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004

G_2-GEOMETRY IN CONTACT GEOMETRY OF SECOND ORDER*

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Abstract. In [13],[14],[15], we formulate the contact equivalence of systems of second order partial differential equations for a scalar function as the Contact Geometry of Second Order or as the geometry of PD-manifolds of second order, generalizing works [3], [4] of E.Cartan. Especially, in [13], generalizing the famous G_2-models in [3], we observed, for each exceptional simple Lie algebra X_ℓ, we could find the overdetermined system (A_ℓ) and the single equation of Goursat type (B_ℓ), whose symmetry algebras are isomporphic with X_ℓ and formulated this fact as the G_2-geometry. The main purpose of the present paper is to construct the (local) models for overdetermined systems (A_ℓ) explicitly for each exceptional simple Lie algebra and also for the classical type analogy for BD type. We will also give parametric descriptions of the single equation of Goursat type (B_ℓ). Our constructions are based on the explicit calculation, in terms of Chevalley basis, of the structure of the Goursat gradation of each exceptional simple Lie algebra and each simple Lie algebra of BD type.

Key words. G_2-geometry, contact transformations, Goursat gradation of exceptional simple Lie algebras.

Mathematics Subject Classification. 53C15, 58A15, 58A20, 58A30.

1. Introduction. In his famous “Five variables paper [3]”, E. Cartan investigated the (local) contact equivalence and integration problems of two classes of second order partial differential equations for a scalar function in two independent variables, following the tradition of geometric theory of partial differential equations of 19th century. One class consists of overdetermined systems, which are involutive, and the other class consists of single equations of Goursat type, i.e., single equations of parabolic type whose Monge characteristic systems are completely integrable. In fact he reduced the contact equivalence of each class of second order equations to the equivalence of differential systems (or Pfaffian systems) on 5-dimensional spaces of rank 2 and 3 respectively. In generic cases he exhibited that the equivalence of differential systems on 5-dimensional spaces of rank 2 and 3 are Parabolic Geometries of G_2-type (see [14], [15]). Especially, as the flat models of these Parabolic Geometries, he found out the following facts in [3]: the symmetry algebras (i.e., the Lie algebras of infinitesimal contact transformations) of the following overdetermined system (A) and the single Goursat type equation (B) are both isomorphic with the 14-dimensional exceptional simple Lie algebra G_2.

$$(A)\qquad \frac{\partial^2 z}{\partial x^2}=\frac{1}{3}\left(\frac{\partial^2 z}{\partial y^2}\right)^3,\quad \frac{\partial^2 z}{\partial x\partial y}=\frac{1}{2}\left(\frac{\partial^2 z}{\partial y^2}\right)^2.$$

$$(B)\qquad 9r^2+12t^2(rt-s^2)+32s^3-36rst=0,$$

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or in parametric form

$$\begin{cases} r = as + \dfrac{1}{6}a^3 \\ s = at - \dfrac{1}{2}a^2. \end{cases}$$

where

$$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

are the classical terminology.

In [13], we formulated the contact equivalence of second order partial differential equations for a scalar function as the Contact Geometry of Second Order (see also [14], [15]) and we observed, for each exceptional simple Lie algebra X_ℓ, we could find the overdetermined system (A_ℓ) and the single equation of Goursat type (B_ℓ), whose symmetry algebras are isomporphic with X_ℓ and formulated this fact as the G_2-geometry. We will first recall this observation in §2.

The main purpose of the present paper is to construct the (local) models for overdetermined systems (A_ℓ) explicitly for each exceptional simple Lie algebra and also for the classical type analogy for BD type, which will be carried out in §3 and §4. We will also give parametric descriptions of the single equation of Goursat type (B_ℓ) in §5.

See [8], for the recent development of this subject. In contrast to [8], our emphasis will be laid on the role of the structure of Goursat gradation of each exceptional simple Lie algebra or of each simple Lie algebra of BD-type. Throughout this paper, we follow the terminology and notations of our previous papers [12], [13], [14] and [15].

2. G_2-geometry.

2.1. Standard Contact Manifolds. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$. Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and choose a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of the root system Φ of $\mathfrak{g}$ relative to $\mathfrak{h}$. Each simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ has the highest root θ. Let Δ_θ denote the subset of Δ consisting of all vertices which are connected to $-\theta$ in the Extended Dynkin diagram of X_ℓ ($\ell \geqq 2$). This subset Δ_θ of Δ, by the construction in §3.3 [12], defines a gradation (or a partition of Φ^+), which distinguishes the highest root θ. Then, this gradation (X_ℓ, Δ_θ) turns out to be a contact gradation, which is unique up to conjugacy (Theorem 4.1 [12]). Explicitly we have $\Delta_\theta = \{\alpha_1, \alpha_\ell\}$ for A_ℓ type and $\Delta_\theta = \{\alpha_\theta\}$ for other types. Here $\alpha_\theta = \alpha_2, \alpha_1, \alpha_2$ for B_ℓ, C_ℓ, D_ℓ types respectively and $\alpha_\theta = \alpha_2, \alpha_1, \alpha_2, \alpha_1, \alpha_8$ for G_2, F_4, E_6, E_7, E_8 types respectively.

Moreover we have the adjoint (or equivalently coadjoint) representation, which has θ as the highest weight. The R-space $J_\mathfrak{g}$ corresponding to (X_ℓ, Δ_θ) can be obtained as the projectivization of the (co-)adjoint orbit of the adjoint group G of $\mathfrak{g}$ passing through the root vector of θ. By this construction, $J_\mathfrak{g}$ has the natural contact structure $C_\mathfrak{g}$ induced from the symplectic structure as the coadjoint orbit, which corresponds to the contact gradation (X_ℓ, Δ_θ) (cf. [12], §4). Standard contact manifolds $(J_\mathfrak{g}, C_\mathfrak{g})$ were first found by Boothby ([1]) as compact simply connected homogeneous complex contact manifolds.

For the explicit description of the standard contact manifolds of the classical type, we refer the reader to §4.3 [12].

2.2. G_2**-geometry.** Let (X_ℓ, Δ_θ) be the (standard) contact gradation. Then we have $\Delta_\theta = \{\alpha_\theta\}$ except for A_ℓ type (see above). As we observed in §6.3 in [13], for the exceptional simple Lie algebras, there exists, without exception, a unique simple root α_G next to α_θ such that the coefficient of α_G in the highest root is 3 (see the diagrams below).

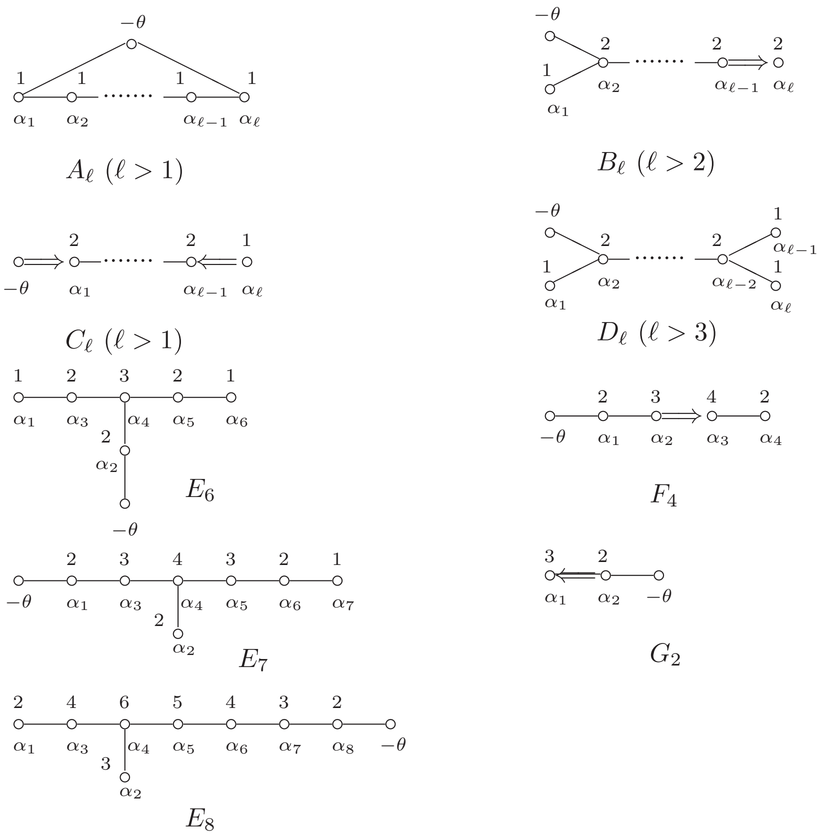

FIG. 1. *Extended Dynkin Diagrams with the coefficient of Highest Root (cf. [2])*

In the classical cases, the pair $\{\alpha_1, \alpha_3\}$ of simple roots plays the role of α_G in BD_ℓ type. We will consider simple graded Lie algebras $(X_\ell, \{\alpha_G\})$ of depth 3 and let $\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ be its negative part. We will call this gradation $\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ the Goursat gradation of type X_ℓ. Moreover we will show that regular differential systems (X, D) of type $\mathfrak{m}$ satisfy the conditions $(X.1)$ to $(X.3)$ in §4.3 [14] so that we can construct PD manifolds $(R(X), D_X^1, D_X^2)$ from (X, D). Our model overdetermined system (A_ℓ) will be the PD manifold of second order constructed from the standard differential system of type $\mathfrak{m}$, where $\mathfrak{m}$ is the Goursat gradation of type X_ℓ.

Explicitly we will here consider the following simple graded Lie algebras of depth 3: $(G_2, \{\alpha_1\})$, $(F_4, \{\alpha_2\})$, $(E_6, \{\alpha_4\})$, $(E_7, \{\alpha_3\})$, $(E_8, \{\alpha_7\})$, $(B_\ell, \{\alpha_1, \alpha_3\})$ $(\ell \geqq 3)$, $(D_\ell, \{\alpha_1, \alpha_3\})$ $(\ell \geqq 5)$ and $(D_4, \{\alpha_1, \alpha_3, \alpha_4\})$. These graded Lie algebras have the common feature with $(G_2, \{\alpha_1\})$ as follows: The Goursat gradation $\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ satisfies $\dim \mathfrak{g}_{-3} = 2$ and $\dim \mathfrak{g}_{-1} = 2 \dim \mathfrak{g}_{-2}$. In fact, in the description of the gradation in terms of the root space decomposition in §3.3 [12], in each case, we can check that $\Phi_3^+ = \{\theta, \theta - \alpha_\theta\}$ such that the coefficient of α_θ in each $\beta \in \Phi_2^+$ is 1 and

Φ_1^+ consists of roots $\theta-\beta$, $\theta-\alpha_\theta-\beta$ for each $\beta\in\Phi_2^+$ (see §3 and §4 for detail). Hence, ignoring the bracket product in $\mathfrak{g}_{-1}$, we can describe the bracket products of other part of $\mathfrak{m}$, in terms of paring, by

$$\mathfrak{g}_{-3}=W,\qquad \mathfrak{g}_{-2}=V\qquad \text{and}\qquad \mathfrak{g}_{-1}=W\otimes V^*,$$

where $\dim W=2$ and $\dim V=s$. Here $s=1,6,9,15,27,2\ell-4$ or $2\ell-5$ corrresponding to $X_\ell=G_2,F_4,E_6,E_7,E_8,B_\ell$ or D_ℓ.

Thus let (X,D) be a regular differential system of type $\mathfrak{m}$, where $\mathfrak{m}$ is the Goursat gradation of type X_ℓ. Then $(X,\partial D)$ is a regular differential system of type $\mathfrak{c}^1(s,2)$, where $\mathfrak{c}^1(s,2)=\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}$, $\mathfrak{g}_{-2}=W$, $\mathfrak{g}_{-1}=V\oplus W\otimes V^*$, is the symbol algebra of the canonical system on the first jet space for 2 dependent and s independent variables. Namely, there exists a coframe $\{\varpi_1,\varpi_2,\omega_1,\dots,\omega_s,\pi_1^1,\dots,\pi_1^s,\pi_2^1,\dots,\pi_2^s\}$ around $x\in X$ such that

$$\partial D=\{\varpi_1=\varpi_2=0\},\qquad D=\{\varpi_1=\varpi_2=\omega_1=\cdots=\omega_s=0\},$$

and

$$\begin{cases} d\varpi_1\equiv\pi_1^1\wedge\omega_1+\cdots+\pi_1^s\wedge\omega_s & (\operatorname{mod}\quad \varpi_1,\varpi_2)\\ d\varpi_2\equiv\pi_2^1\wedge\omega_1+\cdots+\pi_2^s\wedge\omega_s & (\operatorname{mod}\quad \varpi_1,\varpi_2)\end{cases}$$

Thus (X,D) satisfies the conditions $(X.1)$ to $(X.3)$ in §4.3 [14]. Hence we can construct the PD manifold $(R(X);D_X^1,D_X^2)$ as follows: Let us consider the collection $R(X)$ of hyperplanes v in each tangent space $T_x(X)$ at $x\in X$ which contains the fibre $\partial D(x)$ of the derived system ∂D of D.

$$R(X)=\bigcup_{x\in X}R_x\subset J(X,3s+1),$$

$$R_x=\{v\in \operatorname{Gr}(T_x(X),3s+1)\mid v\supset\partial D(x)\}\cong\mathbb{P}(T_x(X)/\partial D(x))=\mathbb{P}^1.$$

Moreover D_X^1 is the canonical system obtained by the Grassmaniann construction and D_X^2 is the lift of D. In fact, D_X^1 and D_X^2 are given by

$$D_X^1(v)=\rho_*^{-1}(v)\supset D_X^2(v)=\rho_*^{-1}(D(x)),$$

for each $v\in R(X)$ and $x=\rho(v)$, where $\rho:R(X)\to X$ is the projection (see §6.2[14] for the precise argument).

Remark 2.1. Let $\mathfrak{c}^1(s,t)=\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}$, $\mathfrak{g}_{-2}=W$, $\mathfrak{g}_{-1}=V\oplus W\otimes V^*$ be the symbol algebra of the canonical system on the first order jet space for t dependent and s independent variables, where $t=\dim W$ and $s=\dim V$. Let (Y,C) be a regular differential system of type $\mathfrak{c}^1(s,t)$. Let $F(y)$ be the subspace of $C(y)$ corresponding to $W\otimes V^*$ under the symbol algebra identification at $y\in Y$. Then, when $t\geqq 2$, F is well defined subbundle of C (a covariant system) and (Y,C) is isomorphic with the canonical system of the first order jet space if and only if F is completely integrable (see Proposition 1.5 [10]). Moreover, when $t\geqq 3$, F is always completely integrable (Theorem 1.6 [10]). So $\mathfrak{c}^1(s,2)$ case is very special and the Goursat gradations give special structures for $F=D$ such that $C=\partial D$.

Moreover, when (X,D) is the model space $(M_\mathfrak{g},D_\mathfrak{g})$ of type $(X_\ell,\{\alpha_G\})$, $R(X)$ can be identified with the model space $(R_\mathfrak{g},E_\mathfrak{g})$ of type $(X_\ell,\{\alpha_\theta,\alpha_G\})$ as follows (here, we understand α_G denotes two simple roots α_1 and α_3 in case of BD_ℓ types and three simple roots α_1,α_3 and α_4 in case of D_4): Let $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ be the standard contact manifold of type $(X_\ell, \{\alpha_\theta\})$. Then we have the double fibration;

$$\begin{array}{ccc} R_{\mathfrak{g}} & \xrightarrow{\pi_c} & J_{\mathfrak{g}} \\ {\scriptstyle \pi_g}\downarrow & & \\ M_{\mathfrak{g}} & & \end{array}$$

Here $(X_\ell, \{\alpha_\theta, \alpha_G\})$ is a graded Lie algebra of depth 5 and satisfies the following: $\dim \check{\mathfrak{g}}_{-5} = \dim \check{\mathfrak{g}}_{-4} = 1$, $\dim \check{\mathfrak{g}}_{-3} = \dim \check{\mathfrak{g}}_{-2} = s$ and $\dim \check{\mathfrak{g}}_{-1} = s + 1$. In fact, comparing with the gradation of $(X_\ell, \{\alpha_G\})$, we can check that $\check{\Phi}_5^+ = \{\theta\}$, $\check{\Phi}_4^+ = \{\theta - \alpha_\theta\}$, $\check{\Phi}_3^+ = \Phi_2^+$, $\check{\Phi}_2^+$ consists of roots $\theta - \beta$ for each $\beta \in \check{\Phi}_3^+$ and $\check{\Phi}_1^+$ consists of roots α_θ and $\theta - \alpha_\theta - \beta$ for each $\beta \in \check{\Phi}_3^+$ (see §6.2 [14] for BD_ℓ-type). Thus we see that $\partial^{(3)} E_{\mathfrak{g}} = (\pi_c)_*^{-1}(C_{\mathfrak{g}})$, $\partial^{(2)} E_{\mathfrak{g}} = (\pi_g)_*^{-1}(\partial D_{\mathfrak{g}})$ and $\partial E_{\mathfrak{g}} = (\pi_g)_*^{-1}(D_g)$. We put $D^1 = \partial^{(3)} E_{\mathfrak{g}}$ and $D^2 = \partial E_{\mathfrak{g}}$. Then $(R_{\mathfrak{g}}; D^1, D^2)$ is a PD manifold of second order. In fact, we have an isomorphism of $(R_{\mathfrak{g}}; D^1, D^2)$ onto $(R(M_{\mathfrak{g}}); D^1_{M_{\mathfrak{g}}}, D^2_{M_{\mathfrak{g}}})$ by the Realization Lemma for $(R_{\mathfrak{g}}, D^1, \pi_g, M_{\mathfrak{g}})$ and an embedding of $R_{\mathfrak{g}}$ into $L(J_{\mathfrak{g}})$ by the Realization Lemma for $(R_{\mathfrak{g}}, D^2, \pi_c, J_{\mathfrak{g}})$. Thus $R_{\mathfrak{g}}$ is identified with a R-space orbit in $L(J_{\mathfrak{g}})$. Moreover, putting $C^* = \partial^{(3)} E_{\mathfrak{g}}$ and $N = \partial^{(2)} E_{\mathfrak{g}}$, $(R_{\mathfrak{g}}; C^*, N)$ is an IG-manifold of corank 1, which is the global model of $(W; C^*, N)$ below.

Let (X, D) be a regular differential system of type $\mathfrak{m}$, where $\mathfrak{m}$ is the Goursat gradation of type X_ℓ. Then $(X, \partial D)$ is a regular differential system of type $\mathfrak{c}^1(s, 2)$. Hence, from $(X, \partial D)$, we can construct an IG manifold $(W(X); C^*, N)$ of corank 1 and a PD manifold $(R(X); D^1_X, D^2_X)$ of second order of Goursat type as is explained in §5 [13] or Theorem 6.1 [15]. Thus, from the standard differential system of type $\mathfrak{m}$, we can obtain the single equation of Grousat type (B_ℓ) as in §5 (see Remark 6.2 (1) and §8.3 [15] for BD_ℓ-type).

2.3. Goursat gradations of Exceptional simple Lie algebras. Let $\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ be the Goursat gradation of type X_ℓ, where X_ℓ is one of the excetional simple Lie algebra. In order to obtain the structure of $\mathfrak{m}$, we will first check, in each case in §4, the following description in terms of the root space decomposition of $\mathfrak{m}$:

$$\Phi_3^+ = \{\theta, \theta - \alpha_\theta\}, \quad \Phi_2^+ = \{\beta_1, \ldots, \beta_s\}, \quad \Phi_1^+ = \{\alpha_\theta + \gamma_1, \ldots, \alpha_\theta + \gamma_s, \gamma_1, \ldots, \gamma_s\}.$$

where $\theta - (2\alpha_\theta + 3\alpha_G)$, $\beta_i - (\alpha_\theta + 2\alpha_G)$ and $\gamma_i - \alpha_G$ are spanned by simple roots other than α_θ and α_G of X_ℓ such that $\theta = \beta_i + \alpha_\theta + \gamma_i$ $(i = 1, \ldots, s)$.

Then we will calculate the structure of $\mathfrak{m}$ explicitly by use of the Chevalley basis of X_ℓ. By adjusting the Chevalley basis (especially, for E_6, E_7, E_8, by changing the orientation of Chevalley basis) suitably (see §4 for detail), we obtain the basis $\{W_1, W_2, Z_1, \ldots, Z_s, Y_1, \ldots, Y_s, X_1, \ldots, X_s\}$ of $\mathfrak{m}$ satisfying the following:

$$\mathfrak{g}_{-3} = \langle\{W_1, W_2\}\rangle, \quad \mathfrak{g}_{-2} = \langle\{Z_1, \ldots, Z_s\}\rangle, \quad \mathfrak{g}_{-1} = \langle\{Y_1, \ldots, Y_s, X_1, \ldots, X_s\}\rangle$$

such that

$$[Z_i, Y_j] = \delta^i_j W_1, \qquad [Z_i, X_j] = \delta^i_j W_2 \qquad [X_i, X_j] = [Y_i, Y_j] = 0 \quad (1 \leqq i, j \leqq s)$$

In these basis, we calculate $[X_j, Y_k]$ for $1 \leqq j, k \leqq s$ in §4.

