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# SURVEY ON DERIVATION LIE ALGEBRAS OF ISOLATED SINGULARITIES*

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*Dedicated to the memory of Professor John Mather*

**Abstract.** Let $V$ be a hypersurface with an isolated singularity at the origin defined by the holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. Let $L(V)$ be the Lie algebra of derivations of the moduli algebra $A(V) := \mathcal{O}_n/(f, \partial f/\partial x_1, \cdots, \partial f/\partial x_n)$, i.e., $L(V) = \mathrm{Der}(A(V), A(V))$. The Lie algebra $L(V)$ is finite dimensional solvable algebra and plays an important role in singularity theory. According to Elashvili and Khimshiashvili ([15], [23]) $L(V)$ is called Yau algebra and the dimension of $L(V)$ is called Yau number. The studies of finite dimensional Lie algebras $L(V)$ that arising from isolated singularities was started by Yau [44] and has been systematically studied by Yau, Zuo and their coauthors. Most studies of Lie algebras $L(V)$ were oriented to classify the isolated singularities. This work surveys the researches on Yau algebras $L(V)$ of isolated singularities.

**Key words.** Fewnomial, Lie algebra, isolated singularity.

**Mathematics Subject Classification.** 32S05, 14B05.

**1. Introduction.** Let $\mathbb{C}[x_1, \cdots, x_n]$ be the algebra of complex polynomials in $n$ indeterminates. The algebra of germs of holomorphic functions at the origin of $\mathbb{C}^n$ is denote as $\mathcal{O}_n$. Clearly, $\mathcal{O}_n$ can be naturally identified with the algebra of convergent power series in $n$ indeterminates with complex coefficients. For a polynomial $f \in \mathbb{C}[x_1, \cdots, x_n]$, denoted by $V = V(f)$ the germ at the origin of $\mathbb{C}^n$ of hypersurface $\{f = 0\} \subset \mathbb{C}^n$. In other words, if the origin is an isolated zero of the gradient of $f$ then $V$ is a germ of isolated hypersurface singularity. The local (function) algebra of $V$ is defined as the (commutative associative) algebra $F(V) \cong \mathcal{O}_n/(f)$, where $(f)$ is the principal ideal generated by the germ of $f$ at the origin. According to Hilbert's Nullstellensatz for an isolated singularity $V = V(f) = \{f = 0\}$ the factor-algebra $A(V) = \mathcal{O}_n/(f, \partial f/\partial x_1, \cdots, \partial f/\partial x_n)$ is finite dimensional. This factor-algebra is called the moduli algebra of $V$ and its dimension $\tau(V)$ is called Tyurina number.

Recall that finite dimensional Lie algebras are semi-direct product of the semi-simple Lie algebras and solvable Lie algebras. Brieskorn gave the connection between simple Lie algebras and simple singularities. It is also important to establish connections between singularities and solvable (nilpotent) Lie algebras. Yau [47] provides a general method of constructing a solvable Lie algebra $L(V)$ by taking a derivations of $A(V)$, i.e., $L(V) := \mathrm{Der}(A(V), A(V))$ and according to [44] this Lie algebra is finite dimensional. The ideas are based on the well-known Mather-Yau Theorem [32]: Let $V_1$ and $V_2$ be two isolated hypersurface singularities and, $A(V_1)$ and $A(V_2)$ be the moduli algebra, then $(V_1, 0) \cong (V_2, 0) \iff A(V_1) \cong A(V_2)$. The Mather-Yau theorem was raised two main questions. The first one is: which kind of commutative local algebra is moduli algebra of an isolated singularity. The second one is: what kind of information do we need from moduli algebra to determine the topological type of singularity. On the one hand, it is well-known that moduli algebras are Artinian algebras and their associated derivation Lie algebras are finite dimensional. According to

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Yau conjecture that proposed in 1983, these derivation Lie algebras are solvable. This gives a necessary conditions for first problem. On the other hand, the Lie algebras $L(V)$ in some sense only a generic topological invariant but it is not a topological invariant. Seeley and Yau [37] shown that Lie algebra $L(V)$ is a useful invariant that can be used to solve the moduli problems of singularities. They also distinguished the complex analytic structure of simple-elliptic singularities $\widetilde{E_7}$ and $\widetilde{E_8}$ by using Lie algebras $L(V)$. According to Yau ([47], [48]), the Lie algebra $L(V)$ is solvable for lower dimensions. Few years later in 1991, Yau proved that the $n$ dimensional Lie algebras $L(V)$ is solvable [49]. The problem of solvability of $n$ dimensional Lie algebras $L(V)$ is divided in two parts: the first part consists of classification of $sl(2,\mathbb{C})$ actions on $\mathcal{O}_n$ via derivations preserving the $m$-adic filtration and second part is the classification of gradient spaces invariants by $sl(2,\mathbb{C})$ action. In [47] Kac-moody Lie algebra was attached to an isolated hypersurface singularity and the generalized Cartan matrix was computed for simple hypersurface singularities. Yau and Benson [7] made a computer program by using C programming language that computes the Lie algebras of derivations and Lie algebra cohomology. Seeley [38] proved that the generalized Cartan matrix of simple-elliptic singularities $\widetilde{E_7}$ and $\widetilde{E_8}$ is not a topological invariant. The order of the lowest nonvanishing term in the power series expansion of $f$ at 0 is called the multiplicity (denoted by $mult(f)$) of the singularity $(V,0)$. It is well-known that a polynomial $f \in \mathbb{C}[x_1,\cdots,x_n]$ is said to be weighted homogeneous if there exist positive rational numbers $w_1,\cdots,w_n$ (weights of $x_1,\cdots,x_n$) and $d$ such that, $\sum a_i w_i = d$ for each monomial $\prod x_i^{a_i}$ appearing in $f$ with nonzero coefficient. The number $d$ is called weighted homogeneous degree ($w$-degree) of $f$ with respect to weights $w_j$. The weight type of $f$ is denoted as $(w_1,\cdots,w_n;d)$. Without loss of generality, we can assume that $w$-deg$f = 1$. The Milnor number of the isolated hypersurface singularity is defined by $\mu = \dim \mathbb{C}[x_1,\cdots,x_n]/(\partial f/\partial x_1,\cdots,\partial f/\partial x_n)$. The Milnor number in case of weighted homogeneous hypersurface singularity is calculated by: $\mu = (\frac{1}{w_1}-1)(\frac{1}{w_2}-1)\cdots(\frac{1}{w_n}-1)$. In 1971, Saito was the first person who computed the necessary and sufficient condition for $V$ to be defined by a weighted homogeneous polynomial. It is well-known that $f$ is a weighted homogeneous polynomial after a biholomorphic change of coordinates $\Longleftrightarrow \mu = \tau$ [36].

