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ON THE NEWTON POLYHEDRONS WITH ONE INNER LATTICE POINT*

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In memory of John Mather

Abstract. Geometric genus is an important invariant in the classification theory for isolated singularities. In this paper we give a complete classification of three-dimensional isolated weighted homogeneous singularities with geometric genus one. This is one of important classes of minimally elliptic singularities. We reduce it to nineteen classes Newton polyhedrons with one inner lattice point.

Key words. Geometric genus, isolated singularity.

Mathematics Subject Classification. 32S05.

1. Introduction. A list of simple zero-dimensional complete intersections has been obtained by M.Giusti in [1]. Zero-dimensional gradient singularities were classified by Alexsandrov and Zuo [2]. The topological classification of simplest Gorenstein non-complete intersection surface singularities was obtain in [3]. Artin [4] first introduced the definition of rational surface singularity. He classified all rational surface singularities embeddable in $\mathbb{C}^3$. These are precisely Du Val singularities in $\mathbb{C}^3$ defined by one of the following polynomial equations:

$$\begin{aligned}
&A_n : x^2+y^2+z^{n+1}, \text{ for } n \geq 1,\\
&D_n : x^2+y^2z+z^{n-1}, \text{ for } n \geq 4,\\
&E_6 : x^2+y^3+z^4,\\
&E_7 : x^2+y^3+yz^3,\\
&E_8 : x^2+y^3+z^5.
\end{aligned}$$

It is well-known that any canonical singularity, i.e., singularity that occurs in a canonical model of a surface of general type, is analytically isomorphic to one of the rational double points listed above.

In [5] Burns defined higher dimensional rational singularity as follows. Let (V,p) be a n-dimensional isolated singularity. Let $\pi : \tilde{V} \to V$ be a resolution of singularity. And p is said to be a rational singularity if $R^i\pi_*(\mathscr{O}_{\tilde{V}}) = 0$ for $1 \leq i \leq n-1$. In a beautiful paper [6], Yau shows for Gorenstein singularity that it is sufficient to require $R^{n-1}\pi_*(\mathscr{O}_{\tilde{V}}) = 0$. Furthermore, it has been proved that $R^{n-1}\pi_*(\mathscr{O}_{\tilde{V}}) \cong H^0(V-\{p\},\Omega^n)/L^2(V-\{p\},\Omega^n)$, where Ω^n is the sheaf of germs of holomorphic n-forms and $L^2(V-\{p\},\Omega^n)$ is the space of holomorphic n-forms on $V-\{p\}$, which are L^2-integrable. The geometric genus p_g of the singularity (V,p) is defined to be

$$p_g := \dim R^{n-1}\pi_*(\mathscr{O}_{\tilde{V}})_p = \dim H^0(V-\{p\},\Omega^n)/L^2(V-\{p\},\Omega^n).$$

It turns out that p_g is an important invariant of (V,p).

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We shall call an isolated hypersurface singularity defined by a weigthed homogeneous polynomial an *isolated weighted homogeneous singularity.* Isolated weighted homogeneous surface singularities have been studied by many authors, see Arnold [7, 8], Orlik and Wagreich [9, 10], Yau and Zuo [11]-[19], Yoshinaga and Suzuki [21], Yoshinaga and Watanabe [23], and references therein. In [21], isolated weighted homogeneous surface singularities (non-degenerate weighted homogeneous polynomials) are classified under the conditions of their inner modalities ≤ 4. Yoshinaga and Watanabe showed that the geometric genus p_g of an isolated weighted homogeneous singularity is completely determined only by its weights. The isolated weighted homogeneous singularities with $p_g = 0$ are the rational double points. Those ones with $p_g = 1$ are classified as a part of the minimally elliptic singularities by Laufer [22] for surface singularities, and also as a part of the isolated weighted homogeneous singularities with inner modalities 1, 2, 3 or 4, by Yoshinaga and Suzuki [21]. Those isolated weighted homogeneous surface singularities with $p_g = 2, 3$ were classified by Yoshinaga and Ohyanagi in [20, 24]. In [25] Yau and Yu classified three-dimensional rational isolated weighted homogeneous singularities. These singularities have important application in four dimensional $N = 2$ superconformal field theories (SCFT) [26]. In fact, these singularities give us a large number of new four dimensional $N = 2$ SCFTs. The next very important step in classifying singularities seems to be the case of three dimensional minimally elliptic (i.e. isolated Gorenstein singularity with $p_g = 1$) weighted homogeneous isolated complete intersection singularities. However, the calculation in [25] is very complicated and is hard to be generalized. In this paper, we develop an new approach to the classification problem for minimally elliptic weighted homogeneous isolated hypersurface singularities, which is mainly based on simple properties of invariant and deformation theory for such singularities. We extend the results in [25] to $p_g = 1$. We give a complete list of three-dimensional isolated weighted homogeneous hypersurface singularities with $p_g = 1$. We state the main theorem here:

Main Theorem. *Let $(V, 0)$ be a three-dimensional isolated weighted homogeneous hypersurface singularity with* $p_g = 1$. *Then $(V, 0)$ is defined by a weighted homogeneous polynomial of one of the nineteen cases of Table 2 such that the corresponding linear form α satisfies $\alpha(x, y, z, w) = 1$.*

If (i)-(a_0, a_1, a_2, a_3) has $p_g = 1$ then (a_0, a_1, a_2, a_3) is one of Table (i) where $i \in \{(I), \cdots, (XIX)\}$.

We only include some short tables at the end of this paper due to the page limit. We refer the interested readers to [41] for the complete list of tables.

In section 2, we shall give a classification of weighted homogeneous polynomials of 4 variables with isolated singularity at the origin. This list was obtained first by Kouchnirenko [27] and Orlik-Randell [28, 29] independently. In section 3, we classify all three-dimensional isolated weighted homogeneous hypersurface singularities with $p_g = 1$.

2. Classification of weighted homogeneous polynomials in four variables with isolated singularities at the origin. In this section, we recall some definitions and theorems for weighted homogeneous polynomials.

Definition 2.1. *suppose that $(w_0, w_1, \cdots, w_n)$ are given positive numbers. A polynomial $f(z_0, z_1, \cdots, z_n)$ is said to be weighted homogeneous of type $(w_0, w_1, \cdots, w_n)$ if it can be expressed as a linear combination of monomials*

$$z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n}$$

for which $\frac{i_o}{w_o} + \frac{i_1}{w_1} + \cdots + \frac{i_n}{w_n} = 1$.

DEFINITION 2.2. *A weighted homogeneous polynomial f is said to be non-degenerate if $(0, 0, \cdots, 0)$ is an isolated critical point, i.e., if its hypersurface has an isolated singularity at $(0, 0, \cdots, 0)$.*

In this paper, we consider 3-dimensional hypersurface isolated singularities, consequently with 4 variables z_0, z_1, z_2 and z_3.