3. Classical Cases (BD_ℓ-type). The structure of the Goursat gradation $\mathfrak{m}$ of type BD_ℓ,

$$\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$$

is given by the following brackets among the basis $\{W_1, W_2, Z_1, \ldots, Z_{p+1}, Y_1, \ldots, Y_{p+1}, X_1, \ldots, X_{p+1}\}$ of $\mathfrak{m}$;

$$\mathfrak{g}_{-3} = \langle \{W_1, W_2\} \rangle, \quad \mathfrak{g}_{-2} = \langle \{Z_1, \ldots, Z_{p+1}\} \rangle, \quad \mathfrak{g}_{-1} = \langle \{Y_1, \ldots, Y_{p+1}, X_1, \ldots, X_{p+1}\} \rangle$$

such that

$$[Z_i, Y_j] = \delta^i_j W_1, \quad [Z_i, X_j] = \delta^i_j W_2, \quad [X_{k_1}, Y_{k_2}] = \delta^{k_1}_{k_2} Z_1, \quad [X_1, Y_k] = [X_k, Y_1] = Z_k, \tag{3.1}$$

$$[X_i, X_j] = [Y_i, Y_j] = 0 \qquad (1 \leqq i, j \leqq p+1, 2 \leqq k, k_1, k_2 \leqq p+1),$$

In fact we obtain these relations through the matrices description of the Goursat gradation of $\mathfrak{o}(p+3, 3)$ (see §6.2 [14] and §8.3 [15]). The following differential system (X, D) on $X = \mathbb{C}^{3p+5}$ describes the standard differential system of type $\mathfrak{m}$

$$D = \{\, \varpi_1 = \varpi_2 = \omega_1 = \cdots = \omega_{p+1} = 0 \,\},$$

where

$$\begin{cases} \varpi_1 = dw_1 - (z_1 + \frac{1}{2}\sum_{k=2}^{p+1} x_k y_k)\, dy_1 - \sum_{k=2}^{p+1} \{z_k + \frac{1}{2}(x_k y_1 + x_1 y_k)\}\, dy_k, \\ \varpi_2 = dw_2 - (z_1 - \frac{1}{2}\sum_{k=2}^{p+1} x_k y_k)\, dx_1 - \sum_{k=2}^{p+1} \{z_k - \frac{1}{2}(x_k y_1 + x_1 y_k)\}\, dx_k, \\ \omega_1 = dz_1 + \frac{1}{2}\sum_{k=2}^{p+1} (y_k\, dx_k - x_k\, dy_k), \\ \omega_k = dz_k + \frac{1}{2}(y_1\, dx_k - x_k\, dy_1) + \frac{1}{2}(y_k\, dx_1 - x_1\, dy_k), \quad (2 \leqq k \leqq p+1), \end{cases}$$

and $(w_1, w_2, z_1, \ldots, z_{p+1}, y_1, \ldots, y_{p+1}, x_1, \ldots, x_{p+1})$ is a coordinate system of $X = \mathbb{C}^{3p+5}$. In fact we have

$$\begin{cases} d\varpi_1 = dy_1 \wedge \omega_1 + \cdots + dy_{p+1} \wedge \omega_{p+1}, \\ d\varpi_2 = dx_1 \wedge \omega_1 + \cdots + dx_{p+1} \wedge \omega_{p+1}, \\ d\omega_1 = dy_2 \wedge dx_2 + \cdots + dy_{p+1} \wedge dx_{p+1}, \\ d\omega_k = dy_1 \wedge dx_k + dy_k \wedge dx_1 \quad (2 \leqq k \leqq p+1), \end{cases} \tag{3.2}$$

which is the dual of (3.1). In particular, we have

$$\partial D = \{\varpi_1 = \varpi_2 = 0\}.$$

Now, utilizing the First Reduction Theorem, we will construct the model equation (A) from the standard differential system (X, D) of type $\mathfrak{m}$ constructed as above, which is the local model corresponding to $(BD_\ell, \{\alpha_1, \alpha_3\})$. As in §2.2, $(R(X); D^1_X, D^2_X)$ is constructed as follows; $R = R(X)$ is the collection of hyperplanes v in each tangent space $T_x(X)$ at $x \in X$ which contains the fobre ∂D of D.

$$R(X) = \bigcup_{x\in X} R_x \subset J(X, 3p+4),$$
$$R_x = \{v \in \mathrm{Gr}(T_x(X), 3p+4) \mid v \supset \partial D(x)\} \cong \mathbb{P}^1,$$

Moreover D^1 is the canonical system obtained by the Grassmaniann construction and D^2 is the lift of D. Precisely, D^1 and D^2 are given by

$$D^1(v) = \rho_*^{-1}(v) \supset D^2(v) = \rho_*^{-1}(D(x)),$$

for each $v \in R(X)$ and $x = \rho(v)$, where $\rho : R(X) \to X$ is the projection.

We introduce a fibre coordinate λ by $\varpi = \varpi_1 + \lambda\varpi_2$, where

$$D^1 = \{\, \varpi = 0 \,\} \qquad \text{and} \qquad \partial D = \{\varpi_1 = \varpi_2 = 0\}.$$

Here $(w_1, w_2, z_1, \ldots, z_{p+1}, y_1, \ldots, y_{p+1}, x_1, \ldots, x_{p+1}, \lambda)$ constitute a coordinate system on $R(X)$. Then we have

$$\begin{aligned} d\varpi &= (dy_1 + \lambda dx_1) \wedge \omega_1 + \cdots + (dy_{p+1} + \lambda dx_{p+1}) \wedge \omega_{p+1} + d\lambda \wedge \varpi_2, \\ \mathrm{Ch}\,(D^1) &= \{\ \varpi = \varpi_2 = \omega_1 = \cdots = \omega_{p+1} = dy_1 \\ &\qquad +\lambda dx_1 = \cdots = dy_{p+1} + \lambda dx_{p+1} = d\lambda = 0\ \}, \\ D^2 &= \{\ \varpi = \varpi_2 = \omega_1 = \cdots = \omega_{p+1} = 0\ \} \quad \text{and} \quad \partial D^2 = \{\ \varpi = \varpi_2 = 0\ \}. \end{aligned}$$

Thus $(R(X); D^1, D^2)$ is a PD-manifold of seconod order. Now we calculate

$$\begin{aligned} \varpi &= \varpi_1 + \lambda\varpi_2 \\ &= dw_1 - (z_1 + \tfrac{1}{2}\sum_{k=2}^{p+1} x_k y_k)\, dy_1 - \sum_{k=2}^{p+1}\{z_k + \tfrac{1}{2}(x_k y_1 + x_1 y_k)\}\, dy_k, \\ &\quad +\lambda\{dw_2 - (z_1 - \tfrac{1}{2}\sum_{k=2}^{p+1} x_k y_k)\, dx_1 - \sum_{k=2}^{p+1}\{z_k - \tfrac{1}{2}(x_k y_1 + x_1 y_k)\}\, dx_k\}, \\ &= dw_1 + \lambda dw_2 - (z_1 + \tfrac{1}{2}\sum_{k=2}^{p+1} x_k y_k)\,(dy_1 + \lambda dx_1) \\ &\quad -\sum_{k=2}^{p+1}\{z_k + \tfrac{1}{2}(x_k y_1 + x_1 y_k)\}\,(dy_k + \lambda dx_k) + \lambda\{(\sum_{k=2}^{p+1} x_k y_k) dx_1 + \sum_{k=2}^{p+1}(x_k y_1 + x_1 y_k) dx_k\} \\ &= d(w_1 + \lambda w_2) - (z_1 + \tfrac{1}{2}\sum_{k=2}^{p+1} x_k y_k)\, d(y_1 + \lambda x_1) - \sum_{k=2}^{p+1}\{z_k + \tfrac{1}{2}(x_k y_1 + x_1 y_k)\}\, d(y_k + \lambda x_k) \\ &\quad -\{w_2 - (z_1 + \tfrac{1}{2}\sum_{k=2}^{p+1} x_k y_k)\, x_1 - \sum_{k=2}^{p+1}\{z_k + \tfrac{1}{2}(x_k y_1 + x_1 y_k)\}\, x_k\} d\lambda \\ &\quad +\lambda\{(\sum_{k=2}^{p+1} x_k y_k) dx_1 + \sum_{k=2}^{p+1}(x_k y_1 + x_1 y_k) dx_k\} \end{aligned}$$

Moreover we calculate

$$\lambda\{(\sum_{k=2}^{p+1} x_k y_k) dx_1 + \sum_{k=2}^{p+1}(x_k y_1 + x_1 y_k) dx_k\}$$

$$
\begin{aligned}
&= \sum_{k=2}^{p+1}\{\lambda y_k d(x_1\, x_k) + \tfrac{1}{2}\lambda y_1 d(x_k)^2\}\\
&= \sum_{k=2}^{p+1}\{d(\lambda x_1 x_k y_k + \tfrac{1}{2}\lambda x_k^2 y_1) - x_1 x_k d(\lambda y_k) - \tfrac{1}{2}x_k^2 d(\lambda y_1)\}\\
&= \sum_{k=2}^{p+1}\{d(\lambda x_1 x_k y_k + \tfrac{1}{2}\lambda x_k^2 y_1) - (x_1 x_k y_k + \tfrac{1}{2}x_k^2 y_1)d\lambda - \lambda x_1 x_k dy_k - \tfrac{1}{2}\lambda x_k^2 dy_1\}\\
&= \sum_{k=2}^{p+1}\{d(\lambda x_1 x_k y_k + \tfrac{1}{2}\lambda x_k^2 y_1) - (x_1 x_k y_k + \tfrac{1}{2}x_k^2 y_1)d\lambda\\
&\quad -\lambda x_1 x_k d(y_k + \lambda x_k) - \tfrac{1}{2}\lambda x_k^2 d(y_1 + \lambda x_1) + \lambda x_1 x_k d(\lambda x_k) + \tfrac{1}{2}\lambda x_k^2 d(\lambda x_1)\}\\
&= \sum_{k=2}^{p+1}\{d(\lambda x_1 x_k y_k + \tfrac{1}{2}\lambda x_k^2 y_1 + \tfrac{1}{2}\lambda^2 x_1 x_k^2) - (x_1 x_k y_k + \tfrac{1}{2}x_k^2 y_1 - \tfrac{1}{2}\lambda x_1 x_k^2)d\lambda\\
&\quad -\lambda x_1 x_k d(y_k + \lambda x_k) - \tfrac{1}{2}\lambda x_k^2 d(y_1 + \lambda x_1)\}.
\end{aligned}
$$

Thus we obtain

$$\varpi = dZ - P_1 dX_1 - \sum_{k=2}^{p+2} P_k dX_k$$

where

$$
\begin{cases}
Z = w_1 + \lambda\, w_2 + \lambda(x_1 \sum_{k=2}^{p+1} x_k y_k + \frac{1}{2}\sum_{k=2}^{p+1} x_k^2 y_1) + \frac{1}{2}\lambda^2\, x_1 \sum_{k=2}^{p+1} x_k^2,\\
P_1 = w_2 - \sum_{i=1}^{p+1} x_i z_i - \frac{1}{2}\lambda\, x_1 \sum_{k=2}^{p+1} x_k^2,\\
P_2 = z_1 + \frac{1}{2}\sum_{k=2}^{p+1} x_k y_k + \frac{1}{2}\lambda \sum_{k=2}^{p+1} x_k^2,\\
P_{k+1} = z_k + \frac{1}{2}(x_k y_1 + x_1 y_k) + \lambda x_1 x_k, \quad (2 \leqq k \leqq p+1),\\
X_1 = \lambda,\\
X_{i+1} = y_i + \lambda x_i \quad (1 \leqq i \leqq p+1).
\end{cases}
$$

Thus

$$D^1 = \{\, dZ - \sum_{i=1}^{p+2} P_i\, dX_i = 0\,\},$$

and $(X_1, \ldots, X_{p+2}, Z, P_1, \ldots, P_{p+2})$ constitutes a canonical coordinate system on $J = R(X)/\mathrm{Ch}\,(D^1)$.

Putting $x_i = a_i$, we solve

$$
\begin{cases}
\lambda = X_1, \quad y_i = X_{i+1} - a_i X_1 \quad (1 \leqq i \leqq p+1),\\
z_1 = P_2 - \frac{1}{2}\sum_{k=2}^{p+1} a_k X_{k+1},\\
z_k = P_{k+1} - \frac{1}{2}(a_k X_2 + a_1 X_{k+1}) \quad (2 \leqq k \leqq p+1),\\
w_2 = P_1 + a_1(P_2 - \frac{1}{2}\sum_{k=2}^{p+1} a_k X_{k+1})\\
\qquad\qquad + \sum_{k=2}^{p+1} a_k\{P_{k+1} - \frac{1}{2}(a_k X_2 + a_1 X_{k+1})\} + \frac{1}{2}a_1 \sum_{p=2}^{p+1} a_k^2\, X_1\\
\quad = P_1 + \sum_{k=1}^{p+1} a_k P_{k+1} + \frac{1}{2}a_1 \sum_{k=2}^{p+1} a_k^2\, X_1 - \frac{1}{2}\sum_{k=2}^{p+1} a_k^2\, X_2 - \sum_{k=2}^{p+1} a_1 a_k\, X_{k+1}.
\end{cases}
$$

Then we calculate

$$\begin{cases} \omega_1 = dP_2 + \frac{1}{2}\sum_{k=2}^{p+1} a_k^2\, dX_1 - \sum_{k=2}^{p+1} a_k\, dX_{k+1}, \\ \omega_k = dP_{k+1} + a_1 a_k\, dX_1 - a_k\, dX_2 - a_1\, dX_{k+1} \qquad (k = 2, \ldots, p+1), \\ \varpi_2 = dP_1 + \sum_{k=2}^{p+1} a_k dP_{k+1} + \frac{1}{2}\sum_{k=2}^{p+1} a_1 a_k^2\, dX_1 - \frac{1}{2}\sum_{k=2}^{p+1} a_k^2\, dX_2 - \sum_{k=2}^{p+1} a_1 a_k\, dX_{k+1}, \\ \quad = \sum_{k=1}^{p+1} a_k \omega_k + dP_1 - \sum_{k=2}^{p+1} a_1 a_k^2\, dX_1 + \frac{1}{2}\sum_{k=2}^{p+1} a_k^2\, dX_2 + \sum_{k=2}^{p+1} a_1 a_k\, dX_{k+1}. \end{cases} \tag{3.3}$$

This implies $R(X)$ is given by the following $\frac{1}{2}(p+1)(p+2)+1$ equations;

$$P_{22} = 0, P_{ij} = \delta_{ij} P_{33} \quad (3 \leqq i, j \leqq p+2),$$

$$P_{11} = P_{33}\sum_{k=2}^{p+1} P_{2,k+1}^2, \quad P_{12} = -\frac{1}{2}\sum_{k=2}^{p+1} P_{2,k+1}^2, \quad P_{1,k+1} = -P_{33}P_{2,k+1} \quad (2 \leqq k \leqq p{+}1),$$

in terms of the canonical coordinate $(X_1, \ldots, X_{p+2}, Z, P_1, \ldots, P_{p+2}, P_{11}, \ldots, P_{p+2,p+2})$ of $L(J)$.

4. Exceptional Cases. Let $\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ be the Goursat gradation of type X_ℓ, where X_ℓ is one of the excetional simple Lie algebra. As in §2.3, we choose the basis $\{W_1, W_2, Z_1, \ldots, Z_s, Y_1, \ldots, Y_s, X_1, \ldots, X_s\}$ of $\mathfrak{m}$ satisfying the following:

$$\mathfrak{g}_{-3} = \langle\{W_1, W_2\}\rangle, \quad \mathfrak{g}_{-2} = \langle\{Z_1, \ldots, Z_s\}\rangle, \quad \mathfrak{g}_{-1} = \langle\{Y_1, \ldots, Y_s, X_1, \ldots, X_s\}\rangle$$

such that

$$[Z_i, Y_j] = \delta^i_j W_1, \qquad [Z_i, X_j] = \delta^i_j W_2 \qquad [X_i, X_j] = [Y_i, Y_j] = 0 \quad (1 \leqq i, j \leqq s).$$

Utilizing the bilinear forms $f_i(x_1, \ldots, x_s, y_1, \ldots, y_s)$ for $i = 1, \ldots, s$, which describes the brackets $[X_j, Y_k]$ (see the following subsections), we can describe the standard differential system of type $\mathfrak{m}$ as follows: Let $(w_1, w_2, z_1, \ldots, z_s, y_1, \ldots, y_s, x_1, \ldots, x_s)$be the linear coordinate of $X = \mathfrak{m}$ given by the above basis of $\mathfrak{m}$. Then (X, D) on $X = \mathbb{C}^{3s+2}$ describes the standard differential system of type $\mathfrak{m}$

$$D = \{\, \varpi_1 = \varpi_2 = \omega_1 = \cdots = \omega_s = 0 \,\},$$

where

$$\varpi_1 = dw_1 - \sum_{i=1}^{s}(z_i + \tfrac{1}{2}f_i)\, dy_i, \qquad \varpi_2 = dw_2 - \sum_{i=1}^{s}(z_i - \tfrac{1}{2}f_i)\, dx_i$$

and

$$\omega_i = dz_i - \tfrac{1}{2}\{f_i(x_k, dy_k) - f_i(dx_k, y_k)\} \qquad (i = 1, \ldots, s).$$

In fact we have

$$d\omega_i = -f_i(dx_j \wedge dy_k) \qquad (i = 1, \ldots, s)$$

and we calculate

$$d\varpi_1 = \sum_{i=1}^{s} dy_i \wedge (dz_i + \tfrac{1}{2}df_i) = \sum_{i=1}^{s} dy_i \wedge \omega_i + \sum_{i=1}^{s} dy_i \wedge f_i(x_k, dy_k),$$
$$d\varpi_2 = \sum_{i=1}^{s} dx_i \wedge (dz_i - \tfrac{1}{2}df_i) = \sum_{i=1}^{s} dx_i \wedge \omega_i - \sum_{i=1}^{s} dx_i \wedge f_i(dx_k, y_k).$$

Then, from $\sum_{i=1}^{s} dx_i \wedge f_i(dx_k, y_k) = 0$ and $\sum_{i=1}^{s} dy_i \wedge f_i(x_k, dy_k) = 0$ (see below), we obtain

$$d\varpi_1 = dy_1 \wedge \omega_1 + dy_2 \wedge \omega_2 + \cdots + dy_s \wedge \omega_s,$$
$$d\varpi_2 = dx_1 \wedge \omega_1 + dx_2 \wedge \omega_2 + \cdots + dx_s \wedge \omega_s,$$

which describes the structure of $\mathfrak{m}$. In particular, we have

$$\partial D = \{\varpi_1 = \varpi_2 = 0\}.$$

By the First Reduction Theorem, as in §2.2, our model overdetermined system $(R(X); D^1_X, D^2_X)$ is constructed from (X, D) as follows:

$$R(X) = \bigcup_{x \in X} R_x \subset J(X, 3s+1),$$
$$R_x = \{v \in \mathrm{Gr}(T_x(X), 3s+1) \mid v \supset \partial D(x)\} \cong \mathbb{P}^1.$$

Moreover D^1_X and D^2_X are given by

$$D^1_X(v) = \rho_*^{-1}(v) \supset D^2_X(v) = \rho_*^{-1}(D(x)),$$

for $v \in R(X)$, $x = \rho(v)$, and $\rho : R(X) \to X$ is the projection.

We introduce a fibre coordinate λ by $\varpi = \varpi_1 + \lambda\varpi_2$, where

$$D^1 = \{\, \varpi = 0 \,\} \qquad \text{and} \qquad \partial D = \{\varpi_1 = \varpi_2 = 0\}.$$

Here $(w_1, w_2, z_1, \ldots, z_s, y_1, \ldots, y_s, x_1, \ldots, x_s, \lambda)$ constitutes a coordinate system on $R(X)$.

Now we first observe, in each case, that $f_i(x_k, y_k)$ is symmetric in x_k and y_k, i.e., $f_i(x_1, \ldots, x_s, y_1, \ldots, y_s) = f_i(y_1, \ldots, y_s, x_1 \ldots, x_s)$ for $i = 1, \ldots, s$ and put

$$\sum_{i=1}^{s} x_i\, f_i = 2\sum_{i=1}^{s} y_i\, g_i$$

where $g_i (i = 1, \ldots, s)$ are quadratic polynomials in $x_1, \ldots, x_s$.

Then we can check in each case

$$g_i(x_1, \ldots, x_s) = \tfrac{1}{2} f_i(x_1, \ldots, x_s, x_1, \ldots, x_s),$$
$$\sum_{i=1}^{s} f_i dx_i = \sum_{i=1}^{s} y_i dg_i \quad \text{and} \quad \sum_{i=1}^{s} g_i\, dx_i = dg,$$

for a cubic polynomial g in $x_1, \ldots, x_s$. Since $g_i = \frac{\partial g}{\partial x_i}$ and g is a cubic polynomial, we have

$$3g = \sum_{i=1}^{s} x_i\, g_i.$$

Moreover, since $f_i(x_k, y_k)$ is symmetric in x_k and y_k, we have

$$\sum_{i=1}^{s} f_i\, dx_i = \sum_{i=1}^{s} y_i\, dg_i, \quad \sum_{i=1}^{s} f_i\, dy_i = \sum_{i=1}^{s} x_i\, dh_i,$$

where $h_i(y_1, \ldots, y_s) = g_i(y_1, \ldots, y_s) = \frac{1}{2} f_i(y_k, y_k)$. Taking the exterior derivative of the first equation and from $df_i = f_i(dx_k, y_k) + f_i(x_k, dy_k)$, we have

$$\sum_{i=1}^{s} \{f_i(dx_k, y_k) + f_i(x_k, dy_k)\} \wedge dx_i = \sum_{i=1}^{s} dy_i \wedge dg_i,$$

This implies $\sum_{i=1}^{s} dx_i \wedge f_i(dx_k, y_k) = 0$. Similarly, from the second equation, we obtain $\sum_{i=1}^{s} dy_i \wedge f_i(x_k, dy_k) = 0$.