In [43] Xu and Yau showed that if an isolated hypersurface singularity admits a weighted homogeneous structure, then its moduli algebra is isomorphic to finite dimensional non-negatively graded algebra.

Another important class of isolated hypersurface singularity is fewnomial singularities which is defined by Elashvili and Khimshiashvili [15]. Fewnomial singularities are those which can be defined by $n$-nomial in $n$ indeterminates, i.e., a weighted homogeneous polynomial $f(x_1,\cdots,x_n)$ is called fewnomial if number of variables coincides with number of monomials ([15], [58]). According to Ebeling and Takahashi [16], the fewnomial singularities are also called invertible singularities.

The nonexistence of negative weight derivation in case of zero dimensional weighted complete intersection was studied in [33, 34]. For positive-dimensional weighted homogeneous singularities the nonexistence of negative weight derivation has been studied in following references ([31], [40, 41, 42]). Kantor ([21, 22]) proved the nonexistence of weight derivation for weighted homogeneous hypersurface singularities and weighted homogeneous curve singularities. Let $R = \mathbb{C}[x_1,\cdots,x_n]/(f)$ and where $f$ is a weighted homogeneous polynomial defining an isolated hypersuface singularity. Then $R$ and $Der(R,R)$ are graded and there is no non-zero negative weight derivation on $Der(R,R)$. Wahl conjectured as this is still true if $f$ replace with $f_1, f_2, \cdots, f_m$ weighted homogeneous polynomials defining an isolated, normal and complete intersection singularity with same weight type $(w_1, w_2, \cdots, w_n)$. Wahl Conjecture and its generalization (without the condition of complete intersection singularity) for $R$ was solved in [9] under the condition that the degree of $f_i, 1 \leq i \leq m$ are bounded below by a constant $C$ depending only on the weights $w_1, w_2, \cdots, w_n$. In case of zero-dimensional quasi-homogeneous singularities the problem of nonexistence of negative weight derivation on moduli algebras was solved in [59]. In [14], in case of weighted homogeneous fewnomial isolated singularity $f \in \mathbb{C}[x_1, x_2, \cdots, x_n]$ with positive weights $w_1, w_2, \cdots, w_n$, the milnor algebra $A(f) = \mathbb{C}[x_1, \cdots, x_n]/(f_1, f_2, \cdots, f_m)$ has no non-zero negative weight derivation on $A(f)$ with the multiplicity at least 5.

Classification of isolated hypersurface singularity through derivation Lie algebra is another interesting topic. According to Elashvili and Khimshiashvili [15] the Lie algebra $L(V)$ is called Yau algebra and its dimension $\lambda(V)$ is called Yau number. They also proved the following main results: Suppose $X$ and $Y$ belong to simple hypersurface singularities, except the pair $A_6$ and $D_5$, then $L(X) \cong L(Y)$, if and only if $X$ and $Y$ are analytically isomorphic. They also defined the natural grading on Yau algebra in case of simple hypersurface singularities and proved that all roots of Poincaré polynomials lie on unit circle. It is interesting to note that Yau algebra of simple hypersurface singularities have a property of completeness and commutative polarization. Khimshiashvili [23] investigated the Yau algebra of binomial singularities and used this algebra to distinguish the analytic isomorphism type of these singularities. Pursell and Shanks [35] defined diffeomorphism type of a manifold by using Lie algebra of smooth vector fields on a smooth manifold. Khimshiashvili [23] proved the analog of Pursell-Shanks's theorem for Yau algebras of binomial singularities. It is follows from ([7], [37]), two non-isomorphic isolated singularities defined by 4-nomial in 3 variables have isomorphic Yau algebra. Therefore, it is not sure whether Pursell-Shanks's theorem holds outside the fewnomial singularities.

It is interesting to bound the Yau number with a number which depend on weight type. In [58], Yau and Zuo proposed the sharp upper estimate conjecture that bound the Yau number. They also proved that this conjecture holds in case of binomial isolated hypersurface singularities. In [19], Yau, Zuo and present author verified this conjecture in case of fewnomial surface singularities. In this article we shall give a survey on derivation Lie algebra and divided this work in to five sections. We shell also gives some open problems related to derivation Lie algebras. Yau and Zuo's series papers are good references for singularity theory and recent progress of the above topics ([54]-[61], [25], [6], [13], [26], [27]-[29], [45], [46], [51], [52], [24], [39], [11], [53]).

**2. construction of derivation Lie algebras.** It is important to know the connection between derivation Lie algebras and isolated hypersurface singularity. The moduli algebra is defined as $A(V) := \mathcal{O}_n/(f, \partial f/\partial x_1, \cdots, \partial f/\partial x_n)$ and where $V$ is an isolated hypersurface singularity defined by the holomorphic function $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$. Derivation Lie algebras $L(V)$ is an algebra of derivations of moduli algebra. It follows from famous Mather-Yau theorem that the natural mapping $(V, 0) \longrightarrow A(V)$ is one to one. Yau was the first person who systematically studied the algebra of derivation of $A(V)$. The Lie algebra $L(V)$ is finite dimensional Lie algebra that contained in the endomorphism of moduli algebra $A(V)$. On the other hand, we have another natural mapping which is defined as:

$$(V, 0) \longrightarrow L(V) = \text{ Derivation Lie algebra of } A(V).$$

Suppose that holomorphic function $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ defining a ring of germ

$\varphi_{V,0}$ at origin, we set $\varphi_{V,0} = \mathcal{O}_n/(f)\mathcal{O}_n$ is the local ring of $V$ at 0. It was proved in [44], the $Der(\varphi_{V,0})$ induces a $Der(A(V))$. Hence, there exist a natural map from the algebra of derivations of $\varphi_{V,0}$ to $L(V)$ and this natural map is not surjective. This fact can be observed from example 1.

Let $f$ be a weighted homogeneous function, i.e., there exists $p_1, p_2, \cdots, p_n, d \in \mathbb{N}$ such that $f(t^{p_1}x_1, \cdots, t^{p_n}x_n) = t^d f(x_1, \cdots, x_n)$ for all $(x_1, \cdots, x_n) \in \mathbb{C}^n$ and $t \in \mathbb{C}^*$. Then $p_1 x_1 \frac{\partial}{\partial x_1} + \cdots + p_n x_n \frac{\partial}{\partial x_n}$ is contained in $Der(\varphi_{V,0})$ and called Euler derivation.