Orlik and Wagreich [9] and Arnold [8] showed that if $h(z_0, z_1, z_2)$ is a weighted homogeneous polynomial in $\mathbb{C}^3$ and $V = \{z \in \mathbb{C}^3 : h(z) = 0\}$ has an isolated singularity at the origin, then V can be deformed into one of the seven classes of weighted homogeneous singularities below, while keeping the differential structure of the link $K_V := S^5 \cap V$ constant. Let $(w_0, w_1, w_2) = (wt(z_0), wt(z_1), wt(z_2))$ be the weight type and μ be the Milnor number. The classification of 3-dimensional hypersurface isolated singularities, keeping the differential structure of the link $K_V = S^5 \cap V$ constant has been list in Table 1.

TABLE 1
The classification of 3-dimensional hypersurface isolated singularities, keeping the differential structure of the link $K_V = S^5 \cap V$ constant.

Class	$h(z_0, z_1, z_2, z_3) = 0$	(w_1, w_2, w_3)	μ
I	$\{z_0^{a_0} + z_1^{a_1} + z_2^{a_2} = 0\}$	$(w_0, w_1, w_2) = (a_0, a_1, a_2)$	$\mu = (a_0 - 1)(a_1 - 1)(a_2 - 1)$
II	$\{z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2} = 0\}$	$(w_0, w_1, w_2) = (a_0, a_1, \frac{a_1 a_2}{a_1 - 1})$	$\mu = (a_0 - 1)(a_1 a_2 - a_1 + 1)$
III	$\{z_0^{a_0} + z_1^{a_1} z_2 + z_1 z_2^{a_2} = 0\}$	$(w_0, w_1, w_2) = (a_0, \frac{a_1 a_2 - 1}{a_2 - 1}, \frac{a_1 a_2 - 1}{a_1 - 1})$	$\mu = (a_0 - 1) a_1 a_2$
IV	$\{z_0^{a_0} + z_1^{a_1} z_2 + z_0 z_2^{a_2} = 0\}$	$(w_0, w_1, w_2) = (a_0, \frac{a_0 a_1 a_2}{a_0 a_2 - a_0 + 1}, \frac{a_0 a_2}{a_0 - 1})$	$\mu = a_0 a_2 (a_1 - 1) + a_0 - 1$
V	$\{z_0^{a_0} z_1 + z_1^{a_1} z_2 + z_0 z_2^{a_2} = 0\}$	$(w_0, w_1, w_2) = (\frac{a_0 a_1 a_2 + 1}{a_1 a_2 - a_2 + 1}, \frac{a_0 a_1 a_2 + 1}{a_0 a_2 - a_0 + 1}, \frac{a_0 a_1 a_2 + 1}{a_0 a_1 - a_1 + 1})$	$\mu = a_0 a_1 a_2$
VI	$\{z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} = 0\}$, $(a_0 - 1)(a_1 b_2 + a_2 b_1) = a_0 a_1 a_2$	$(w_0, w_1, w_2) = (a_0, \frac{a_0 a_1}{a_0 - 1}, \frac{a_0 a_2}{a_0 - 1})$	$\mu = \frac{(a_0 a_1 - a_0 + 1)(a_0 a_2 - a_0 + 1)}{a_0 - 1}$
VII	$\{z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} = 0\}$, $(a_0 - 1)(a_1 b_2 + a_2 b_1) = a_2 (a_0 a_1 - 1)$	$(w_0, w_1, w_2) = (\frac{a_0 a_1 - 1}{a_1 - 1}, \frac{a_0 a_1 - 1}{a_0 - 1}, \frac{a_2 (a_0 a_1 - 1)}{a_1 (a_0 - 1)})$	$\mu = \frac{a_0 (a_0 a_1 a_2 - a_0 a_1 + a_1 - a_2)}{a_0 - 1}$

Recall that two isolated hypersurface singularities $(V, 0), (W, 0)$ in $\mathbb{C}^{n+1}$ are said to have the same topological type if $(\mathbb{C}^{n+1}, V, 0)$ is homeomorphic to $(\mathbb{C}^{n+1}, W, 0)$ [30].

In [31], the authors proved that the above deformation is actually a deformation that preserves the weights and embedded topological type. Therefore, any weighted homogeneous singularity should have the same topological type of one of the seven classes in Table 1.

If $h(z_0, z_1, z_2, z_3)$ is a weighted homogeneous polynomial in $\mathbb{C}^4$ and $V = \{z \in \mathbb{C}^4 : h(z) = 0\}$ has an isolated singularity at the origin, then Kouchnirenko [27] and Orlik and Randell [28]observed that V can be deformed into one of the following nineteen classes of weighted homogeneous singularities below, while keeping the differential structure of the link $K_V := S^7 \cap V$ constant. The classification are in Table 2. The meaning of the linear forms $\alpha(x, y, z, w)$ in Table 2 will be clarified in the proof of Theorem 3.2 later.

THEOREM 2.1 ([25]). *Suppose $h(z_0, z_1, z_2, z_3)$ is a polynomial and*

$$V_k = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 : h(z_0, z_1, z_2, z_3) = 0\}$$

has an isolated singularity at 0. Then $h(z_0, z_1, z_2, z_3) = f(z_0, z_1, z_2, z_3) + g(z_0, z_1, z_2, z_3)$, where f is one of the nineteen classes (see Table 2) with only an *isolated singularity at* 0, *and* f *and* g *have no monomial in common. If* h *is weighted homogeneous of type* (w_0, w_1, w_2, w_3), *then so are* f *and* g. *Let* $V_f = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 : f(z_0, z_1, z_2, z_3) = 0\}$ *and let*

$$K_f = V_f \cap S^7, \quad K_h = V_h \cap S^7.$$

Then K_f *is equivariantly diffeomorphic to* K_h.

Proof. If none of the monomials in $\{z_0^{a_0}, z_0^{a_0} z_1, z_0^{a_0} z_2, z_0^{a_0} z_3\}$ appears in $h(z_0, z_1, z_2, z_3)$, then $\frac{\partial h}{\partial z_j}(z_0, 0, 0, 0) = 0$, $0 \le j \le 3$. This contradicts the fact that h has an isolated singularity at 0. Therefore, one of the monomials in $\{z_0^{a_0}, z_0^{a_0} z_1, z_0^{a_0} z_2, z_0^{a_0} z_3\}$ does appear in h. Similarly, one of the monomials in each of the following sets appears in h : $\{z_0 z_1^{a_1}, z_1^{a_1}, z_1^{a_1} z_2, z_1^{a_1} z_3\}$, $\{z_0 z_2^{a_2}, z_1 z_2^{a_2}, z_2^{a_2}, z_2^{a_2} z_3\}$, $\{z_0 z_3^{a_3}, z_1 z_3^{a_3}, z_2 z_3^{a_3}, z_3^{a_3}\}$. Taking a monomial from each of the 4 sets above, we get 256 polynomials. One can check that these 256 polynomials are equivalent to one of the nineteen classes above up to permutation of coordinates. Notice that in Type VIII, for example, the monomial $z_2^p z_3^q$ is added to make sure that f has an isolated singularity at 0. Obviously if h is weighted homogeneous of type (w_0, w_1, w_2, w_3), then so are f and g.