Now, for $\varpi = \varpi_1 + \lambda\varpi_2$, we calculate

$$\begin{aligned}
d\varpi &= (dy_1 + \lambda dx_1) \wedge \omega_1 + \cdots + (dy_s + \lambda dx_s) \wedge \omega_s + d\lambda \wedge \varpi_2, \\
\varpi &= \varpi_1 + \lambda\varpi_2 \\
&= dw_1 - \sum_{i=1}^{s} (z_i + \tfrac{1}{2} f_i) dy_i + \lambda\{dw_2 - \sum_{i=1}^{s} (z_i - \tfrac{1}{2} f_i) dx_i\} \\
&= dw_1 + \lambda dw_2 - \sum_{i=1}^{s} (z_i + \tfrac{1}{2} f_i)(dy_i + \lambda dx_i) + \lambda \sum_{i=1}^{s} f_i dx_i \\
&= d(w_1 + \lambda w_2) - \{w_2 - \sum_{i=1}^{s} (x_i\, z_i + \tfrac{1}{2} x_i\, f_i)\} d\lambda \\
&\qquad - \sum_{i=1}^{s} (z_i + \tfrac{1}{2} f_i) d(y_i + \lambda x_i) + \lambda \sum_{i=1}^{s} f_i dx_i.
\end{aligned}$$

Moreover we calculate

$$\begin{aligned}
\lambda \sum_{i=1}^{s} f_i dx_i = \lambda \sum_{i=1}^{s} y_i dg_i &= \lambda \sum_{i=1}^{s} \{d(y_i\, g_i) - g_i\, dy_i\} \\
&= d(\lambda \sum_{i=1}^{s} y_i\, g_i) - (\sum_{i=1}^{s} y_i\, g_i) d\lambda - \lambda \sum_{i=1}^{s} g_i\, dy_i \\
&= d(\lambda \sum_{i=1}^{s} y_i\, g_i) - (\tfrac{1}{2} \sum_{i=1}^{s} x_i\, f_i) d\lambda - \lambda \sum_{i=1}^{s} g_i\, dy_i,
\end{aligned}$$

$$\begin{aligned}
\lambda \sum_{i=1}^{s} g_i dy_i &= \lambda \sum_{i=1}^{s} g_i\, d(y_i + \lambda x_i) - \lambda(\sum_{i=1}^{s} x_i\, g_i) d\lambda - \lambda^2 \sum_{i=1}^{s} g_i dx_i \\
&= \lambda \sum_{i=1}^{s} g_i\, d(y_i + \lambda x_i) - 3g\lambda d\lambda - \lambda^2 dg \\
&= \lambda \sum_{i=1}^{s} g_i\, d(y_i + \lambda x_i) - \lambda g d\lambda - d(\lambda^2 g).
\end{aligned}$$

Thus we obtain

$$\varpi = dZ - \sum_{i=1}^{s+1} P_i dX_i$$

where

$$\begin{cases} Z = w_1 + \lambda\, w_2 + \lambda \sum_{i=1}^{s} y_i\, g_i + \lambda^2\, g \\ P_1 = w_2 - \sum_{i=1}^{s} x_i\, z_i - \lambda\, g, \\ P_{i+1} = z_i + \frac{1}{2} f_i + \lambda\, g_i \qquad (i = 1, \dots, s), \\ X_1 = \lambda, \\ X_{i+1} = y_i + \lambda x_i \qquad (i = 1, \dots, s), \end{cases}$$

Hence we have

$$D^1 = \{\, dZ - P_1\, dX_1 - \cdots - P_{s+1}\, dX_{s+1} = 0 \,\},$$

and $(X_1, \dots, X_{s+1}, Z, P_1, \dots, P_{s+1})$ constitutes a canonical coordinate system on $J = R(X)/\mathrm{Ch}\,(D^1)$.

Putting $x_i = a_i \quad (i = 1, \dots, s)$, we solve

$$\lambda = X_1, \quad y_i = X_{i+1} - a_i X_1 \quad (i = 1, \dots, s).$$

From $f_i(a_k, X_{k+1}) = f_i(a_k, y_k + \lambda a_k) = f_i(a_k, y_k) + 2\lambda g_i(a_k)$, we have

$$z_i = P_{i+1} - \tfrac{1}{2} f_i - \lambda g_i = P_{i+1} - \tfrac{1}{2} f_i(a_k, X_{k+1}) \qquad (i = 1, \dots, s).$$

From $\sum_{k=1}^{s} x_k(\frac{1}{2} f_k + \lambda g_k) = \sum_{k=1}^{s} (y_k + \lambda x_k) g_k$, we have

$$w_2 = P_1 + \sum_{k=1}^{s} a_k P_{k+1} + \hat{g} X_1 - \sum_{k=1}^{s} \hat{g}_k X_{k+1}$$

where $\hat{g} = g(a_1, \dots, a_s)$, and $\hat{g}_k = g_k(a_1, \dots, a_s) \quad (k = 1, \dots, s)$.

Moreover we calculate

$$\begin{aligned} \omega_i &= dz_i - \tfrac{1}{2}\{f_i(x_k, dy_k) - f_i(dx_k, y_k)\} \\ &= d\{P_{i+1} - \tfrac{1}{2} f_i(a_k, X_{k+1})\} - \tfrac{1}{2}\{f_i(a_k, d(X_{k+1} - a_k X_1)) - f_i(da_k, X_{k+1} - a_k X_1)\} \\ &= dP_{i+1} - f_i(a_k, dX_{k+1}) + \tfrac{1}{2} f_i(a_k, a_k) dX_1 \\ &= dP_{i+1} - f_i(a_k, dX_{k+1}) + g_i(a_k, a_k) dX_1 \qquad (i = 1, \dots, s), \end{aligned}$$

$$\begin{aligned} \varpi_2 &= dw_2 - \sum_{i=1}^{s} (z_i - \tfrac{1}{2} f_i) dx_i \\ &= d(P_1 + \sum_{k=1}^{s} a_k P_{k+1} + \hat{g} X_1 - \sum_{k=1}^{s} \hat{g}_k X_{k+1}) - \sum_{k=1}^{s} P_{k+1} da_k + \sum_{i=1}^{s} f_i dx_i + X_1 d\hat{g} \\ &= dP_1 + \sum_{k=1}^{s} a_k dP_{k+1} + \hat{g} dX_1 - \sum_{k=1}^{s} \hat{g}_k dX_{k+1} \\ &= dP_1 + (\sum_{k=1}^{s} a_k \omega_k + 2 \sum_{k=1}^{s} \hat{g}_k dX_{k+1} - 3\hat{g} dX_1) + \hat{g} dX_1 - \sum_{k=1}^{s} \hat{g}_k dX_{k+1} \end{aligned}$$

$$= \sum_{k=1}^{s} a_k \omega_k + dP_1 - 2\hat{g}dX_1 + \sum_{k=1}^{s} \hat{g}_k dX_{k+1}.$$

Thus we obtain

$$\begin{cases} \omega_i = dP_{i+1} - f_i(a_k, dX_{k+1}) + \hat{g}_i dX_1 \qquad (i = 1, \ldots, s), \\ \varpi_2 = \sum_{k=1}^{s} a_k \omega_k + dP_1 - 2\hat{g}dX_1 + \sum_{k=1}^{s} \hat{g}_k dX_{k+1}, \\ \quad = dP_1 + \sum_{k=1}^{s} a_k dP_{k+1} + \hat{g}dX_1 - \sum_{k=1}^{s} \hat{g}_k dX_{k+1}. \end{cases} \tag{4.1}$$

We will utilize the above formulae to describe our model system for each X_ℓ.

In the following subsections, we wil follow Bourbaki [2] for the numbering of simple roots and descriptions of positive roots. Let us take a Chevalley basis $\{x_\alpha(\alpha \in \Phi); h_i(1 \leqq i \leqq \ell)\}$ of the exceptional simple Lie algebra of type X_ℓ and put $y_\beta = x_{-\beta}$ for $\beta \in \Phi^+$ (cf. Chapter VII [5]). We will describe the structure of the Goursat gradation $\mathfrak{m}$ in terms of $\{y_\beta\}_{\beta \in \Phi^+}$. Moreover we will regard simple Lie algebras E_6 and E_7 as regular subalgebras of E_8 and utilize the root space decomposition of E_8 to describe the structure of the Goursat gradation of E_6 and E_7.

Remark 4.1. In §3, in the case of BD_ℓ-type, from (3.1), the bilinear forms $f_i(x_1, \ldots, x_{p+1}, y_1, \ldots, y_{p+1})$ for $i = 1, \ldots, p+1$ are given by

$$f_1 = \sum_{k=2}^{p+1} x_k y_k, \qquad f_k = x_k y_1 + x_1 y_k \quad (k = 2, \ldots, p+1).$$

Thus $f_i(x_k, y_k)$ $(i = 1, \ldots, p+1)$ are symmetric in x_k and y_k and satisfy

$$\sum_{i=1}^{p+1} f_i\, dx_i = \sum_{i=1}^{p+1} y_i\, dg_i, \quad \sum_{i=1}^{p+1} g_i\, dx_i = dg, \quad 3g = \sum_{i=1}^{p+1} x_i\, g_i,$$

where

$$g_1 = \frac{1}{2}\sum_{k=2}^{p+1} x_k^2, \qquad g_k = x_1 x_k \quad (k = 2, \ldots, p+1), \qquad \text{and} \qquad g = \frac{1}{2}\sum_{k=2}^{p+1} x_1 x_k^2.$$

Moreover $g_i(x_k) = \frac{1}{2} f_i(x_k, x_k)$ for $i = 1, \ldots, p+1$. Hence the actual calculations in §3 are covered by those in §4. Thus we obtain the uniform description (4.1) of G_2-Geometry, which gives us the parametric form of overdetermined systems (A_ℓ) (see the following subsections in §4), and also gives us the parametric form of the single equation of Goursat type (B_ℓ) in §5.

4.1. Goursat gradation and model system of type F_4. Let $\mathfrak{m}$ be the Goursat gradation of type F_4. For $(F_4, \{\alpha_2\})$, we have

$\Phi_3^+ = \{\beta_{24} = 2342, \beta_{23} = 1342\}$,
$\Phi_2^+ = \{\beta_{22} = 1242, \beta_{21} = 1232, \beta_{20} = 1222, \beta_{19} = 1231, \beta_{18} = 1221, \beta_{16} = 1220\}$,
$\Phi_1^+ = \{\beta_5 = 1100, \beta_8 = 1110, \beta_{11} = 1120, \beta_{12} = 1111, \beta_{15} = 1121, \beta_{17} = 1122,$

$$\beta_2 = 0100, \beta_6 = 0110, \beta_9 = 0120, \beta_{10} = 0111, \beta_{13} = 0121, \beta_{14} = 0122\}$$

where $a_1a_2a_3a_4$ stands for the root $\beta = \sum_{i=1}^4 a_i\alpha_i \in \Phi^+$.

We fix the orientation (or sign) of y_β as in the following: First we choose the orientation of y_{α_i} for simple roots by fixing the root vectors $y_i = y_{\alpha_i} \in \mathfrak{g}_{-\alpha_i}$ for $i = 1, 2, 3, 4$. For $\beta_i \in \Phi^+ (i = 1, \dots, 24)$, we put $y_i = y_{\beta_i}$ and fix the orientation by the following order;

$$\begin{aligned} &y_5 = [y_1, y_2], \quad y_6 = [y_2, y_3], \quad y_8 = [y_3, y_5], \quad 2y_9 = [y_3, y_6], \quad y_{10} = [y_4, y_6], \\ &y_{11} = [y_1, y_9], \quad y_{12} = [y_4, y_8], \quad y_{13} = [y_3, y_{10}], \quad 2y_{14} = [y_4, y_{13}], \quad y_{15} = [y_4, y_{11}], \\ &y_{16} = [y_2, y_{11}], \quad y_{17} = [y_2, y_{14}], \quad y_{18} = [y_4, y_{16}], \quad y_{19} = [y_3, y_{18}], \quad y_{20} = [y_2, y_{17}], \\ &y_{21} = [y_4, y_{19}], \quad 2y_{22} = [y_3, y_{21}], \quad y_{23} = [y_2, y_{22}], \quad y_{24} = [y_1, y_{23}]. \end{aligned}$$

Then, for example, we calculate

$$[y_1, y_6] = [y_1, [y_2, y_3]] = [[y_1, y_2], y_3] = [y_5, y_3] = -y_8.$$

In this way, by the repeated application of Jacobi identities, we obtain

$$2y_{24} = [-2y_{22}, y_5] = [y_{21}, y_8] = [2y_{20}, y_{11}] = [y_{19}, -y_{12}] = [-y_{18}, y_{15}] = [2y_{16}, y_{17}],$$

and

$$2y_{23} = [-2y_{22}, y_2] = [y_{21}, -y_6] = [2y_{20}, y_9] = [y_{19}, y_{10}] = [-y_{18}, y_{13}] = [2y_{16}, y_{14}].$$

Thus, putting

$$\begin{aligned} &W_1 = 2y_{24}, \quad W_2 = 2y_{23}, \\ &Z_1 = -2y_{22}, \quad Z_2 = y_{21}, \quad Z_3 = 2y_{20}, \quad Z_4 = y_{19}, \quad Z_5 = -y_{18}, \quad Z_6 = 2y_{16}, \\ &Y_1 = y_5, \quad Y_2 = y_8, \quad Y_3 = y_{11}, \quad Y_4 = -y_{12}, \quad Y_5 = y_{15}, \quad Y_6 = y_{17}, \\ &X_1 = y_2, \quad X_2 = -y_6, \quad X_3 = y_9, \quad X_4 = y_{10}, \quad X_5 = y_{13}, \quad X_6 = y_{14} \end{aligned}$$

we obtain the basis $\{W_1, W_2, Z_1, \dots, Z_6, Y_1, \dots, Y_6, X_1, \dots, X_6\}$ of $\mathfrak{m}$ satisfying the following:

$$\mathfrak{g}_{-3} = \langle\{W_1, W_2\}\rangle, \quad \mathfrak{g}_{-2} = \langle\{Z_1, \dots, Z_6\}\rangle, \quad \mathfrak{g}_{-1} = \langle\{Y_1, \dots, Y_6, X_1, \dots, X_6\}\rangle$$

such that

$$[Z_i, Y_j] = \delta^i_j W_1, \qquad [Z_i, X_j] = \delta^i_j W_2 \qquad [X_i, X_j] = [Y_i, Y_j] = 0 \quad (1 \leqq i, j \leqq 6).$$

Then we calculate $[X_j, Y_k]$ for $1 \leqq j, k \leqq 6$ and obtain

$$\begin{aligned} Z_1 &= \ \ 2[X_3, Y_6] = -[X_5, Y_5] = 2[X_6, Y_3], \\ Z_2 &= \ \ [X_2, Y_6] = [X_4, Y_5] = [X_5, Y_4] = [X_6, Y_2], \\ Z_3 &= \ \ 2[X_1, Y_6] = [X_4, Y_4] = 2[X_6, Y_1], \\ Z_4 &= \ \ [X_2, Y_5] = [X_3, Y_4] = [X_4, Y_3] = [X_5, Y_2], \\ Z_5 &= -[X_1, Y_5] = [X_2, Y_4] = [X_4, Y_2] = -[X_5, Y_1], \\ Z_6 &= \ \ 2[X_1, Y_3] = [X_2, Y_2] = 2[X_3, Y_1]. \end{aligned}$$

Here we define the bilinear forms $f_i(x_1,\ldots,x_6,y_1,\ldots,y_6)(i=1,\ldots,6)$ as follows;

$$\begin{cases} f_1 = 2x_3\,y_6 - x_5\,y_5 + 2x_6\,y_3, \\ f_2 = x_2\,y_6 + x_4\,y_5 + x_5\,y_4 + x_6\,y_2, \\ f_3 = 2x_1\,y_6 + x_4\,y_4 + 2x_6\,y_1, \\ f_4 = x_2\,y_5 + x_3\,y_4 + x_4\,y_3 + x_5\,y_2, \\ f_5 = -x_1\,y_5 + x_2\,y_4 + x_4\,y_2 - x_5\,y_1, \\ f_6 = 2x_1\,y_3 + x_2\,y_2 + 2x_3\,y_1. \end{cases}$$

Observe that $f_i(x_k,y_k)$ is symmetric in x_k and y_k for $i=1,\ldots,6$. Moreover we put

$$\sum_{i=1}^{6} x_i\,f_i = 2\sum_{i=1}^{6} y_i\,g_i$$

where the quadratic forms $g_i(x_1,\ldots,x_6)(i=1,\ldots,6)$ are given by

$$\begin{cases} g_1 = 2x_3\,x_6 - \frac{1}{2}x_5^2, \quad g_2 = x_2\,x_6 + x_4\,x_5, \quad g_3 = 2x_1\,x_6 + \frac{1}{2}x_4^2, \\ g_4 = x_2\,x_5 + x_3\,x_4, \quad g_5 = -x_1\,x_5 + x_2\,x_4, \quad g_6 = 2x_1\,x_3 + \frac{1}{2}x_2^2. \end{cases}$$

Thus we have $g_i(x_1,\ldots,x_6) = g_i(x_k) = \frac{1}{2}f_i(x_k,x_k)$ for $i=1,\cdots,6$. Moreover we have

$$\sum_{i=1}^{6} f_i\,dx_i = \sum_{i=1}^{6} y_i\,dg_i, \quad \sum_{i=1}^{6} g_i\,dx_i = dg, \quad 3g = \sum_{i=1}^{6} x_i\,g_i,$$

where $g(x_1,\ldots,x_6)$ is the cubic form given by

$$g = 2x_1\,x_3\,x_6 + x_2\,x_4\,x_5 + \tfrac{1}{2}(-x_1\,x_5^2 + x_2^2x_6 + x_3x_4^2).$$

Thus, by (4.1), we obtain

$$\begin{aligned}
\omega_1 &= dP_2 + (2a_3a_6 - \tfrac{1}{2}a_5^2)dX_1 - 2a_6dX_4 + a_5dX_6 - 2a_3dX_7, \\
\omega_2 &= dP_3 + (a_2a_6 + a_4a_5)dX_1 - a_6dX_3 - a_5dX_5 - a_4dX_6 - a_2dX_7, \\
\omega_3 &= dP_4 + (2a_1a_6 + \tfrac{1}{2}a_4^2)dX_1 - 2a_6dX_2 - a_4dX_5 - 2a_1dX_7, \\
\omega_4 &= dP_5 + (a_2a_5 + a_3a_4)dX_1 - a_5dX_3 - a_4dX_4 - a_3dX_5 - a_2dX_6, \\
\omega_5 &= dP_6 + (-a_1a_5 + a_2a_4)dX_1 + a_5dX_2 - a_4dX_3 - a_2dX_5 + a_1dX_6, \\
\omega_6 &= dP_7 + (2a_1a_3 + \tfrac{1}{2}a_2^2)dX_1 - 2a_3dX_2 - a_2dX_3 - 2a_1dX_4, \\
\varpi_2 &= a_1\omega_1 + a_2\omega_2 + \cdots + a_6\omega_6 \\
&\quad + dP_1 - (4a_1a_3a_6 + 2a_2a_4a_5 - a_1a_5^2 + a_2^2a_6 + a_3a_4^2)dX_1 \\
&\quad + (2a_3a_6 - \tfrac{1}{2}a_5^2)dX_2 + (a_2a_6 + a_4a_5)dX_3 + (2a_1a_6 + \tfrac{1}{2}a_4^2)dX_4 \\
&\quad + (a_2a_5 + a_3a_4)dX_5 + (-a_1a_5 + a_2a_4)dX_6 + (2a_1a_3 + \tfrac{1}{2}a_2^2)dX_7.
\end{aligned}$$

This implies that our model system $R(X)$ of type F_4 is given by the following 22 equations;

$$-P_{66} = \tfrac{1}{2}P_{47}\ (= a_1),\ P_{37} = P_{56}\ (= a_2),\ P_{55} = \tfrac{1}{2}P_{27}\ (= a_3),$$

$$P_{36} = P_{45}\ (= a_4),\ P_{35} = -P_{26}\ (= a_5),\ P_{33} = \tfrac{1}{2}P_{24}\ (= a_6),$$

$$P_{22} = P_{23} = P_{25} = P_{34} = P_{44} = P_{46} = P_{57} = P_{67} = P_{77} = 0,$$

$$P_{17} = 2P_{55}P_{66} - \tfrac{1}{2}P_{37}^2,\ P_{16} = -P_{35}P_{66} - P_{36}P_{37},\ P_{15} = -P_{35}P_{37} - P_{36}P_{55},$$

$$P_{14} = 2P_{33}P_{66} - \tfrac{1}{2}P_{36}^2,\ P_{13} = -P_{33}P_{37} - P_{35}P_{36},\ P_{12} = \tfrac{1}{2}P_{35}^2 - 2P_{33}P_{55},$$

$$P_{11} = -4P_{33}P_{55}P_{66} + 2P_{35}P_{36}P_{37} + P_{35}^2P_{66} + P_{33}P_{37}^2 + P_{36}^2P_{55}$$

in terms of the canonical coordinate $(X_1, \ldots, X_7, Z, P_1, \ldots, P_7, P_{11}, \ldots, P_{77})$ of $L(J)$.

4.2. Goursat gradation and model system of type E_6. In the following subsections , let us fix the root space decomposition of Simple Lie algebra E_8 and regard E_6 and E_7 as the regular subalgebras of E_8.

