exact, area-preserving twist map. Generically, there does not exist invariant circle with rational rotation number. Corresponding to each rational number $\frac{k}{k'}$ such that k' is not large there is a Birkhoff instability region looks like an annulus. Its width is of order $O(\sqrt{\epsilon})$ if the Hamiltonian takes the form

$$H(p, q, t) = h_1(p_1, q_1) + h_2(p_2) + \epsilon P(p, q, t)$$

where h_1, h_2 represent the pendulum and the rotator respectively. In contrast with the width of the Birkhoff instability region, the splitting of stable and unstable manifold of each invariant circle is upper bounded by the order $O(\epsilon)$, much smaller than $O(\sqrt{\epsilon})$. It appears unclear whether the unstable manifold of the circle on a side of the Birkhoff region intersects the stable manifold of a circle on the other side, namely, the transition chain may break down. It is so-called the *gap* problem, see the right of Figure 1.

To cross the gap, let us recall a fact in the theory for twist maps. In each Birkhoff instability region, there exist some orbits which connect the invariant circles on both sides (cf.[M91]), namely, there exist orbits lying on the cylinder which cross the gap. Such orbits shadow a sequence of successively connected orbits $(\gamma_i(t), \dot\gamma_i(t))$ which are minimal in the sense that

$$A_{L'}(\gamma_i(t))|_{[t_0,t_1]} = \min_{\substack{\xi(t_0)=\gamma_i(t_0)\\ \xi(t_1)=\gamma_i(t_1)}} A_{L'}(\xi(t))|_{[t_0,t_1]} \tag{3.1}$$

holds for any $t_0 < t_1$, where the action of L' along a curve $\xi(t)$ is defined as

$$A_{L'}(\xi(t))|_{[t_0,t_1]} = \int_{t_0}^{t_1} L'(\xi(t), \dot{\xi}(t), t)dt,$$

where $L' = L + L''$, L is related to the Hamiltonian H via Legendre transformation, L'' is carefully chosen such that $\frac{d}{dt}\frac{\partial L''}{\partial \dot{q}} - \frac{\partial L''}{\partial q}$ vanishes in a neighborhood of the curve $\gamma_i(t)$. For local connecting orbits in Birkhoff instability region, the principle of cohomology equivalence is applied to construct L''. In Arnold's example, the orbit connecting the circle $\Gamma_{p_{2,i}}$ to the circle $\Gamma_{p_{2,i+1}}$ is also minimal. The choice of L'' in this case makes use of the fact that the minimal points of the barrier function is totally disconnected, see [CY04, CY09]. In this way, one obtains a variational version of (generalized) transition chain: along the path of diffusion there is a sequence of minimal orbits, successively connected. The diffusion orbit is constructed shadowing these local connecting orbits and dynamically connects two invariant circles. Two

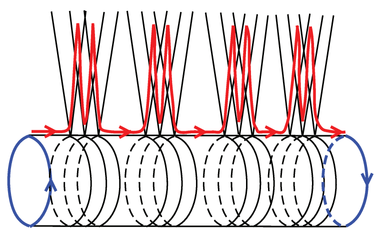


FIG. 2. *The red curve shows the projection of diffusion orbit in the normal direction, its projection to the center manifold turns around the cylinder.*

invariant sets S_1 and S_2 are said to be *dynamically connected* if there is an orbit whose α-limit set is contained in S_1 and the ω-limit set is contained in S_2, or vice versa.

Until now we have taken it as granted in generic case that, for every invariant circle, the stable manifold intersects the unstable manifold transversally. However, it is a challenging problem to verify the condition, we call it the *transversality* problem.

It would be trivial if we check the transversality condition for one, and consequently for countably many circles only. The intersection of countably many open-dense sets makes up a residual set in the function space. But one needs to check the condition for a set of circles with positive Lebesgue measure.

Recall that the transversal intersection of the stable and unstable manifold implies the non-degeneracy of the minimal point of the barrier function. If the barrier function B_σ has a non-degenerate minimal point, then for σ' sufficiently close to σ, the function $B_{\sigma'}$ is non-degenerate at its minimal point either. However, one does not know how small $|\sigma - \sigma'|$ needs to be such that the non-degeneracy holds. The situation will be changed if the barrier is parameterized by some number σ so that

$$\sup_q |B_\sigma(q) - B_{\sigma'}(q)| \leq C|\sigma - \sigma'|^\mu. \tag{3.2}$$

In this case, the size of the neighborhood of σ is of order $O(|d|^{1/\mu}|)$, where $|d|$ is the size of the non-degeneracy. Therefore, one is able to solve the transversality problem once certain modulus of continuity of barrier functions is established.

The modulus of continuity is also a hard issue. Fortunately, one obtains it in the *a priori* unstable case, i.e. some NHIC already exists before the system is perturbed. To do it, we arbitrarily choose one circle Γ_0 and parameterize another circle Γ_σ by the algebraic area between Γ_0 and Γ_σ,

$$\sigma = \int (\Gamma_\sigma(q_2) - \Gamma_0(q_2))dq_2.$$

This integration is in the sense that we pull it back to the standard cylinder Π_0. In this way, we obtain one-parameter family of circles $\Gamma_\sigma : \mathbb{T} \times \mathbb{S} \to \Pi_0$ where $\mathbb{S} \subset [A', B']$ is a Cantor set with positive Lebesgue measure. One can think Γ_σ as a map to function space C^0 equipped with supremum norm

$$\|\Gamma_\sigma - \Gamma_{\sigma'}\| = \max_{q_2} |\Gamma_\sigma(q_2) - \Gamma_{\sigma'}(q_2)|.$$

Since all circles are Lipschitz curve with uniform Lipschitz constant C_L, the annulus bounded by Γ_σ and $\Gamma_{\sigma'}$ contains a diamond, the length of its diagonals is $\|\Gamma_\sigma - \Gamma_{\sigma'}\|$ and $C_L^{-1}\|\Gamma_\sigma - \Gamma_{\sigma'}\|$ respectively. As the area of the diamond is not larger than $|\sigma - \sigma'|$, one obtains

$$\|\Gamma_\sigma - \Gamma_{\sigma'}\| \le \sqrt{2C_L|\sigma - \sigma'|}.$$

The modulus continuity of barrier functions is then obtained by applying the theorem of normally hyperbolic invariant manifold (NHIM). Certain modulus continuity also holds for Aubry-Mather sets of twist map [Zm].

A sub-manifold Π, invariant for a diffeomorphism Φ, is called normally hyperbolic if the tangent bundle over Π admits $D\Phi$-invariant splitting

$$T_zM = T_zN^+ \oplus T_z\Pi \oplus T_zN^-$$

some $\Lambda_2 > \Lambda_1 \ge 1$ such that the following hold

$$\begin{aligned} \Lambda_1^{-1} < \frac{\|D\Phi(z)v\|}{\|v\|} &< \Lambda_1, \qquad \forall\, v \in T_z\Pi, \\ \frac{\|D\Phi(z)v\|}{\|v\|} &\le \Lambda_2, \qquad \forall\, v \in T_zN^+, \\ \frac{\|D\Phi(z)v\|}{\|v\|} &\ge \Lambda_2^{-1}, \qquad \forall\, v \in T_zN^-. \end{aligned}$$

The NHIM survives small perturbation to the diffeomorphism, the stable (unstable) manifold admits a foliation of stable (unstable) fibers $\Upsilon_z^{u,s}$, each of which depends on its base point $z \in \Pi$ smoothly. For each $z' \in \Upsilon_z^{u,s}$, $\|\Phi^k(z') - \Phi^k(z)\| \to 0$ exponentially fast as $k \to \pm\infty$. With this observation, we find that the modulus of continuity of the invariant circles induces the modulus of continuity of the barrier function.

Not all invariant circles in the cylinder are smooth. Nevertheless, they are Lipschitz curve. Consequently, the stable (unstable) manifold can only be assumed of Lipschitz. It does not make trouble for the approach. When an invariant circle disappears, some invariant set still exists, looks like a Cantor set embedded on circle, called the Aubry-Mather set. It has its stable (unstable) set, the pseudo-graph of the differential of the weak KAM solutions (certain viscosity solution of the Hamilton-Jacobi equation), which can be treated as the substitute of the generating function of stable (unstable) manifold, also denoted by u^u, u^s respectively. Since H is assumed to be positive definite in p, each weak KAM solution is of Lipschitz. At each differential point of u^u (u^s), there exists an orbit emanating from $(p = \partial u^u(q), q)$ $((p = \partial u^s(q), q))$ and approaching the Aubry-Mather set as $t \to -\infty$ $(t \to +\infty)$.