The proof of Theorem 3.1.4 in [9] shows that K_f is equivariantly diffeomorphic to K_h. □

TABLE 2

The classification of 3-dimensional hypersurface isolated singularities, keeping the differential structure of the link $K_V = S^7 \cap V$ *constant.*

Type	$h(z_0, z_1, z_2, z_3) = 0$	(w_0, w_1, w_2, w_3),μ and $\alpha(x, y, z, \omega)$
I	$\{z_0^{a_0} + z_1^{a_1} + z_2^{a_2} + z_3^{a_3} = 0\}$	$(w_0, w_1, w_2, w_3) = (a_0, a_1, a_2, a_3)$
		$\mu = (a_0 - 1)(a_1 - 1)(a_2 - 1)(a_3 - 1)$
		$\alpha(x, y, z, w) = \frac{x}{a_0} + \frac{y}{a_1} + \frac{z}{a_2} + \frac{w}{a_3}$
II	$\{z_0^{a_0} + z_1^{a_1} + z_2^{a_2} + z_2 z_3^{a_3} = 0\}$	$(w_0, w_1, w_2, w_3) = (a_0, a_1, a_2, \frac{a_2 a_3}{a_2 - 1})$
		$\mu = (a_0 - 1)(a_1 - 1)[a_2(a_3 - 1) + 1]$
		$\alpha(x, y, z, w) = \frac{x}{a_0} + \frac{y}{a_1} + \frac{z}{a_2} + \frac{(a_2 - 1)w}{a_2 a_3}$
III	$\{z_0^{a_0} + z_1^{a_1} + z_2^{a_3} z_3 + z_2 z_3^{a_4} = 0\}$	$(w_0, w_1, w_2, w_3) = (a_0, a_1, \frac{a_2 a_3 - 1}{a_3 - 1}, \frac{a_2 a_3 - 1}{a_2 - 1})$
		$\mu = (a_0 - 1)(a_1 - 1) a_2 a_3$
		$\alpha(x, y, z, w) = \frac{x}{a_0} + \frac{y}{a_1} + \frac{(a_3 - 1)z}{a_2 a_3 - 1} + \frac{(a_2 - 1)w}{a_2 a_3 - 1}$
IV	$\{z_0^{a_0} + z_0 z_1^{a_1} + z_2^{a_2} + z_2 z_3^{d} = 0\}$	$(w_0, w_1, w_2, w_3) = (a_0, \frac{a_0 a_1}{a_0 - 1}, a_2, \frac{a_2 a_3}{a_2 - 1})$
		$\mu = [a_0(a_1 - 1) + 1][a_2(a_3 - 1) + 1]$
		$\alpha(x, y, z, w) = \frac{x}{a_0} + \frac{(a_0 - 1)y}{a_0 a_1} + \frac{z}{a_2} + \frac{(a_2 - 1)w}{a_2 a_3}$
V	$\{z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_2^{a_2} + z_2 z_3^{a_3} = 0\}$	$(w_0, w_1, w_2, w_3) = (\frac{a_0 a_1 - 1}{a_1 - 1}, \frac{a_0 a_1 - 1}{a_0 - 1}, a_2, \frac{a_2 a_3}{a_2 - 1})$
		$\mu = a_0 a_1 [a_2(a_3 - 1) + 1]$
		$\alpha(x, y, z, w) = \frac{(a_1 - 1)x}{a_0 a_1 - 1} + \frac{(a_0 - 1)y}{a_0 a_1 - 1} + \frac{z}{a_2} + \frac{(a_2 - 1)w}{a_2 a_3}$
VI	$\{z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_2^{a_2} z_3 + z_2 z_3^{a_3} = 0\}$	$(w_0, w_1, w_2, w_3) = (\frac{a_0 a_1 - 1}{a_1 - 1}, \frac{a_0 a_1 - 1}{a_0 - 1}, \frac{a_2 a_3 - 1}{a_3 - 1}, \frac{a_2 a_3 - 1}{a_2 - 1})$
		$\mu = a_0 a_1 a_2 a_3$
		$\alpha(x, y, z, w) = \frac{(a_1 - 1)x}{a_0 a_1 - 1} + \frac{(a_0 - 1)y}{a_0 a_1 - 1} + \frac{(a_3 - 1)z}{a_2 a_3 - 1} + \frac{(a_2 - 1)w}{a_3 a_2 - 1}$
VII	$\{z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2} + z_2 z_3^{a_3} = 0\}$	$(w_0, w_1, w_2, w_3) = (a_0, a_1, \frac{a_1 a_2}{a_1 - 1}, \frac{a_1 a_2 a_3}{a_1(a_2 - 1) + 1})$
		$\mu = (a_0 - 1)[a_1 a_2 (a_3 - 1) + a_1 - 1]$
		$\alpha(x, y, z, w) = \frac{x}{a_0} + \frac{y}{a_1} + \frac{(a_1 - 1)z}{a_1 a_2} + \frac{[a_1(a_2 - 1) + 1]w}{a_1 a_2 a_3}$
VIII	$\{z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2} + z_1 z_3^{a_3} + z_2^p z_3^q = 0,$ $\frac{p(a_1 - 1)}{a_1 a_2} + \frac{q(a_1 - 1)}{a_1 a_3} = 1\}$	$(w_0, w_1, w_2, w_3) = (a_0, a_1, \frac{a_1 a_2}{a_1 - 1}, \frac{a_1 a_3}{a_1 - 1})$
		$\mu = \frac{(a_0 - 1)[a_1(a_2 - 1) + 1][a_1(a_3 - 1) + 1]}{a_1 - 1}$
		$\alpha(x, y, z, w) = \frac{x}{a_0} + \frac{y}{a_1} + \frac{(a_1 - 1)z}{a_1 a_2} + \frac{(a_1 - 1)w}{a_1 a_3}$
IX	$\{z_0^{a_0} + z_1^{a_1} z_3 + z_2^{a_2} z_3 + z_1 z_3^{a_3} + z_1^p z_2^q = 0$ $\frac{p(a_3 - 1)}{a_1 a_3 - 1} + \frac{q a_1 (a_3 - 1)}{a_2(a_1 a_3 - 1)} = 1\}$	$(w_0, w_1, w_2, w_3) = (a_0, \frac{a_1 a_3 - 1}{a_3 - 1}, \frac{a_2(a_1 a_3 - 1)}{a_1(a_3 - 1)}, \frac{a_1 a_3 - 1}{a_1 - 1})$
		$\mu = \frac{(a_0 - 1) a_3 [a_2(a_1 a_3 - 1) - a_1(a_3 - 1)]}{a_3 - 1}$
		$\alpha(x, y, z, w) = \frac{x}{a_0} + \frac{(a_3 - 1)y}{a_1 a_3 - 1} + \frac{a_1(a_3 - 1)z}{a_2(a_1 a_3 - 1)} + \frac{(a_1 - 1)w}{a_1 a_3 - 1}$
X	$\{z_0^{a_0} + z_1^{a_1} z_2 + z_2^{a_2} z_3 + z_1 z_3^{a_3} = 0\}$	$(w_0, w_1, w_2, w_3) = (a_0, \frac{a_1 a_2 a_3 + 1}{a_3(a_2 - 1) + 1},$ $\frac{a_1 a_2 a_3 + 1}{a_1(a_3 - 1) + 1}, \frac{a_1 a_2 a_3 + 1}{a_2(a_1 - 1) + 1})$
		$\mu = (a_0 - 1) a_1 a_2 a_3$
		$\alpha(x, y, z, w) = \frac{x}{a_0} + \frac{[a_3(a_2 - 1) + 1]y}{a_1 a_2 a_3 + 1}$ $+ \frac{[a_1(a_3 - 1) + 1]z}{a_1 a_2 a_3 + 1} + \frac{[a_2(a_1 - 1) + 1]w}{a_1 a_2 a_3 + 1}$