Let $\mathfrak{m}$ be the Goursat gradation of type E_6. For $(E_6, \{\alpha_4\})$, we have

$$\begin{aligned}
\Phi_3^+ = \{&\gamma_{38} = \begin{smallmatrix}1&2&3&2&1\\&&2&&\end{smallmatrix}, \gamma_{37} = \begin{smallmatrix}1&2&3&2&1\\&&1&&\end{smallmatrix}\},\\
\Phi_2^+ = \{&\gamma_{36} = \begin{smallmatrix}1&2&2&2&1\\&&1&&\end{smallmatrix}, \gamma_{35} = \begin{smallmatrix}1&2&2&1&1\\&&1&&\end{smallmatrix}, \gamma_{34} = \begin{smallmatrix}1&1&2&2&1\\&&1&&\end{smallmatrix}, \gamma_{33} = \begin{smallmatrix}1&1&2&1&1\\&&1&&\end{smallmatrix}, \gamma_{32} = \begin{smallmatrix}1&2&2&1&0\\&&1&&\end{smallmatrix},\\
&\gamma_{31} = \begin{smallmatrix}0&1&2&2&1\\&&1&&\end{smallmatrix}, \gamma_{29} = \begin{smallmatrix}1&1&2&1&0\\&&1&&\end{smallmatrix}, \gamma_{28} = \begin{smallmatrix}0&1&2&1&1\\&&1&&\end{smallmatrix}, \gamma_{24} = \begin{smallmatrix}0&1&2&1&0\\&&1&&\end{smallmatrix}\},\\
\Phi_1^+ = \{&\gamma_{10} = \begin{smallmatrix}0&0&1&0&0\\&&1&&\end{smallmatrix}, \gamma_{14} = \begin{smallmatrix}0&0&1&1&0\\&&1&&\end{smallmatrix}, \gamma_{15} = \begin{smallmatrix}0&1&1&0&0\\&&1&&\end{smallmatrix}, \gamma_{19} = \begin{smallmatrix}0&1&1&1&0\\&&1&&\end{smallmatrix}, \gamma_{20} = \begin{smallmatrix}0&0&1&1&1\\&&1&&\end{smallmatrix},\\
&\gamma_{21} = \begin{smallmatrix}1&1&1&0&0\\&&1&&\end{smallmatrix}, \gamma_{25} = \begin{smallmatrix}0&1&1&1&1\\&&1&&\end{smallmatrix}, \gamma_{26} = \begin{smallmatrix}1&1&1&1&0\\&&1&&\end{smallmatrix}, \gamma_{30} = \begin{smallmatrix}1&1&1&1&1\\&&1&&\end{smallmatrix},\\
&\gamma_{4} = \begin{smallmatrix}0&0&1&0&0\\&&0&&\end{smallmatrix}, \gamma_{11} = \begin{smallmatrix}0&0&1&1&0\\&&0&&\end{smallmatrix}, \gamma_{12} = \begin{smallmatrix}0&1&1&0&0\\&&0&&\end{smallmatrix}, \gamma_{16} = \begin{smallmatrix}0&1&1&1&0\\&&0&&\end{smallmatrix}, \gamma_{17} = \begin{smallmatrix}0&0&1&1&1\\&&0&&\end{smallmatrix},\\
&\gamma_{18} = \begin{smallmatrix}1&1&1&0&0\\&&0&&\end{smallmatrix}, \gamma_{22} = \begin{smallmatrix}0&1&1&1&1\\&&0&&\end{smallmatrix}, \gamma_{23} = \begin{smallmatrix}1&1&1&1&0\\&&0&&\end{smallmatrix}, \gamma_{27} = \begin{smallmatrix}1&1&1&1&1\\&&0&&\end{smallmatrix}\}
\end{aligned}$$

where $\begin{smallmatrix}a_1&a_3&a_4&a_5&a_6\\&&a_2&&\end{smallmatrix}$ stands for the root $\gamma = \sum_{i=1}^{6} a_i\alpha_i \in \Phi^+$. Also we put $\gamma_9 = \begin{smallmatrix}0&0&0&1&1\\&&0&&\end{smallmatrix}$ and $\gamma_{13} = \begin{smallmatrix}1&1&0&0&0\\&&0&&\end{smallmatrix}$ for later use.

We fix the orientation (or sign) of y_γ as in the following: First we choose the orientation of y_{α_i} for simple roots by fixing the root vectors $y_i = y_{\alpha_i} \in \mathfrak{g}_{-\alpha_i}$ for $i = 1, \ldots, 8$. For $\gamma_i \in \Phi^+ (i = 1, \ldots, 38)$, we put $y_i = y_{\gamma_i}$ and fix the orientation by the following order;

$$\begin{array}{lllll}
y_9 = [y_5, y_6], & y_{10} = [y_2, y_4], & y_{11} = [y_4, y_5], & y_{12} = [y_3, y_4], & y_{13} = [y_1, y_3],\\
y_{14} = [y_2, y_{11}], & y_{15} = [y_2, y_{12}], & y_{16} = [y_3, y_{11}], & y_{17} = [y_4, y_9], & y_{18} = [y_4, y_{13}],\\
y_{19} = [y_2, y_{16}], & y_{20} = [y_2, y_{17}], & y_{21} = [y_2, y_{18}], & y_{22} = [y_3, y_{17}], & y_{23} = [y_1, y_{16}],\\
y_{24} = [y_4, y_{19}], & y_{25} = [y_2, y_{22}], & y_{26} = [y_2, y_{23}], & y_{27} = [y_1, y_{22}], & y_{28} = [y_4, y_{25}],\\
y_{29} = [y_2, y_{26}], & y_{30} = [y_2, y_{27}], & y_{31} = [y_5, y_{28}], & y_{32} = [y_3, y_{29}], & y_{33} = [y_4, y_{30}],\\
y_{34} = [y_1, y_{31}], & y_{35} = [y_3, y_{33}], & y_{36} = [y_3, y_{34}], & y_{37} = [y_4, y_{36}], & y_{38} = [y_2, y_{37}],
\end{array}$$

Then, by the repeated application of Jacobi identities, we obtain

$$\begin{aligned}
y_{38} = [-y_{36}, y_{10}] = [-y_{35}, y_{14}] = [y_{34}, y_{15}] = [y_{33}, y_{19}] = [y_{32}, y_{20}]\\
= [y_{31}, y_{21}] = [-y_{29}, y_{25}] = [-y_{28}, y_{26}] = [y_{24}, y_{30}],
\end{aligned}$$

$$\begin{aligned}
y_{37} = [-y_{36}, y_4] = [-y_{35}, y_{11}] = [y_{34}, y_{12}] = [y_{33}, y_{16}] = [y_{32}, y_{17}]\\
= [y_{31}, y_{18}] = [-y_{29}, y_{22}] = [-y_{28}, y_{23}] = [y_{24}, y_{27}],
\end{aligned}$$

Thus, putting

$$W_1 = y_{38}, \quad W_2 = y_{37},$$

$$Z_1 = -y_{36}, \quad Z_2 = -y_{35}, \quad Z_3 = y_{34}, \quad Z_4 = y_{33}, \quad Z_5 = y_{32},$$
$$Z_6 = y_{31}, \quad Z_7 = -y_{29}, \quad Z_8 = -y_{28}, \quad Z_9 = y_{24},$$
$$Y_1 = y_{10}, \quad Y_2 = y_{14}, \quad Y_3 = y_{15}, \quad Y_4 = y_{19}, \quad Y_5 = y_{20},$$
$$Y_6 = y_{21}, \quad Y_7 = y_{25}, \quad Y_8 = y_{26}, \quad Y_9 = y_{30},$$
$$X_1 = y_4, \quad X_2 = y_{11}, \quad X_3 = y_{12}, \quad X_4 = y_{16}, \quad X_5 = y_{17},$$
$$X_6 = y_{18}, \quad X_7 = y_{22}, \quad X_8 = y_{23}, \quad X_9 = y_{27},$$

we obtain the basis $\{W_1, W_2, Z_1, \ldots, Z_9, Y_1, \ldots, Y_9, X_1, \ldots, X_9\}$ of $\mathfrak{m}$ satisfying the following:

$$\mathfrak{g}_{-3} = \langle\{W_1, W_2\}\rangle, \quad \mathfrak{g}_{-2} = \langle\{Z_1, \ldots, Z_9\}\rangle, \quad \mathfrak{g}_{-1} = \langle\{Y_1, \ldots, Y_9, X_1, \ldots, X_9\}\rangle$$

such that

$$[Z_i, Y_j] = \delta^i_j W_1, \qquad [Z_i, X_j] = \delta^i_j W_2 \qquad [X_i, X_j] = [Y_i, Y_j] = 0 \quad (1 \leqq i, j \leqq 9).$$

Then we calculate $[X_j, Y_k]$ for $1 \leqq j, k \leqq 9$ and obtain

$$\begin{aligned}
Z_1 &= [X_4, Y_9] = -[X_7, Y_8] = -[X_8, Y_7] = [X_9, Y_4],\\
Z_2 &= -[X_3, Y_9] = -[X_6, Y_7] = -[X_7, Y_6] = -[X_9, Y_3],\\
Z_3 &= -[X_2, Y_9] = [X_5, Y_8] = [X_8, Y_5] = -[X_9, Y_2],\\
Z_4 &= [X_1, Y_9] = [X_5, Y_6] = [X_6, Y_5] = [X_9, Y_1],\\
Z_5 &= [X_3, Y_8] = [X_4, Y_6] = [X_6, Y_4] = [X_8, Y_3],\\
Z_6 &= -[X_2, Y_7] = [X_4, Y_5] = [X_5, Y_4] = -[X_7, Y_2],\\
Z_7 &= -[X_1, Y_8] = -[X_2, Y_6] = -[X_6, Y_2] = -[X_8, Y_1],\\
Z_8 &= -[X_1, Y_7] = [X_3, Y_5] = [X_5, Y_3] = -[X_7, Y_1],\\
Z_9 &= [X_1, Y_4] = -[X_2, Y_3] = -[X_3, Y_2] = [X_4, Y_1].
\end{aligned}$$

Here we define the bilinear forms $f_i(x_1, \ldots, x_9, y_1, \ldots, y_9)(i = 1, \ldots, 9)$ as follows;

$$\begin{cases}
f_1 = x_4\, y_9 - x_7\, y_8 - x_8\, y_7 + x_9\, y_4,\\
f_2 = -x_3\, y_9 - x_6\, y_7 - x_7\, y_6 - x_9\, y_3,\\
f_3 = -x_2\, y_9 + x_5\, y_8 + x_8\, y_5 - x_9\, y_2,\\
f_4 = x_1\, y_9 + x_5\, y_6 + x_6\, y_5 + x_9\, y_1,\\
f_5 = x_3\, y_8 + x_4\, y_6 + x_6\, y_4 + x_8\, y_3,\\
f_6 = -x_2\, y_7 + x_4\, y_5 + x_5\, y_4 - x_7\, y_2,\\
f_7 = -x_1\, y_8 - x_2\, y_6 - x_6\, y_2 - x_8\, y_1,\\
f_8 = -x_1\, y_7 + x_3\, y_5 + x_5\, y_3 - x_7\, y_1,\\
f_9 = x_1\, y_4 - x_2\, y_3 - x_3\, y_2 + x_4\, y_1.
\end{cases}$$

Observe that $f_i(x_k, y_k)$ is symmetric in x_k and y_k for $i = 1, \ldots, 9$. Moreover we put

$$\sum_{i=1}^{9} x_i\, f_i = 2 \sum_{i=1}^{9} y_i\, g_i$$

where the quadratic forms $g_i(x_1, \ldots, x_9)(i = 1, \ldots, 9)$ are given by

$$\begin{cases}
g_1 = x_4\, x_9 - x_7\, x_8, \quad g_2 = -x_3\, x_9 - x_6\, x_7, \quad g_3 = x_5\, x_8 - x_2\, x_9,\\
g_4 = x_1\, x_9 + x_5\, x_6, \quad g_5 = x_3\, x_8 + x_4\, x_6, \quad g_6 = x_4\, x_5 - x_2\, x_7,\\
g_7 = -x_1\, x_8 - x_2\, x_6, \quad g_8 = x_3\, x_5 - x_1\, x_7, \quad g_9 = x_1\, x_4 - x_2\, x_3.
\end{cases}$$

Thus we have $g_i(x_1,\dots,x_9) = g_i(x_k) = \frac{1}{2}f_i(x_k,x_k)$ for $i = 1,\cdots,9$. Moreover we have

$$\sum_{i=1}^{9} f_i\,dx_i = \sum_{i=1}^{9} y_i\,dg_i, \quad \sum_{i=1}^{9} g_i\,dx_i = dg, \quad 3g = \sum_{i=1}^{9} x_i\,g_i,$$

where $g(x_1,\dots,x_9)$ is the cubic form given by

$$g = x_1\,x_4\,x_9 - x_1\,x_7\,x_8 - x_2\,x_3\,x_9 - x_2\,x_6\,x_7 + x_3\,x_5\,x_8 + x_4\,x_5\,x_6.$$

Thus, by (4.1), we obtain

$$\begin{aligned}
\omega_1 &= dP_2 + (a_4a_9 - a_7a_8)dX_1 - a_9dX_5 + a_8dX_8 + a_7dX_9 - a_4dX_{10},\\
\omega_2 &= dP_3 - (a_3a_9 + a_6a_7)dX_1 + a_9dX_4 + a_7dX_7 + a_6dX_8 + a_3dX_{10},\\
\omega_3 &= dP_4 + (a_5a_8 - a_2a_9)dX_1 + a_9dX_3 - a_8dX_6 - a_5dX_9 + a_2dX_{10},\\
\omega_4 &= dP_5 + (a_1a_9 + a_5a_6)dX_1 - a_9dX_2 - a_6dX_6 - a_5dX_7 - a_1X_{10},\\
\omega_5 &= dP_6 + (a_3a_8 + a_4a_6)dX_1 - a_8dX_4 - a_6dX_5 - a_4dX_7 - a_3dX_9,\\
\omega_6 &= dP_7 + (a_4a_5 - a_2a_7)dX_1 + a_7dX_3 - a_5dX_5 - a_4dX_6 + a_2X_8,\\
\omega_7 &= dP_8 - (a_1a_8 + a_2a_6)dX_1 + a_8dX_2 + a_6dX_3 + a_2dX_7 + a_1dX_9,\\
\omega_8 &= dP_9 + (a_3a_5 - a_1a_7)dX_1 + a_7dX_2 - a_5dX_4 - a_3dX_6 + a_1dX_8,\\
\omega_9 &= dP_{10} + (a_1a_4 - a_2a_3)dX_1 - a_4dX_2 + a_3dX_3 + a_2dX_4 - a_1dX_5,\\
\varpi_2 &= a_1\omega_1 + a_2\omega_2 + \cdots + a_9\omega_9\\
&\quad + dP_1 - 2(a_1a_4a_9 - a_1a_7a_8 - a_2a_3a_9 - a_2a_6a_7 + a_3a_5a_8 + a_4a_5a_6)dX_1\\
&\quad + (a_4a_9 - a_7a_8)dX_2 - (a_3a_9 + a_6a_7)dX_3 + (a_5a_8 - a_2a_9)dX_4\\
&\quad + (a_1a_9 + a_5a_6)dX_5 + (a_3a_8 + a_4a_6)dX_6 + (a_4a_5 - a_2a_7)dX_7\\
&\quad - (a_1a_8 + a_2a_6)dX_8 + (a_3a_5 - a_1a_7)dX_9 + (a_1a_4 - a_2a_3)dX_{10}.
\end{aligned}$$

This implies $R(X)$ is given by the following 46 equations;

$$\begin{aligned}
&P_{5,10} = -P_{89}\ (= a_1),\ P_{78} = P_{4,10}\ (= -a_2),\ P_{69} = -P_{3,10}\ (= a_3),\\
&P_{67} = P_{2,10}\ (= a_4),\ P_{49} = P_{57}\ (= a_5),\ P_{56} = -P_{38}\ (= a_6),\\
&P_{29} = P_{37}\ (= -a_7),\ P_{46} = -P_{28}\ (= a_8),\ P_{25} = -P_{34}\ (= a_9)\\
&P_{22} = P_{23} = P_{24} = P_{26} = P_{27} = P_{33} = P_{35} = P_{36}\\
&\qquad = P_{39} = P_{44} = P_{45} = P_{47} = P_{48} = P_{55} = 0,\\
&P_{58} = P_{59} = P_{66} = P_{68} = P_{6,10} = P_{77} = P_{79} = P_{7,10}\\
&\qquad = P_{88} = P_{8,10} = P_{99} = P_{9,10} = P_{10,10} = 0,\\
&P_{1,10} = -P_{78}P_{69} - P_{5,10}P_{67},\ P_{19} = -P_{5,10}P_{29} - P_{69}P_{49},\\
&\qquad P_{18} = P_{5,10}P_{46} - P_{78}P_{56},\\
&P_{17} = -P_{78}P_{29} - P_{67}P_{49},\ P_{16} = -P_{69}P_{46} - P_{67}P_{56},\\
&\qquad P_{15} = -P_{5,10}P_{25} - P_{49}P_{56},\\
&P_{14} = -P_{78}P_{25} - P_{49}P_{46},\ P_{13} = P_{69}P_{25} - P_{56}P_{29},\ P_{12} = -P_{29}P_{46} - P_{67}P_{25},\\
&P_{11} = 2(P_{5,10}P_{67}P_{25} + P_{5,10}P_{29}P_{46} + P_{78}P_{69}P_{25}\\
&\qquad - P_{78}P_{56}P_{29} + P_{69}P_{49}P_{46} + P_{67}P_{49}P_{56}).
\end{aligned}$$

in coordinates $(X_1,\dots,X_{10},Z,P_1,\dots,P_{10},P_{11},\dots,P_{10,10})$ of $L(J)$.

4.3. Goursat gradation and model system of type E_7. Let $\mathfrak{m}$ be the Goursat gradation of type E_7. For $(E_7, \{\alpha_3\})$, we have

$$
\begin{aligned}
\Phi_3^+ = \{&\gamma_{64} = \substack{2\,3\,4\,3\,2\,1\\2}, \gamma_{63} = \substack{1\,3\,4\,3\,2\,1\\2}\},\\
\Phi_2^+ = \{&\gamma_{62} = \substack{1\,2\,4\,3\,2\,1\\2}, \gamma_{61} = \substack{1\,2\,3\,3\,2\,1\\2}, \gamma_{60} = \substack{1\,2\,3\,3\,2\,1\\1}, \gamma_{59} = \substack{1\,2\,3\,2\,2\,1\\2},\\
&\gamma_{58} = \substack{1\,2\,3\,2\,2\,1\\1}, \gamma_{57} = \substack{1\,2\,3\,2\,1\,1\\2}, \gamma_{56} = \substack{1\,2\,2\,2\,2\,1\\1}, \gamma_{55} = \substack{1\,2\,3\,2\,1\,1\\1},\\
&\gamma_{53} = \substack{1\,2\,2\,2\,1\,1\\1}, \gamma_{50} = \substack{1\,2\,2\,1\,1\,1\\1}, \gamma_{38} = \substack{1\,2\,3\,2\,1\,0\\2}, \gamma_{37} = \substack{1\,2\,3\,2\,1\,0\\1},\\
&\gamma_{36} = \substack{1\,2\,2\,2\,1\,0\\1}, \gamma_{35} = \substack{1\,2\,2\,1\,1\,0\\1}, \gamma_{32} = \substack{1\,2\,2\,1\,0\,0\\1}, \},\\
\Phi_1^+ = \{&\gamma_{13} = \substack{1\,1\,0\,0\,0\,0\\0}, \gamma_{18} = \substack{1\,1\,1\,0\,0\,0\\0}, \gamma_{21} = \substack{1\,1\,1\,0\,0\,0\\1}, \gamma_{23} = \substack{1\,1\,1\,1\,0\,0\\0},\\
&\gamma_{26} = \substack{1\,1\,1\,1\,0\,0\\1}, \gamma_{27} = \substack{1\,1\,1\,1\,1\,0\\0}, \gamma_{29} = \substack{1\,1\,2\,1\,0\,0\\1}, \gamma_{30} = \substack{1\,1\,1\,1\,1\,0\\1},\\
&\gamma_{33} = \substack{1\,1\,2\,1\,1\,0\\1}, \gamma_{34} = \substack{1\,1\,2\,2\,1\,0\\1}, \gamma_{45} = \substack{1\,1\,1\,1\,1\,1\\0}, \gamma_{47} = \substack{1\,1\,1\,1\,1\,1\\1},\\
&\gamma_{48} = \substack{1\,1\,2\,1\,1\,1\\1}, \gamma_{51} = \substack{1\,1\,2\,2\,1\,1\\1}, \gamma_{54} = \substack{1\,1\,2\,2\,2\,1\\1},\\
&\gamma_{3} = \substack{0\,1\,0\,0\,0\,0\\0}, \gamma_{12} = \substack{0\,1\,1\,0\,0\,0\\0}, \gamma_{15} = \substack{0\,1\,1\,0\,0\,0\\1}, \gamma_{16} = \substack{0\,1\,1\,1\,0\,0\\0},\\
&\gamma_{19} = \substack{0\,1\,1\,1\,0\,0\\1}, \gamma_{22} = \substack{0\,1\,1\,1\,1\,0\\0}, \gamma_{24} = \substack{0\,1\,2\,1\,0\,0\\1}, \gamma_{25} = \substack{0\,1\,1\,1\,1\,0\\1},\\
&\gamma_{28} = \substack{0\,1\,2\,1\,1\,0\\1}, \gamma_{31} = \substack{0\,1\,2\,2\,1\,0\\1}, \gamma_{43} = \substack{0\,1\,1\,1\,1\,1\\0}, \gamma_{44} = \substack{0\,1\,1\,1\,1\,1\\1},\\
&\gamma_{46} = \substack{0\,1\,2\,1\,1\,1\\1}, \gamma_{49} = \substack{0\,1\,2\,2\,1\,1\\1}, \gamma_{52} = \substack{0\,1\,2\,2\,2\,1\\1}, \},
\end{aligned}
$$

where $\substack{a_1\ a_3\ a_4\ a_5\ a_6\ a_7\\ a_2}$ stands for the root $\gamma = \sum_{i=1}^{7} a_i\alpha_i \in \Phi^+$.