LEMMA 2.1 ([44]). *Let $\varphi_{V,0} = \mathcal{O}_n/(f)\mathcal{O}_n$, where $V$ is an isolated hypersurface singularity defined by weighted homogeneous holomorphic function $f$. Then the algebra of $Der(\varphi_{V,0})$ is induced as an $\varphi_{V,0}$ module by Euler derivation and following derivations*

$$\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}.$$

EXAMPLE 1. Let $V = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1^3 + x_2^3 + x_3^3 = 0\}$. Then the moduli algebra

$$A(V) =< 1, x_1, x_2, x_3, x_1x_2, x_2x_3, x_3x_1, x_1x_2x_3 > .$$

The derivation Lie algebras $L(V)$ is defined as

$$\begin{aligned} L(V) = < & x_1\partial_1, x_1x_2\partial_1, x_3x_1\partial_1, x_1x_2x_3\partial_1, x_2\partial_2, x_1x_2\partial_2, x_2x_3\partial_2, x_1x_2x_3\partial_2, x_3\partial_3, \\ & x_2x_3\partial_3, x_1x_3\partial_3, x_1x_2x_3\partial_3 > . \end{aligned}$$

It is observed that the natural mapping from the algebra of $Der(\varphi_{V,0})$ to $Der(A(V))$ is highly nonsurjective in the context of above lemma.

It follows from following proposition that the derivation Lie algebra $L(V)$ is nontrivial invariant.

PROPOSITION 2.1 ([47]). *Let $V = \{(x_1, \cdots, x_n) \in \mathbb{C}^n : f(x_1, \cdots, x_n) = 0\}$ be an isolated singularity at origin defined by $f(x_1, \cdots, x_n)$ weighted homogeneous function. Then derivation Lie algebras $L(V)$ is abelian if and only if $(V, 0)$ is either $A_1$ or $A_2$.*

**2.1. Derivations.** A linear endomorphism $D$ of commutative associative algebra $A$ that satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$ is called derivation of $A$. The set $\mathrm{Der}(A, A)$ denoted the derivation on $A$ (sometimes use as $\mathrm{Der}A$). The $Der(A)$ has natural Lie algebras structure with Lie bracket defined by the commutator of linear endo-morphisms. In case of moduli algebra $A(V) = \mathbb{C}[x_1, \cdots, x_n]/(f, \partial f/\partial x_1, \cdots, \partial f/\partial x_n)$, one can consider the Lie algebras $\mathrm{Der}(A(V), A(V))$. The Lie algebras of other kinds are also studied in ([3], [5], [7]). According to Yu [50] and Khimshiashvili [23] the Lie algebras $L(V)$ is called Yau algebra of $V$ and its dimension $\lambda(V)$ is called Yau number. The Yau number is an analytic invariant. Yau algebras of many concrete singularities can be computed by using the following basic concepts and results.

Let $B_1, B_2$ be a associative algebras over $\mathbb{C}$ and $M(B_1)$ is multiplication algebra of $B_1$. The algebra $M(B_1)$ is the subalgebra of endomorphisms of $A$ generated by the left and right multiplications by elements of $B_1$ and identity element. The centroid algebra $C(B_1)$ is defined as the set of endomorphisms of $B_1$ which commute with all elements of $M(B_1)$. Therefore, the algebra $C(B_1)$ is a unital subalgebra of $\mathrm{End}(B_1)$. Follow the proposition 1.2 of [8], we have following statement.

Let $W = B_1 \otimes B_2$ be a tensor product of finite dimensional associative algebras with units. Then

$$\mathrm{Der}W \cong (\mathrm{Der}B_1) \otimes C(B_2) + C(B_1) \otimes (\mathrm{Der}B_2).$$

It is noted that this result is only used for commutative associative algebras with unit, when the centroid coincides with the algebra itself. Thus for commutative associative algebras $B_1, B_2$ we have following:

THEOREM 2.1 ([8]). *For commutative associative algebras $B_1, B_2$,*

$$DerW \cong (DerB_1) \otimes B_2 + B_1 \otimes (DerB_2).$$

DEFINITION 2.1. *Let $J$ be an ideal in an analytic algebra $W$. Then $Der_J W \subseteq Der_{\mathbb{C}} W$ is Lie subalgebra of all $\sigma \in Der_{\mathbb{C}} W$ for which $\sigma(J) \subset J$.*

We use the following well-known result to compute the derivations.

THEOREM 2.2 ([58]). *Let $J$ be an ideal in $R = \mathbb{C}\{x_1, \cdots, x_n\}$. Then there is a natural isomorphism of Lie algebras*

$$(Der_J R)/(J \cdot Der_{\mathbb{C}} R) \cong Der_{\mathbb{C}}(R/J).$$

## 3. Solvability of Derivation Lie algebras .

DEFINITION 3.1. *Let $L$ be a Lie algebra and this Lie algebra is called nilpotent if lower central series: $L_{(*)} = \{L_{(i)}\}, L_{(0)} = L, L_{(1)} = [L, L], \cdots, L_{(i)} = [L, L_{(i-1)}]$ is terminate for $i = 2, 3, \cdots, L$. If upper central series: $L^{(*)} = \{L^{(i)}\}, L^{(0)} = L, L^{(1)} = [L, L], \cdots, L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$ for $i = 2, 3, \cdots, L$ is vanished then $L$ is called solvable.*

It is natural question that under what conditions an Artinian algebra can be a moduli algebra. The following main theorem gives the positive answer to this question.

THEOREM 3.1 ([44]). *Let $(V, 0)$ be an isolated singularity. Then finite dimensional Lie algebras $L(V)$ is solvable.*

In following theorems solvability of $L(V)$ for $n \leq 3$ and $n \leq 5$ was proved by Yau.

THEOREM 3.2 ([47]). *Let $V = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : f(x_1, x_2, x_3) = 0\}$ has an isolated singularity at $(0, 0, 0)$. Then the Lie algebra $L(V)$ with finite dimension is solvable.*

In general, to prove the solvability of the Lie algebras $L(V)$, we need to assume that the multiplicity of $f > 2$.

THEOREM 3.3 ([48]). *Let $V = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{C}^5 : f(x_1, x_2, x_3, x_4, x_5) = 0\}$ has an isolated singularity at $(0, 0, 0, 0, 0)$. Then the finite dimensional Lie algebras $L(V)$ associated to the singularity is solvable.*

It is natural question whether the finite dimensional Lie algebras $L(V)$ is solvable for general $n$. The proof of this more general problem consist of two main parts. First part consists of classification of $sl(2,\mathbb{C})$ actions on power series ring $\mathcal{O}_n$, which was proved in [48] under the derivations preserving the m-adic filtration. The second one is the characterization of gradient spaces invariants under $sl(2,\mathbb{C})$ action and this part was proved in [50] under the condition $n \leq 5$.

THEOREM 3.4 ([49]). *Let $V = \{(x_1, \cdots, x_n) \in \mathbb{C}^n : f(x_1, \cdots, x_n) = 0\}$ be a hypersurface with an isolated singularity at $(0, \cdots, 0)$. Then the finite dimensional Lie algebras $L(V)$ associated to $V$ is solvable.*

**4. characterization of isolated singularities.** Characterization of isolated singularities is an important problem in singularity theory. Since Lie algebra $L(V)$ is a complete invariant and this invariant was studied to classify the isolated singularities. By using this invariant we can also introduce some new invariants, which we discuss later.

