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GAMMA STRUCTURES ON SURFACES*

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Abstract. Presented here is a version of my talk at the Tsinghua Sanya International Mathematics Conference on Singularities in memory of John Mather. This article is partly expository. I will briefly recount the rise of the modern theory of foliations, describe John Mather's contributions and then allow the discussion to lead to a report of work, old and new, on Real Analytic Gamma Structures.

Key words. Classifying spaces, homology, real analytic.

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1. Introduction. Presented here is a version of my talk at the Tsinghua Sanya International Mathematics Conference on Singularities in memory of John Mather. This article is partly expository. I will briefly recount the rise of the modern theory of foliations, describe John Mather's contributions, and then allow the discussion to lead to a report of work, old and new, on Real Analytic Γ-Structures. Although the greater part of John Mather's research was focussed on Singularity Theory, in the early 1970's, while at Harvard, during a time of intense activity in Foliations, he worked on the topology of Haefliger's Classifying Spaces. I met John Mather in 1971, while I was a graduate student at Dartmouth College attending Raoul Bott's Foliations seminar. Bott introduced me to Mather who subsequently became my thesis advisor. I am indebted to Mather for the generosity he showed by taking me on as his graduate student, and for his mentorship.

2. The Reeb Foliation. The modern theory of foliations began when George Reeb produced a global example of a smooth co-dimension 1 foliation on the 3-sphere, [13]. He conjectured that no co-dimension 1 foliation on a simply connected closed manifold could be real analytic. Several years later Andre Haefliger proved it, [2], and in the process initiated the homotopy theory of foliations. The constructions introduced singularities in the definition of foliations and manifold structures. He eventually expanded the notion of foliations to include more general structures, Γ-Structures, which could be classified like topological groups.

Before elaborating on Haefliger's classification I'll describe the Reeb Foliation of the 3-sphere. The construction begins by foliating a vertical strip by parabola-like curves, then spinning the foliation around the central axis to obtain a foliation of the infinite cylinder. Identifying vertically by the action of the integers gives a co-dimension 1 foliation of the solid torus, with the torus itself as one of the leaves, as shown in Figure 1.

The Clifford Sphere is the 3-sphere described as a union of two solid tori identified along a torus which forms a common boundary. In Figure 2 the blue torus is the common boundary. The viewer of the picture is standing inside the outside torus. The z axis, identified at infinity, is the core of the outer torus. Foliating each torus as before produces the Reeb Foliation. It can be constructed smoothly and has co-dimension 1.

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FIG. 1. *Foliated Torus*

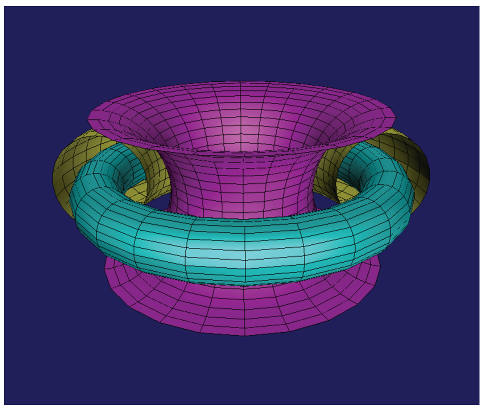

FIG. 2. *Clifford Sphere*

What makes this foliation not real analytic? Referring to the annulus in Figure 3, inside the inner circle the foliation produces a center. There is a transversal to this central circle across which the leaves do not vary analytically, for the holonomy, obtained by a rotating the transversal arc around the central circle in a neighborhood of the central circle is the identity on one side and not on the other. Such a transition cannot be analytic. This behavior is typical. There is always a closed transversal to the leaves of a smooth codimension 1 foliation on a compact manifold. In a neighborhood of the transversal Poincaré-Bendixson Theory implies that the structure must look like a spiral toward a closed orbit. Inside the closed orbit the foliation produces a singularity obtained by a local projection to the reals, which is a center. This contradicts analyticity.

3. Haefliger's Classifying Spaces. What Haefliger uncovered in his proof was a homotopy theory of foliations. The appropriate generalization of foliation is Γ-structure. A Γ-structure can be classified in the same way as a G-structure, where G is a topological group. The discovery of the classifying space $B\Gamma$ led naturally to attempts to determine its algebraic invariants. For example, as we will observe, the obstruction to constructing a real analytic foliation of the 3-sphere is an element of the fundamental group of the classifying space associated to the pseudogroup of orientation preserving local, real analytic homeomorphisms of the reals.

The process of classification proceeds as follows, [3].

Classically a G-bundle on a space X is represented by a 1-coycle with values in

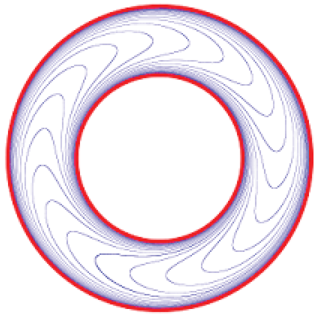

FIG. 3. *Transverse Structure*

topological group G. A Γ-structure on a space X is represented by a 1-coycle with values in a topological groupoid Γ. Relevant to foliations is the case where Γ is the topological groupoid of germs of elements of a pseudogroup of local homeomorphisms of a space X. More specifically a manifold or foliation is represented by a 1-coycle on a manifold X with values in the topological groupoid Γ_n^r of germs of elements, with the sheaf topology, of the pseudogroup of local C^r homeomorphisms of n-dimensional Euclidean Space. The local projections should be submersions. The dimension n is the codimension of the foliation. When n is the dimension of X, then a Γ-structure provides X with a manifold structure.