We fix the orientation (or sign) of y_γ as in the following: For $\gamma_i \in \Phi^+ (i = 39, \dots, 64)$, we put $y_i = y_{\gamma_i}$ and fix the orientation by the following order;

$$
\begin{aligned}
&y_{39} = [y_6, y_{11}],\ y_{40} = [y_5, y_{39}],\ y_{41} = [y_4, y_{40}],\ y_{42} = [y_2, y_{41}],\ y_{43} = [y_7, y_{22}],\\
&y_{44} = [y_2, y_{43}],\ y_{45} = [y_7, y_{27}],\ y_{46} = [y_4, y_{44}],\ y_{47} = [y_2, y_{45}],\ y_{48} = [y_4, y_{47}],\\
&y_{49} = [y_7, y_{31}],\ y_{50} = [y_7, y_{35}],\ y_{51} = [y_7, y_{34}],\ y_{52} = [y_6, y_{49}],\ y_{53} = [y_5, y_{50}],\\
&y_{54} = [y_6, y_{51}],\ y_{55} = [y_4, y_{53}],\ y_{56} = [y_6, y_{53}],\ y_{57} = [y_2, y_{55}],\ y_{58} = [y_4, y_{56}],\\
&y_{59} = [y_2, y_{58}],\ y_{60} = [y_5, y_{58}],\ y_{61} = [y_2, y_{60}],\ y_{62} = [y_4, y_{61}],\ y_{63} = [y_3, y_{62}],\\
&y_{64} = [y_1, y_{63}].
\end{aligned}
$$

Then, by the repeated application of Jacobi identities, we obtain

$$
\begin{aligned}
y_{64} &= [-y_{62}, y_{13}] = [y_{61}, y_{18}] = [y_{60}, -y_{21}] = [-y_{59}, y_{23}] = [y_{58}, y_{26}] = [-y_{57}, y_{27}]\\
&= [-y_{56}, y_{29}] = [y_{55}, y_{30}] = [-y_{53}, y_{33}] = [y_{50}, y_{34}] = [y_{38}, y_{45}] = [-y_{37}, y_{47}]\\
&= [y_{36}, y_{48}] = [-y_{35}, y_{51}] = [-y_{32}, y_{54}],\\
y_{63} &= [-y_{62}, y_3] = [y_{61}, -y_{12}] = [y_{60}, y_{15}] = [-y_{59}, y_{16}] = [y_{58}, y_{19}] = [-y_{57}, y_{22}]\\
&= [-y_{56}, y_{24}] = [y_{55}, y_{25}] = [-y_{53}, y_{28}] = [y_{50}, y_{31}] = [y_{38}, y_{43}] = [-y_{37}, y_{44}]\\
&= [y_{36}, y_{46}] = [-y_{35}, y_{49}] = [-y_{32}, y_{52}].
\end{aligned}
$$

Thus, putting

$$
\begin{aligned}
&W_1 = y_{64}, \quad W_2 = y_{63},\\
&Z_1 = -y_{62}, \quad Z_2 = y_{61}, \quad Z_3 = y_{60}, \quad Z_4 = -y_{59}, \quad Z_5 = y_{58}, \quad Z_6 = -y_{57},\\
&\qquad Z_7 = -y_{56}, \quad Z_8 = y_{55}, \quad Z_9 = -y_{53}, \quad Z_{10} = y_{50}, \quad Z_{11} = y_{38}, \quad Z_{12} = -y_{37},\\
&\qquad\qquad Z_{13} = y_{36}, \quad Z_{14} = -y_{35}, \quad Z_{15} = -y_{32},\\
&Y_1 = y_{13}, \quad Y_2 = y_{18}, \quad Y_3 = -y_{21}, \quad Y_4 = y_{23}, \quad Y_5 = y_{26}, \quad Y_6 = y_{27}, \quad Y_7 = y_{29},\\
&\qquad Y_8 = y_{30}, \quad Y_9 = y_{33}, \quad Y_{10} = y_{34}, \quad Y_{11} = y_{45}, \quad Y_{12} = y_{47}, \quad Y_{13} = y_{48},
\end{aligned}
$$

$$
\begin{gathered}
Y_{14} = y_{51}, \quad Y_{15} = y_{54}, \\
X_1 = y_3, \quad X_2 = -y_{12}, \quad X_3 = y_{15}, \quad X_4 = y_{16}, \quad X_5 = y_{19}, \quad X_6 = y_{22}, \\
X_7 = y_{24}, \quad X_8 = y_{25}, \quad X_9 = y_{28}, \quad X_{10} = y_{31}, \quad X_{11} = y_{43}, \quad X_{12} = y_{44}, \\
X_{13} = y_{46}, \quad X_{14} = y_{49}, \quad X_{15} = y_{52}.
\end{gathered}
$$

We obtain the basis $\{W_1, W_2, Z_1, \ldots, Z_{15}, Y_1, \ldots, Y_{15}, X_1, \ldots, X_{15}\}$ of $\mathfrak{m}$ satisfying the following:

$$
\mathfrak{g}_{-3} = \langle \{W_1, W_2\} \rangle, \quad \mathfrak{g}_{-2} = \langle \{Z_1, \ldots, Z_{15}\} \rangle, \quad \mathfrak{g}_{-1} = \langle \{Y_1, \ldots, Y_{15}, X_1, \ldots, X_{15}\} \rangle
$$

such that

$$
[Z_i, Y_j] = \delta^i_j W_1, \qquad [Z_i, X_j] = \delta^i_j W_2 \qquad [X_i, X_j] = [Y_i, Y_j] = 0 \quad (1 \leqq i, j \leqq 15).
$$

Then we calculate $[X_j, Y_k]$ for $1 \leqq j, k \leqq 15$ and obtain

$$
\begin{aligned}
Z_1 &= -[X_7, Y_{15}] = -[X_9, Y_{14}] = [X_{10}, Y_{13}] = [X_{13}, Y_{10}] = -[X_{14}, Y_9] = -[X_{15}, Y_7], \\
Z_2 &= [X_5, Y_{15}] = [X_8, Y_{14}] = -[X_{10}, Y_{12}] = -[X_{12}, Y_{10}] = [X_{14}, Y_8] = [X_{15}, Y_5], \\
Z_3 &= [X_4, Y_{15}] = [X_6, Y_{14}] = -[X_{10}, Y_{11}] = -[X_{11}, Y_{10}] = [X_{14}, Y_6] = [X_{15}, Y_4], \\
Z_4 &= [X_3, Y_{15}] = -[X_8, Y_{13}] = [X_9, Y_{12}] = [X_{12}, Y_9] = -[X_{13}, Y_8] = [X_{15}, Y_3], \\
Z_5 &= [X_2, Y_{15}] = [X_6, Y_{13}] = -[X_9, Y_{11}] = -[X_{11}, Y_9] = [X_{13}, Y_6] = [X_{15}, Y_2], \\
Z_6 &= [X_3, Y_{14}] = [X_5, Y_{13}] = -[X_7, Y_{12}] = -[X_{12}, Y_7] = [X_{13}, Y_5] = [X_{14}, Y_3], \\
Z_7 &= -[X_1, Y_{15}] = -[X_6, Y_{12}] = [X_8, Y_{11}] = [X_{11}, Y_8] = -[X_{12}, Y_6] = -[X_{15}, Y_1], \\
Z_8 &= [X_2, Y_{14}] = -[X_4, Y_{13}] = [X_7, Y_{11}] = [X_{11}, Y_7] = -[X_{13}, Y_4] = [X_{14}, Y_2], \\
Z_9 &= -[X_1, Y_{14}] = [X_4, Y_{12}] = -[X_5, Y_{11}] = -[X_{11}, Y_5] = [X_{12}, Y_4] = -[X_{14} Y_1], \\
Z_{10} &= [X_1, Y_{13}] = -[X_2, Y_{12}] = -[X_3, Y_{11}] = -[X_{11}, Y_3] = -[X_{12}, Y_2] = [X_{13}, Y_1], \\
Z_{11} &= -[X_3, Y_{10}] = -[X_5, Y_9] = [X_7, Y_8] = [X_8, Y_7] = -[X_9, Y_5] = -[X_{10}, Y_3], \\
Z_{12} &= -[X_2, Y_{10}] = [X_4, Y_9] = -[X_6, Y_7] = -[X_7, Y_6] = [X_9, Y_4] = -[X_{10}, Y_2], \\
Z_{13} &= [X_1, Y_{10}] = -[X_4, Y_8] = [X_5, Y_6] = [X_6, Y_5] = -[X_8, Y_4] = [X_{10}, Y_1], \\
Z_{14} &= -[X_1, Y_9] = [X_2, Y_8] = [X_3, Y_6] = [X_6, Y_3] = [X_8, Y_2] = -[X_9, Y_1], \\
Z_{15} &= -[X_1, Y_7] = [X_2, Y_5] = [X_3, Y_4] = [X_4, Y_3] = [X_5, Y_2] = -[X_7, Y_1].
\end{aligned}
$$

Here we define the bilinear forms $f_i(x_1,\ldots,x_{15},y_1,\ldots,y_{15})(i=1,\ldots,15)$ as follows;

$$\begin{cases} f_1 = -x_7\,y_{15} - x_9\,y_{14} + x_{10}\,y_{13} + x_{13}\,y_{10} - x_{14}\,y_9 - x_{15}\,y_7, \\ f_2 = \quad x_5\,y_{15} + x_8\,y_{14} - x_{10}\,y_{12} - x_{12}\,y_{10} + x_{14}\,y_8 + x_{15}\,y_5, \\ f_3 = \quad x_4\,y_{15} + x_6\,y_{14} - x_{10}\,y_{11} - x_{11}\,y_{10} + x_{14}\,y_6 + x_{15}\,y_4, \\ f_4 = \quad x_3\,y_{15} - x_8\,y_{13} + x_9\,y_{12} + x_{12}\,y_9 - x_{13}\,y_8 + x_{15}\,y_3, \\ f_5 = \quad x_2\,y_{15} + x_6\,y_{13} - x_9\,y_{11} - x_{11}\,y_9 + x_{13}\,y_6 + x_{15}\,y_2, \\ f_6 = \quad x_3\,y_{14} + x_5\,y_{13} - x_7\,y_{12} - x_{12}\,y_7 + x_{13}\,y_5 + x_{14}\,y_3, \\ f_7 = -x_1\,y_{15} - x_6\,y_{12} + x_8\,y_{11} + x_{11}\,y_8 - x_{12}\,y_6 - x_{15}\,y_1, \\ f_8 = \quad x_2\,y_{14} - x_4\,y_{13} + x_7\,y_{11} + x_{11}\,y_7 - x_{13}\,y_4 + x_{14}\,y_2, \\ f_9 = -x_1\,y_{14} + x_4\,y_{12} - x_5\,y_{11} - x_{11}\,y_5 + x_{12}\,y_4 - x_{14}\,y_1, \\ f_{10} = \quad x_1\,y_{13} - x_2\,y_{12} - x_3\,y_{11} - x_{11}\,y_3 - x_{12}\,y_2 + x_{13}\,y_1, \\ f_{11} = -x_3\,y_{10} - x_5\,y_9 + x_7\,y_8 + x_8\,y_7 - x_9\,y_5 - x_{10}\,y_3, \\ f_{12} = -x_2\,y_{10} + x_4\,y_9 - x_6\,y_7 - x_7\,y_6 + x_9\,y_4 - x_{10}\,y_2, \\ f_{13} = \quad x_1\,y_{10} - x_4\,y_8 + x_5\,y_6 + x_6\,y_5 - x_8\,y_4 + x_{10}\,y_1, \\ f_{14} = -x_1\,y_9 + x_2\,y_8 + x_3\,y_6 + x_6\,y_3 + x_8\,y_2 - x_9 y_1, \\ f_{15} = -x_1\,y_7 + x_2\,y_5 + x_3\,y_4 + x_4\,y_3 + x_5\,y_2 - x_7\,y_1. \end{cases}$$

Observe that $f_i(x_k,y_k)$ is symmetric in x_k and y_k for $i=1,\ldots,15$. Moreover we put

$$\sum_{i=1}^{15} x_i\,f_i = 2\sum_{i=1}^{15} y_i\,g_i$$

where the quadratic forms $g_i(x_1,\ldots,x_{15})(i=1,\ldots,15)$ are given by

$$\begin{cases} g_1 = -x_7\,x_{15} - x_9\,x_{14} + x_{10}\,x_{13},\ \ g_2 = x_5\,x_{15} + x_8\,x_{14} - x_{10}\,x_{12}, \\ g_3 = x_4\,x_{15} + x_6\,x_{14} - x_{10}\,x_{11},\ \ g_4 = x_3\,x_{15} - x_8\,x_{13} + x_9\,x_{12}, \\ g_5 = x_2\,x_{15} + x_6\,x_{13} - x_9\,x_{11},\ \ g_6 = x_3\,x_{14} + x_5\,x_{13} - x_7\,x_{12}, \\ g_7 = -x_1\,x_{15} - x_6\,x_{12} + x_8\,x_{11},\ \ g_8 = x_2\,x_{14} - x_4\,x_{13} + x_7\,x_{11}, \\ g_9 = -x_1\,x_{14} + x_4\,x_{12} - x_5\,x_{11},\ \ g_{10} = x_1\,x_{13} - x_2\,x_{12} - x_3\,x_{11}, \\ g_{11} = -x_3\,x_{10} - x_5\,x_9 + x_7\,x_8,\ \ g_{12} = -x_2\,x_{10} + x_4\,x_9 - x_6\,x_7, \\ g_{13} = x_1\,x_{10} - x_4\,x_8 + x_5\,x_6,\ \ g_{14} = -x_1\,x_9 + x_2\,x_8 + x_3\,x_6, \\ g_{15} = -x_1\,x_7 + x_2\,x_5 + x_3\,x_4. \end{cases}$$

Thus we have $g_i(x_1,\ldots,x_{15}) = g_i(x_k) = \frac{1}{2}f_i(x_k,x_k)$ for $i=1,\cdots,15$. Moreover we have

$$\sum_{i=1}^{15} f_i\,dx_i = \sum_{i=1}^{15} y_i\,dg_i, \quad \sum_{i=1}^{15} g_i\,dx_i = dg, \quad 3g = \sum_{i=1}^{15} x_i\,g_i,$$

where $g(x_1,\ldots,x_{15})$ is the cubic form given by

$$\begin{aligned} g = &-x_1\,x_7\,x_{15} - x_1\,x_9\,x_{14} + x_1\,x_{10}\,x_{13} + x_2\,x_5\,x_{15} + x_2\,x_8\,x_{14} - x_2\,x_{10}\,x_{12}. \\ &+x_3\,x_4\,x_{15} + x_3\,x_6\,x_{14} - x_3\,x_{10}\,x_{11} - x_4\,x_8\,x_{13} + x_4\,x_9\,x_{12} + x_5\,x_6\,x_{13} - x_5\,x_9\,x_{11} \\ &-x_6\,x_7\,x_{12} + x_7\,x_8\,x_{11}. \end{aligned}$$

Thus, by (4.1), we obtain

$$
\begin{aligned}
\omega_1 &= dP_2 + \hat{g}_1 dX_1 - (-a_{15}dX_8 - a_{14}dX_{10} + a_{13}dX_{11} + a_{10}dX_{14} - a_9 dX_{15} - a_7 dX_{16}),\\
\omega_2 &= dP_3 + \hat{g}_2 dX_1 - (a_{15}dX_6 + a_{14}dX_9 - a_{12}dX_{11} - a_{10}dX_{13} + a_8 dX_{15} + a_5 dX_{16}),\\
\omega_3 &= dP_4 + \hat{g}_3 dX_1 - (a_{15}dX_5 + a_{14}dX_7 - a_{11}dX_{11} - a_{10}dX_{12} + a_6 dX_{15} + a_4 dX_{16}),\\
\omega_4 &= dP_5 + \hat{g}_4 dX_1 - (a_{15}dX_4 - a_{13}dX_9 + a_{12}dX_{10} + a_9 dX_{13} - a_8 dX_{14} + a_3 dX_{16}),\\
\omega_5 &= dP_6 + \hat{g}_5 dX_1 - (a_{15}dX_3 + a_{13}dX_7 - a_{11}dX_{10} - a_9 X_{12} + a_6 dX_{14} + a_2 dX_{16}),\\
\omega_6 &= dP_7 + \hat{g}_6 dX_1 - (a_{14}dX_4 + a_{13}dX_6 - a_{12}dX_8 - a_7 dX_{13} + a_5 dX_{14} + a_3 dX_{15}),\\
\omega_7 &= dP_8 + \hat{g}_7 dX_1 - (-a_{15}dX_2 - a_{12}dX_7 + a_{11}dX_9 + a_8 dX_{12} - a_6 dX_{13} - a_1 dX_{16}),\\
\omega_8 &= dP_9 + \hat{g}_8 dX_1 - (a_{14}dX_3 - a_{13}dX_5 + a_{11}dX_8 + a_7 dX_{12} - a_4 dX_{14} + a_2 dX_{15}),\\
\omega_9 &= dP_{10} + \hat{g}_9 dX_1 - (-a_{14}dX_2 + a_{12}dX_5 - a_{11}dX_6 - a_5 dX_{12} + a_4 dX_{13} - a_1 dX_{15}),\\
\omega_{10} &= dP_{11} + \hat{g}_{10} dX_1 - (a_{13}dX_2 - a_{12}dX_3 - a_{11}dX_4 - a_3 dX_{12} - a_2 dX_{13} + a_1 dX_{14}),\\
\omega_{11} &= dP_{12} + \hat{g}_{11} dX_1 - (-a_{10}dX_4 - a_9 dX_6 + a_8 dX_8 + a_7 dX_9 - a_5 dX_{10} - a_3 dX_{11}),\\
\omega_{12} &= dP_{13} + \hat{g}_{12} dX_1 - (-a_{10}dX_3 + a_9 dX_5 - a_7 dX_7 - a_6 dX_8 + a_4 dX_{10} - a_2 dX_{11}),\\
\omega_{13} &= dP_{14} + \hat{g}_{13} dX_1 - (a_{10}dX_2 - a_8 dX_5 + a_6 dX_6 + a_5 dX_7 - a_4 dX_9 + a_1 dX_{11}),\\
\omega_{14} &= dP_{15} + \hat{g}_{14} dX_1 - (-a_9 dX_2 + a_8 dX_3 + a_6 dX_4 + a_3 dX_7 + a_2 dX_9 - a_1 dX_{10}),\\
\omega_{15} &= dP_{16} + \hat{g}_{15} dX_1 - (-a_7 dX_2 + a_5 dX_3 + a_4 dX_4 + a_3 dX_5 + a_2 dX_6 - a_1 dX_8),\\
\varpi_2 &= a_1\omega_1 + a_2\omega_2 + \cdots + a_{15}\omega_{15} + dP_1 - 2\hat{g}dX_1 + \hat{g}_1 dX_2. + \cdots + \hat{g}_{15} dX_{16}.
\end{aligned}
$$

This implies $R(X)$ is given by the following 121 equations;

$$
\begin{aligned}
&-P_{8,16} = -P_{10,15} = P_{11,14}\ (= a_1), \quad P_{6,16} = P_{9,15} = -P_{11,13}\ (= a_2),\\
&P_{5,16} = P_{7,15} = -P_{11,12}\ (= a_3), \quad P_{4,16} = -P_{9,14} = P_{10,13}\ (= a_4),\\
&P_{3,16} = P_{7,14} = -P_{10,12}\ (= a_5), \quad P_{4,15} = P_{6,14} = -P_{8,13}\ (= a_6),\\
&-P_{2,16} = -P_{7,13} = P_{9,12}\ (= a_7), \quad P_{3,15} = -P_{5,14} = P_{8,12}\ (= a_8),\\
&-P_{2,15} = P_{5,13} = -P_{6,12}\ (= a_9), \quad P_{2,14} = -P_{3,13} = -P_{4,12}\ (= a_{10}),\\
&-P_{4,11} = -P_{6,10} = P_{8,9}\ (= a_{11}), \quad -P_{3,11} = P_{5,10} = -P_{7,8}\ (= a_{12}),\\
&P_{2,11} = -P_{5,9} = P_{6,7}\ (= a_{13}), \quad -P_{2,10} = P_{3,9} = P_{4,7}\ (= a_{14}),\\
&-P_{2,8} = P_{3,6} = P_{4,5}\ (= a_{15}),
\end{aligned}
$$

$$
\begin{aligned}
P_{2,2} &= P_{2,3} = P_{2,4} = P_{2,5} = P_{2,6} = P_{2,7} = P_{2,9} = P_{2,12} = P_{2,13} = P_{3,3}\\
&= P_{3,4} = P_{3,5} = P_{3,7} = 0,\\
P_{3,8} &= P_{3,10} = P_{3,12} = P_{3,14} = P_{4,4} = P_{4,6} = P_{4,8} = P_{4,9} = P_{4,10} = P_{4,13}\\
&= P_{4,14} = P_{5,5} = 0,\\
P_{5,6} &= P_{5,7} = P_{5,8} = P_{5,11} = P_{5,12} = P_{5,15} = P_{6,6} = P_{6,8} = P_{6,9} = P_{6,11}\\
&= P_{6,13} = P_{6,15} = 0,\\
P_{7,7} &= P_{7,9} = P_{7,10} = P_{7,11} = P_{7,12} = P_{7,16} = P_{8,8} = P_{8,10} = P_{8,11} = P_{8,14}\\
&= P_{8,15} = 0,\\
P_{9,9} &= P_{9,10} = P_{9,11} = P_{9,13} = P_{9,16} = P_{10,10} = P_{10,11} = P_{10,14} = P_{10,16} = 0,\\
P_{11,11} &= P_{11,15} = P_{11,16} = P_{12,12} = P_{12,13} = P_{12,14} = P_{12,15} = P_{12,16} = 0,\\
P_{13,13} &= P_{13,14} = P_{13,15} = P_{13,16} = P_{14,14} = P_{14,15} = P_{14,16} = P_{15,15} = P_{15,16}\\
&= P_{16,16} = 0,
\end{aligned}
$$

$$P_{1,16} = P_{2,16}P_{8,16} - P_{3,16}P_{6,16} - P_{4,16}P_{5,16}, \quad P_{1,15} = -P_{2,15}P_{8,16} - P_{3,15}P_{6,16} - P_{4,15}P_{5,16},$$
$$P_{1,14} = P_{2,14}P_{8,16} + P_{3,15}P_{4,16} - P_{3,16}P_{4,15}, \quad P_{1,13} = P_{2,14}P_{5,16} + P_{2,15}P_{4,16} - P_{2,16}P_{4,15},$$
$$P_{1,12} = P_{2,14}P_{5,16} - P_{2,15}P_{3,16} + P_{2,15}P_{2,16}, \quad P_{1,11} = P_{2,11}P_{8,16} - P_{3,11}P_{6,16} + P_{4,11}P_{5,16},$$
$$P_{1,10} = P_{2,10}P_{8,16} + P_{3,11}P_{4,16} - P_{3,16}P_{4,11}, \quad P_{1,9} = P_{2,10}P_{6,16} + P_{2,11}P_{4,16} - P_{2,16}P_{4,11},$$
$$P_{1,8} = P_{2,8}P_{8,16} - P_{2,11}P_{4,15} + P_{3,15}P_{8,9}, \quad P_{1,7} = P_{2,10}P_{5,16} - P_{2,11}P_{3,16} + P_{2,16}P_{3,11},$$
$$P_{1,6} = P_{2,8}P_{6,16} - P_{2,11}P_{4,15} + P_{2,16}P_{4,11}, \quad P_{1,5} = P_{2,8}P_{5,16} + P_{2,11}P_{3,15} - P_{2,15}P_{3,11},$$
$$P_{1,4} = P_{2,8}P_{4,16} + P_{2,10}P_{4,15} - P_{2,14}P_{4,11}, \quad P_{1,3} = P_{2,8}P_{3,16} + P_{2,10}P_{3,15} - P_{2,14}P_{3,11},$$
$$P_{1,2} = P_{2,8}P_{2,16} + P_{2,10}P_{2,15} - P_{2,11}P_{2,14},$$

$$\begin{aligned} P_{1,1} = 2(&P_{2,8}P_{2,16}P_{8,16} + P_{2,10}P_{2,15}P_{8,16} - P_{2,11}P_{2,14}P_{8,16} - P_{2,8}P_{3,16}P_{6,16} \\ &- P_{2,10}P_{3,15}P_{6,16} + P_{2,14}P_{3,11}P_{6,16} - P_{2,8}P_{4,16}P_{5,16} - P_{2,10}P_{4,15}P_{5,16} \\ &+ P_{2,14}P_{4,11}P_{5,16} - P_{2,11}P_{3,15}P_{4,16} + P_{2,15}P_{3,11}P_{4,16} + P_{2,11}P_{3,16}P_{4,15} \\ &- P_{2,15}P_{3,16}P_{4,11} - P_{2,16}P_{3,11}P_{4,15} + P_{2,16}P_{3,15}P_{4,11}), \end{aligned}$$

in coordinates $(X_1, \ldots, X_{16}, Z, P_1, \ldots, P_{16}, P_{11}, \ldots, P_{16,16})$ of $L(J)$.