Type	$h(z_0, z_1, z_2, z_3) = 0$	(w_0, w_1, w_2, w_3), μ and $\alpha(x, y, z, \omega)$
XI	$\{z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} + z_2 z_3^{a_3} = 0\}$	$(w_0, w_1, w_2, w_3) = (a_0, \frac{a_0 a_1}{a_0-1}, \frac{a_0 a_1 a_2}{a_0(a_1-1)+1}, \frac{a_0 a_1 a_2 a_3}{a_0 a_1(a_2-1)+(a_0-1)})$ $\mu = a_0 a_1 a_2(a_3 - 1) + a_0(a_1 - 1) + 1$ $\alpha(x, y, z, w) = \frac{x}{a_0} + \frac{(a_0-1)y}{a_0 a_1} + \frac{[a_0(a_1-1)+1]z}{a_0 a_1 a_2} + \frac{[a_0 a_1(a_2-1)+(a_0-1)]w}{a_0 a_1 a_2 a_3}$
XII	$\{z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1 z_3^{a_3} + z_1^p z_2^q = 0,$ $\frac{p(a_0-1)}{a_0 a_1} + \frac{q(a_0-1)}{a_0 a_2} = 1\}$	$(w_0, w_1, w_2, w_3) = (a_0, \frac{a_0 a_1}{a_0-1}, \frac{a_0 a_2}{a_0-1}, \frac{a_0 a_1 a_3}{a_0(a_1-1)+1})$ $\mu = \frac{(a_0(a_2-1)+1)(a_0 a_1(a_3-1)+a_0-1)}{a_0-1}$ $\alpha(x, y, z, w) = \frac{x}{a_0} + \frac{(a_0-1)y}{a_0 a_1} + \frac{(a_0-1)z}{a_0 a_2} + \frac{[a_0(a_1-1)+1]w}{a_0 a_1 a_3}$
XIII	$\{z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} + z_1 z_3^{a_3} + z_2^p z_3^q = 0,$ $\frac{p[a_0(a_1-1)+1]}{a_0 a_1 a_2} + \frac{q[a_0(a_1-1)+1]}{a_0 a_1 a_3} = 1\}$	$(w_0, w_1, w_2, w_3) = (a_0, \frac{a_0 a_1}{a_0-1}, \frac{a_0 a_1 a_2}{a_0(a_1-1)+1}, \frac{a_0 a_1 a_3}{a_0(a_1-1)+1})$ $\mu = \frac{[a_0 a_1(a_2-1)+a_0-1][a_0 a_1(a_3-1)+a_0-1]}{a_0(a_1-1)+1}$ $\alpha(x, y, z, w) = \frac{x}{a_0} + \frac{(a_0-1)y}{a_0 a_1} + \frac{[a_0(a_1-1)+1]z}{a_0 a_1 a_2} + \frac{[a_0(a_1-1)+1]w}{a_0 a_1 a_3}$
XIV	$\{z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_0 z_3^{a_3} + z_1^p z_2^q + z_2^r z_3^s = 0,$ $\frac{p(a_0-1)}{a_0 a_1} + \frac{q(a_0-1)}{a_0 a_2} = 1,$ $\frac{r(a_0-1)}{a_0 a_2} + \frac{s(a_0-1)}{a_0 a_3} = 1\}$	$(w_0, w_1, w_2, w_3) = (a_0, \frac{a_0 a_1}{a_0-1}, \frac{a_0 a_2}{a_0-1}, \frac{a_0 a_3}{a_0-1})$ $\mu = \frac{[a_0(a_1-1)+1][a_0(a_2-1)+1][a_0(a_3-1)+1]}{(a_0-1)^2}$ $\alpha(x, y, z, w) = \frac{x}{a_0} + \frac{(a_0-1)y}{a_0 a_1} + \frac{(a_0-1)z}{a_0 a_2} + \frac{(a_0-1)w}{a_0 a_3}$
XV	$\{z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_2 z_3^{a_3} + z_1^p z_2^q = 0,$ $\frac{p(a_0-1)}{a_0 a_1-1} + \frac{q a_1(a_0-1)}{a_2(a_0 a_1-1)} = 1\}$	$(w_0, w_1, w_2, w_3) = (\frac{a_0 a_1-1}{a_1-1}, \frac{a_0 a_1-1}{a_0-1}, \frac{a_2(a_0 a_1-1)}{a_1(a_0-1)}, \frac{a_2 a_3(a_0 a_1-1)}{a_2(a_0 a_1-1)-a_1(a_0-1)})$ $\mu = \frac{a_0[a_2(a_3-1)(a_0 a_1-1)+a_1(a_0-1)]}{a_0-1}$ $\alpha(x, y, z, w) = \frac{(a_1-1)x}{a_0 a_1-1} + \frac{(a_0-1)y}{a_0 a_1-1} + \frac{a_1(a_0-1)z}{a_2(a_0 a_1-1)} + \frac{[a_2(a_0 a_1-1)-a_1(a_0-1)]w}{a_2 a_3(a_0 a_1-1)}$
XVI	$\{z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_0 z_3^{a_3} + z_1^p z_2^q + z_2^r z_3^s = 0,$ $\frac{p(a_0-1)}{a_0 a_1-1} + \frac{q a_1(a_0-1)}{a_2(a_0 a_1-1)} = 1,$ $\frac{r a_1(a_0-1)}{a_2(a_0 a_1-1)} + \frac{s a_1(a_0-1)}{a_3(a_0 a_1-1)} = 1\}$	$(w_0, w_1, w_2, w_3) = (\frac{a_0 a_1-1}{a_1-1}, \frac{a_0 a_1-1}{a_0-1}, \frac{a_2(a_0 a_1-1)}{a_1(a_0-1)}, \frac{a_3(a_0 a_1-1)}{a_1(a_0-1)})$ $\mu = \frac{a_0[a_2(a_0 a_1-1)-a_1(a_0-1)][a_3(a_0 a_1-1)-a_1(a_0-1)]}{a_1(a_0-1)^2}$ $\alpha(x, y, z, w) = \frac{(a_1-1)x}{a_0 a_1-1} + \frac{(a_0-1)y}{a_0 a_1-1} + \frac{a_1(a_0-1)z}{a_2(a_0 a_1-1)} + \frac{a_1(a_0-1)w}{a_3(a_0 a_1-1)}$
XVII	$\{z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_1 z_2^{a_2} + z_0 z_3^{a_3} + z_1^p z_3^q + z_0^r z_2^s = 0,$ $\frac{p(a_0-1)}{a_0 a_1-1} + \frac{q a_1(a_0-1)}{a_3(a_0 a_1-1)} = 1$ $\frac{r(a_1-1)}{a_0 a_1-1} + \frac{s a_0(a_1-1)}{a_2(a_0 a_1-1)} = 1\}$	$(w_0, w_1, w_2, w_3) = (\frac{a_0 a_1-1}{a_1-1}, \frac{a_0 a_1-1}{a_0-1}, \frac{a_2(a_0 a_1-1)}{a_0(a_1-1)}, \frac{a_3(a_0 a_1-1)}{a_1(a_0-1)})$ $\mu = \frac{[a_2(a_0 a_1-1)-a_0(a_1-1)][a_3(a_0 a_1-1)-a_1(a_0-1)]}{(a_0-1)(a_1-1)}$ $\alpha(x, y, z, w) = \frac{(a_1-1)x}{a_0 a_1-1} + \frac{(a_0-1)y}{a_0 a_1-1} + \frac{a_0(a_1-1)z}{a_2(a_0 a_1-1)} + \frac{a_1(a_0-1)w}{a_3(a_0 a_1-1)}$
XVIII	$\{z_0^{a_0} z_2 + z_0 z_1^{a_1} + z_1 z_2^{a_2} + z_1 z_3^{a_3} + z_2^p z_3^q = 0,$ $\frac{p[a_0(a_1-1)+1]}{a_0 a_1 a_2+1} + \frac{q a_2[a_0(a_1-1)+1]}{a_3(a_0 a_1 a_2+1)} = 1\}$	$(w_0, w_1, w_2, w_3) = (\frac{a_0 a_1 a_2+1}{a_1(a_2-1)+1}, \frac{a_0 a_1 a_2+1}{a_2(a_0-1)+1}, \frac{a_0 a_1 a_2+1}{a_0(a_1-1)+1}, \frac{a_3(a_0 a_1 a_2+1)}{a_2[a_0(a_1-1)+1]})$ $\mu = \frac{a_0 a_1[a_0 a_1 a_2(a_3-1)+a_2(a_0-1)+a_3]}{a_0(a_1-1)+1}$ $\alpha(x, y, z, w) = \frac{[a_1(a_2-1)+1]x}{a_0 a_1 a_2+1} + \frac{[a_2(a_0-1)+1]y}{a_0 a_1 a_2+1} + \frac{[a_0(a_1-1)+1]z}{a_0 a_1 a_2+1} + \frac{a_2[a_0(a_1-1)+1]w}{a_3(a_0 a_1 a_2+1)}$
XIX	$\{z_0^{a_0} z_2 + z_0 z_1^{a_1} + z_2^{a_2} z_3 + z_1 z_3^{a_3} = 0\}$	$(w_0, w_1, w_2, w_3) = (\frac{a_0 a_1 a_2 a_3-1}{a_1[a_3(a_2-1)+1]-1}, \frac{a_0 a_1 a_2 a_3-1}{a_3[a_2(a_0-1)+1]-1}, \frac{a_0 a_1 a_2 a_3-1}{a_0[a_1(a_3-1)+1]-1}, \frac{a_0 a_1 a_2 a_3-1}{a_2[a_0(a_1-1)+1]-1})$ $\mu = a_0 a_1 a_2 a_3$ $\alpha(x, y, z, w) = \frac{[a_1(a_3(a_2-1)+1)-1]x}{a_0 a_1 a_2 a_3-1} + \frac{[a_3(a_2(a_0-1)+1)-1]y}{a_0 a_1 a_2 a_3-1} + \frac{[a_0(a_1(a_3-1)+1)-1]z}{a_0 a_1 a_2 a_3-1} + \frac{[a_2(a_0(a_1-1)+1)-1]w}{a_0 a_1 a_2 a_3-1}$