P. Griffiths studied the Torelli type problem by using given family of complex projective hypersurface in $\mathbb{C}P^n$ and his school asks an interesting question: when a period map is injective on that family. In other words whether the family of complex hypersurface can be distinguished by using their Hodge structure. It is well-know that complex projective hypersurface in $\mathbb{C}P^n$ can be viewed as a complex hypersurface isolated singularity in $\mathbb{C}P^{n+1}$. Seeley and Yau [38] distinguished the complex analytic structures of isolated singularities and also constructed the continuous numerical invariants by using the Yau algebra.

Recall that the class of simple (Kleinian, rational double point) singularities have following series $A_l : \{x_1^{l+1} = 0\} \subset \mathbb{C}, l \geq 1$, $D_l : \{x_1^2 x_2 + x_2^{l-1} = 0\} \subset \mathbb{C}^2, l \geq 4$, and other three singularities $E_6 : \{x_1^3 + x_2^4 = 0\}, E_7 : \{x_1^3 + x_1 x_2^3 = 0\}$ and $E_8 : \{x_1^3 + x_2^5 = 0\}$. It is well-knows that simple elliptic singularities consist of three types $\widetilde{E_6}$, $\widetilde{E_7}$ and $\widetilde{E_8}$. The first type defined as $\widetilde{E_6} : \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1^3 + x_2^3 + x_3^3 = 0\}$ and it follows from [44] that the $(\mu, \tau)$-constant family of $\widetilde{E_6}$ is defined as:

$$V_t = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid f_t(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_2^3 + t x_1 x_2 x_3 = 0\}$$

with $t^3 + 27 \neq 0$. The second type of simple elliptic singularity defined as $\widetilde{E_7}$ : $\{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1^4 + x_2^4 + x_2^3 = 0\}$ and in [37], the authors showed that its $(\mu, \tau)-$ constant family is defined as:

$$V_t = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid f_t(x_1, x_2, x_3) = x_1^4 + x_2^4 + t x_1^2 x_2^2 + x_3^2 = 0\}$$

with $t^2 \neq 4$. The simple elliptic singularity $\widetilde{E_8}$ is defined by $\{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1^6 + x_2^3 + x_3^2 = 0\}$ and in [37], the authors had studied the $(\mu, \tau)-$ constant family of $\widetilde{E_8}$, which is defined as:

$$V_t = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid f_t = x_1^6 + x_2^3 + x_3^2 + t x_1^4 x_2 = 0\}$$

with $4t^3 + 27 \neq 0$. It is natural to present the following question. Find out what type of singularities such that $L(V)$ is a complete invariants, i.e., if $V_1, V_2$ are two such type of singularities, then $L(V_1) \cong L(V_2)$ if and only if $V_1 \cong V_2$.

THEOREM 4.1 ([37]). *Simple elliptic singularities $\tilde{E}_7$ satisfies the Torelli-type theorem, i.e., $L(V_{t_1}) \cong L(V_{t_2})$ as Lie algebras, for $t_1 \neq t_2$ in $\mathbb{C} - \{\pm 2\}$, if and only if $V_{t_1}$ and $V_{t_2}$ are analytically isomorphic.*

THEOREM 4.2 ([37]). *The Torelli-type theorem holds for simple elliptic singularities $\tilde{E}_8$, i.e., $L(V_{t_1}) \cong L(V_{t_2})$ as Lie algebras, for $t_1 \neq t_2$ in $\mathbb{C} - \{t \in \mathbb{C} : 4t^3 + 27 = 0\}$, if and only if $V_{t_1}$ and $V_{t_2}$ are analytically isomorphic (i.e., $t_1^3 = t_2^3$).*

It is easy to see that for $t \neq 0$ and $216 - \frac{t^6}{27} + 7t^3 \neq 0$, the Lie algebra $L(\tilde{E}_6)$ are isomorphic. Thus $L(\tilde{E}_6)$ is a trivial family. M. Benson and Yau construct the one parameter family of all inequivalent representation of $L(\tilde{E}_6)$. This family of representations can not be constructed through action of automorphism group of $L(\tilde{E}_6)$ on a representation. It is interesting to study the following question which is proposed by Yau, Zuo and present author.

QUESTION 4.1. *Does there exist another derivation Lie algebra that distinguish the complex analytic structures of $\tilde{E}_6$ singularity.*

In [47] the isolated hypersurface singularities are attached with Kac-Moody Lie algebra and compute the GCM(generalized Cartan matrix) for simple hypersurface singularities. This new invariant was computed by using maximal ideal of Lie algebra $L(V)$ which is consisting of nilpotent elements. GCM is another analytic invariant of isolated singularities. Seeley and Yau [38] show that the GCM is not a topological invariant of singularity. They also show that the Yau algebra of solvable Lie algebras is not topological invariant.

An isolated hypersurface singularity $(V, 0) = \{(x_1, \cdots, x_n) : f(x_1, \cdots, x_n) = 0\} \subseteq \mathbb{C}^n$ is quasi-homogeneous if $f$ belong to the jacobian ideal of $f$ (i.e., $f \in (\partial f/\partial x_1, \cdots, \partial f/\partial x_n)$). According to beautiful result of Saito [36], after a biholomorphic change of coordinates the quasi-homogeneous polynomial $f$ with an isolated critical point at zero becomes a weighted homogeneous polynomial. It is natural question what is necessary condition for a complex analytic isolated hypersurface singularity to be a quasi-homogeneous in terms of its moduli algebra. The following theorem give the answer of this question.

THEOREM 4.3 ([43]). *Let $(V, 0) = \{(x_1, \cdots, x_n) : f(x_1, \cdots, x_n) = 0\} \subseteq \mathbb{C}^n$ be an isolated hypersurface singularity. Then isolated hypersuface singularity has a quasi-homogeneous structure $\Longleftrightarrow A(V) \cong \oplus_{j \geq 0} A_j$, with $A_0 = \mathbb{C}$.*

A derivation $D$ of graded algebra $\oplus_{j=0}^{\infty} A_j$ have a weight $k$ if $D$ sends $A_j$ to $A_{j+k}$ for all $j$. Le Dung Trang asks a natural question whether it is possible to characterize the quasi-homogeneous isolated hypersurface singularity in terms of its Lie algebra. The following theorem have micro-local characterization of quasi-homogeneous isolated hypersurface singularity.