4.4. Goursat gradation and model system of type E_8. Let $\mathfrak{m}$ be the Goursat gradation of type E_8. For $(E_8, \{\alpha_7\})$, we have

$$\begin{aligned}
\Phi_3^+ = \{&\gamma_{120} = \begin{smallmatrix}2&4&6&5&4&3&2\\&&3\end{smallmatrix}, \gamma_{119} = \begin{smallmatrix}2&4&6&5&4&3&1\\&&3\end{smallmatrix}\}, \\
\Phi_2^+ = \{&\gamma_{118} = \begin{smallmatrix}2&4&6&5&4&2&1\\&&3\end{smallmatrix}, \gamma_{117} = \begin{smallmatrix}2&4&6&5&3&2&1\\&&3\end{smallmatrix}, \gamma_{116} = \begin{smallmatrix}2&4&6&4&3&2&1\\&&3\end{smallmatrix}, \gamma_{115} = \begin{smallmatrix}2&4&5&4&3&2&1\\&&3\end{smallmatrix}, \\
&\gamma_{114} = \begin{smallmatrix}2&4&5&4&3&2&1\\&&2\end{smallmatrix}, \gamma_{113} = \begin{smallmatrix}2&3&5&4&3&2&1\\&&3\end{smallmatrix}, \gamma_{112} = \begin{smallmatrix}2&3&5&4&3&2&1\\&&2\end{smallmatrix}, \gamma_{111} = \begin{smallmatrix}1&3&5&4&3&2&1\\&&3\end{smallmatrix}, \\
&\gamma_{110} = \begin{smallmatrix}2&3&4&4&3&2&1\\&&2\end{smallmatrix}, \gamma_{109} = \begin{smallmatrix}1&3&5&4&3&2&1\\&&2\end{smallmatrix}, \gamma_{108} = \begin{smallmatrix}1&3&4&4&3&2&1\\&&2\end{smallmatrix}, \gamma_{107} = \begin{smallmatrix}2&3&4&3&3&2&1\\&&2\end{smallmatrix}, \\
&\gamma_{106} = \begin{smallmatrix}1&2&4&4&3&2&1\\&&2\end{smallmatrix}, \gamma_{105} = \begin{smallmatrix}1&3&4&3&3&2&1\\&&2\end{smallmatrix}, \gamma_{104} = \begin{smallmatrix}2&3&4&3&2&2&1\\&&2\end{smallmatrix}, \gamma_{103} = \begin{smallmatrix}1&2&4&3&3&2&1\\&&2\end{smallmatrix}, \\
&\gamma_{102} = \begin{smallmatrix}1&3&4&3&2&2&1\\&&2\end{smallmatrix}, \gamma_{100} = \begin{smallmatrix}1&2&3&3&3&2&1\\&&2\end{smallmatrix}, \gamma_{99} = \begin{smallmatrix}1&2&4&3&2&2&1\\&&2\end{smallmatrix}, \gamma_{97} = \begin{smallmatrix}1&2&3&3&3&2&1\\&&1\end{smallmatrix}, \\
&\gamma_{96} = \begin{smallmatrix}1&2&3&3&2&2&1\\&&2\end{smallmatrix}, \gamma_{94} = \begin{smallmatrix}1&2&3&3&2&2&1\\&&1\end{smallmatrix}, \gamma_{93} = \begin{smallmatrix}1&2&3&2&2&2&1\\&&2\end{smallmatrix}, \gamma_{91} = \begin{smallmatrix}1&2&3&2&2&2&1\\&&1\end{smallmatrix}, \\
&\gamma_{88} = \begin{smallmatrix}1&2&2&2&2&2&1\\&&1\end{smallmatrix}, \gamma_{85} = \begin{smallmatrix}1&1&2&2&2&2&1\\&&1\end{smallmatrix}, \gamma_{82} = \begin{smallmatrix}0&1&2&2&2&2&1\\&&1\end{smallmatrix}, \}, \\
\Phi_1^+ = \{&\gamma_{65} = \begin{smallmatrix}0&0&0&0&0&1&1\\&&0\end{smallmatrix}, \gamma_{66} = \begin{smallmatrix}0&0&0&0&1&1&1\\&&0\end{smallmatrix}, \gamma_{67} = \begin{smallmatrix}0&0&0&1&1&1&1\\&&0\end{smallmatrix}, \gamma_{68} = \begin{smallmatrix}0&0&1&1&1&1&1\\&&0\end{smallmatrix}, \\
&\gamma_{69} = \begin{smallmatrix}0&0&1&1&1&1&1\\&&1\end{smallmatrix}, \gamma_{70} = \begin{smallmatrix}0&1&1&1&1&1&1\\&&0\end{smallmatrix}, \gamma_{71} = \begin{smallmatrix}0&1&1&1&1&1&1\\&&1\end{smallmatrix}, \gamma_{72} = \begin{smallmatrix}1&1&1&1&1&1&1\\&&0\end{smallmatrix}, \\
&\gamma_{73} = \begin{smallmatrix}0&1&2&1&1&1&1\\&&1\end{smallmatrix}, \gamma_{74} = \begin{smallmatrix}1&1&1&1&1&1&1\\&&1\end{smallmatrix}, \gamma_{75} = \begin{smallmatrix}1&1&2&1&1&1&1\\&&1\end{smallmatrix}, \gamma_{76} = \begin{smallmatrix}0&1&2&2&1&1&1\\&&1\end{smallmatrix}, \\
&\gamma_{77} = \begin{smallmatrix}1&2&2&1&1&1&1\\&&1\end{smallmatrix}, \gamma_{78} = \begin{smallmatrix}1&1&2&2&1&1&1\\&&1\end{smallmatrix}, \gamma_{79} = \begin{smallmatrix}0&1&2&2&2&1&1\\&&1\end{smallmatrix}, \gamma_{80} = \begin{smallmatrix}1&2&2&2&1&1&1\\&&1\end{smallmatrix}, \\
&\gamma_{81} = \begin{smallmatrix}1&1&2&2&2&1&1\\&&1\end{smallmatrix}, \gamma_{83} = \begin{smallmatrix}1&2&3&2&1&1&1\\&&1\end{smallmatrix}, \gamma_{84} = \begin{smallmatrix}1&2&2&2&2&1&1\\&&1\end{smallmatrix}, \gamma_{86} = \begin{smallmatrix}1&2&3&2&1&1&1\\&&2\end{smallmatrix}, \\
&\gamma_{87} = \begin{smallmatrix}1&2&3&2&2&1&1\\&&1\end{smallmatrix}, \gamma_{89} = \begin{smallmatrix}1&2&3&2&2&1&1\\&&2\end{smallmatrix}, \gamma_{90} = \begin{smallmatrix}1&2&3&3&2&1&1\\&&1\end{smallmatrix}, \gamma_{92} = \begin{smallmatrix}1&2&3&3&2&1&1\\&&2\end{smallmatrix}, \\
&\gamma_{95} = \begin{smallmatrix}1&2&4&3&2&1&1\\&&2\end{smallmatrix}, \gamma_{98} = \begin{smallmatrix}1&3&4&3&2&1&1\\&&2\end{smallmatrix}, \gamma_{101} = \begin{smallmatrix}2&3&4&3&2&1&1\\&&2\end{smallmatrix}, \\
&\gamma_{7} = \begin{smallmatrix}0&0&0&0&0&1&0\\&&0\end{smallmatrix}, \gamma_{39} = \begin{smallmatrix}0&0&0&0&1&1&0\\&&0\end{smallmatrix}, \gamma_{40} = \begin{smallmatrix}0&0&0&1&1&1&0\\&&0\end{smallmatrix}, \gamma_{41} = \begin{smallmatrix}0&0&1&1&1&1&0\\&&0\end{smallmatrix}, \\
&\gamma_{42} = \begin{smallmatrix}0&0&1&1&1&1&0\\&&1\end{smallmatrix}, \gamma_{43} = \begin{smallmatrix}0&1&1&1&1&1&0\\&&0\end{smallmatrix}, \gamma_{44} = \begin{smallmatrix}0&1&1&1&1&1&0\\&&1\end{smallmatrix}, \gamma_{45} = \begin{smallmatrix}1&1&1&1&1&1&0\\&&0\end{smallmatrix}, \\
&\gamma_{46} = \begin{smallmatrix}0&1&2&1&1&1&0\\&&1\end{smallmatrix}, \gamma_{47} = \begin{smallmatrix}1&1&1&1&1&1&0\\&&1\end{smallmatrix}, \gamma_{48} = \begin{smallmatrix}1&1&2&1&1&1&0\\&&1\end{smallmatrix}, \gamma_{49} = \begin{smallmatrix}0&1&2&2&1&1&0\\&&1\end{smallmatrix}, \\
&\gamma_{50} = \begin{smallmatrix}1&2&2&1&1&1&0\\&&1\end{smallmatrix}, \gamma_{51} = \begin{smallmatrix}1&1&2&2&1&1&0\\&&1\end{smallmatrix}, \gamma_{52} = \begin{smallmatrix}0&1&2&2&2&1&0\\&&1\end{smallmatrix}, \gamma_{53} = \begin{smallmatrix}1&2&2&2&1&1&0\\&&1\end{smallmatrix}, \\
&\gamma_{54} = \begin{smallmatrix}1&1&2&2&2&1&0\\&&1\end{smallmatrix}, \gamma_{55} = \begin{smallmatrix}1&2&3&2&1&1&0\\&&1\end{smallmatrix}, \gamma_{56} = \begin{smallmatrix}1&2&2&2&2&1&0\\&&1\end{smallmatrix}, \gamma_{57} = \begin{smallmatrix}1&2&3&2&1&1&0\\&&2\end{smallmatrix}, \\
&\gamma_{58} = \begin{smallmatrix}1&2&3&2&2&1&0\\&&1\end{smallmatrix}, \gamma_{59} = \begin{smallmatrix}1&2&3&2&2&1&0\\&&2\end{smallmatrix}, \gamma_{60} = \begin{smallmatrix}1&2&3&3&2&1&0\\&&1\end{smallmatrix}, \gamma_{61} = \begin{smallmatrix}1&2&3&3&2&1&0\\&&2\end{smallmatrix}, \\
&\gamma_{62} = \begin{smallmatrix}1&2&4&3&2&1&0\\&&2\end{smallmatrix}, \gamma_{63} = \begin{smallmatrix}1&3&4&3&2&1&0\\&&2\end{smallmatrix}, \gamma_{64} = \begin{smallmatrix}2&3&4&3&2&1&0\\&&2\end{smallmatrix}\},
\end{aligned}$$

where $\begin{smallmatrix}a_1&a_3&a_4&a_5&a_6&a_7&a_8\\&&a_2\end{smallmatrix}$ stands for the root $\gamma = \sum_{i=1}^{8} a_i\alpha_i \in \Phi^+$.

We fix the orientation (or sign) of y_γ as in the following: For $\gamma_i \in \Phi^+ (i = 65, \dots, 120)$, we put $y_i = y_{\gamma_i}$ and fix the orientation by the following order;

$$
\begin{aligned}
&y_{65} = [y_7, y_8], \quad y_{66} = [y_6, y_{65}], \quad y_{67} = [y_5, y_{66}], \quad y_{68} = [y_4, y_{67}], \quad y_{69} = [y_2, y_{68}],\\
&y_{70} = [y_3, y_{68}], \quad y_{71} = [y_2, y_{70}], \quad y_{72} = [y_1, y_{70}], \quad y_{73} = [y_4, y_{71}], \quad y_{74} = [y_1, y_{71}],\\
&y_{75} = [y_1, y_{73}], \quad y_{76} = [y_5, y_{73}], \quad y_{77} = [y_3, y_{75}], \quad y_{78} = [y_5, y_{75}], \quad y_{79} = [y_6, y_{76}],\\
&y_{80} = [y_5, y_{77}], \quad y_{81} = [y_6, y_{78}], \quad y_{82} = [y_7, y_{79}], \quad y_{83} = [y_7, y_{80}], \quad y_{84} = [y_6, y_{80}],\\
&y_{85} = [y_1, y_{82}], \quad y_{86} = [y_2, y_{83}], \quad y_{87} = [y_6, y_{83}], \quad y_{88} = [y_3, y_{85}], \quad y_{89} = [y_2, y_{87}],\\
&y_{90} = [y_5, y_{87}], \quad y_{91} = [y_4, y_{88}], \quad y_{92} = [y_7, y_{90}], \quad y_{93} = [y_7, y_{91}], \quad y_{94} = [y_5, y_{91}],\\
&y_{95} = [y_4, y_{92}], \quad y_{96} = [y_2, y_{94}], \quad y_{97} = [y_6, y_{94}], \quad y_{98} = [y_3, y_{95}], \quad y_{99} = [y_4, y_{96}],\\
&y_{100} = [y_2, y_{97}],\\
&y_{101} = [y_1, y_{98}], \quad y_{102} = [y_3, y_{99}], \quad y_{103} = [y_4, y_{100}], \quad y_{104} = [y_1, y_{102}],\\
&y_{105} = [y_3, y_{103}], \quad y_{106} = [y_5, y_{103}], \quad y_{107} = [y_1, y_{105}], \quad y_{108} = [y_5, y_{105}],\\
&y_{109} = [y_4, y_{108}], \quad y_{110} = [y_1, y_{108}], \quad y_{111} = [y_2, y_{109}], \quad y_{112} = [y_4, y_{110}],\\
&y_{113} = [y_2, y_{112}], \quad y_{114} = [y_3, y_{112}], \quad y_{115} = [y_2, y_{114}], \quad y_{116} = [y_4, y_{115}],\\
&y_{117} = [y_5, y_{116}], \quad y_{118} = [y_6, y_{117}], \quad y_{119} = [y_7, y_{118}], \quad y_{120} = [y_8, y_{119}].
\end{aligned}
$$

Then, by the repeated application of Jacobi identities, we obtain

$$
\begin{aligned}
y_{120} &= [-y_{118}, y_{65}] = [y_{117}, y_{66}] = [-y_{116}, y_{67}] = [y_{115}, y_{68}] = [y_{114}, -y_{69}] = [-y_{113}, y_{70}]\\
&= [y_{112}, y_{71}] = [y_{111}, y_{72}] = [y_{110}, -y_{73}] = [y_{109}, -y_{74}] = [y_{108}, y_{75}] = [y_{107}, y_{76}]\\
&= [y_{106}, -y_{77}] = [y_{105}, -y_{78}] = [y_{104}, -y_{79}] = [y_{103}, y_{80}] = [y_{102}, y_{81}] = [y_{100}, -y_{83}]\\
&= [y_{99}, -y_{84}] = [y_{97}, y_{86}] = [y_{96}, y_{87}] = [y_{94}, -y_{89}] = [y_{93}, -y_{90}] = [y_{91}, y_{92}]\\
&= [y_{88}, -y_{95}] = [y_{85}, y_{98}] = [y_{82}, -y_{101}],\\
y_{119} &= [-y_{118}, y_7] = [y_{117}, y_{39}] = [-y_{116}, y_{40}] = [y_{115}, y_{41}] = [y_{114}, y_{42}] = [-y_{113}, y_{43}]\\
&= [y_{112}, -y_{44}] = [y_{111}, -y_{45}] = [y_{110}, y_{46}] = [y_{109}, y_{47}] = [y_{108}, -y_{48}] = [y_{107}, -y_{49}]\\
&= [y_{106}, y_{50}] = [y_{105}, y_{51}] = [y_{104}, y_{52}] = [y_{103}, -y_{53}] = [y_{102}, -y_{54}] = [y_{100}, y_{55}]\\
&= [y_{99}, y_{56}] = [y_{97}, -y_{57}] = [y_{96}, -y_{58}] = [y_{94}, y_{59}] = [y_{93}, y_{60}] = [y_{91}, -y_{61}]\\
&= [y_{88}, y_{62}] = [y_{85}, -y_{63}] = [y_{82}, y_{64}].
\end{aligned}
$$

Thus, putting

$$
\begin{aligned}
&W_1 = y_{120}, \quad W_2 = y_{119},\\
&Z_1 = -y_{118}, \quad Z_2 = y_{117}, \quad Z_3 = -y_{116}, \quad Z_4 = y_{115}, \quad Z_5 = y_{114}, \quad Z_6 = -y_{113},\\
&Z_7 = y_{112}, \quad Z_8 = y_{111}, \quad Z_9 = y_{110}, \quad Z_{10} = y_{109}, \quad Z_{11} = y_{108}, \quad Z_{12} = y_{107},\\
&Z_{13} = y_{106}, \quad Z_{14} = y_{105}, \quad Z_{15} = y_{104}, \quad Z_{16} = y_{103}, \quad Z_{17} = y_{102}, \quad Z_{18} = y_{100},\\
&Z_{19} = y_{99}, \quad Z_{20} = y_{97}, \quad Z_{21} = y_{96}, \quad Z_{22} = y_{94}, \quad Z_{23} = y_{93}, \quad Z_{24} = y_{91},\\
&Z_{25} = y_{88}, \quad Z_{26} = y_{85}, \quad Z_{27} = y_{82},\\
&Y_1 = y_{65}, \quad Y_2 = y_{66}, \quad Y_3 = y_{67}, \quad Y_4 = y_{68}, \quad Y_5 = -y_{69}, \quad Y_6 = y_{70}, \quad Y_7 = y_{71},\\
&Y_8 = y_{72}, \quad Y_9 = -y_{73}, \quad Y_{10} = -y_{74}, \quad Y_{11} = y_{75}, \quad Y_{12} = y_{76}, \quad Y_{13} = -y_{77},\\
&Y_{14} = -y_{78}, \quad Y_{15} = -y_{79}, \quad Y_{16} = y_{80}, \quad Y_{17} = y_{81}, \quad Y_{18} = -y_{83}, \quad Y_{19} = -y_{84},\\
&Y_{20} = y_{86}, \quad Y_{21} = y_{87}, \quad Y_{22} = -y_{89}, \quad Y_{23} = -y_{90}, \quad Y_{24} = y_{92}, \quad Y_{25} = -y_{95},
\end{aligned}
$$

$Y_{26} = y_{98}, \quad Y_{27} = -y_{101},$
$X_1 = y_7, \quad X_2 = y_{39}, \quad X_3 = y_{40}, \quad X_4 = y_{41}, \quad X_5 = y_{42}, \quad X_6 = y_{43},$
$X_7 = -y_{44}, \quad X_8 = -y_{45}, \quad X_9 = y_{46}, \quad X_{10} = y_{47}, \quad X_{11} = -y_{48}, \quad X_{12} = -y_{49},$
$X_{13} = y_{50}, \quad X_{14} = y_{51}, X_{15} = y_{52}, \quad X_{16} = -y_{53}, \quad X_{17} = -y_{54}, \quad X_{18} = y_{55},$
$X_{19} = y_{56}, \quad X_{20} = -y_{57}, \quad X_{21} = -y_{58}, \quad X_{22} = y_{59}, \quad X_{23} = y_{60}, \quad X_{24} = -y_{61},$
$X_{25} = y_{62}, \quad X_{26} = -y_{63}, \quad X_{27} = y_{64},$

we obtain the basis $\{W_1, W_2, Z_1, \ldots, Z_{27}, Y_1, \ldots, Y_{27}, X_1, \ldots, X_{27}\}$ of $\mathfrak{m}$ satisfying the following:

$$\mathfrak{g}_{-3} = \langle\{W_1, W_2\}\rangle, \quad \mathfrak{g}_{-2} = \langle\{Z_1, \ldots, Z_{27}\}\rangle, \quad \mathfrak{g}_{-1} = \langle\{Y_1, \ldots, Y_{27}, X_1, \ldots, X_{27}\}\rangle$$

such that

$$[Z_i, Y_j] = \delta^i_j W_1, \qquad [Z_i, X_j] = \delta^i_j W_2 \qquad [X_i, X_j] = [Y_i, Y_j] = 0 \quad (1 \leqq i, j \leqq 27).$$