We shall use the theory developed in [31] and [32] to show that $(V_f, 0)$ and $(V_h, 0)$ have the same embedded topological type.

Definition 2.3. *Given a real manifold B of dimension m, and a family $\{(M_t, N_t) : t \in B$, N_t is a closed submanifold of a compact differentiable manifold $M_t\}$, we say that (M_t, N_t) depends $\mathbb{C}^\infty$ on t and that $\{(M_t, N_t) : t \in B\}$ is a $\mathbb{C}^\infty$ family of compact manifolds with submanifolds, if there is a $\mathbb{C}^\infty$ manifold $\mathcal{M}$, a closed* *submanifold $\mathcal{N}$ and a $\mathbb{C}^\infty$ map w from $\mathcal{M}$ onto B such that $\overline{w} := w|\mathcal{N}$ is also a $\mathbb{C}^\infty$ map from $\mathcal{N}$ onto B satisfying the following conditions*

(i) $M_t = w^{-1}(t) \supseteq N_t = \overline{w}^{-1}(t)$

(ii) *The rank of the Jacobian of w (respectively $\overline{w}$) is equal to m at every point of $\mathcal{M}$ (respectively $\mathcal{N}$).*

THEOREM 2.2 ([31]). *Let $((\mathcal{M},\mathcal{N}),(w,\overline{w}))$ be a $\mathbb{C}^\infty$ family of compact manifolds with submanifolds, with B connected. Then $(M_t, N_t) = (w^{-1}(t), \overline{w}^{-1}(t))$ is diffeomorphic to (M_{t_0}, N_{t_0}) for any $t, t_0 \in B$.*

Now we fix weights $(w_0, \ldots, w_n)$ with $w_i \geq 2$. Suppose that there is a weighted homogeneous polynomial $f(z_0, \ldots, z_n)$ with the weights $(w_0, \ldots, w_n)$ such that f has an isolated singularity at the origin. Let Δ be the intersection of the plane $\sum\limits_{i=0}^{n} \frac{x_i}{w_i} = 1$ with the first quadrant of $\mathbb{R}^{n+1}$. Let $\mathbb{C}[\Delta] = \{f \in \mathbb{C}[z_0, \ldots, z_n] : \operatorname{supp} f \subset \Delta\}$ where $\operatorname{supp} f = \{(d_0, \ldots, d_n) \in \mathbb{R}^{n+1} : z_0^{d_0} z_1^{d_1} \cdots z_n^{d_n} \text{ occurs in } f\}$. Let N be the number of the integer points which are in Δ. There is a canonical correspondence between $\mathbb{C}[\Delta]$ and $\mathbb{C}^N$. Thus we may introduce a Zariski topology on $\mathbb{C}[\Delta]$.

THEOREM 2.3 ([31]). *Let*

$$U = \{f \in \mathbb{C}[\Delta] : f \text{ has an isolated singularity at the origin}\}$$

Then U is a nonempty Zariski open set of $\mathbb{C}[\Delta]$.