Although the barrier function is only Lipschitz, it is differential at it minimal point. Indeed, the backward weak KAM solution u^u is semi-concave and the forward weak KAM solution u^s is semi-convex. A function is said to be semi-concave (semi-convex) if it is the sum of a C^2-function plus a concave (convex) function. Therefore, the barrier function $u^u - u^s$ is semi-concave which is differentiable at its minimal point, where both u^u and u^s must be differentiable (see [Fa]).

4. Reduction of normal form and diffusion path. Towards the resolution of Arnold's conjecture, let us consider small perturbation of integrable Hamiltonian with three degrees of freedom

$$H(p, q) = h(p) + \epsilon P(p, q), \qquad (p, q) \in \mathbb{R}^3 \times \mathbb{T}^3, \tag{4.1}$$

where $\partial^2 h(p)$ is positive definite, both h and P are C^r-differentiable with $r \geq 6$.

As the first step to answer the question, we search for normal hyperbolic invariant cylinder (NHIC) along resonant path. Once a NHIC is found, around which the system turns out to be *a priori* unstable. In the system with three degrees of freedom, an irreducible integer vector $k' \in \mathbb{Z}^3 \backslash \{0\}$ determines a resonant path

$$\Gamma' = \{p \in h^{-1}(E) : \langle \partial h(p), k' \rangle = 0\}.$$

Along the path Γ', there are countably many points $\{p'' \in \Gamma'\}$ where the frequency vector $\partial h(p'')$ satisfies additional resonant condition, i.e. $\exists\, k'' \in \mathbb{Z}^3 \backslash \{0\}$ independent of k' such that $\langle \partial h(p''), k'' \rangle = 0$. We call p'' double resonant point.

One can choose the path so that k' is totally irreducible. A vector $k = (k_1, k_2, k_3) \in \mathbb{Z} \backslash \{0\}$ is said to be *totally irreducible* if the greatest common divisor of k_i and k_j is equal to 1 for any $i \neq j$ and $i, j = 1, 2, 3$. It is based on the observation that, for any $k \in \mathbb{Z}^3 \backslash \{0\}$, one can choose totally irreducible $k' \in \mathbb{Z}^3 \backslash \{0\}$ such that $\langle k, k' \rangle / \|k\| \|k'\|$ is close to 1.

To reduce the Hamiltonian (4.1) into a normal form, small disks $\{\|p - p''_i\| < K_i^{-1} \epsilon^{\kappa}\}$ are chosen to cover the resonant path Γ', where $\{p''_i \in \Gamma'\}$ are double resonant points, $0 < \kappa < \frac{1}{6}$, $K_i \leq K^* \epsilon^{-\frac{1}{3}(1-3\kappa)}$ is the period of the double resonance at p''_i, namely, $K_i \partial h(p''_i) \in \mathbb{Z}^3$ and $K \partial h(p''_i) \notin \mathbb{Z}^3$ for any $K < K_i$, K^* is a constant independent of ϵ (see Chapter 3 of [Lo]). The radius of each disk is between $O(\epsilon^{\frac{1}{3}})$ and $O(\epsilon^{\frac{1}{7}})$.

Fix a disk $\{\|p - p''_i\| < K_i^{-1} \epsilon^{\kappa'}\}$. To get the normal form around a double resonance, we introduce a coordinate transformation $\Phi_{\epsilon F}$ which is defined as the time-2π-map $\Phi_{\epsilon F} = \Phi^t_{\epsilon F}|_{t=2\pi}$ of the Hamiltonian flow generated by the function $\epsilon F(p, q)$, where F solves the homological equation

$$\left\langle \omega''_i, \frac{\partial F}{\partial q} \right\rangle = -P(p, q) + Z(p, q)$$

where $\omega''_i = \frac{\partial h}{\partial p}(p''_i)$ and

$$Z(p, q) = \sum_{\langle k, \omega''_i \rangle = 0} P_k(p) e^{i\langle k, q \rangle},$$

in which P_k represents the Fourier coefficient of P. Expanding F into Fourier series and comparing both sides of the equation we obtain

$$F(p,q) = \sum_{\langle k,\omega_i''\rangle\neq 0} \frac{iP_k(p)}{\langle k,\omega_i''\rangle} e^{i\langle k,q\rangle}.$$

Under the transformation $\Phi_{\epsilon F}$ we obtain a new Hamiltonian

$$\begin{aligned}\Phi_{\epsilon F}^* H =& h(p) + \epsilon Z(p,q) + \epsilon\Big\langle \frac{\partial h}{\partial p}(p) - \frac{\partial h}{\partial p}(p_i''), \frac{\partial F}{\partial q}\Big\rangle \\ &+ \frac{\epsilon^2}{2}\int_0^1 (1-t)\{\{H,F\},F\}\circ \Phi_{\epsilon F}^t dt.\end{aligned}$$

The function $\Phi_{\epsilon F}^* H(p,q)$ determines its Hamiltonian equation

$$\frac{dq}{dt} = \frac{\partial}{\partial p}\Phi_{\epsilon F}^* H, \qquad \frac{dp}{dt} = -\frac{\partial}{\partial q}\Phi_{\epsilon F}^* H. \tag{4.2}$$

For this equation, by choosing carefully a matrix M (see Section 3 of [C18]) we introduce a linear transformation

$$q = M^{-1}u, \qquad p = M^t v,$$

followed by another transformation

$$\tilde G_\epsilon = \frac{1}{\epsilon}\Phi_{\epsilon F}^* H, \qquad \tilde y = \frac{1}{\sqrt{\epsilon}}\Big(v - M^{-t}p_i''\Big), \qquad \tilde x = u, \qquad s = \sqrt{\epsilon}t, \tag{4.3}$$

with $\tilde x = (x, x_3)$, $\tilde y = (y, y_3)$, $x = (x_1, x_2)$, $y = (y_1, y_2)$. In the new canonical variables $(\tilde x, \tilde y)$ and the new time s, Equation (4.2) turns out to be the Hamiltonian equation with the generating function as the following:

$$\tilde G_\epsilon(\tilde x, \tilde y) = \frac{1}{\epsilon}\Big(h(p_i'' + \sqrt{\epsilon}\tilde y) - h(p_i'')\Big) - V(x) + \sqrt{\epsilon}\tilde R_\epsilon(\tilde x, \tilde y),$$

where $V = -Z(p_i'', \langle Mk', \tilde x\rangle, \langle Mk'', \tilde x\rangle)$ which is independent of x_3, 2π-periodic in x_1 and $2\pi/m$-periodic in x_2 with some $m\in\mathbb{N}$ determined by M, the term $\|\sqrt{\epsilon}\tilde R\|_{C^{r-2}}$ is bounded by a small number of order $O(\sqrt{\epsilon})$ when $(\tilde x, \tilde y)$ is restricted in the domain

$$\Omega_\epsilon = \Big\{(\tilde x, \tilde y) : |\tilde y| \le \epsilon^{\kappa-\frac12}, \tilde x \in \mathbb{T}^3\Big\}, \quad \text{with} \ \ 0<\kappa<\frac{1}{6}.$$

Notice that M is set such that $M^t\omega_i'' = (0,0,\omega_3)$. Let $(I,\theta) = (\frac{\omega_3}{\sqrt{\epsilon}}y_3, \frac{\sqrt{\epsilon}}{\omega_3}x_3)$ be another symplectic coordinate rescaling, the function $I = G_\epsilon(x, y, \frac{\omega_3}{\sqrt{\epsilon}}\theta)$ with

$$G_\epsilon\Big(x, y, \frac{\omega_3}{\sqrt{\epsilon}}\theta\Big) = \frac{1}{2}\langle By, y\rangle - V(x) + \sqrt{\epsilon}R_\epsilon\Big(x, y, \frac{\omega_3}{\sqrt{\epsilon}}\theta\Big). \tag{4.4}$$

solves the equation $\tilde G_\epsilon(x, y, \frac{\omega_3}{\sqrt{\epsilon}}\theta, I) = 0$, where B is a positive definite matrix. In the domain $\{|y| \le O(\epsilon^{\kappa-\frac12}), |I| \le O(\epsilon^{\kappa-1})\}$, the term $\sqrt{\epsilon}R_\epsilon$ is bounded by a quantity of order $O(\sqrt{\epsilon})$ in C^{r-2}-topology, see Section 3 of [C18] for the details of the reduction. In the spirit of [B07], one believes in the correspondence between the set of minimal orbits of the Hamiltonians H and the set of G_ϵ.