THEOREM 4.4 ([43]). *Let $(V, 0) = \{(x_1, \cdots, x_n) \in \mathbb{C}^n : f(x_1, \cdots, x_n) = 0\}$ be an isolated hypersurface singularity. Then isolated hypersurface singularity is a quasi-homogeneous singularity, i.e., $f \in (\partial f/\partial x_1, \cdots, \partial f/\partial x_n)$ if*
(1) *$L(V) \cong \oplus_{j=0}^{k} L_j$ without center.*
(2) *There exist $H \in L_0$ such that $[H, D_j] = jD_j$ for any $D_j \in L_j$.*
(3) *For any $\beta \in u - u^2$ where $u$ is maximal ideal of $A(V)$, then $\beta H \notin L_0$.*

It is noted that in above theorem the Lie algebra $\oplus_{j=0}^{k} L_j$ is non-negative graded Lie algebra. It follows from above theorem the conditions (2) and (3) are necessary for isolated singularity to be a quasi-homogeneous. We also believe that condition (1) is also necessary. The above necessary condition (1) is a special case of Halperin Conjecture. The Halperin Conjecture has a important applications in rational homotopy theory.

Halperin Conjecture ([30]). Let $A = F[x_1, \cdots, x_n]/(f_1, \cdots, f_n)$ where $f_1, \cdots, f_n$ are weighted homogeneous polynomials and $F$ is a field of characteristic zero. Then both $A$ and derivation Lie algebra $L(V)$ are graded.

Let $A$ be a local Artinian algebra and $m$ is maximal ideal, then complex vector subspace Soc $A = \{a \in A : a.m = 0\}$ defined a socle of $A$. Dimension of vector space Soc $A$ defined a type of $A$. The algebra $A$ is Gorenstein if type of $A$ is equal to one.

Proposition 4.1 ([43]). *Let $\oplus_{j=0}^{t} A_j$ be a graded commutative Artinian local algebra with $A_0 = \mathbb{C}$ and $t \geq 1$. Then $dim_{\mathbb{C}}\ L(A) \geq dim_{\mathbb{C}} A - dim_{\mathbb{C}}\ SocA$.*

Proposition 4.2 ([43]). *Let $A$ be a commutative Artinian local algebra. Let $D$ be any derivation of $A$. Then $D$ preserve the $m$-adic filtration of $A$, i.e., $D(m) \subseteq m$ where $m$ is the maximal ideal of $A$*

Another way to define a wighted homogeneous isolated singularities is as follows: Let $w_1, \cdots, w_n$ (weights of $x_1, \cdots, x_n$) be a positive rational numbers and $d$ is weighted homogeneous degree ($w$-degree) of polynomial $f \in \mathbb{C}[x_1, \cdots, x_n]$ with respect to weights $w_j$. Then $f$ is said be the weighted homogeneous if $\sum a_i w_i = d$ for each monomial $\prod x_i^{a_i}$ appearing in $f$ with nonzero coefficient. The collection $(w_1, w_2, \cdots, w_n; d)$ denote the weight type of $f$. We can compute the Milnor number of weighted homogeneous isolated hypersurface singularities in following way $\mu = (\frac{1}{w_1} - 1)(\frac{1}{w_2} - 1) \cdots (\frac{1}{w_n} - 1)$. In 1971, Saito gave a complete description of isolated singularity which is defined by weighted homogeneous polynomial. The weighted homogeneous polynomial $f(x_1, x_2, \cdots, x_n)$ is called fewnomial if number of variables coincides with number of monomials ([15], [14]), i.e., fewnomial singularities are those which can be defined by $n$-nomial in $n$ indeterminates. The fewnomial singularities are also called invertible singularities [16]. Khimshiashvili [23] distinguished the analytic isomorphism type of binomial singularities by using the Yau algebra. At the early stage of the mirror symmetry fewnomial singularities have extensively been studied for a long time and also applied to give a lot of topological mirror pairs of Calabi-Yau manifolds.

According to Pursell-Shanks's theorem the diffeomorphism type of a manifold can be determined by using the Lie algebra of smooth vector fields on a smooth manifold. In [23] the analog of Pursell-Shanks's theorem was proved for Yau algebras of binomial singularities. It was given in ([47], [7]) that there exist isolated singularities defined by four nomials in three variables which are analytically non-isomorphic but have isomorphic Yau algebra [37]. It is interesting to raise a question whether the Pursell-Shanks's theorem hold outside fewnomial singularities. It is conformed from ([47], [7], [37]) that the analytically non-isomorphic isolated singularities which is defined by four nomial in three variable have isomorphic Yau algebra. Therefore outside the fewnomial singularities, it is not sure whether the Pursell-Shanks's theorem holds. The following results yield the analog of Pursell-Shanks's theorem for certain classes of isolated singularities.

Theorem 4.5 ([15]). *Let $X$ and $Y$ are simple singularities except the pair $A_6$ and $D_5$, then $L(X) \cong L(Y)$ as Lie algebras $\Longleftrightarrow$ $X$ and $Y$ are analytically isomorphic.*

Khimshiashvili [23] extend the method and main results of [15] to wider classes of singularities and proved following main results:

Theorem 4.6 ([23]). *The Lie algebras $L(V)$ completely characterizes the binomial isolated singularities when $\mu > 6$ .*

Theorem 4.7 ([23]). *X The Lie algebras $L(V)$ completely characterizes the decomposable fewnomial singularities when $\mu > 6$.*

The following open problem is proposed by Yau, Zuo and present author.

Question 4.2. *Can we find such a derivation Lie algebra which completely characterizes the simple hypersurface singularities.*

**5. structural properties of derivation lie algebras.** In this section we shell discuss the some natural properties of derivation Lie algebras. A Lie algebras which have trivial center and all of its derivations are inner is called complete Lie algebras. In [15] the simple singularities have only inner derivation.

Theorem 5.1 ([15]). *The Lie algebras $L(V)$ of simple hypersurface singularities except $E_6$ singularity is complete when $\mu \geq 8$.*

In general, the Lie algebra $L(V)$ needs not to be complete [15]. The present author proposed another open problem for readers.

Question 5.1. *Can we find another derivation Lie algebra of simple hypersurface singularities which have a property of completeness without any exceptional case.*

A Lie algebra $L(V)$ has a maximal commutative polarization if its commutative subalgebra has dimension equal to $\frac{1}{2}(\dim\ L{+}\text{ind}\ L)$, where ind $L$ is indices of Lie algebra $L(V)$.