Then we calculate $[X_j, Y_k]$ for $1 \leqq j, k \leqq 27$ and obtain

$$\begin{aligned}
Z_1 &= -[X_{13}, Y_{27}] = [X_{16}, Y_{26}] = -[X_{18}, Y_{25}] = [X_{20}, Y_{24}] = -[X_{22}, Y_{23}] \\
&= -[X_{23}, Y_{22}] = [X_{24}, Y_{20}] = -[X_{25}, Y_{18}] = [X_{26}, Y_{16}] = -[X_{27}, Y_{13}], \\
Z_2 &= -[X_{11}, Y_{27}] = [X_{14}, Y_{26}] = -[X_{17}, Y_{25}] = [X_{19}, Y_{24}] = -[X_{21}, Y_{22}] \\
&= -[X_{22}, Y_{21}] = [X_{24}, Y_{19}] = -[X_{25}, Y_{17}] = [X_{26}, Y_{14}] = -[X_{27}, Y_{11}], \\
Z_3 &= -[X_9, Y_{27}] = [X_{12}, Y_{26}] = -[X_{15}, Y_{25}] = [X_{19}, Y_{23}] = -[X_{20}, Y_{21}] \\
&= -[X_{21}, Y_{20}] = [X_{23}, Y_{19}] = -[X_{25}, Y_{15}] = [X_{26}, Y_{12}] = -[X_{27}, Y_9], \\
Z_4 &= -[X_7, Y_{27}] = [X_{10}, Y_{26}] = -[X_{15}, Y_{24}] = [X_{17}, Y_{23}] - [X_{18}, Y_{21}] \\
&= -[X_{21}, Y_{18}] = [X_{23}, Y_{17}] = -[X_{24}, Y_{15}] = [X_{26}, Y_{10}] = -[X_{27}, Y_7], \\
Z_5 &= -[X_6, Y_{27}] = -[X_{10}, Y_{25}] = [X_{12}, Y_{24}] = -[X_{14}, Y_{23}] = [X_{16}, Y_{21}] \\
&= [X_{21}, Y_{16}] = -[X_{23}, Y_{14}] = [X_{24}, Y_{12}] = -[X_{25}, Y_{10}] = -[X_{27}, Y_6], \\
Z_6 &= -[X_5, Y_{27}] = [X_8, Y_{26}] = -[X_{15}, Y_{22}] = [X_{17}, Y_{20}] = -[X_{18}, Y_{19}] \\
&= -[X_{19}, Y_{18}] = [X_{20}, Y_{17}] = -[X_{22}, Y_{15}] = [X_{26}, Y_8] = -[X_{27}, Y_5], \\
Z_7 &= -[X_4, Y_{27}] = [X_8, Y_{25}] = -[X_{12}, Y_{22}] = [X_{14}, Y_{20}] = -[X_{16}, Y_{19}] \\
&= -[X_{19}, Y_{16}] = [X_{20}, Y_{14}] = -[X_{22}, Y_{12}] = [X_{25}, Y_8] = -[X_{27}, Y_4], \\
Z_8 &= [X_6, Y_{26}] = [X_7, Y_{25}] = -[X_9, Y_{24}] = [X_{11}, Y_{23}] = -[X_{13}, Y_{21}] \\
&= -[X_{21}, Y_{13}] = [X_{23}, Y_{11}] = -[X_{24}, Y_9] = [X_{25}, Y_7] = [X_{26}, Y_6], \\
Z_9 &= -[X_3, Y_{27}] = -[X_8, Y_{24}] = [X_{10}, Y_{22}] = -[X_{14}, Y_{18}] = [X_{16}, Y_{17}] \\
&= [X_{17}, Y_{16}] = -[X_{18}, Y_{14}] = [X_{22}, Y_{10}] = -[X_{24}, Y_8] = -[X_{27}, Y_3], \\
Z_{10} &= [X_4, Y_{26}] = -[X_5, Y_{25}] = [X_9, Y_{22}] = -[X_{11}, Y_{20}] = [X_{13}, Y_{19}] \\
&= [X_{19}, Y_{13}] = -[X_{20}, Y_{11}] = [X_{22}, Y_9] = -[X_{25}, Y_5] = [X_{26}, Y_4], \\
Z_{11} &= -[X_2, Y_{27}] = [X_8, Y_{23}] = -[X_{10}, Y_{20}] = [X_{12}, Y_{18}] = -[X_{15}, Y_{16}] \\
&= -[X_{16}, Y_{15}] = [X_{18}, Y_{12}] = -[X_{20}, Y_{10}] = [X_{23}, Y_8] = -[X_{27}, Y_2], \\
Z_{12} &= [X_3, Y_{26}] = [X_5, Y_{24}] = -[X_7, Y_{22}] = [X_{11}, Y_{18}] = -[X_{13}, Y_{17}] \\
&= -[X_{17}, Y_{13}] = [X_{18}, Y_{11}] = -[X_{22}, Y_7] = [X_{24}, Y_5] = [X_{26}, Y_3], \\
Z_{13} &= -[X_1, Y_{27}] = -[X_8, Y_{21}] = [X_{10}, Y_{19}] = -[X_{12}, Y_{17}] = [X_{14}, Y_{15}] \\
&= [X_{15}, Y_{14}] = -[X_{17}, Y_{12}] = [X_{19}, Y_{10}] = -[X_{21}, Y_8] = -[X_{27}, Y_1],
\end{aligned}$$

$$
\begin{aligned}
Z_{14} &= [X_2, Y_{26}] = -[X_5, Y_{23}] = [X_7, Y_{20}] = -[X_9, Y_{18}] = [X_{13}, Y_{15}] \\
&= [X_{15}, Y_{13}] = -[X_{18}, Y_9] = [X_{20}, Y_7] = -[X_{23}, Y_5] = [X_{26}, Y_2], \\
Z_{15} &= -[X_3, Y_{25}] = -[X_4, Y_{24}] = -[X_6, Y_{22}] = -[X_{11}, Y_{16}] = [X_{13}, Y_{14}] \\
&= [X_{14}, Y_{13}] = -[X_{16}, Y_{11}] = -[X_{22}, Y_6] = -[X_{24}, Y_4] = -[X_{25}, Y_3], \\
Z_{16} &= [X_1, Y_{26}] = [X_5, Y_{21}] = -[X_7, Y_{19}] = [X_9, Y_{17}] = -[X_{11}, Y_{15}] \\
&= -[X_{15}, Y_{11}] = [X_{17}, Y_9] = -[X_{19}, Y_7] = [X_{21}, Y_5] = [X_{26}, Y_1], \\
Z_{17} &= -[X_2, Y_{25}] = [X_4, Y_{23}] = [X_6, Y_{20}] = [X_9, Y_{16}] = -[X_{12}, Y_{13}] \\
&= -[X_{13}, Y_{12}] = [X_{16}, Y_9] = [X_{20}, Y_6] = [X_{23}, Y_4] = -[X_{25}, Y_2], \\
Z_{18} &= -[X_1, Y_{25}] = -[X_4, Y_{21}] = -[X_6, Y_{19}] = -[X_9, Y_{14}] = [X_{11}, Y_{12}] \\
&= [X_{12}, Y_{11}] = -[X_{14}, Y_9] = -[X_{19}, Y_6] = -[X_{21}, Y_4] = -[X_{25}, Y_1], \\
Z_{19} &= [X_2, Y_{24}] = [X_3, Y_{23}] = -[X_6, Y_{18}] = -[X_7, Y_{16}] = [X_{10}, Y_{13}] \\
&= [X_{13}, Y_{10}] = -[X_{16}, Y_7] = -[X_{18}, Y_6] = [X_{23}, Y_3] = [X_{24}, Y_2], \\
Z_{20} &= [X_1, Y_{24}] = -[X_3, Y_{21}] = [X_6, Y_{17}] = [X_7, Y_{14}] = -[X_{10}, Y_{11}] \\
&= -[X_{11}, Y_{10}] = [X_{14}, Y_7] = [X_{17}, Y_6] = -[X_{21}, Y_3] = [X_{24}, Y_1], \\
Z_{21} &= -[X_2, Y_{22}] = -[X_3, Y_{20}] = -[X_4, Y_{18}] = [X_5, Y_{16}] = -[X_8, Y_{13}] \\
&= -[X_{13}, Y_8] = [X_{16}, Y_5] = -[X_{18}, Y_4] = -[X_{20} Y_3] = -[X_{22}, Y_2], \\
Z_{22} &= -[X_1, Y_{23}] = -[X_2, Y_{21}] = -[X_6, Y_{15}] = -[X_7, Y_{12}] = [X_9, Y_{10}] \\
&= [X_{10}, Y_9] = -[X_{12}, Y_7] = -[X_{15}, Y_6] = -[X_{21}, Y_2] = -[X_{23}, Y_1], \\
Z_{23} &= -[X_1, Y_{22}] = [X_3, Y_{19}] = [X_4, Y_{17}] = -[X_5, Y_{14}] = [X_8, Y_{11}] \\
&= [X_{11}, Y_8] = -[X_{14}, Y_5] = [X_{17}, Y_4] = [X_{19}, Y_3] = -[X_{22}, Y_1], \\
Z_{24} &= [X_1, Y_{20}] = [X_2, Y_{19}] = -[X_4, Y_{15}] = [X_5, Y_{12}] = -[X_8, Y_9] \\
&= -[X_9, Y_8] = [X_{12}, Y_5] = -[X_{15}, Y_4] = [X_{19}, Y_2] = [X_{20}, Y_1], \\
Z_{25} &= -[X_1, Y_{18}] = -[X_2, Y_{17}] = -[X_3, Y_{15}] = -[X_5, Y_{10}] = [X_7, Y_8] \\
&= [X_8, Y_7] = -[X_{10}, Y_5] = -[X_{15}, Y_3] = -[X_{17}, Y_2] = -[X_{18}, Y_1], \\
Z_{26} &= [X_1, Y_{16}] = [X_2, Y_{14}] = [X_3, Y_{12}] = [X_4, Y_{10}] = [X_6, Y_8] \\
&= [X_8, Y_6] = [X_{10}, Y_4] = [X_{12}, Y_3] = [X_{14}, Y_2] = [X_{16}, Y_1], \\
Z_{27} &= -[X_1, Y_{13}] = -[X_2, Y_{11}] - -[X_3, Y_9] = -[X_4, Y_7] = -[X_5, Y_6] \\
&= -[X_6, Y_5] = -[X_7, Y_4] = -[X_9, Y_3] = -[X_{11}, Y_2] = -[X_{13}, Y_1].
\end{aligned}
$$

Here we define the bilinear forms $f_i(x_1, \ldots, x_{27}, y_1, \ldots, y_{27})(i = 1, \ldots, 27)$ as follows;

$$
\begin{aligned}
f_1 = &-x_{13}\, y_{27} + x_{16}\, y_{26} - x_{18}\, y_{25} + x_{20}\, y_{24} - x_{22}\, y_{23} - x_{23}\, y_{22} + x_{24}\, y_{20} - x_{25}\, y_{18} \\
&+x_{26}\, y_{16} - x_{27}\, y_{13}, \\
f_2 = &-x_{11}\, y_{27} + x_{14}\, y_{26} - x_{17}\, y_{25} + x_{19}\, y_{24} - x_{21}\, y_{22} - x_{22}\, y_{21} + x_{24}\, y_{19} - x_{25}\, y_{17} \\
&+x_{26}\, y_{14} - x_{27}\, y_{11}, \\
f_3 = &-x_9\, y_{27} + x_{12}\, y_{26} - x_{15}\, y_{25} + x_{19}\, y_{23} - x_{20}\, y_{21} - x_{21}\, y_{20} + x_{23}\, y_{19} - x_{25}\, y_{15} \\
&+x_{26}\, y_{12} - x_{27}\, y_9, \\
f_4 = &-x_7\, y_{27} + x_{10}\, y_{26} - x_{15}\, y_{24} + x_{17}\, y_{23} - x_{18}\, y_{21} - x_{21}\, y_{18} + x_{23}\, y_{17} - x_{24}\, y_{15} \\
&+x_{26}\, y_{10} - x_{27}\, y_7, \\
f_5 = &-x_6\, y_{27} - x_{10}\, y_{25} + x_{12}\, y_{24} - x_{14}\, y_{23} + x_{16}\, y_{21} + x_{21}\, y_{16} - x_{23}\, y_{14} + x_{24}\, y_{12} \\
&-x_{25}\, y_{10} - x_{27}\, y_6,
\end{aligned}
$$

$$
\begin{aligned}
f_6 &= -x_5\,y_{27} + x_8\,y_{26} - x_{15}\,y_{22} + x_{17}\,y_{20} - x_{18}\,y_{19} - x_{19}\,y_{18} + x_{20}\,y_{17} - x_{22}\,y_{15}\\
&\quad + x_{26}\,y_8 - x_{27}\,y_5,\\
f_7 &= -x_4\,y_{27} + x_8\,y_{25} - x_{12}\,y_{22} + x_{14}\,y_{20} - x_{16}\,y_{19} - x_{19}\,y_{16} + x_{20}\,y_{14} - x_{22}\,y_{12}\\
&\quad + x_{25}\,y_8 - x_{27}\,y_4,\\
f_8 &= x_6\,y_{26} + x_7\,y_{25} - x_9\,y_{24} + x_{11}\,y_{23} - x_{13}\,y_{21} - x_{21}\,y_{13} + x_{23}\,y_{11} - x_{24}\,y_9\\
&\quad + x_{25}\,y_7 - x_{26}\,y_6,\\
f_9 &= -x_3\,y_{27} - x_8\,y_{24} + x_{10}\,y_{22} - x_{14}\,y_{18} + x_{16}\,y_{17} + x_{17}\,y_{16} - x_{18}\,y_{14} + x_{22}\,y_{10}\\
&\quad - x_{24}\,y_8 - x_{27}\,y_3,\\
f_{10} &= x_4\,y_{26} - x_5\,y_{25} + x_9\,y_{22} - x_{11}\,y_{20} + x_{13}\,y_{19} + x_{19}\,y_{13} - x_{20}\,y_{11} + x_{22}\,y_9\\
&\quad - x_{25}\,y_5 - x_{26}\,y_4,\\
f_{11} &= -x_2\,y_{27} + x_8\,y_{23} - x_{10}\,y_{20} + x_{12}\,y_{18} - x_{15}\,y_{16} - x_{16}\,y_{15} + x_{18}\,y_{12} - x_{20}\,y_{10}\\
&\quad + x_{23}\,y_8 - x_{27}\,y_2,\\
f_{12} &= x_3\,y_{26} + x_5\,y_{24} - x_7\,y_{22} + x_{11}\,y_{18} - x_{13}\,y_{17} - x_{17}\,y_{13} + x_{18}\,y_{11} - x_{22}\,y_7\\
&\quad + x_{24}\,y_5 + x_{26}\,y_3,\\
f_{13} &= -x_1\,y_{27} - x_8\,y_{21} + x_{10}\,y_{19} - x_{12}\,y_{17} + x_{14}\,y_{15} + x_{15}\,y_{14} - x_{17}\,y_{12} + x_{19}\,y_{10}\\
&\quad - x_{21}\,y_8 - x_{27}\,y_1,\\
f_{14} &= x_2\,y_{26} - x_5\,y_{23} + x_7\,y_{20} - x_9\,y_{18} + x_{13}\,y_{15} + x_{15}\,y_{13} - x_{18}\,y_9 + x_{20}\,y_7\\
&\quad - x_{23}\,y_5 + x_{26}\,y_2,\\
f_{15} &= -x_3\,y_{25} - x_4\,y_{24} - x_6\,y_{22} - x_{11}\,y_{16} + x_{13}\,y_{14} + x_{14}\,y_{13} - x_{16}\,y_{11} - x_{22}\,y_6\\
&\quad - x_{24}\,y_4 - x_{25}\,y_3,\\
f_{16} &= x_1\,y_{26} + x_5\,y_{21} - x_7\,y_{19} + x_9\,y_{17} - x_{11}\,y_{15} - x_{15}\,y_{11} + x_{17}\,y_9 - x_{19}\,y_7\\
&\quad + x_{21}\,y_5 + x_{26}\,y_1,\\
f_{17} &= -x_2\,y_{25} + x_4\,y_{23} + x_6\,y_{20} + x_9\,y_{16} - x_{12}\,y_{13} - x_{13}\,y_{12} + x_{16}\,y_9 + x_{20}\,y_6\\
&\quad + x_{23}\,y_4 - x_{25}\,y_2,\\
f_{18} &= -x_1\,y_{25} - x_4\,y_{21} - x_6\,y_{19} - x_9\,y_{14} + x_{11}\,y_{12} + x_{12}\,y_{11} - x_{14}\,y_9 - x_{19}\,y_6\\
&\quad - x_{21}\,y_4 - x_{25}\,y_1,\\
f_{19} &= x_2\,y_{24} + x_3\,y_{23} - x_6\,y_{18} - x_7\,y_{16} + x_{10}\,y_{13} + x_{13}\,y_{10} - x_{16}\,y_7 - x_{18}\,y_6\\
&\quad + x_{23}\,y_3 + x_{24}\,y_2,\\
f_{20} &= x_1\,y_{24} - x_3\,y_{21} + x_6\,y_{17} + x_7\,y_{14} - x_{10}\,y_{11} - x_{11}\,y_{10} + x_{14}\,y_7 + x_{17}\,y_6\\
&\quad - x_{21}\,y_3 + x_{24}\,y_1,\\
f_{21} &= -x_2\,y_{22} - x_3\,y_{20} - x_4\,y_{18} + x_5\,y_{16} - x_8\,y_{13} - x_{13}\,y_8 + x_{16}\,y_5 - x_{18}\,y_4\\
&\quad - x_{20}\,y_3 - x_{22}\,y_2,\\
f_{22} &= -x_1\,y_{23} - x_2\,y_{21} - x_6\,y_{15} - x_7\,y_{12} + x_9\,y_{10} + x_{10}\,y_9 - x_{12}\,y_7 - x_{15}\,y_6\\
&\quad - x_{21}\,y_2 - x_{23}\,y_1,\\
f_{23} &= -x_1\,y_{22} + x_3\,y_{19} + x_4\,y_{17} - x_5\,y_{14} + x_8\,y_{11} + x_{11}\,y_8 - x_{14}\,y_5 + x_{17}\,y_4\\
&\quad + x_{19}\,y_3 - x_{22}\,y_1,\\
f_{24} &= x_1\,y_{20} + x_2\,y_{19} - x_4\,y_{15} + x_5\,y_{12} - x_8\,y_9 - x_9\,y_8 + x_{12}\,y_5 - x_{15}\,y_4\\
&\quad + x_{19}\,y_2 + x_{20}\,y_1,\\
f_{25} &= -x_1\,y_{18} - x_2\,y_{17} - x_3\,y_{15} - x_5\,y_{10} + x_7\,y_8 + x_8\,y_7 - x_{10}\,y_5 - x_{15}\,y_3\\
&\quad - x_{17}\,y_2 - x_{18}\,y_1,
\end{aligned}
$$

$$
\begin{aligned}
f_{26} &= x_1\, y_{16} + x_2\, y_{14} + x_3\, y_{12} + x_4\, y_{10} + x_6\, y_8 + x_8\, y_6 + x_{10}\, y_4 + x_{12}\, y_3 \\
&\quad + x_{14}\, y_2 + x_{16}\, y_1, \\
f_{27} &= -x_1\, y_{13} - x_2\, y_{11} - x_3\, y_9 - x_4\, y_7 - x_5\, y_6 - x_6\, y_5 - x_7\, y_4 - x_9\, y_3 - x_{11}\, y_2 \\
&\quad - x_{13}\, y_1.
\end{aligned}
$$

Observe that $f_i(x_k, y_k)$ is symmetric in x_k and y_k for $i = 1, \ldots, 27$. Moreover we put

$$\sum_{i=1}^{27} x_i\, f_i = 2 \sum_{i=1}^{27} y_i\, g_i$$

where the quadratic forms $g_i(x_1, \ldots, x_{27})(i = 1, \ldots, 27)$ are given by

$$
\begin{aligned}
g_1 &= -x_{13}x_{27} + x_{16}x_{26} - x_{18}x_{25} + x_{20}x_{24} - x_{22}x_{23}, \\
g_2 &= -x_{11}x_{27} + x_{14}x_{26} - x_{17}x_{25} + x_{19}x_{24} - x_{21}x_{22}, \\
g_3 &= -x_9x_{27} + x_{12}x_{26} - x_{15}x_{25} + x_{19}x_{23} - x_{20}x_{21}, \\
g_4 &= -x_7x_{27} + x_{10}x_{26} - x_{15}x_{24} + x_{17}x_{23} - x_{18}x_{21}, \\
g_5 &= -x_6x_{27} - x_{10}x_{25} + x_{12}x_{24} - x_{14}x_{23} + x_{16}x_{21}, \\
g_6 &= -x_5x_{27} + x_8x_{26} - x_{15}x_{22} + x_{17}x_{20} - x_{18}x_{19}, \\
g_7 &= -x_4x_{27} + x_8x_{25} - x_{12}x_{22} + x_{14}x_{20} - x_{16}x_{19}, \\
g_8 &= x_6x_{26} + x_7x_{25} - x_9x_{24} + x_{11}x_{23} - x_{13}x_{21}, \\
g_9 &= -x_3x_{27} - x_8x_{24} + x_{10}x_{22} - x_{14}x_{18} + x_{16}x_{17}, \\
g_{10} &= x_4x_{26} - x_5x_{25} + x_9x_{22} - x_{11}x_{20} + x_{13}x_{19}, \\
g_{11} &= -x_2x_{27} + x_8x_{23} - x_{10}x_{20} + x_{12}x_{18} - x_{15}x_{16}, \\
g_{12} &= x_3x_{26} + x_5x_{24} + x_7x_{20} - x_9x_{18} + x_{13}x_{15}, \\
g_{13} &= -x_1x_{27} - x_8x_{21} + x_{10}x_{19} - x_{12}x_{17} + x_{14}x_{15}, \\
g_{14} &= x_2x_{26} - x_5x_{23} + x_7x_{20} - x_9x_{18} + x_{13}x_{15}, \\
g_{15} &= -x_3x_{25} - x_4x_{24} - x_6x_{22} - x_{11}x_{16} + x_{13}x_{14}, \\
g_{16} &= x_1x_{26} + x_5x_{21} - x_7x_{19} + x_9x_{17} - x_{11}x_{15}, \\
g_{17} &= -x_2x_{25} + x_4x_{23} + x_6x_{20} + x_9x_{16} - x_{12}x_{13}, \\
g_{18} &= -x_1x_{25} - x_4x_{21} - x_6x_{19} - x_9x_{14} + x_{11}x_{12}, \\
g_{19} &= x_2x_{24} + x_3x_{23} - x_6x_{18} - x_7x_{16} + x_{10}x_{13}, \\
g_{20} &= x_1x_{24} - x_3x_{21} + x_6x_{17} + x_7x_{14} - x_{10}x_{11}, \\
g_{21} &= -x_2x_{22} - x_3x_{20} - x_4x_{18} + x_5x_{16} - x_8x_{13}, \\
g_{22} &= -x_1x_{23} - x_2x_{21} - x_6x_{15} - x_7x_{12} + x_9x_{10}, \\
g_{23} &= -x_1x_{22} + x_3x_{19} + x_4x_{17} - x_5x_{14} + x_8x_{11}, \\
g_{24} &= x_1x_{20} + x_2x_{19} - x_4x_{15} + x_5x_{12} - x_8x_9, \\
g_{25} &= -x_1x_{18} - x_2x_{17} - x_3x_{15} - x_5x_{10} + x_7x_8, \\
g_{26} &= x_1x_{16} + x_2x_{14} + x_3x_{12} + x_4x_{10} + x_6x_8, \\
g_{27} &= -x_1x_{13} - x_2x_{11} - x_3x_9 - x_4x_7 - x_5x_6.
\end{aligned}
$$