The main part of G_ϵ is a classical system of mechanics with two degrees of freedom which was obtained in [A66] already

$$G(x,y)=\frac{1}{2}\langle By,y\rangle - V(x), \qquad (x,y)\in\mathbb{T}^2\times\mathbb{R}^2. \tag{4.5}$$

It is generic to assume that V attains its minimum only at one point, by a translation of coordinates, we assume this point is $x=0$ with $V(0)=0$. It is generic either to assume that the Hessian matrix $\partial^2 V(0)$ is positive definite.

The classical system $G(x,y)$ is far from integrable, perturbation method does not apply any more. Given a class $g\in H_1(\mathbb{T}^2,\mathbb{Z})$, it is possible that there does not exist cylinder which admits a foliation of periodic orbits associated with the class g. There might be a singular cylinder with hole, taking homoclinic orbits as its boundary, it does not survive small time-periodic perturbation. It is a main issue of double resonance which will be discussed later.

Next, let us find the diffusion path connecting neighborhoods of two action variables p and p' with $h(p)=h(p')=E$. Given any small $\delta>0$, there exist two irreducible integer vectors $k,k'\in\mathbb{Z}^3\backslash\{0\}$ such that the circles of resonance $\Gamma_k=\{p\in\mathbb{R}^3: h(p)=E,\langle k,\partial h(p)\rangle=0\}$, $\Gamma_{k'}=\{p\in\mathbb{R}^3: h(p)=E,\langle k',\partial h(p)\rangle=0\}$ passes through a δ-neighborhood of p and p' respectively. These two circles are either coincide or intersect at two points. In both cases, one obtains a resonant path

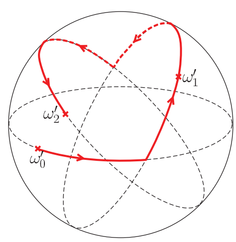


FIG. 3. *The resonant path in the surface of $h^{-1}(E)$, $\omega=\partial h(p)$.*

connecting the δ-neighborhood of p and p'. This path is covered by finitely many disks $\{\|p-p_i''\|<K_i^{-1}\epsilon^{\kappa'}\}$ where each p_i'' is a double resonant point on the path. A double resonance is called *strong* if there does not exist NHIC around the double resonant point. Although the number of the points $\{p_i''\}$ depends on ϵ, the number of strong double resonant points is finite, independent of ϵ for C^r-generic P. Indeed, if we expand P into Fourier series, then

$$Z(p,q)=Z_k(p,\langle k,q\rangle)+Z_{k,k_i}(p,\langle k,q\rangle,\langle k_i,q\rangle)$$

where

$$\begin{aligned} Z_k &= \sum_{j\in\mathbb{Z}\backslash\{0\}} P_{jk}(p)e^{j\langle k,q\rangle i},\\ Z_{k,k_i} &= \sum_{(j,l)\in\mathbb{Z}^2,l\neq 0} P_{jk+lk_i}(p)e^{(j\langle k,q\rangle+l\langle k_i,q\rangle)i}. \end{aligned}$$

As $|P_k|$ decrease fast as $|k|$ increases $|P_k|\le O(|k|^{-r})$, the term Z_{k,k_i} is treated as a small perturbation to Z_k for big $|k_i|$. Indeed, notice $\langle k,q\rangle=x_1$, the system

$$\frac{1}{2}\langle By,y\rangle+Z_k(p_i'',x_1)$$

is integrable, there exists a cylinder foliated into periodic orbits $\{y = \text{const.}, x_1 = x_1^*\}$ where x_1^* is a maximal point of Z_k. For C^r-generic Z_k, the second derivative of Z_k in x_1 at x_1^* is negative. It implies the cylinder is normally hyperbolic, which survives small perturbation. The term Z_{k,k_i} will be small enough provided $|k_i|$ is large enough. Therefore, the number of strong double resonances is independent of ϵ. However, it is unavoidable to encounter strong double resonance, the number depends on generic P.

5. NHICs of classical systems. Unlike *a priori* unstable case, the main part of the normal form (4.5) is a classical system of mechanics, far from integrable

$$G(x,y) = \frac{1}{2}\langle By, y\rangle - V(x), \qquad x \in \mathbb{T}^2,\ y \in \mathbb{R}^2.$$

To construct normally hyperbolic invariant cylinders we apply the variational method. The details can be found in [CZ16]. Let $A = B^{-1}$, the Hamiltonian G determines a Lagrangian via the Legendre transformation

$$L(\dot{x}, x) = \frac{1}{2}\langle A\dot{x}, \dot{x}\rangle + V(x).$$

Notice each closed curve γ: $[0,T] \to \mathbb{T}^2$ is associated with a class $[\gamma] \in H_1(\mathbb{T}^2, \mathbb{Z})$. Given a class $g \in H_1(\mathbb{T}^2, \mathbb{Z})$, we wish to find minimal periodic orbits with rotation vector λg where $\lambda \in [\lambda_0, \lambda_1]$ $(0 < \lambda_0 < \lambda_1 < \infty)$. An orbit $(\gamma, \dot{\gamma})$ is called λg periodic if $[\gamma] = g$, $\gamma(t) = \gamma(t + \lambda^{-1})$ and $\gamma(t) \neq \gamma(t + \nu^{-1})$ for any $\nu > \lambda$.

We look for periodic orbits by considering the minimal curves

$$F(x, \lambda) = \min_{\substack{\gamma(0)=\gamma(\lambda^{-1})=x \\ [\gamma]=g}} \int_0^{\lambda^{-1}} L(\gamma(t), \dot{\gamma}(t))dt. \tag{5.1}$$

If the action along the closed curve γ_λ minimizes the function, $A[\gamma_\lambda] = \min_x F(x, \lambda)$, $(\gamma_\lambda(t), \dot{\gamma}_\lambda(t))$ is a periodic orbit of the Lagrange flow ϕ_L^t, namely, γ_λ solves the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0.$$

To see why, we notice that each minimal curve with fixed boundary solves the Euler-Lagrange equation. we only need to prove $\dot{\gamma}(0) = \dot{\gamma}(\lambda^{-1})$. If it was not true, for small $\delta > 0$ we join two points $\gamma_\lambda(\delta)$ and $\gamma_\lambda(\lambda^{-1} - \delta)$ by a minimal curve γ'. The action along γ' is obviously smaller than action along $\gamma_\lambda|_{[0,\delta]\cup[\lambda^{-1}-\delta,\lambda^{-1}]}$. It follows that $F(\gamma'(0), \lambda) < F(x, \lambda)$, but it is absurd. The periodic orbit obtained in this way is called λg-minimal.

As $F(\cdot, \lambda)$ keeps constant along the minimal curve, it is sufficient to restrict $F(x, \lambda)$ on $\zeta \times [\lambda_0, \lambda_1]$ where ζ is a smooth circle on $\mathbb{T}^2$ such that $[\zeta]$ is independent of g. Restrict suitably smaller sub-interval of $[\lambda_0, \lambda_1]$, we can also choose a smooth circle ζ such that each λg-minimal curve intersects ζ only once when the parameter λ is restrict on the subinterval.

To answer the question whether the orbit is hyperbolic, one needs to examine the smoothness of the function F. In general, $F(\cdot, \lambda)$ is not differentiable on the whole circle. In particular, if x lies in the "cut locus" of itself, namely, there are two or more minimal curve of $F(x, \lambda)$. Fortunately, F is differentiable in x when it is restricted a neighborhood of its minimal point, see Lemma 2.1 in [CZ16]. This property allows us to examine the hyperbolicity of the λg-minimal circle by studying the non-degeneracy of F its minimal point, see Theorem 4.1 in [CZ16]. A minimal point of F is said to be non-degenerate if its second derivative is positive.