Theorem 5.2 ([15]). *The indices of Lie algebra $L(V)$ is 0, if dim $L(V)$ is even and if dim $L(V)$ is odd then indices of Lie algebra is 1. Moreover, the Lie algebras $L(V)$ has commutative polarization.*

There are some natural gradings on associated Lie algebras which we shell discuss in detail. The first grading is $\mathbb{Z}^n$-grading which is also called Cartan grading. The second grading is $\mathbb{Z}$-grading which is also called Euler grading. Using the vector field notation for elements of Lie algebras $L(V)$, we can establish following correspondence: $x_k^{a_k}\partial_j \mapsto (a_1, \cdots, a_n; j)$. To describe the Cartan grading we first need to define the Cartan subalgebra, which is maximal commutative subalgebra $H$ such that, for each $h \in H$ and ad $H$ is semisimple. It follows that each element of Cartan subalgebra have following form $h_i = (0, \cdots, 1, \cdots, 0; i)$ with $i$ staying on the $i$th place. The basis vectors are eigenvectors for each of operator ad $h_i$. The eigenvalues of those operators define the Cartan grading. Therefore, a basis elements of the form $(a; j)$ has Cartan grading $(a_1, \cdots, a_{j-1}, a_j - 1, \cdots, a_n)$. For an element of the form $(a; j)$, the Euler height is defined as $h(a, j) = -1 + \sum a_i$. The correspondence $(a; j) \mapsto h(a, j)$ defines the Lie algebras $\mathbb{Z}$-grading called Euler grading. Thus a Euler grading is sum of component of Cartan grading. Another natural grading can be defined on $L(V)$ by setting the weight of $\partial_j$ equal to $-w_j$. Therefore the vector field $x_k^m\partial_j$ has a weight $mw_k - w_j$. Thus this weight defines a grading on Lie algebra. This grading is called quasi-homogeneous grading on $L(V)$.

Definition 5.1. *The Poincaré polynomial $P_A$ of graded algebra $A = \oplus_{\beta\in\mathbb{Z}^n} A_\beta$ with respect to $\mathbb{Z}^n$ grading is defined as*

$$P_A(t_1, \cdots, t_n) = \sum_\beta dim(A_\beta) t_1^{\beta_1}, \cdots, t_n^{\beta_n}.$$

*In case of moduli algebra $A(V)$, the Poincaré polynomial $P(V)$ is equal to the product of Poincaré polynomials $P(V_i)(t_i)$ of $A(V_i)$:*

$$P(V)(\boldsymbol{t}) = \prod_{i=1}^{n} P(V_i)(t_i), \boldsymbol{t} = (t_1, \cdots, t_n) \in \mathbb{C}^n.$$

*The derivation Lie algebra $L(V_i)$ admit a $\mathbb{Z}$-grading under following convention $deg(\partial/\partial_x) = -deg(x)$ for all involved variables $x$. Similarly, the $\mathbb{Z}^n$-grading of moduli algebra $A(V_i)$ induces a $\mathbb{Z}^n$-grading on the derivation Lie algebra $L(V)$. Let $P_L(V_i)$ be a Poincaré polynomials of derivation Lie algebra $L(V_i)$ corresponding these gradings. The Poincaré polynomial of $L(V)$ is defined as*

$$P_L(V)(\boldsymbol{t}) = \sum_{i=1}^{n} \frac{P_L(V_i)(t_i)}{P(V_i)(t_i)} P(V)(\boldsymbol{t}).$$

It is interesting to note that Cartan grading induces a various $\mathbb{Z}$-gradings of $A(V)$ and $L(V)$ via linear functionals $\varphi : \mathbb{Z}^n \longrightarrow \mathbb{Z}$. In other words, we can define a $\mathbb{Z}$-grading on $A(V)$ and $L(V)$ as linear combination with integer weights $w_i$ of the $\mathbb{Z}$ grading for $V_i$. The Poincaré polynomials $P_L^{\varphi}(V)(t)$ of $L(V)$ with respect to the resulting grading will be just $P_L(V)(t_1^{w_1}, \cdots, t_n^{w_n})$. The Poincaré polynomials of moduli algebra with respect to quasi-homogeneous grading was computed in [17].

Recall that the Pham and $D_{k_1,k_2}$ series defined by polynomials $P_k = \sum x_j^{k_j+1}, k = (k_1, \cdots, k_n)$, and $D_{k_1,k_2} = x_1^{k_1}x_2 + x_2^{k_2}$, where $k_i$ are arbitrary natural numbers bigger than 1. A real polynomial $P = \sum a_i t^i$ is called unimodal if, for some $i$ the coefficients $a_k$ monotonously increase up to $k = i$ and monotonously decrease for $k > i$. If for each $i$, one has $a_i = a_{n-i}$, then real polynomial $P$ is called palindromic. A polynomial $P$ is called unimodular, if all its roots lie on unit circle. A polynomial $P$ which is palindromic and unimodal simultaneously is called unipalindromic.

Theorem 5.3 ([15]). *The Poincaré polynomials for Euler grading that arise from Lie algebra $L(V)$ of Pham singularity and simple singularity are unipalindromic.*

Since semisimple singularity is the direct sum of simple singularity. Jibladze and Novikov [20] determined a large class of semisimple singularities with the property that the roots of Poincaré polynomials of derivation Lie algebras lie on unit circle. To prove the unimodularity of Poincaré polynomials they use the quasi homogeneous grading. They also gave the correct version of theorem 4.5 from [15].

Theorem 5.4 ([20]). *The Poincaré polynomials for quasi-homogeneous grading that arise from Lie algebra of semisimple singularity $V$ of type $A \oplus D$ has a all roots on unit circle $\{|t| = 1\}$.*

Let $R = \mathbb{C}[x_1, \cdots, x_n]/(f)$ and where $f$ is weighted homogeneous polynomial defining an isolated singularity at the origin. It has been a natural and long-standing problem whether the $Der(R, R)$ is non-negative graded. Actually this problem has been motivated from both algebraic topology and singularities. Since it is well known that both algebras $R$ and $Der(R, R)$ are graded. The derivation of $R$ has no negatively graded component [40] and according to Wahl conjecture this fact is still true, if we let $R = \mathbb{C}[x_1, \cdots, x_n]/(f_1, \cdots, f_m)$ to be an isolated, normal and complete intersection singularity. In above higher codimensional case, $f_1, \cdots, f_m$ are weighted homogeneous polynomials with same weight type $(w_1, \cdots, w_n)$. Wahl conjecture for singular cones give a beautiful cohomological characterization of complex projective space ([42], [31]). In case of quasi-homogeneous normal isolated complete intersection singularity (ICIS) Wahl conjecture is defined as:

Wahl Conjecture ([18]). Any quasi-homogeneous ICIS with dimension $\geq 2$ has no negative weight derivations under the consideration of some positive grading.

Aleksandrov [1] proved the Wahl Conjecture in case of complete intersections.