The standard classifying space constructions work in this context to give the same result as for topological groups.

$$\Gamma(X) = [X, B\Gamma].$$

The left hand side is the set of homotopy classes of Γ-structures on X; the right hand side is the set of homotopy classes of maps from X to the classifying space of Γ.

4. Mather's Legacy. Mather was essentially the first to produce a general result describing the homology of the classifying spaces for foliations in a way that allowed for some computations. His main contributions are to be found in the following three works.

a) Integrability in codimension 1, [10],
b) The vanishing of the homology of certain groups of homeomorphisms, [11],
c) Commutators of diffeomorphisms, [12].

4.1. Remarks. In the first paper Mather proved that there is a homology equivalence from the discrete group of C^r homeomorphisms of the real line with compact support, G^r, $r = 0, 1, 2, ..., \infty$, to the loop space of $B\Gamma_1^r$. Every space has functorially the homology of a discrete group, [9], so the power of his result is the explicit nature of the equivalence together with the fact that it provides a context in which some homology calculations can be made.

In the second paper Mather showed that G^0 is acyclic which implies that $B\Gamma_1^0$ is contractible. Mather's proof has been formalized and generalized, and applied extensively. We refer the reader to J. Berrick's works, in particular, [1], which has an extensive bibliography.

In c) Mather proved that the first homology group of G^r is trivial for $r \geq 3$. Otherwise almost nothing is known about its homology in terms of explicit calculations.

4.2. Calculations, a conjecture, and some questions. That $B\Gamma_1^r$ is simply connected for $0 \leq r \leq \infty$ is an elementary result following from the fact that Γ_1^r is connected as a topological space. However the latter condition is not necessary, for the classifying space in the holomorphic co-dimension 1 case is also simply connected, [6].

The main result of c) above implies, in particular, that $H_2(B\Gamma_1^\infty) = 0$.

William Thurston proved that $H^3(B\Gamma_1^r, Z) \to \mathbb{R}$ is onto for both $r = \infty$, and $r = \omega =$ real analytic, [14]. On the other hand $B\Gamma_1^\omega$ is an Eilenberg Maclane Space of type $(G, 1)$ with $H_1(G, Z) = 0$, [3]. These results led to speculation that $B\Gamma_1^\infty$ and $B\Gamma_1^\omega$ were homology equivalent.

Alberto Verjovsky and I have shown that $H_2(B\Gamma_1^\omega)$ is uncountable, [4]. The remainder of this paper will be devoted to a discussion of this result.

Deep and fundamental questions arising from Reeb's example and Haefliger's counterexample remain.
What is the relationship between smooth and real analytic foliations?
What is the relationship between the smooth and real analytic classifying spaces?
What is $H_2(B\Gamma_1^\omega)$?

5. The Real Analytic Classifying Space.

5.1. The Basic Structure of $G = \pi_1(B\Gamma_1^\omega)$. The group $H_2(B\Gamma_1^r)$ classifies cobordism classes of C^r, co-dimension 1, Γ-structures on closed surfaces, [15]. The theory changes drastically in the transition from smooth to real analytic. What was a simply connected classifying space becomes a $K(G, 1)$, and geometry and topology turns into combinatorial group theory.

All the information about the classifying space for codimension 1 real analytic Γ-structures is incorporated in the discrete group G, which has the following natural presentation coming from the simplicial construction of $B\Gamma_1^\omega$, [5].

$G = G(S)$ is the free group with generating set S = maximally extended, local, orientation preserving, real analytic diffeomorphisms of the real line with the relations $f \cdot g = f \circ g$ whenever the composition is defined.

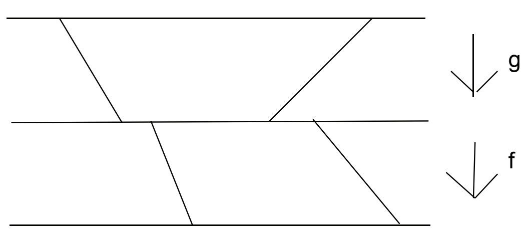

FIG. 4. *Composition*

Composition in S has the following dynamics. Consider the composite $h = f \circ g$ of two elements of K. Suppose the domains of g and f are respectively, (a, b) and (c, d). Then the domain of h is g^{-1}(domain $f \cap$ range g), unless $g(b) = d$ in which case the right end point of the domain of h may be to the right of b, or $g(a) = c$ in which case the left end point of the domain of h may be to the left of a.

The structure of S is that of an *associative partial group,* and $G(S)$ is its universal group. The definition of partial group, and a description of several contexts in which partial goups arise can be found in [8].