Since F is obviously Lipschitz in λ, we have the following result. For the details of the proof, one refers to the proof for Theorem 2.1 of [CZ16].

THEOREM 5.1. *There is an open-sense set $\mathfrak{O} \subset C^r(\mathbb{T}^2, \mathbb{R})$ with $r \geq 5$, for each $V \in \mathfrak{O}$ it holds simultaneously for all $\lambda \in [\lambda_0, \lambda_1]$ that all minimal points of $F(\cdot, \lambda)$ are non-degenerate. Indeed, there are finitely many parameters $\lambda_i \in [\lambda_0, \lambda_1]$ such that $F(\cdot, \lambda)$ has only one minimal point if $\lambda \neq \lambda_i$ and $F(\cdot, \lambda_i)$ has exactly two minimal points.*

Generically, a C^2-function is Morse-function. However, it is no longer the case for a family of functions. Here is an example. Let $F(x, \lambda) = x^3 - \lambda x$ when it is restricted on $(x, \lambda) \in [-d, d] \times [-\delta, \delta]$. This function can be smoothly extended to $\mathbb{T}^2 \times [-1, 1]$. The point $x = 0$ is a degenerate critical point of $F(\cdot, 0)$. Under any small C^2-perturbation $F(x, \lambda) \to F'(x, \lambda) = F(x, \lambda) + f(x)$, $F'(\cdot, \lambda)$ still has a degenerate point for some λ close to 0. We mention the smoothness condition. A C^r-perturbation on the potential induces C^{r-1}-perturbation on $F(\cdot, \lambda)$. The theorem for a family of functions $F(\cdot, \lambda_i)$ is under C^4-smooth condition.

Each minimal point corresponds to a λg-minimal orbit, since all minimal points are non-degenerate, these minimal periodic orbits make up finitely many pieces of NHICs. Although the number is finite, but it may approach infinite as $\lambda_1 \to \infty$. Since we are considering the NHICs in the region where $|y| \leq O(\epsilon^{-\kappa})$, it seems possible the number of cylinders depends on ϵ. It would make trouble for the construction of the transition chain. Fortunately, it is not the case as one has the following result.

THEOREM 5.2. *There exists an open-dense set $\mathfrak{V}_\infty \subset C^r(\mathbb{T}^2, \mathbb{R})$ with $r \geq 2$, for each $V \in \mathfrak{V}_\infty$ and each $g \in H_1(\mathbb{T}^2, \mathbb{Z})$ certain $E_0 > 0$ exists such that there exists exactly one λg-minimal periodic orbit lying in the energy level set $G^{-1}(E)$ provided $E \geq E_0$. The hyperbolicity is lower bounded uniformly above zero.*

To illustrate the proof, we recall that the classical system G is obtained around the double resonant point p''. On the resonant path $\Gamma' \cap \{|p - p''| \leq \epsilon^\kappa\}$ we chose points $\{p'_i\}$ so that $\partial_1 h(p'_i) = Ki\sqrt{\epsilon}$, where $K \in \mathbb{Z}$. The number of these points is bounded by a quantity of $O([K^{-1}\epsilon^{\kappa - \frac{1}{2}}])$. We consider the disks $\{|p - p'_i| \leq K\sqrt{\epsilon}\}$ which are "quite away from" the double resonant point in the sense that $Ki \gg 1$. Let $Ki = \Omega_i$, we obtain a normal form

$$G_i(x, y) = \Omega_i y_1 + \frac{1}{2}\langle By, y\rangle - V(x), \tag{5.2}$$

where $G_i(x, y) = G(x, y) + \frac{1}{2\epsilon}\langle B(p'_i - p''), (p'_i - p'')\rangle$. We claim that the cylinders in these disks look like the cylinders in the case of single resonance when $\Omega_i \to \infty$, and for typical potential V, the normal hyperbolicity of the cylinders are uniformly lower bounded away from zero, independent of ϵ.

A set $\mathfrak{V}_\infty \subset C^r(\mathbb{T}^2, \mathbb{R})$ is defined such that $\forall\, V \in \mathfrak{V}_\infty$, the function $[V]$

$$[V](x_2) = \frac{1}{2\pi}\int_0^{2\pi} V(x_1, x_2)dx_1.$$

has a unique minimal point which is non-degenerate, i.e. $\frac{d^2}{dx_2^2}[V](x_2) > 0$ holds at the minimal point. Clearly, the set $\mathfrak{V}_\infty$ is open-dense in $C^r(\mathbb{T}^2, \mathbb{R})$ with $r \geq 2$. For each

$V \in \mathfrak{V}_\infty$, the normal hyperbolicity of the cylinders are lower bounded by the quantity approximately equal to $\frac{d^2}{dx_2^2}[V](x_2)$.

To show it, we introduce a coordinate transformation

$$(x_1, x_2, y_1, y_2) \to \Big(\frac{x_1}{\Omega_i}, x_2, \Omega_i y_1, y_2\Big),$$

which reduces the Hamiltonian G_i turns out to be

$$G_i' = y_1 + \frac{1}{2\Omega_i^2} B_{11} y_1^2 + \frac{B_{12}}{\Omega_i} y_1 y_2 + \frac{1}{2} B_{22} y_2^2 - V\Big(\Omega_i x_1, x_2\Big),$$

where B_{ij} is the ij-th entry of B. The equation $G_i'(x_1, x_2, y_1(x_1, x_2, y_2), y_2) = E\Omega_i$ is solved by the function

$$y_1 = E\Omega_i - \frac{1}{2} B_{11} E - \frac{1}{2} B_{22} y_2^2 - B_{12} E y_2 + V(\Omega_i x_1, x_2) + \Omega_i^{-1} R_H,$$

where $E \in [0, \bar{K}]$ with $\bar{K}$ independent of Ω_i, the remainder $\Omega_i^{-1} R_H$ is of order $O(\Omega_i^{-1})$ in C^r-topology. Let $\tau = x_1$ be the new "time", the Hamiltonian $-y_1$ induces a Lagrangian up to an additive constant

$$L_1 = \frac{1}{2B_{22}} \Big(\frac{dx_2}{d\tau}\Big)^2 - \frac{B_{12} E}{B_{22}} \frac{dx_2}{d\tau} + V + \frac{1}{\Omega_i} R_L,$$

where R_L is C^r-bounded for any large Ω_i. The periodic orbit with rotation vector $(\nu, 0)$ for $\Phi^t_{G_i}$ is converted to be periodic orbit of $\phi^\tau_{L_1}$. Since $\Omega_i \in \mathbb{N}$, the hyperbolicity of such minimal periodic orbit is uniquely determined by the nondegeneracy of the minimal point of the following function (see [C17a])

$$F(x_2, \Omega_i, E) = \inf_{\gamma(0)=\gamma(2\pi)=x_2} \int_0^{2\pi} L_1\Big(\dot\gamma(\tau), \gamma(\tau), \Omega_i \tau, E\Big) d\tau.$$

Due to the condition $\gamma(0) = \gamma(2\pi) = x_2$, the term $\frac{B_{12} E}{B_{22}} \dot x_2$ does not contribute to F (an exact form), so it can be dropped.

The Lagrangian L_1 depend on Ω_i in a singular way as $\Omega_i \to \infty$, the function F appears regular in Ω_i^{-1} as $\Omega_i \to \infty$. Indeed, F admits a decomposition

$$F(x_2, \Omega_i, E) = F_0(x_2, \Omega_i, E) + \frac{1}{\Omega_i} F_R(x_2, \Omega_i, E)$$

where

$$\begin{aligned}
F_0 &= \int_0^{2\pi} \Big(\frac{1}{2B_{22}} (\dot\gamma_{\Omega_i,E}(\tau, x_2))^2 + [V](\gamma_{\Omega_i,E}(\tau, x_2))\Big) d\tau, \\
F_R &= \int_0^{2\pi} \Omega_i (V - [V])(-\Omega_i \tau, \gamma_{\Omega_i,E}(\tau, x_2)) d\tau \\
&\quad + \int_0^{2\pi} R_L(\gamma_{\Omega_i,E}(\tau, x_2), \dot\gamma_{\Omega_i,E}(\tau, x_2), \Omega_i \tau) d\tau.
\end{aligned}$$

Because $V - [V]$ is $\frac{1}{\Omega_i}$-periodically depends on τ with zero average, by using the method to prove Riemann-Lebesgue's Theorem, one can show that F_R is uniformly bounded in C^2-topology as $\Omega_i \to \infty$. Let $\gamma_{\Omega_i,E}(\tau, x_2)$ be the minimal curve of $F(x_2, \Omega_i, E)$.