Theorem 5.5 ([2]). *Let $(V, 0)$ be a positive-dimensional quasi-homogeneous ICIS which is defined by $f_1, f_2, \cdots, f_m \in \mathbb{C}[x_1, \cdots, x_n]$. Then*

$$A := \mathbb{C}[x_1, \cdots, x_n]/(f_1, f_2, \cdots, f_m)$$

*has no negative weight derivation except the following two cases: 1). $m = 1$ and $f_1$ has multiplicity 2; 2). $m \geq 2, n \geq 3m$, $\dim V \geq 4$ and $f_i$ has multiplicity 2 for every $i \in \{1, 2, \cdots, m\}$. The grading in first exceptional case is not unique and this grading always be chosen such that the singularity has no derivations of negative weight. In the second case, the grading is defined uniquely and for such a singularity there may be derivations of negative weight.*

Generalized Wahl Conjecture ([41]). Let $P = \mathbb{C}[x_1, \cdots, x_n]$ be the weighted polynomial ring of $n$ weighted variables $x_1, x_2, \cdots, x_n$ with positive integer weights $w_1 \geq w_2 \geq \cdots \geq w_n$, $(n \geq 2)$. Let $(V, 0)$ be a positive-dimensional variety which is defined by weighted homogeneous polynomials $f_1, f_2, \cdots, f_m \in P$. Suppose $(V, 0)$ is an isolated singularity. Then the graded ring $R = P/(f_1, f_2, \cdots, f_m)$ has no negative weight derivations if the (weighted) degrees of $f_i, 1 \leq i \leq m$, are large.

Yau gave a conjecture that if $f$ is weighted homogeneous polynomial then moduli algebra $A(V) = \mathbb{C}[x_1, \cdots, x_n]/(\partial f/\partial x_1, \cdots, \partial f/\partial x_n)$ has no negatively weighted derivation. By assuming this conjecture, Yau gave the characterization of of quasi-homogeneous hypersurface singularities by using the derivation Lie algebra $L(V)$. This Yau conjectured can be view as Artinian equivalent to Wahl conjecture. It was also proved that Yau conjecture holds in case of lower and higher dimension singularities. According to Yau and Zuo [59], Yau conjecture is still true under the consideration the lowest weight is greater than or equal to half of the highest weight.

Yau Conjecture ([59]). Let $(V, 0) = \{(x_1, \cdots, x_n) \in \mathbb{C}^n : f(x_1, \cdots, x_n) = 0\}$ is an isolated singularity and suppose that $f(x_1, \cdots, x_n)$ is a weighted homogeneous polynomial. Then the moduli algebra $A(V) = \mathbb{C}[x_1, \cdots, x_n]/(\partial f/\partial x_1, \cdots, \partial f/\partial x_n)$ has no non-negative weighted derivation.

It was shown in [43] a graded Lie algebra $L(V)$ has no negative weight in case of homogeneous polynomial $f$.

Theorem 5.6 ([43]). *Let $A = \oplus_{i=0}^{t} A_i$ be a commutative Artinian local algebra with $A_0 = \mathbb{C}$. Assume that for some $j$ greater than zero the maximal ideal of $A$ is generated by $A_j$. Then Lie algebras $L(A)$ has no negative weight derivation.*

The Yau conjecture studied in ([12], [10]) for $n \leq 4$.

Theorem 5.7 ([12]). *Let $(V, 0) = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : f(x_1, x_2, x_3) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, x_3)$ of type $(w_1, w_2, w_3; d)$. Suppose that $d \geq 2w_1 \geq 2w_2 \geq 2w_3$. Let $D$ be a derivation of the moduli algebra*

$$\mathbb{C}[x_1, x_2, x_3]/(\partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3).$$

*Then $D \equiv 0$ if $D$ is negatively weighted.*

THEOREM 5.8 ([10]). *Let $(V,0) = \{(x_1,x_2,x_3,x_4) \in \mathbb{C}^4 : f(x_1,x_2,x_3,x_4) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1,x_2,x_3,x_4)$ of type $(w_1,w_2,w_3,w_4;d)$. Suppose that $d \geq 2w_1 \geq 2w_2 \geq 2w_3 \geq 2w_4$. If $D$ is negatively weighted then $D \equiv 0$ and where $D$ is a derivation of moduli algebra*

$$\mathbb{C}[x_1,x_2,x_3,x_4]/(\partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3, \partial f/\partial x_4).$$

In following theorem Yau and Zuo prove the above conjecture for high dimensional singularities.

THEOREM 5.9 ([59]). *Let $(V,0) = \{(x_1,\cdots,x_n) \in \mathbb{C}^n : f(x_1,x_2,\cdots,x_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1,x_2,\cdots,x_n)$ of type $(w_1,w_2,\cdots,w_n;d)$. Suppose that $d \geq 2w_1 \geq 2w_2 \geq, \cdots, \geq 2w_n$ without loss of generality. Let $D$ be a derivation of the moduli algebra*

$$\mathbb{C}[x_1,\cdots,x_n]/(\partial f/\partial x_1,\cdots,\partial f/\partial x_n).$$

*If $w_n \geq \frac{w_1}{2}$, then $Der^{<0}(A(V)) = 0$.*

In [9] the Wahl Conjecture and its generalization for $R = \mathbb{C}[x_1,x_2,\cdots,x_n]/(f_1,f_2,\cdots,f_m)$ has a positive answer without the condition of complete and intersection singularity. In fact, this is true when degree of $f_i, 1 \leq i \leq m$ are bounded below by constant $C = (m-1+w_1)(w_1w_2)^{n-1}$. Moreover the bound $C = (m-1+w_1)(w_1w_2)^{n-1}$ is improved when any two weights are co-prime. In case of low degree of $f_i$, there are counter examples for Wahl Conjecture and its generalization. So the main result of [9], is more or less optimal in the sense that the only lower bound constant can be improved.

It is well-known that in case of positive-dimensional isolated complete intersection singularity the derivation algebra is generated by Euler derivation and trivial derivations. Thus the generators of derivations are completely known. But there is no known description of all holomorphic vector fields in case of non complete intersection singularity. Consequently, the generalized Wahl Conjecture is more difficult than Wahl conjecture for isolated complete intersection singularity.

THEOREM 5.10 ([9]). *Let $R = \mathbb{C}[x_1,x_2,\cdots,x_n]/(f_1,f_2,\cdots,f_m)$, where $\mathbb{C}[x_1,x_2,\cdots,x_n]$ is weighted polynomial ring with positive integer weight $w_1 \geq w_2 \geq \cdots \geq w_n$, $(n \geq 2)$. Suppose $f_1,f_2,\cdots,f_m$ are weighted homogeneous polynomials defining a positive-dimensional isolated singularity at the origin with degrees greater than $(m-1+w_1)(w_1w_2)^{n-1}$. Then $R$ has no non-zero negative weight derivations.*

It follows from counter examples given in [9], the non-existence of negative weight derivation for positive dimensional singularities can not be expected for large degree. But constant $C = (m-1+w_1)(w_1w_2)^{n-1}$ may not be sharp. The following theorem tells us that this bound can be improved under the condition that any two of the weights $w_1,w_2,\cdots,w_n$ are coprime.