5.2. Poincaré-Bendixson = Word Problem in $G(S)$**.** A fundamental observation is that every element of S corresponds to a real analytic manifold structure on the circle. The essential step in Haefliger's proof of the non-existence of real analytic co-dimension 1 foliations on the 3-sphere is that no real analytic manifold structure on the circle and be extended to a Γ_1^ω-structure on the disk. This translates in the language of G to showing that no element of S other than the identity is trivial in G.

So the proof of his result can be reduced to solving the identity problem in G; a combinatorial problem. That step effectively replaces the appliction of Poincaré-Bendixson in Haefliger's proof.

A combinatorial analysis of the partial group S and its universal group $G(S)$ leads to the following results, which, as remarked, imply the non-existence of co-dimension 1, real analytic foliations on simply connected compact manifolds, [5].

Theorem.

a) $G = G(S)$ is uncountable and every generator represents a distinct element of infinite order.
b) All elements of S other than the identity are conjugate in G.
c) $H_1(G) = 0$.

6. $H_2(G)$ **is uncountable.** In this final section we present the ideas behind the construction of an uncountable number of homologically distinct Γ_1^ω-structures on surfaces. The proof is in [4].

First we give an algebraic context for the result.

6.1. Ideal extensions.

Definition. An exact sequence of non-trivial groups $1 \to A \to E \xrightarrow{e} G \to 1$ is an *ideal extension* of G if $e_* : H_1(E, \mathbb{Z}) \to H_1(G, \mathbb{Z})$ is an isomorphism.

The extension gives rise to a homomorphism $H_2(G, \mathbb{Z}) \xrightarrow{\delta} H_1(A)^E \to 0.$ where δ is the connecting homomorphism of the five term exact sequence associated to the short exact sequence, and $H_1(A)^E$ is the quotient of $H_1(A)$ by the conjugacy action of E.

Ideal Extensions are designed to detect elements in the second homology of a group which are represented by closed surfaces of genus greater than 1. These appear in $H_1(A)^E$.

An ideal extension exists for

a) $G =$ fundamental group of a surface of genus greater than 1,
b) $G =$ orientation preserving homeomorphisms of the circle, and
c) $G = G(S)$, the fundamental group for real analytic structures.

The invariant in a) is the fundamental class. In b) it's the Euler class for flat circle bundles, [7]. We now introduce the invariant in c).

6.2. An ideal extension of $G(S)$**.** Let Π_0 denote the subgroup of G consisting of those elements $f \in S$ whose domain contains an interval of the form $(a, 0)$ and which map $(a, 0)$ to $(f(a), 0)$. Let Π_1 denote the subgroup of π consisting of those elements $f \in K$ whose domain contains an interval of the form $(1, b)$ and which map $(1, b)$ to $(1, f(b))$.

Consider the free product $\Pi_0 * \Pi_1$, and define the *evaluation* $\epsilon : \Pi_0 * \Pi_1 \to G$ to be the homomorphism induced by the inclusions $\Pi_0 \to G$, and $\Pi_1 \to G$ If $f \in \Pi_a \subset G$, where $a \in \{0, 1\}$, the corresponding generator of $\Pi_0 * \Pi_1$ will be denoted by $(f)_a$, and "a" will be called its *distinguished point.* An element of $A = ker(\epsilon)$ is a *cycle.*

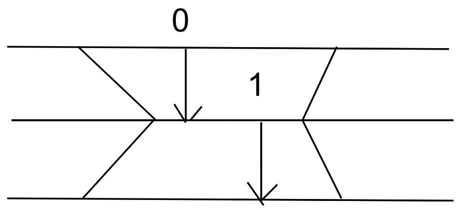

FIG. 5. *A cycle of length 2*

A general *cycle* has the form

$$F = (f_{n+1})_{a_{n+1}} * (f_n)_{a_n} * \cdots * (f_1)_{a_1}.$$

Each f_i has a fixed point at a_i, and f_{n+1} is given by $f_{n+1}^{-1} = f_n \circ \cdots \circ f_1$. If F is reduced the a_i alternate between 0 and 1.

The shortest non-trivial cycles have length 2 and take the form $(f^{-1})_{a_2} * (f)_{a_1}$, (Figure 5). Note, if $(f^{-1})_{a_2} * (f)_{a_1}$ is such a cycle then it is analytic, at both 0 and 1, for it must extend from the left of 0 to the right of 1. This leads to the invariant constructed in [4].

In [4] we show that the following extension is ideal, which leads to the construction of an uncountable number of elements in $H_1(A)^E$.

$$1 \to A \to \Pi_0 * \Pi_1 \xrightarrow{\epsilon} \pi \to 1.$$

THEOREM. The group $H_2(G)$ has a subgroup which maps onto $\mathbb{R}$. Moreover there are an uncountable number of non-vanishing classes represented by cycles of length 2.

6.3. A fundamental fact, and a question. The homotopy theory of real analytic Γ structures in co-dimension 1 reveals that there is only one real analytic foliation locally transverse to a circle up to free homotopy, but there are uncountably many up to base point preserving homotopy.

Can the global differences between real analytic and smooth foliations be explained by this fact, and if not, what else is involved, algebraically and geometrically?

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