As F_0 dominates the action and the minimal curve of F_0 is obviously a straight line $\dot\gamma = 0$, one can see that $|\dot\gamma^*_{\Omega_i,E}(\tau)| \to 0$ as $\Omega_i \to \infty$. The non-degeneracy of the minimal circle is obviously lower bounded by $\frac{d^2}{dx_2^2}[V](x_2)$, refer to Section 3 of [C17a] for details.

Let us return to the coordinates before the transformation. That the new coordinate x_1 goes around the circle $\mathbb{T}$ once amounts to that the old coordinate x_1 sweeps out an angle of Ω_i. In the original coordinate system, we have $\frac{dx_1}{dt} = \Omega_i + O(1)$. Therefore, the normal hyperbolicity we obtain for $\tau = 2\pi$-map is almost the same as the time $t = 2\pi$-map determined by the Hamiltonian flow $\Phi^t_{G_i}$. Because there is only one λg-minimal periodic orbit in each energy level set which is hyperbolic, these circles make up one piece of NHIC.

In [B10, BKZ] a piece of NHIC was shown exist along single resonance. But it keeps $O(\epsilon^{1/4})$-away from double resonance. Transformed to classical system, the cylinder is $O(\epsilon^{-1/4})$ far away from the double resonant point, due to the transformation (4.3).

6. The topological transitivity around double resonance. The way we construct to cross double resonance is based on the understanding of the dynamics around the double resonant point. Avoid being involved in too much details about the variational method based on the Mather theory, we try to explain here what are the geometrical counterparts behind.

Generically, there is only one fixed point $\{(x,y)=0\} \in G^{-1}(0)$, which is hyperbolic. Its stable and unstable manifold $W^s(0)$, $W^u(0)$ intersect "transversally" in the sense that, at each intersection point z, one has

$$\mathrm{span}\{T_zW^s(0), T_zW^u(0)\} = T_zG^{-1}(0).$$

The intersection induces a Smale horseshoe, a uniformly hyperbolic invariant set where the system is conjugate to a Bernoulli shift. Each horseshoe is strongly mixing, contains infinitely many different periodic orbits which are approached by infinitely many dense orbits. It hints that the Hamilton-Jacobi equation $G(x, \partial_x u) = E$ does not have smooth solution for $E = 0$. It follows from the upper semi-continuity that some small $E_0 > 0$ exists such that, $\forall\ E \in (0, E_0)$, the Hamilton-Jacobi equation does not have classical solution either. One can refer to Section 3.3 of [C17b] for the details of the proof which was finished in a completely different way.

The dynamics on the energy level set $G^{-1}(E)$ with $E > 0$ appears similar to twist map. For each irreducible class $g \in H_1(\mathbb{T}^2, \mathbb{Z})$, certain $\lambda > 0$ exists such that $G^{-1}(E)$ contains a λg-minimal periodic orbit. Let $\lambda_i g_i$ be such a sequence of rotation vectors such that $\lambda_i g_i$ converges to an irrational one if we treat each class $\rho \in H_1(\mathbb{T}^2, \mathbb{R})$ as a vector in $\mathbb{R}^2$. These $\lambda_i g_i$-minimal periodic orbits approaches either to an invariant torus or to a Cantor set. The latter one is the suspension of Aubry-Mather set for twist map. To understand what is Aubry-Mather set of area-preserving twist diffeomorphism, let us recall Denjoy's theorem. For each homeomorphism f on circle is associated with its rotation number $\rho(f)$. If it is irrational and variation of the differential of the map is unbounded, then there is a Cantor set in the circle, restricted on which, the map is semi-conjugate to a rigid rotation $x \to x + \rho(f)$. This set is called Denjoy set. One can refer to Section I of Chapter 3 in [A78] for details. The Aubry-Mather set for the rotation number ρ is exactly the Denjoy set for the same rotation number, embedded in a circle of degree one.

Therefore, for each $E \in (0, E_0)$, there does not exist invariant torus, because such torus is the graph of the differential of the smooth solution of the Hamilton-Jacobi equation. Consequently, the whole energy level set $G^{-1}(E)$ with $E \in (0, E_0)$ appears similar to a Birkhoff instability region of area-preserving twist map. Any two minimal periodic orbits (Aubry-Mather sets) are connected by minimal orbits [M91]. Indeed, the set of all minimal periodic orbits and Aubry-Mather set is topolically transitive, as proved in Section 6 of [C17b].

The topological transitivity was studied by Birkhoff for the geodesic flow of a closed surface with genus 2 [Bir]. The curvature is endowed negative everywhere, the system is then uniformly hyperbolic. The curvature on torus can not be negative everywhere, restricted by Gauss-Bonnet formula. However, the idea of Birkhoff still can be applied if each minimal closed geodesic is disconnected others. We shall explain later the idea of the variational construction.

7. A way to "cross" double resonance. Given a class $g \in H_1(\mathbb{T}^2, \mathbb{Z})$ and any small number $E' > 0$, by the result obtained in [CZ16, C17a], there are finitely many pieces of NHIC only, made up by λg-minimal periodic orbits of Φ_G^t, which extend from the level set with very high energy $E = \epsilon^{-\kappa}$ to the level set with very lower energy $E = E'$. Therefore, these cylinders penetrate deeply into a layer of energy level sets, each of which is topological transitive. Namely, we have the following

(1) some $E_0 > 0$ exists such that $\forall\ E \in (0, E_0]$ the set of minimal periodic orbits and Aubry-Mather sets on $G^{-1}(E)$ is topologically transitive;
(2) given $E' \in (0, E_0)$ and for any two classes $g, g' \in H_1(\mathbb{T}^2, \mathbb{Z})$, all λg, $\lambda g'$ periodic orbits are hyperbolic provided they are located in the energy level set $G^{-1}(E)$ with $E \geq E'$. These periodic orbits make up finitely many pieces of NHICs.

To consider the time-periodic perturbation $G \to G_\epsilon$, we work in the extended phase space $\mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{T}$. Because G_ϵ is reduced from the Hamiltonian H with three degrees of freedom, the extra-dimension corresponds to the coordinate x_3. As the Hamiltonian G_ϵ is no longer autonomous, there does not exist a foliation of invariant energy level sets in general. Nevertheless, it is also uncertain that the whole layer is topologically transitive either. Fortunately, it is proved in [C17b] that the topological transitivity exists among those Aubry sets if they share the same average action. In autonomous case, the average action is nothing else but the energy. We shall explain later what is Aubry set.

On the other hand, because of the normally hyperbolicity, the NHICs survive the small perturbation and still extend into the layer admitting a "foliation" of Aubry sets with the same average action. In the extended phase space, the NHIC is 3-dimensional.

The phenomena allow us to construct diffusion orbits to cross double resonance in a way as follows. Given any Aubry set $\tilde{\mathcal{A}}_i$ on the cylinder Π_i for $i = 1, 2$. Each Aubry set $\tilde{\mathcal{A}}_i$ is dynamically connected to an Aubry set $\tilde{\mathcal{A}}'_i \subset \Pi_i$ with average action less than E_0. We choose $\tilde{\mathcal{A}}'_1$ and $\tilde{\mathcal{A}}'_2$ such that they share the same average action, which implies that they are dynamically connected. Therefore, one obtains diffusion orbits from $\tilde{\mathcal{A}}_1$ to $\tilde{\mathcal{A}}_2$, or vice versa.

The way we constructed to cross double resonance is essentially different from what Mather suggested. The following figures show the paths in the first cohomology class space. We shall explain a bit more after we introduce some concepts of Mather theory.