THEOREM 5.11 ([9]). *Let $P = \mathbb{C}[x_1,x_2,\cdots,x_n]$ be the weighted polynomial ring of $n$ weighted variables $x_1,x_2,\cdots,x_n$ with positive integer weights $w_1 \geq w_2 \geq \cdots \geq w_n$ $(n \geq 2)$ and $f_1,f_2,\cdots,f_m$ be $m$ weighted homogeneous polynomials of degrees greater than $(m-1+w_1)w_1w_2$. Suppose that any two of the original weights*

*$w_1, w_2, \cdots, w_n$ are co-prime and $f_1, f_2, \cdots, f_m$ define a positive-dimensional isolated singularity at the origin. Then there are no non-zero negative weight derivations on $R = P/(f_1, f_2, \cdots, f_m)$.*

Since dimension of derivation Lie algebra $L(V)$ is called Yau number and it is denoted as $\lambda(V)$. The Yau number is an analytic invariant. Yau and Zuo [58] proposed the following sharp upper estimate conjecture:

Yau-Zuo Conjecture ([58]). Let $(V, 0) = \{(x_1, x_2, \cdots, x_n) \in \mathbb{C}^n : f(x_1, x_2, \cdots, x_n) = 0\}$ $(n \geq 2)$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, \cdots, x_n)$ of weight type $(w_1, w_2, \cdots, w_n; 1)$. Then the Yau number

$$\lambda(V) \leq n\mu - \sum_{i}^{n} (\frac{1}{w_1} - 1)(\frac{1}{w_2} - 1) \cdots \widehat{(\frac{1}{w_i} - 1)} \cdots (\frac{1}{w_n} - 1),$$

where $\widehat{(\frac{1}{w_i} - 1)}$ means that $\frac{1}{w_i} - 1$ is omitted and $\mu$ is the Milnor number.

Finally, we get

$$\lambda(V) \leq n \prod_{i=1}^{n} (\frac{1}{w_i} - 1) - \sum_{i}^{n} (\frac{1}{w_1} - 1)(\frac{1}{w_2} - 1) \cdots \widehat{(\frac{1}{w_i} - 1)} \cdots (\frac{1}{w_n} - 1).$$

In [58] this conjecture was proved for a binomial isolated hypersurface singularities.

Proposition 5.1 ([58]). *Let $f$ be a weighted homogeneous fewnomial isolated singularity with mult$(f) \geq 3$. Then $f$ analytically equivalent to a linear combination of the following three series:*
*Type A.* $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}$, $n \geq 1$,
*Type B.* $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n}$, $n \geq 2$,
*Type C.* $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1$, $n \geq 2$.

Proposition 5.1 has an immediate corollary.

Corollary 5.1 ([58]). *Each binomial isolated singularity is analytically equivalent to one from the three series: A) $x_1^{a_1} + x_2^{a_2}$, B) $x_1^{a_1} x_2 + x_2^{a_2}$, C) $x_1^{a_1} x_2 + x_2^{a_2} x_1$.*

Theorem 5.12 ([58]). *Let $(V, 0)$ be a binomial singularity defined by the weighted homogeneous polynomial $f(x_1, x_2)$ with weight type $(w_1, w_2; 1)$. Then $\lambda(V) \leq 2(\frac{1}{w_1} - 1)(\frac{1}{w_2} - 1) - (\frac{1}{w_1}) - (\frac{1}{w_2}) + 2$.*

Let $f(x_1, x_2, \cdots, x_n) = 0$ of weight type $(w_1, w_2, \cdots, w_n; 1)$ and $g(y_1, y_2, \cdots, y_m) = 0$ of weight type $(w_{n+1}, w_{n+2}, \cdots, w_{n+m}; 1)$ be two weighted homogeneous polynomials which define two isolated hypersurface singularities $(V_f, 0) \subset (\mathbb{C}^n, 0)$ and $(V_g, 0) \subset (\mathbb{C}^m, 0)$. It is clear that $f(x_1, \cdots, x_n) + g(y_1, \cdots, y_m) = 0$ (which is called addition of Thom-Sebastiani) has weight type $(w_1, w_2, \cdots, w_{n+m}; 1)$ and define an weighted homogeneous isolated singularity $(V_{f+g}, 0) \subset (\mathbb{C}^{m+n}, 0)$.

Theorem 5.13 ([58]). *Let $(V_f, 0) \subset (\mathbb{C}^n, 0)$ and $(V_g, 0) \subset (\mathbb{C}^m, 0)$ be defined by weighted homogeneous polynomials $f(x_1, x_2, \cdots, x_n) = 0$ of weight type $(w_1, w_2, \cdots, w_n; 1)$ and $g(y_1, y_2, \cdots, y_m) = 0$ of weight type*

*$(w_{n+1}, w_{n+2}, \cdots, w_{n+m}; 1)$ respectively. Let $\mu(V_f)$, $\mu(V_g)$, $A(V_f)$ and $A(V_g)$ be the Milnor numbers and moduli algebras of $(V_f, 0)$ and $(V_g, 0)$ respectively. Then*

$$\lambda(V_{f+g}) = \mu(V_f)\lambda(V_g) + \mu(V_g)\lambda(V_f). \tag{5.1}$$

*Furthermore if both $f$ and $g$ satisfy the Yau-Zuo conjecture, then $f+g$ also satisfies the Yau-Zuo conjecture.*

THEOREM 5.14 ([58]). *Let $f \in \mathbb{C}\{x_1, \cdots, x_n\}$ be a weighted homogeneous isolated singularity which is a Thom-Sebastiani summation of the following three types with mult$(f) \geq 3$:*
1) $x_1^{a_1} + x_2^{a_2} + \cdots + x_{m-1}^{a_{m-1}} + x_m^{a_m}$, $m \geq 1$,
2) $x_1^{a_1}x_2 + x_2^{a_2}$,
3) $x_1^{a_1}x_2 + x_2^{a_2}x_1$.
*Then $f$ satisfies the Yau-Zuo conjecture.*

Wolfgang and Atsushi [16] give the following classification of weighted homogeneous fewnomial singularities in case of three variables.

PROPOSITION 5.2 ([16]). *Let $f(x_1, x_2, x_3)$ be a weighted homogeneous fewnomial isolated singularity with mult$(f) \geq 3$. Then $f$ is analytically equivalent to following five types:*
*Type 1.* $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$,
*Type 2.* $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}$,
*Type 3.* $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_1$,
*Type 4.* $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}x_2$,
*Type 5.* $x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}$.

According to [4], classification of weighted homogeneous function of three variable consists of seven types. The first five types are fewnomial singularities and remaining two types are non-fewnomial singularities. In [19] Yau, Zuo and present author proved the Yau-Zuo conjecture in case of fewnomial surface singularities. But in case of non-fewnomial surface singularities the verification of Yau-Zuo conjecture is still open. The Yau-Zuo Conjecture is also open for higher-dimensional weighted homogeneous singularities. Zuo proposed the following open problem.

QUESTION 5.2. *Whether we can use other invariants of singularities to construct a sharp upper or lower bound of the Yau number for isolated hypersurface singularity.*

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