THEOREM 2.4 ([31]). *Suppose that $f(z_0, \ldots, z_n)$ and $g(z_0, \ldots, z_n)$ are weighted homogeneous polynomials with the same weights $(w_0, \ldots, w_n)$. If the variety V of f and the variety W of g have an isolated singularity at the origin, then $(\mathbb{C}^{n+1}, V, 0)$ is homeomorphically equivalent to $(\mathbb{C}^{n+1}, W, 0)$.*

COROLLARY 2.1. *Suppose that $h(z_0, \cdots, z_3)$ is a weighted homogeneous polynomial with weights $(w_0, \cdots, w_3)$ and the variety $h^{-1}(0)$ has an isolated singularity at the origin. Then $h = f + g$ where f and g have no monomials in common, f is one of the nineteen classes above and f and g are weighted homogeneous of type $(w_0, \cdots, w_3)$. Moreover $h^{-1}(0)$ and $f^{-1}(0)$ have the same embedded topological type.*

3. Classification of three-dimensional isolated weighted homogeneous singularities with $p_g = 1$. The classification of rational three-dimensional weighted homogeneous isolated singularities can be found in [25]. In this paper we shall classify all three-dimensional weighted homogeneous isolated singularities with $p_g = 1$.

Let $f(z_0, \ldots, z_n)$ be a germ of an analytic function at the origin such that $f(0) = 0$. Suppose that f has an isolated critical point at the origin. f can be developed in a convergent Taylor series $\sum_\lambda a_\lambda z^\lambda$ where $z^\lambda = z_0^{\lambda_0} \cdots z_n^{\lambda_n}$. Recall that the Newton boundary $\Gamma(f)$ of f is the union of compact faces of $\Gamma_+(f)$, where $\Gamma_+(f)$ is the convex hull of the union of the subsets $\{\lambda + (\mathbb{R}^+)^{n+1}\}$ for λ such that $a_\lambda \neq 0$. Finally, let $\Gamma_-(f)$, the Newton polyhedron of f, be the cone over $\Gamma(f)$ with vertex at 0. For any closed face Δ of $\Gamma(f)$, we associate it with the polynomial $f_\Delta(z) = \sum_{\lambda \in \Delta} a_\lambda z^\lambda$. We say that f is nondegenerate if f_Δ has no critical point in $(\mathbb{C}^*)^{n+1}$ for any $\Delta \in \Gamma(f)$ where $\mathbb{C}^* = \mathbb{C} - \{0\}$.

THEOREM 3.1 (Merle and Teissier, [33]). *Let $(V, 0)$ be an isolated hypersurface singularity defined by a nondegenerate holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Then the geometric genus $p_g(V, 0) = \#\{p \in \mathbb{Z}^{n+1} \cap \Gamma_-(f) : p \text{ is positive}\}$.*

The above theorem plays an important role in the study of Yau's geometric conjecture and Yau's number theoretic conjecture [34]-[40].

Corollary 3.1. *suppose that $(V, 0)$ is an isolated hypersurface singularity dofned by a nondegenerate weighted homogeneous polynomial of type (w_0, w_1, w_2, w_3). Let $r_i = \frac{1}{w_i}$, $i = 0, \cdots, 3$. Then the geometric genus of the (X, x) is given by*

$$\sharp \left\{ (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{N}^4 \,|1 - (r_0 + r_1 + r_2 + r_3) \geq \lambda_0 r_0 + \lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3 \right\},$$

where $\mathbb{N}$ is the set of all non-negative integers.

Corollary 3.2. *With the same notations in Corollary 3.1, a weighted homogeneous isolated singularity has*

(1) *$p_g = 0$ if and only if $1 < r_0 + r_1 + r_2 + r_3$;*

(2) *$p_g = 1$ if and only if $0 \leq 1 - (r_0 + r_1 + r_2 + r_3) < \min_{i=0,\cdots,3} r_i$.*

Now we are ready to give the classification of three-dimensional isolated weighted homogeneous hypersurface singularities with $p_g = 1$.

Theorem 3.2. *Let $(V, 0)$ be a three-dimensional isolated weighted homogeneous hypersurface singularity with $p_g = 1$. Then $(V, 0)$ is defined by a weighted homogeneous polynomial of one of the nineteen cases of Table 2 such that the corresponding linear form α satisfies $\alpha(x, y, z, w) = 1$.*

Proof. In view of Corollary 2.1 and Theorem 3.1, it is clear that a three-dimensional isolated weighted homogeneous hypersurface singularity with $p_g = 1$ is defined by one of the nineteen types in section 2. The equations of the Γ_- hyperplanes of these nineteen types are respectively given by $\alpha(x, y, z, w) = 1$ in Table 2. □

In order to find all hypersurfaces among these nineteen types with $p_g = 1$, by Corollary 3.2, we only need to find all solutions of $0 \leq 1 - \alpha(1, 1, 1, 1) < \min\{\frac{1}{w_0}, \cdots, \frac{1}{w_3}\}$ among these nineteen types. We have used the MAPLE program to perform the computations. The solutions are listed as following.

Let $h(z_0, z_1, z_2, z_3)$ be a non-degenerate weighted homogeneous polynomial of class (I) in Table 2 (resp. (II), $\cdots$, (XIX)) and denote the locus of it by (I)-(a_0, a_1, a_2, a_3), $\cdots$, (XIX)-(a_0, a_1, a_2, a_3).

Theorem 3.3. *If (i)-(a_0, a_1, a_2, a_3) has $p_g = 1$ then (a_0, a_1, a_2, a_3) is one of Table (i) where $i \in \{(I), \cdots, (XIX)\}$.*

We only append six short tables, i.e. Table 3-6 corresponding to Type (I), (VI), (XIV), (XVII), (XVI) and (XIX), respectively. The complete tables have been put in [41] due to the page limit.

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Table 3

Type I. Assume $a_0 \le a_1 \le a_2 \le a_3$. Total=314.

a_0	a_1	a_2	a_3
2	3	7	42
2	3	7	⋮
2	3	7	83
2	3	8	24
2	3	8	⋮
2	3	8	47
2	3	9	18
2	3	9	⋮
2	3	9	35
2	3	10	15
2	3	10	⋮
2	3	10	29
2	3	11	14
2	3	11	⋮
2	3	11	26
2	3	12	12
2	3	12	⋮
2	3	12	23
2	3	13	13
2	3	13	⋮
2	3	13	22
2	3	14	14
2	3	14	⋮
2	3	14	20
2	3	15	15
2	3	15	⋮
2	3	15	19
2	3	16	16
2	3	16	⋮
2	3	16	19
2	3	17	17
2	3	17	18
2	4	5	20
2	4	5	⋮
2	4	5	39
2	4	6	12
2	4	6	⋮
2	4	6	23
2	4	7	10
2	4	7	⋮
2	4	7	18
2	4	8	8
2	4	8	⋮
2	4	8	15
2	4	9	9
2	4	9	⋮
2	4	9	14
2	4	10	10
2	4	10	⋮
2	4	10	13
2	4	11	11
2	4	11	12
2	5	5	10
2	5	5	⋮
2	5	5	19
2	5	6	8
2	5	6	⋮
2	5	6	14
2	5	7	7
2	5	7	⋮
2	5	7	12
2	5	8	8
2	5	8	⋮
2	5	8	11
2	5	9	9
2	5	9	10
2	6	6	6
2	6	6	⋮
2	6	6	11
2	6	7	7
2	6	7	⋮
2	6	7	10
2	6	8	8
2	6	8	9
2	7	7	7
2	7	7	8
2	7	7	9
2	7	8	8
3	3	4	12
3	3	4	⋮
3	3	4	23
3	3	5	8
3	3	5	⋮
3	3	5	14
3	3	6	6
3	3	6	⋮
3	3	6	11
3	3	7	7
3	3	7	⋮
3	3	7	10
3	3	8	8
3	3	8	9
3	4	4	6
3	4	4	⋮
3	4	4	11
3	4	5	5
3	4	5	⋮
3	4	5	9
3	4	6	6
3	4	6	7
3	4	7	7
3	5	5	5
3	5	5	6
3	5	5	7
3	5	6	6
4	4	4	4
4	4	4	⋮
4	4	4	7
4	4	5	5
4	4	5	6
4	5	5	5