Since the single resonant path is covered by finitely many disks $\{\|p - p''_i\| < K_i^{-1}\epsilon^{\kappa'}\}$, we find that a transition chain is constructed connecting small neighborhood of two action variables $p', p'' \in h^{-1}(e)$ with $e > \min h$. It moves along single resonance

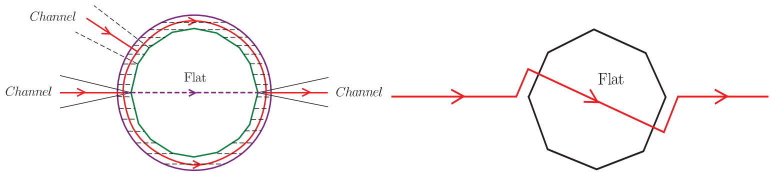


FIG. 4. *The left figure: our way to skirt around double resonance; the right one, Mather's way to pass through double resonance.*

path and turns around when it encounters strong double resonant points. Along the chain, the diffusion orbits are constructed by the variational method, they shadow a sequence of local minimal orbits, successively connected.

Therefore, the conjecture of Arnold diffusion for positive definite Hamiltonian with three degrees of freedom is proved in the C^k-smooth category in the sense of cusp-residual genericity with $k \geq 6$.

8. Brief introduction to the variational proof. A function $L : T\mathbb{T}^n \times \mathbb{T} \to \mathbb{T}$ is called Tonelli Lagrangian if it is convex in $\dot{x}$ with superlinear growth on each tangent fiber: $L/\|\dot{x}\| \to \infty$ as $\|\dot{x}\| \to \infty$, and any solution of the Euler-Lagrange equation is defined for $t \in \mathbb{R}$. Notice that $H^1(\mathbb{T}^n, \mathbb{Z}) = \mathbb{R}^n$, each $c \in \mathbb{R}^n$ is thought as a first cohomology class. Let $L_c = L - \langle c, \dot{x}\rangle$, we see that the Euler-Lagrange equation for L_c is the same as it for L:

$$\frac{d}{dt}\Big(\frac{\partial L_c}{\partial \dot{x}}\Big) - \frac{\partial L_c}{\partial x} = \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}}\Big) - \frac{\partial L}{\partial x}.$$

A curve $\gamma(t)$ (an orbit $(\gamma(t), \dot{\gamma}(t))$ is called c-minimal if for any $t < t_1, t_2$ one has

$$\int_t^{t_1} L_c(\gamma(s), \dot{\gamma}(s), s)ds = \min_{\substack{t_1 = t_2 \bmod T \\ \zeta(t) = \gamma(t) \\ \zeta(t_2) = \gamma(t_1)}} \int_t^{t_2} L_c(\zeta(s), \dot{\zeta}(s), s)ds + (t_2 - t_1)\alpha(c),$$

where T is the period of L in t, $\alpha(c)$ is the average action, to be defined below. An orbit $(x(t), y(t))$ of a Hamiltonian flow is called c-minimal if $x(t)$ is c-minimal. To see why a class c is related to minimal orbit let us consider an example $L = \frac{1}{2}\langle A\dot{x}, \dot{x}\rangle$. In this case, $L_c = \frac{1}{2}\langle A(\dot{x} - A^{-1}c), (\dot{x} - A^{-1}c)\rangle - \frac{1}{2}\langle c, A^{-1}c\rangle$, every quasi-periodic orbit $\dot{x} = A^{-1}c$ is c-minimal.

In general, not every orbit is minimal for certain first cohomology class. However, the set of minimal orbits for each class c is non-empty, invariant for the Lagrange flow. Such set usually supports a probability invariant measure, called c-minimal measure. The average action of the Lagrangian on minimal measure μ for class c is defined as

$$\alpha(c) = -\int L_c d\mu = \liminf_{T\to\infty} -\frac{1}{2T}\int_{-T}^{T} L_c(x(t), \dot{x}(t), t)dt$$

where the limit holds for almost every $(x(0), \dot{x}(0), 0) \in \mathrm{supp}\mu$, guaranteed by Birkhoff ergodic theorem. For autonomous system, the average action is nothing else but the energy. It is proved by Mather that α-function is convex in c, finite everywhere with super-linear growth.

The set of all c-minimal orbits is denoted by $\tilde{\mathcal{N}}(c) = \cup(x(t), \dot{x}(t), t)$, its projection along tangent fiber is denoted by $\mathcal{N}(c) \subset \mathbb{T}^n \times \mathbb{T}$. We call it Mañé set.

Aubry set $\tilde{\mathcal{A}}(c)$ is defined to be a set of so-called regular c-minimal orbits. Denoted by $\mathcal{A}(c) = \pi\tilde{\mathcal{A}}(c)$ its projection down to $\mathbb{T}^{n+1}$, the inverse of π is of Lipschitz when it is restricted on $\mathcal{A}(c)$. Roughly speaking, a point $(x,t) \in \mathcal{A}(c)$ if there exists a sequence of closed curves γ_i starting from (x,t) such that the c-action along the curves approaches zero as $i \to \infty$ see [M93].

Two ways have been found to connect $\tilde{\mathcal{N}}(c)$ to $\tilde{\mathcal{N}}(c')$ if $|c-c'| \ll 1$. One is Arnold's mechanism applied in [A64], the variational version appears more applicable, another one is based on cohomology equivalence. In both cases, one has $H_1(\mathbb{T}^{n+1}, \mathcal{N}(c), \mathbb{Z}) \neq 0$.

The Arnold's mechanism has been explained already. To establish the cohomology equivalence, we need to choose a non-degenerately embedded section Σ_c of $\mathbb{T}^{n+1}$, i.e. $\exists$ a smooth injection φ: $\mathbb{T}^n \to \mathbb{T}^{n+1}$ such that Σ_c is the image of φ, and the induced map φ_*: $H_1(\mathbb{T}^n, \mathbb{Z}) \to H_1(\mathbb{T}^{n+1}, \mathbb{Z})$ is an injection. A class $\tilde{c} \in H^1(\mathbb{T}^{n+1}, \mathbb{Z})$ is defined to be $\tilde{c} = (c, -\alpha(c))$.

Given a class $\tilde{c}$, we assume that $\exists$ a non-degenerate embedded n-dimensional torus $\Sigma_c \subset \mathbb{T}^{n+1}$ such that each c-minimal curve transversally intersects Σ_c. Let

$$\mathbb{V}_c = \bigcap_U \{ i_{U*} H_1(U, \mathbb{R}) : U \text{ is a neighborhood of} \mathcal{N}(c) \cap \Sigma_c \},$$

here i_U: $U \to \mathbb{T}^{n+1}$ denotes inclusion map. $\mathbb{V}_c^{\perp}$ is defined to be the annihilator of $\mathbb{V}_c$, i.e. $\tilde{c}' \in \mathbb{V}_c^{\perp}$ if and only if $\langle \tilde{c}', h \rangle = 0$ for all $h \in \mathbb{V}_c$. There exists a neighborhood U of $\mathcal{N}(c) \cap \Sigma_c$ such that $\mathbb{V}_c = i_{U*} H_1(U, \mathbb{R})$, $\mathbb{V}_c^{\perp} = \ker i_U^*$.

A class c' with $|c' - c| \ll 1$ is said to be cohomologically equivalent to c if $\tilde{c}' - \tilde{c} \in \mathbb{V}_c^{\perp}$. In this case, some function u: $\mathbb{T}^{n+1} \to \mathbb{R}$ exists such that $du = \langle \tilde{c}' - \tilde{c}, (dx, dt) \rangle$ when it is restricted on a neighborhood U of $\mathcal{N}(c) \cap \Sigma_c$. With this property, one can show that $\tilde{\mathcal{A}}(c)$ is dynamically connected to the Aubry set $\tilde{\mathcal{A}}(c')$.

To illustrate how they are connected by local minimal orbits, we obtain from the cohomlogy equivalence that the Lagrange multiplier $\eta = \langle \tilde{c}' - \tilde{c}, (\dot{x}, 1) \rangle - \sum_{i=1}^n \partial_{x_i} u \dot{x}_i - \partial_t u = 0$ when it is restricted in the neighborhood U of $\mathcal{N}(c) \cap \Sigma_c$ and

$$\frac{d}{dt} \frac{\partial \eta}{\partial \dot{x}} - \frac{\partial \eta}{\partial x} \equiv 0.$$

To introduce a modified Lagrangian, we consider a covering space $\bar{M}$ of $\mathbb{T}^{n+1}$ homeomorphic to $\mathbb{R} \times \mathbb{T}^n$ such that the lift of Σ_c to $\bar{M}$ consists of infinitely many compact components $\bar{\Sigma}_c = \cup \Sigma_{c,i}$. For instance, $\bar{\Sigma}_c = \cup_i \{x_1 = i\}$ if $\Sigma_c = \{x_1 = 0\}$, $\bar{M} = \mathbb{R} \times \mathbb{T}^n$.

We choose one section $\Sigma_{c,0}$ which separates $\bar{M}$ into two parts $\bar{M} = \bar{M}^- \cup \bar{M}^+$, the lift of each c-minimal curve extends to infinity through $\bar{M}^{\pm}$ as $t \to \pm\infty$. Let ρ: $\bar{M} \to \mathbb{R}$ be a C^2-function so that $\rho = 0$ as $(x,t) \in \bar{M}^- \backslash V_0$ and $\rho = 1$ as $(x,t) \in \bar{M}^+ \backslash V_0$, where V_0 is a neighborhood of $\Sigma_{c,0}$. With the preparation work we are able to introduce a modified Lagrangian

$$L_{c,c'} = L - \langle \tilde{c}, (\dot{x}, 1) \rangle - \rho\eta,$$

choose $(x^{\pm}, t^{\pm}) \in \bar{M}^{\pm}$ and consider the minimal curve of

$$A_{c,c'}((x^-, t^-), (x^+, t^+)) = \min_{\substack{\zeta(t^-) = x^- \\ \zeta(t^+) = x^+}} \int_{t^-}^{t^+} L_{c,c'}(\zeta(t), \dot{\zeta}(t), t) dt.$$

Let $\mathrm{dist}((x_i^{\pm}, t_i^{\pm})), \Sigma_{c,0}) \to \infty$ with $t_i^{\pm} \to \pm\infty$, we consider the sequence of the minimal curves γ_i along which the action approaches the limit infimum

$$\liminf_{i\to\infty} A_{c,c'}((x_i^-, t_i^-), (x_i^+, t_i^+)).$$

If $c' = c$, the curves approach the set of c-minimal curves. It follows that they do not touch the support of the 1-form $\eta = \langle \tilde{c}' - \tilde{c}, (dx, dt)\rangle - \sum_{i=1}^n \partial_{x_i} u dx_i - \partial_t u dt$ when they pass through the neighborhood V_0. Recall the upper-semi continuity of the set of minimal curves with respect to Lagrangian. For c' sufficiently close to c, each minimal curve of $L_{c,c'}$ also does not touch the support of the 1-form η when it passes through V_0. Since $\rho = 0$ for $(x,t) \in \bar{M}^-\backslash V_0$ and $\rho = 1$ for $(x,t) \in \bar{M}^+\backslash V_0$ while η is closed,

$$\frac{d}{dt}\frac{\partial \rho\eta}{\partial \dot{x}} - \frac{\partial \rho\eta}{\partial x} = 0$$

holds along each minimal curve of $L_{c,c'}$. It implies that every minimal curve of $L_{c,c'}$ solves the Euler-Lagrange equation of L. Clearly, the α-limit set of the minimal orbit lies in $\tilde{\mathcal{N}}(c)$ and the ω-limit set lies in $\tilde{\mathcal{N}}(c')$. Indeed, $L_{c.c'} = L_c$ when it is restricted on $T\bar{M}^-\backslash V_0$ and $L_{c,c'} = L_{c'} - du$ when it is restricted on $T\bar{M}^+\backslash V_0$. The term du does not contribute to the average action, its action along any closed curve equals zero.

Two classes c, c' are said to be equivalent if there is sequence of classes $\{c_i\}_{0\le i\le K}$ such that $c_0 = c$, $c_K = c'$, $|c_i - c_{i+1}| \ll 1$ and c_i is equivalent to c_{i+1} for $0 \le i < K$.

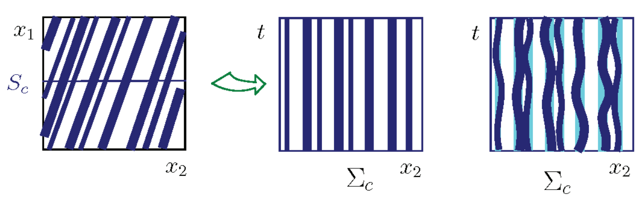
In [M93], Σ_c is chosen as the time-section $\mathbb{T}^n \times \{t = 0\}$. It is very restrictive, e.g. it does not apply in autonomous system to obtain interesting result. In fact, for the problem to cross double resonance, the time-section does not work either.

Let us consider the classical system G first. In the energy level set $G^{-1}(E)$ with $E \in (0, E_0)$, each curve in the suspension of Aubry-Mather set or minimal periodic points is minimal for certain c with $\alpha(c) \equiv E$. Given $c \in \alpha^{-1}(E)$, all c-minimal curves have the same rotation vector $(\omega_{c,1}, \omega_{c,2})$.

If $\omega_{c,1} \neq 0$, there exists a section S_c of $\mathbb{T}^2$ homotopic to $\{(x_1, x_2) : x_1 = 0\}$ such that $S_c \cap \mathcal{N}(c) \subset \mathrm{int} \cup I_{c,i}$ where $\{I_{c,i}\}$ denote disjoint closed intervals, see the left figure shown below. In the extended configuration space $\mathbb{T}^3$ we choose $\Sigma_c = S_c \times \mathbb{T}$, where $\mathbb{T}$ is for the time variable. Recall $\tilde{c} = (c, -\alpha(c))$, one has $V_c = \mathrm{span}\{(0, 0, 1)\}$, from which one obtains

$$(c', -\alpha(c')) - (c, -\alpha(c)) \in V_c^{\perp}, \qquad \text{if } \alpha(c') = \alpha(c),$$

namely, c' is equivalent to c provided $\alpha(c) = \alpha(c')$, see the figure in the middle.



Next, let us consider the time-periodic perturbation G_ϵ of G. Because of the upper semi-continuity of the Mañé set in the Lagrangian, there exist $\epsilon_0 > 0$ such that for each $\epsilon \in [0, \epsilon_0]$ the following holds

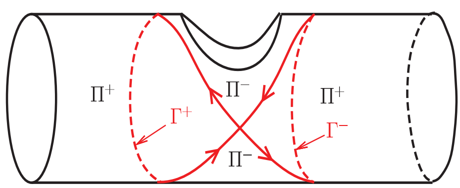
$$\mathcal{N}(c) \cap \Sigma_c \subset (\cup I_{c,i}) \times \mathbb{T},$$

i.e. c is equivalent to c' if $\alpha(c) = \alpha(c') \in (0, E_0)$, see the figure on the right. Therefore, the set of c-minimal orbits is dynamically connected to the set of c'-minimal orbits.

Let us return back to Figure 4 to describe more precisely the way to cross double resonance. Because the fixed point $\{z = 0\}$ is hyperbolic, there exists a flat such that $\alpha(c) = \min \alpha$ holds for any c lying in the flat. The flat is surrounded by an annulus $\{c : 0 < \alpha(c) \leq E_0\}$ which admits a foliation of circles $\{c : \alpha(c) = \text{constant} \leq E_0\}$. Each circle establishes cohomology equivalence. The NHICs for a class $g \in H_1(\mathbb{T}^2, \mathbb{Z})$ corresponds a channel which extends into the annulus. For each c in the channel, the Aubry set lies on NHIC. Therefore, we get a path in the first cohomology class space, emanating from a point in one channel, it moves in the channel until it reaches one of the circles of cohomology equivalence, then it skirts along the circle until it reaches another channel, then it moves up. The Mather's path in the first cohomology class space passes through the flat, instead of skirting around the flat.