TABLE 4

Type VI. Assume $a_0 \leq \min(a_1, a_2)$, $a_2 \leq a_3$. *Total=102.*

a_0	a_1	a_2	a_3
2	2	4	7
2	2	4	⋮
2	2	4	15
2	2	5	5
2	2	5	⋮
2	2	5	10
2	2	6	6
2	2	6	⋮
2	2	6	9
2	2	7	7
2	2	7	8
2	3	3	7
2	3	3	⋮
2	3	3	16
2	3	4	4
2	3	4	⋮
2	3	4	8
2	3	5	5
2	3	5	6
2	3	5	7
2	3	6	6
2	4	3	5
2	4	3	⋮
2	4	3	11
2	4	4	4
2	4	4	⋮
2	4	4	7
2	4	5	5
2	4	5	6
2	5	3	5
2	5	3	⋮
2	5	3	10
2	5	4	4
2	5	4	5
2	5	4	6
2	6	3	4
2	6	3	⋮
2	6	3	9
2	6	4	4
2	6	4	5
2	7	3	4
2	7	3	⋮
2	7	3	8
2	7	4	4
2	8	3	4
2	⋮	3	4
2	16	3	4
2	8	3	5
2	8	3	6
2	9	3	5
2	10	3	5
3	3	3	3
3	3	3	4
3	3	3	5
3	3	3	6
3	3	4	4
3	3	4	5
3	4	3	3
3	4	3	4
3	4	3	5
3	4	4	4
3	5	3	3
3	5	3	4
3	6	3	3

TABLE 5

Type XIV. Assume $a_1 \leq a_2 \leq a_3$. *Total=186.*

a_0	a_1	a_2	a_3
≥ 2	2	3	6
≥ 2	2	4	4
≥ 2	3	3	3
2	2	3	7
⋮	2	3	7
42	2	3	7
2	2	3	8
⋮	2	3	8
24	2	3	8
2	2	3	9
⋮	2	3	9
18	2	3	9
2	2	3	10
⋮	2	3	10
15	2	3	10
2	2	3	11
⋮	2	3	11
14	2	3	11
2	2	4	5
⋮	2	4	5
20	2	4	5
2	2	4	6
⋮	2	4	6
12	2	4	6
2	2	4	7
⋮	2	4	7
10	2	4	7
2	2	5	5
⋮	2	5	5
10	2	5	5
2	2	5	6
⋮	2	5	6
8	2	5	6
2	3	3	4
⋮	3	3	4
12	3	3	4
2	3	3	5
⋮	3	3	5
8	3	3	5
2	3	4	4
⋮	3	4	4
6	3	4	4

Table 6

Type XVII. Assume $a_0 \leq a_1$, and if $a_0 = a_1$, then assume $a_2 \leq a_3$. Total=384

a_0	a_1	a_2	a_3
2	2	3	6
2	2	3	⋮
2	2	3	11
2	2	4	4
2	2	4	⋮
2	2	4	7
2	2	5	5
2	2	5	6
2	3	3	5
2	3	3	⋮
2	3	3	8
2	3	4	3
2	3	⋮	3
2	3	7	3
2	3	4	4
2	3	4	5
2	3	5	4
2	3	6	4
2	3	8	2
2	3	⋮	2
2	3	15	2
2	4	3	4
2	4	3	⋮
2	4	3	7
2	4	4	3
2	4	⋮	3
2	4	7	3
2	4	4	4
2	4	4	5
2	4	5	4
2	4	6	2
2	4	⋮	2
2	4	11	2
2	5	3	4
2	5	3	⋮
2	5	3	7
2	5	4	3
2	5	4	4
2	5	5	3
2	5	6	2
2	5	⋮	2
2	5	10	2
2	6	3	4
2	6	3	⋮
2	6	3	7
2	6	4	3
2	6	5	2
2	6	⋮	2
2	6	9	2
2	7	3	4
2	⋮	3	4
2	15	3	4
2	7	5	2
2	⋮	5	2
2	15	5	2
2	7	3	5
2	7	3	6
2	7	4	3
2	7	6	2
2	7	7	2
2	8	3	5
2	8	4	3
2	8	6	2
2	9	3	5
2	9	6	2
3	3	2	6
3	3	2	⋮
3	3	2	11
3	3	3	3
3	3	3	4
3	3	3	5
3	3	4	4
3	4	2	6
3	4	2	⋮
3	4	2	10
3	4	3	3
3	4	3	4
3	4	3	5
3	4	4	3
3	4	4	4
3	4	5	3
3	4	5	2
3	4	⋮	2
3	4	8	2
3	5	2	5
3	5	2	⋮
3	5	2	9
3	5	3	3
3	5	3	4
3	5	4	3
3	5	4	2
3	5	⋮	2
3	5	7	2
3	6	2	5
3	6	2	⋮
3	6	2	9
3	6	3	3
3	6	4	2
3	6	⋮	2
3	6	7	2
3	7	2	5
3	7	2	⋮
3	7	2	9
3	7	3	3
3	7	4	2
3	⋮	4	2
3	12	4	2
3	7	5	2
3	8	2	5
3	8	2	⋮
3	8	2	9
3	8	3	3
3	8	5	2
3	9	2	5
3	9	2	⋮
3	9	2	8
3	10	2	5
3	⋮	2	5
3	24	2	5
3	10	2	6
3	⋮	2	6
3	14	2	6
3	10	2	7
3	11	2	7
4	4	2	4
4	4	2	⋮
4	4	2	7
4	4	3	3
4	4	3	4
4	5	2	4
4	5	2	⋮
4	5	2	7
4	5	3	3
4	5	3	4
4	5	4	2
4	5	⋮	2
4	5	7	2
4	6	2	4
4	6	2	⋮
4	6	2	7
4	6	3	3
4	6	4	2
4	⋮	4	2
4	9	4	2
4	6	5	2
4	6	6	2
4	7	2	4
4	⋮	2	4
4	15	2	4
4	7	2	5
4	7	2	6
4	7	5	2
4	8	2	5
4	9	2	5
4	10	3	2
4	⋮	3	2
4	27	3	2
5	5	2	4
5	⋮	2	4
5	10	2	4
5	5	2	5
5	5	2	6
5	5	3	3
5	6	2	5
5	6	2	6
5	6	3	3
5	6	4	2
5	6	5	2
5	6	6	2
5	7	2	5
5	7	3	2
5	⋮	3	2
5	18	3	2
5	7	4	2
5	8	4	2
5	9	2	3
5	⋮	2	3
5	32	2	3
6	6	2	3
6	⋮	2	3
6	20	2	3
6	6	2	4
6	⋮	2	4
6	9	2	4
6	6	2	5
6	7	3	2
6	⋮	3	2
6	15	3	2
6	7	4	2
6	8	4	2
7	7	2	3
7	⋮	2	3
7	16	2	3
7	7	2	4
7	8	2	4
7	8	3	2
7	⋮	3	2
7	14	3	2
7	8	4	2
8	8	2	3
8	⋮	2	3
8	14	2	3
8	9	3	2
8	⋮	3	2
8	13	3	2
9	9	2	3
9	⋮	2	3
9	13	2	3
9	10	3	2
9	⋮	3	2
9	12	3	2
10	10	2	3
10	11	2	3
10	11	3	2
10	12	2	3
10	12	3	2
11	11	2	3
11	12	2	3
11	12	3	2