9. Discussions on another way to cross double resonance. Our way to pass through double resonance illustrated in Section 7 is different from what was suggested by Mather. The former does not touch the zero energy level of G, it turns out to be applicable even for the systems with arbitrary degrees of freedom [CX]. Along Mather's way one needs to understand the dynamics on the zero energy level set where some singularity emerges.

In recent years, there are other works for the problem of double resonance such as [KZ, Mar]. They suggest that the classical system (4.5) exhibits an invariant cylinder with hole $\Pi = \Pi^+ \cup \Gamma^+ \cup \Gamma^- \cup \Pi^-$ where Π^+ is made up by periodic orbits with energy ranging over the interval $(0, E]$, Π^- is made up by periodic orbits with negative energy ranging over $[-E, 0)$ with small $E > 0$. Π^+ and Π^- are glued together along the figure eight curve $\Gamma^+ \cup \Gamma^-$. The following picture was shown in [KZ, Mar] for classical system with two degrees of freedom.



If the cylinder is smooth and normally hyperbolic, it survives small perturbation (4.4). In this case, one is able to apply the method for *a priori* unstable system such as [CY04, CY09] to construct diffusion orbits crossing the original system. From the view point of cohomological class, the diffusion path is the same as Mather suggested, see the right picture in Figure 4.

To reach the goal, we feel that the following issues need to be clarified.

First, one needs to consider all λg-minimal orbits which approach the homoclinic orbit with the same class g as $\lambda \downarrow 0$. Each λg-minimal curve corresponds to a minimal point of the action (5.1). Although only finitely many bifurcation points exist for each closed interval $[\lambda_0, \lambda_1]$ with $0 < \lambda_0 < \lambda_1 < \infty$, it is necessary to exclude the possibility that they may accumulate at $\lambda = 0$. The same problem needs to be considered for negative energy.

Second, if there do not exist infinitely many bifurcations, a cylinder with a hole will exist, passing through the double resonance, invariant for the Hamiltonian flow. In this case, we need to check the smoothness of the cylinder along the homoclinic orbits $\Gamma^+ \cup \Gamma^-$.

Acknowledgement. The authors are supported by NNSF of China (No.11790272 and No.11631006). The first author wishes to thank Stephen Yau and his colleagues for organizing "*International Conference on Singularity Theory and Dynamical Systems, in Memory of John Mather*" at Tsinghua Sanya International Mathematics Forum for their hospitality.

REFERENCES

[A64] V. I. Arnold, *Instability of dynamical systems with several degrees of freedom,* Sov. Math., Dokl., 5 (1964), pp. 581–585; translation from Dokl. Akad. Nauk SSSR, 156 (1964), pp. 9–12.

[A66] V. I. Arnold, *The stability problem and ergodic properties for classical dynamical systems*, Proceedings of International Congress of Mathematicians, Moscow (1966) pp. 5–11, in V. I. Arnold-Collected Works. Springer Berlin Heidelberg (2014), pp. 107-*-113.

[A78] V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag (1978).

[A94] V. I. Arnold, *Mathematical problems in classical physics*, in Trends and Perspectives in Applied Mathematics, Appl. Math. Sc. Series 100, Springer-Verlag, New York, (1994), pp. 1–20.

[B07] P. Bernard, *Symplectic aspects of Mather theory*, Duke Math. J., 136 (2007), pp. 401–420.

[B08] P. Bernard, *The dynamics of pseudographs in convex Hamiltonian systems*, J. Amer. Math. Soc., 21 (2008), pp. 615–669.

[B10] P. Bernard, *Large normally hyperbolic cylinders in a priori stable Hamiltonian systems*, Annales H. Poincare, 11 (2010), pp. 929–942.

[BKZ] P. Bernard, V. Kaloshin, and K. Zhang, *Arnold diffusion in arbitrary degrees of freedom and normally hyperbolic invariant cylinders*, Acta Math., 217 (2016), pp. 1–79.

[Bir] G. D. Birkhoff, *Dynamical Systems*, Colloquium Publications of AMS, 9 (1927).

[Bs] U. Bessi, *An approach to Arnold's diffusion through the calculus of variations*, Nonlinear Anal., 26:6 (1996), pp. 1115–1135.

[C13] C.-Q. Cheng, *Arnold diffusion in nearly integrable Hamiltonian systems*, arXiv:1207.4016v2 (2013).

[C17a] C.-Q. Cheng, *Uniform hyperbolicity of invariant cylinder*, J. Differential Geometry, 106 (2017), pp. 1–43.

[C17b] C.-Q. Cheng, *Dynamics around the double resonance*, Cambridge J. Math., 5:2 (2017), pp. 153–228.

[C18] C.-Q. Cheng, *The genericity of Arnold diffusion in nearly integrable Hamiltonian systems*, Asian J. Math. (2018) to appear.

[CY04] C.-Q. Cheng and J. Yan, *Existence of diffusion orbits in a priori unstable Hamiltonian systems,* J. Differential Geometry, 67 (2004), pp. 457–517.

[CY09] C.-Q. Cheng and J. Yan, *Arnold diffusion in Hamiltonian Systems: a priori Unstable Case,* J. Differential Geometry, 82 (2009), pp. 229–277.

[CX] C.-Q. Cheng and J. Xue, *Arnold diffusion in nearly integrable Hamiltonian systems with arbitrary degrees of freedom*, arXiv:1503.04153 (2015).

[CZ16] C.-Q. Cheng and M. Zhou, *Global normally hyperbolic cylinders in Lagrangian systems*, Math. Res. Lett., 23 (2016), pp. 685–705.

[DLS06] A. Delshams, R. de la Llave R., and T. M. Seara, *A geometric mechanism for diffusion in Hamiltonian systems overcoming the large gap problem: heuristics and rigorous verification on a model*, Mem. Amer. Math. Soc, 179 (2006), 844.

[DLS08] A. Delshams, R. de la Llave R., and T. M. Seara, *Geometric properties of the scattering map of a normally hyperbolic invariant manifold*, Advances in Mathematics, 217 (2008), pp. 1096–1153.

[Fa] A. Fathi, *Weak KAM Theorem in Lagrangian Dynamics*, preprint (7-th preliminary version) (2005).

[GSV] V. Gelfreich, C. Simó, and A. Vieiro, *Dynamics of 4D symplectic maps near a double resonance*, Physica D., 243 (2013), pp. 92–110.

[KZ] V. Kaloshin and K. Zhang, *A strong form of Arnold diffusion for two and a half degrees of freedom*, arXiv:1212.1150v2 (2013) and arXiv:1212.1150v3 (2018).

[LC] X. Li and C.-Q. Cheng, *Connecting orbits of autonomous Lagrangian systems*, Nonlinearity, 23 (2009), pp. 119–141.

[Lo] P. Lochak, *Canonical perturbation theory via simultaneous approximation*, Russian Math. Surveys, 47 (1992), pp. 57–133.

[M91] J. Mather, *Variational construction of orbits of twist diffeomorphisms,* J. Amer. Math. Soc., 4 (1991), pp. 169–207.

[M93] J. Mather, *Variational construction of connecting orbits,* Ann. Inst. Fourier (Grenoble), 43:5 (1993), pp. 1349–1386.

[M03] J. Mather, *Arnold diffusion, I: Announcement of results,* J. Mathematical Sciences, 124:5 (2004), pp. 5275–5289. (Russian translation in Sovrem. Mat. Fundam. Napravl, 2 (2003), pp. 116–130.)

[M09] J. Mather, *Near double resonance*, talks at conferences on dynamical systems at Nice (2009), Toronto, Nanjing, Oberwolfach and Edinburgh (2011).

[M11] J. Mather, *Shortest curve asscoiated to a degenerate Jacobi metric on* $\mathbb{T}^2$, Progress in Variational Methods, Nankai Ser. Pure Appl. Math. Theoret. Phys, 7 (2011), pp. 126–168.

[Mar] J. P. Marco, *Generic hyperbolic propertires of classical systems on the torus* $\mathbb{T}^2$, preprint (2013).

[T04] D. V. Treschev, *Evolution of slow variables in a priori unstable Hamiltonian systems*, Nonlinearity, 17 (2004), pp. 1803–1841.

[Zm] M. Zhou, *Hölder regularity of weak KAM solutions in a priori unstable systems*, Math. Res. Lett., 18 (2011), pp. 75–92.