TABLE 7
Type XVI. Assume $a_2 \leq a_3$. Total=349.

a_0	a_1	a_2	a_3
≥ 1	2	3	6
≥ 1	2	4	4
≥ 1	3	2	6
≥ 1	3	3	3
≥ 1	4	2	4
≥ 1	6	2	3
2	2	3	7
$\vdots$	2	3	7
21	2	3	7
2	2	3	8
$\vdots$	2	3	8
12	2	3	8
2	2	3	9
$\vdots$	2	3	9
9	2	3	9
2	2	3	10
$\vdots$	2	3	10
8	2	3	10
2	2	3	11
$\vdots$	2	3	11
7	2	3	11
2	2	4	5
$\vdots$	2	4	5
10	2	4	5
2	2	4	6
$\vdots$	2	4	6
6	2	4	6
2	2	4	7
$\vdots$	2	4	7
5	2	4	7
2	2	5	5
$\vdots$	2	5	5
5	2	5	5
2	2	5	6
3	2	5	6
4	2	5	6
2	3	2	7
$\vdots$	3	2	7
28	3	2	7
2	3	2	8
$\vdots$	3	2	8
16	3	2	8
2	3	2	9
$\vdots$	3	2	9
12	3	2	9
2	3	2	10
$\vdots$	3	2	10
10	3	2	10
2	3	2	11
$\vdots$	3	2	11
9	3	2	11
2	3	3	4
$\vdots$	3	3	4
8	3	3	4
2	3	3	5
$\vdots$	3	3	5
5	3	3	5
2	3	4	4
3	3	4	4
4	3	4	4
2	4	2	5
$\vdots$	4	2	5
15	4	2	5
2	4	2	6
$\vdots$	4	2	6
9	4	2	6
2	4	2	7
$\vdots$	4	2	7
7	4	2	7
2	4	3	3
$\vdots$	4	3	3
9	4	3	3
2	4	3	4
$\vdots$	4	3	4
5	4	3	4
2	5	2	4
$\vdots$	5	2	4
16	5	2	4
2	5	2	5
$\vdots$	5	2	5
8	5	2	5
2	5	2	6
$\vdots$	5	2	6
6	5	2	6
2	5	3	3
$\vdots$	5	3	3
6	5	3	3
2	6	2	4
$\vdots$	6	2	4
10	6	2	4
2	6	2	5
$\vdots$	6	2	5
7	6	2	5
2	7	2	3
$\vdots$	7	2	3
36	7	2	3
2	7	2	4
$\vdots$	7	2	4
8	7	2	4
2	8	2	3
$\vdots$	8	2	3
21	8	2	3
2	9	2	3
$\vdots$	9	2	3
16	9	2	3
2	10	2	3
$\vdots$	10	2	3
14	10	2	3
2	11	2	3
$\vdots$	11	2	3
12	11	2	3

TABLE 8
Type XIX. Assume a_0 is the smallest one. Total=249.

a_0	a_1	a_2	a_3
2	2	4	9
2	2	4	$\vdots$
2	2	4	19
2	2	5	6
2	2	5	$\vdots$
2	2	5	12
2	2	6	5
2	2	6	$\vdots$
2	2	6	10
2	2	7	5
2	2	7	$\vdots$
2	2	7	9
2	2	8	5
2	2	8	$\vdots$
2	2	8	8
2	2	9	4
2	2	9	$\vdots$
2	2	9	8
2	2	10	4
2	2	10	$\vdots$
2	2	10	7
2	2	11	4
2	2	$\vdots$	4
2	2	21	4
2	2	11	5
2	2	$\vdots$	5
2	2	14	5
2	2	11	6
2	3	3	6
2	3	3	$\vdots$
2	3	3	12
2	3	4	4
2	3	4	$\vdots$
2	3	4	8
2	3	5	4
2	3	5	$\vdots$
2	3	5	7
2	3	6	4
2	3	$\vdots$	4
2	3	10	4
2	3	6	5
2	3	6	6
2	3	7	5
2	3	7	6
2	3	8	5
2	3	8	3
2	3	$\vdots$	3
2	3	20	3
2	4	2	9
2	4	2	$\vdots$
2	4	2	21
2	4	3	4
2	4	3	$\vdots$
2	4	3	8
2	4	4	4
2	4	4	$\vdots$
2	4	4	7
2	4	5	3
2	4	5	$\vdots$
2	4	5	6
2	4	6	4
2	4	6	5
2	4	7	4
2	4	6	3
2	4	$\vdots$	3
2	4	13	3
2	5	2	6
2	5	2	$\vdots$
2	5	2	14
2	5	3	4
2	5	$\vdots$	4
2	5	6	4
2	5	3	5
2	5	3	6
2	5	3	7
2	5	4	5
2	5	4	3
2	5	$\vdots$	3
2	5	10	3
2	6	2	5
2	6	2	$\vdots$
2	6	2	11
2	6	3	4
2	6	3	5
2	6	3	6
2	6	4	4
2	6	4	3
2	6	$\vdots$	3
2	6	9	3
2	7	2	5
2	7	2	$\vdots$
2	7	2	10
2	7	3	4
2	7	4	3
2	7	$\vdots$	3
2	7	8	3
2	8	2	5
2	8	2	$\vdots$
2	8	2	9
2	8	3	3
2	$\vdots$	3	3
2	18	3	3
2	8	3	4
2	8	4	3
2	$\vdots$	4	3
2	11	4	3
2	8	5	3
2	8	6	3
2	9	2	4
2	$\vdots$	2	4
2	19	2	4
2	9	2	5
2	$\vdots$	2	5
2	12	2	5
2	9	2	6
2	9	2	7
2	9	5	3
3	3	3	3
3	3	3	$\vdots$
3	3	3	6
3	3	4	3
3	3	4	4
3	3	4	5
3	3	5	3
3	3	5	4
3	3	6	3
3	4	3	3
3	4	3	4
3	4	3	5
3	4	4	3
3	4	4	4
3	4	5	3
3	5	3	3
3	5	3	4
3	5	4	3
3	6	3	3