Thus we have $g_i(x_1, \ldots, x_{27}) = g_i(x_k) = \frac{1}{2} f_i(x_k, x_k)$ for $i = 1, \cdots, 27$. Moreover we have

$$\sum_{i=1}^{27} f_i\, dx_i = \sum_{i=1}^{27} y_i\, dg_i, \quad \sum_{i=1}^{27} g_i\, dx_i = dg, \quad 3g = \sum_{i=1}^{27} x_i\, g_i,$$

where $g(x_1, \dots, x_{27})$ is the cubic form given by

$$\begin{aligned} g = &-x_1 x_{13} x_{27} + x_1 x_{16} x_{26} - x_1 x_{18} x_{25} + x_1 x_{20} x_{24} - x_1 x_{22} x_{23} - x_2 x_{11} x_{27} \\ &+x_2 x_{14} x_{26} - x_2 x_{17} x_{25} + x_2 x_{19} x_{24} - x_2 x_{21} x_{22} - x_3 x_9 x_{27} + x_3 x_{12} x_{26} \\ &-x_3 x_{15} x_{25} + x_3 x_{19} x_{23} - x_3 x_{20} x_{21} - x_4 x_7 x_{27} + x_4 x_{10} x_{26} - x_4 x_{15} x_{24} \\ &+x_4 x_{17} x_{23} - x_4 x_{18} x_{21} - x_5 x_6 x_{27} - x_5 x_{10} x_{25} + x_5 x_{12} x_{24} - x_5 x_{14} x_{23} \\ &+x_5 x_{16} x_{21} + x_6 x_8 x_{26} - x_6 x_{15} x_{22} + x_6 x_{17} x_{20} - x_6 x_{18} x_{19} + x_7 x_8 x_{25} \\ &-x_7 x_{12} x_{22} + x_7 x_{14} x_{20} - x_7 x_{16} x_{19} - x_8 x_9 x_{24} + x_8 x_{11} x_{23} - x_8 x_{13} x_{21} \\ &+x_9 x_{10} x_{22} - x_9 x_{14} x_{18} + x_9 x_{16} x_{17} - x_{10} x_{11} x_{20} + x_{10} x_{13} x_{19} + x_{11} x_{12} x_{18} \\ &-x_{11} x_{15} x_{16} - x_{12} x_{13} x_{17} + x_{13} x_{14} x_{15}. \end{aligned}$$

Thus, by (4.1), we obtain

$$\begin{aligned} \omega_1 &= dP_2 + \hat{g}_1 dX_1 - (-a_{27} dX_{14} + a_{26} dX_{17} - a_{25} dX_{19} + a_{24} dX_{21} - a_{23} dX_{23} \\ &\qquad - a_{22} dX_{24} + a_{20} dX_{25} - a_{18} dX_{26} + a_{16} dX_{27} - a_{13} dX_{28}), \\ \omega_2 &= dP_3 + \hat{g}_2 dX_1 - (-a_{27} dX_{12} + a_{26} dX_{15} - a_{25} dX_{18} + a_{24} dX_{20} - a_{22} dX_{22} \\ &\qquad - a_{21} dX_{23} + a_{19} dX_{25} - a_{17} dX_{26} + a_{14} dX_{27} - a_{11} dX_{28}), \\ \omega_3 &= dP_4 + \hat{g}_3 dX_1 - (-a_{27} dX_{10} + a_{26} dX_{13} - a_{25} dX_{16} + a_{23} dX_{20} - a_{21} dX_{21} \\ &\qquad - a_{20} dX_{22} + a_{19} dX_{24} - a_{15} dX_{26} + a_{12} dX_{27} - a_9 dX_{28}), \\ \omega_4 &= dP_5 + \hat{g}_4 dX_1 - (-a_{27} dX_8 + a_{26} dX_{11} - a_{24} dX_{16} + a_{23} dX_{18} - a_{21} dX_{19} \\ &\qquad - a_{18} dX_{22} + a_{17} X_{24} - a_{15} dX_{25} + a_{10} dX_{27} - a_7 dX_{28}), \\ \omega_5 &= dP_6 + \hat{g}_5 dX_1 - (-a_{27} dX_7 - a_{25} dX_{11} + a_{24} dX_{13} - a_{23} dX_{15} + a_{21} dX_{17} \\ &\qquad + a_{16} dX_{22} - a_{14} dX_{24} + a_{12} dX_{25} - a_{10} dX_{26} - a_6 dX_{28}), \\ \omega_6 &= dP_7 + \hat{g}_6 dX_1 - (-a_{27} dX_6 + a_{26} dX_9 - a_{22} dX_{16} + a_{20} dX_{18} - a_{19} dX_{19} \\ &\qquad - a_{18} dX_{20} + a_{17} dX_{21} - a_{15} dX_{23} + a_8 dX_{27} - a_5 dX_{28}), \\ \omega_7 &= dP_8 + \hat{g}_7 dX_1 - (-a_{27} dX_5 + a_{25} dX_9 - a_{22} dX_{13} + a_{20} dX_{15} - a_{19} dX_{17} \\ &\qquad - a_{16} dX_{20} + a_{14} dX_{21} - a_{12} dX_{23} + a_8 dX_{26} - a_4 dX_{28}), \\ \omega_8 &= dP_9 + \hat{g}_8 dX_1 - (a_{26} dX_7 + a_{25} dX_8 - a_{24} dX_{10} + a_{23} dX_{12} - a_{21} dX_{14} \\ &\qquad - a_{13} dX_{22} + a_{11} dX_{24} - a_9 dX_{25} + a_7 dX_{26} + a_6 dX_{27}), \\ \omega_9 &= dP_{10} + \hat{g}_9 dX_1 - (-a_{27} dX_4 - a_{24} dX_9 + a_{22} dX_{11} - a_{18} dX_{15} + a_{17} dX_{17} \\ &\qquad + a_{16} dX_{18} - a_{14} dX_{19} + a_{10} dX_{23} - a_8 dX_{25} - a_3 dX_{28}), \\ \omega_{10} &= dP_{11} + \hat{g}_{10} dX_1 - (a_{26} dX_5 - a_{25} dX_6 + a_{22} dX_{10} - a_{20} dX_{12} + a_{19} dX_{14} \\ &\qquad + a_{13} dX_{20} - a_{11} dX_{21} + a_9 dX_{23} - a_5 dX_{26} + a_4 dX_{27}), \\ \omega_{11} &= dP_{12} + \hat{g}_{11} dX_1 - (-a_{27} dX_3 + a_{23} dX_9 - a_{20} dX_{11} + a_{18} dX_{13} - a_{16} dX_{16} \\ &\qquad - a_{15} dX_{17} + a_{12} dX_{19} - a_{10} dX_{21} + a_8 dX_{24} - a_2 dX_{28}), \\ \omega_{12} &= dP_{13} + \hat{g}_{12} dX_1 - (a_{26} dX_4 + a_{24} dX_6 - a_{22} dX_8 + a_{18} dX_{12} - a_{17} dX_{14} \\ &\qquad - a_{13} dX_{18} + a_{11} dX_{19} - a_7 dX_{23} + a_5 dX_{25} + a_3 dX_{27}), \\ \omega_{13} &= dP_{14} + \hat{g}_{13} dX_1 - (-a_{27} dX_2 - a_{21} dX_9 + a_{19} dX_{11} - a_{17} dX_{13} + a_{15} dX_{15} \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + a_{14}dX_{16} - a_{12}dX_{18} + a_{10}dX_{20} - a_8dX_{22} - a_1dX_{28}),\\
\omega_{14} &= dP_{15} + \hat{g}_{14}dX_1 - (a_{26}dX_3 - a_{23}dX_6 + a_{20}dX_8 - a_{18}dX_{10} + a_{15}dX_{14}\\
&\qquad\qquad + a_{13}dX_{16} - a_9dX_{19} + a_7dX_{21} - a_5dX_{24} + a_2dX_{27}),\\
\omega_{15} &= dP_{16} + \hat{g}_{15}dX_1 - (-a_{25}dX_4 - a_{24}dX_5 - a_{22}dX_7 - a_{16}dX_{12} + a_{14}dX_{14}\\
&\qquad\qquad + a_{13}dX_{15} - a_{11}dX_{17} - a_6dX_{23} - a_4dX_{25} - a_3dX_{26}),\\
\omega_{16} &= dP_{17} + \hat{g}_{16}dX_1 - (a_{26}dX_2 + a_{21}dX_6 - a_{19}dX_8 + a_{17}dX_{10} - a_{15}dX_{12}\\
&\qquad\qquad - a_{11}dX_{16} + a_9dX_{18} - a_7dX_{20} + a_5dX_{22} + a_1dX_{27}),\\
\omega_{17} &= dP_{18} + \hat{g}_{17}dX_1 - (-a_{25}dX_3 + a_{23}dX_5 + a_{20}dX_7 + a_{16}dX_{10} - a_{13}dX_{13}\\
&\qquad\qquad - a_{12}dX_{14} + a_9dX_{17} + a_6dX_{21} + a_4dX_{24} - a_2dX_{26}),\\
\omega_{18} &= dP_{19} + \hat{g}_{18}dX_1 - (-a_{25}dX_2 - a_{21}dX_5 - a_{19}dX_7 - a_{14}dX_{10} + a_{12}dX_{12}\\
&\qquad\qquad + a_{11}dX_{13} - a_9dX_{15} - a_6dX_{20} - a_4dX_{22} - a_1dX_{26}),\\
\omega_{19} &= dP_{20} + \hat{g}_{19}dX_1 - (a_{24}dX_3 + a_{23}dX_4 - a_{18}dX_7 - a_{16}dX_8 + a_{13}dX_{11}\\
&\qquad\qquad + a_{10}dX_{14} - a_7dX_{17} - a_6dX_{19} + a_3dX_{24} + a_2dX_{25}),\\
\omega_{20} &= dP_{21} + \hat{g}_{20}dX_1 - (a_{24}dX_2 - a_{21}dX_4 + a_{17}dX_7 + a_{14}dX_8 - a_{11}dX_{11}\\
&\qquad\qquad - a_{10}dX_{12} + a_7dX_{15} + a_6dX_{18} - a_3dX_{22} + a_1dX_{25}),\\
\omega_{21} &= dP_{22} + \hat{g}_{21}dX_1 - (-a_{22}dX_3 - a_{20}dX_4 - a_{18}dX_5 + a_{16}dX_6 - a_{13}dX_9\\
&\qquad\qquad - a_8dX_{14} + a_5dX_{17} - a_4dX_{19} - a_3dX_{21} - a_2dX_{23}),\\
\omega_{22} &= dP_{23} + \hat{g}_{22}dX_1 - (-a_{23}dX_2 - a_{21}dX_3 - a_{15}dX_7 - a_{12}dX_8 + a_{10}dX_{10}\\
&\qquad\qquad + a_9dX_{11} - a_7dX_{13} - a_6dX_{16} - a_2dX_{22} - a_1dX_{24}),\\
\omega_{23} &= dP_{24} + \hat{g}_{23}dX_1 - (-a_{22}dX_2 + a_{19}dX_4 + a_{17}dX_5 - a_{14}dX_6 + a_{11}dX_9\\
&\qquad\qquad + a_8dX_{12} - a_5dX_{15} + a_4dX_{18} + a_3dX_{20} - a_1dX_{23}),\\
\omega_{24} &= dP_{25} + \hat{g}_{24}dX_1 - (a_{20}dX_2 + a_{19}dX_3 - a_{15}dX_5 + a_{12}dX_6 - a_9dX_9\\
&\qquad\qquad - a_8dX_{10} + a_5dX_{13} - a_4dX_{16} + a_2dX_{20} + a_1dX_{21}),\\
\omega_{25} &= dP_{26} + \hat{g}_{25}dX_1 - (-a_{18}dX_2 - a_{17}dX_3 - a_{15}dX_4 - a_{10}dX_6 + a_8dX_8\\
&\qquad\qquad + a_7dX_9 - a_5dX_{11} - a_3dX_{16} - a_2dX_{18} - a_1dX_{19}),\\
\omega_{26} &= dP_{27} + \hat{g}_{26}dX_1 - (a_{16}dX_2 + a_{14}dX_3 + a_{12}dX_4 + a_{10}dX_5 + a_8dX_7\\
&\qquad\qquad + a_6X_9 + a_4dX_{11} + a_3dX_{13} + a_2dX_{15} + a_1dX_{17}),\\
\omega_{27} &= dP_{28} + \hat{g}_{27}dX_1 - (-a_{13}dX_2 - a_{11}dX_3 - a_9dX_4 - a_7dX_5 - a_6dX_6\\
&\qquad\qquad - a_5dX_7 - a_4dX_8 - a_3dX_{10} - a_2dX_{12} - a_1dX_{14}),\\
\varpi_2 &= a_1\omega_1 + \cdots + a_{27}\omega_{27} + dP_1 - 2\hat{g}dX_1 + \hat{g}_1dX_2 + \cdots + \hat{g}_{27}dX_{28}.
\end{aligned}
$$

This implies $R(X)$ is given by the following 379 equations;

$$
\begin{aligned}
-P_{14,28} = P_{17,27} = -P_{19,26} = P_{21,25} = -P_{23,24}\ (= a_1),\\
-P_{12,28} = P_{15,27} = -P_{18,26} = P_{20,25} = -P_{22,23}\ (= a_2),\\
-P_{10,28} = P_{13,27} = -P_{16,26} = P_{20,24} = -P_{21,22}\ (= a_3),\\
-P_{8,28} = P_{11,27} = -P_{16,25} = P_{18,24} = -P_{19,22}\ (= a_4),\\
-P_{7,28} = -P_{11,26} = P_{13,25} = -P_{15,24} = P_{17,22}\ (= a_5),\\
-P_{6,28} = P_{9,27} = -P_{16,23} = P_{18,21} = -P_{19,20}\ (= a_6),\\
-P_{5,28} = P_{9,26} = -P_{13,23} = P_{15,21} = -P_{17,20}\ (= a_7),\\
P_{7,27} = P_{8,26} = -P_{10,25} = P_{12,24} = -P_{14,22}\ (= a_8),
\end{aligned}
$$

$$-P_{4,28} = -P_{9,25} = P_{11,23} = -P_{15,19} = P_{17,18} \ (= a_9),$$
$$P_{5,27} = -P_{6,26} = P_{10,23} = -P_{12,21} = P_{14,20} \ (= a_{10}),$$
$$-P_{3,28} = P_{9,24} = -P_{11,21} = P_{13,19} = -P_{16,17} \ (= a_{11}),$$
$$P_{4,27} = P_{6,25} = -P_{8,23} = P_{12,19} = -P_{14,18} \ (= a_{12}),$$
$$-P_{2,28} = -P_{9,22} = P_{11,20} = -P_{13,18} = P_{15,16} \ (= a_{13}),$$
$$P_{3,27} = -P_{6,24} = P_{8,21} = -P_{10,19} = P_{14,16} \ (= a_{14}),$$
$$-P_{4,25} = -P_{5,25} = -P_{7,23} = -P_{12,17} = P_{14,15} \ (= a_{15}),$$
$$P_{2,27} = P_{6,22} = -P_{8,20} = P_{10,18} = -P_{12,16} \ (= a_{16}),$$
$$-P_{3,26} = P_{5,24} = P_{7,21} = P_{10,17} = -P_{13,14} \ (= a_{17}),$$
$$-P_{2,26} = -P_{5,22} = -P_{7,20} = -P_{10,15} = P_{12,13} \ (= a_{18}),$$
$$P_{3,25} = P_{4,24} = -P_{7,19} = -P_{8,17} = P_{11,14} \ (= a_{19}),$$
$$P_{2,25} = -P_{4,22} = P_{7,18} = P_{8,15} = -P_{11,12} \ (= a_{20}),$$
$$-P_{3,23} = -P_{4,21} = -P_{5,19} = P_{6,17} = -P_{9,14} \ (= a_{21}),$$
$$-P_{2,24} = -P_{3,22} = -P_{7,16} = -P_{8,13} = P_{10,11} \ (= a_{22}),$$
$$-P_{2,23} = P_{4,20} = P_{5,18} = -P_{6,15} = P_{9,12} \ (= a_{23}),$$
$$P_{2,21} = P_{3,20} = -P_{5,16} = P_{6,13} = -P_{9,10} \ (= a_{24}),$$
$$-P_{2,19} = -P_{3,18} = -P_{4,16} = -P_{6,11} = P_{8,9} \ (= a_{25}),$$
$$P_{2,17} = P_{3,15} = P_{4,13} = P_{5,11} = P_{7,9} \ (= a_{26}),$$
$$-P_{2,14} = -P_{3,12} = -P_{4,10} = -P_{5,8} = -P_{6,7} \ (= a_{27}),$$

$$P_{2,2} = P_{2,3} = P_{2,4} = P_{2,5} = P_{2,6} = P_{2,7} = P_{2,8} = P_{2,9} = P_{2,10} = P_{2,11} = P_{2,12} = P_{2,13} = 0,$$
$$P_{2,15} = P_{2,16} = P_{2,18} = P_{2,20} = P_{2,22} = P_{3,3} = P_{3,4} = P_{3,5} = P_{3,6} = P_{3,7} = P_{3,8} = P_{3,9} = 0,$$
$$P_{3,10} = P_{3,11} = P_{3,13} = P_{3,14} = P_{3,16} = P_{3,17} = P_{3,19} = P_{3,21} = P_{3,24} = P_{4,4} = P_{4,5} = P_{4,6} = 0,$$
$$P_{4,7} = P_{4,8} = P_{4,9} = P_{4,11} = P_{4,12} = P_{4,14} = P_{4,15} = P_{4,17} = P_{4,18} = P_{4,19} = P_{4,23} = P_{4,25} = 0,$$
$$P_{5,5} = P_{5,6} = P_{5,7} = P_{5,9} = P_{5,10} = P_{5,12} = P_{5,13} = P_{5,14} = P_{5,15} = P_{5,17} = P_{5,20} = P_{5,21} = 0,$$
$$P_{5,23} = P_{5,26} = P_{6,6} = P_{6,8} = P_{6,9} = P_{6,10} = P_{6,12} = P_{6,14} = P_{6,16} = P_{6,18} = P_{6,19} = P_{6,20} = 0,$$
$$P_{6,21} = P_{6,23} = P_{6,27} = P_{7,7} = P_{7,8} = P_{7,10} = P_{7,11} = P_{7,12} = P_{7,13} = P_{7,14} = P_{7,15} = P_{7,17} = 0,$$
$$P_{7,22} = P_{7,24} = P_{7,25} = P_{7,26} = P_{8,8} = P_{8,10} = P_{8,11} = P_{8,12} = P_{8,14} = P_{8,16} = P_{8,18} = P_{8,19} = 0,$$
$$P_{8,22} = P_{8,24} = P_{8,25} = P_{8,27} = P_{99} = P_{9,11} = P_{9,13} = P_{9,15} = P_{9,16} = P_{9,17} = P_{9,18} = P_{9,19} = 0,$$
$$P_{9,20} = P_{9,21} = P_{9,23} = P_{9,28} = P_{10,10} = P_{10,12} = P_{10,13} = P_{10,14} = P_{10,16} = P_{10,20} = P_{10,21} = 0,$$

$$P_{10,22} = P_{10,24} = P_{10,26} = P_{10,27} = P_{11,11} = P_{11,13} = P_{11,15} = P_{11,16} = P_{11,17} = P_{11,18} = 0,$$
$$P_{11,19} = P_{11,22} = P_{11,24} = P_{11,25} = P_{11,28} = P_{12,12} = P_{12,14} = P_{12,15} = P_{12,18} = P_{12,20} = 0,$$
$$P_{12,22} = P_{12,23} = P_{12,25} = P_{12,26} = P_{12,27} = P_{13,13} = P_{13,15} = P_{13,16} = P_{13,17} = P_{13,20} = 0,$$
$$P_{13,21} = P_{13,22} = P_{13,24} = P_{13,26} = P_{13,28} = P_{14,14} = P_{14,17} = P_{14,19} = P_{14,21} = P_{14,23} = 0,$$
$$P_{14,24} = P_{14,25} = P_{14,26} = P_{14,27} = P_{15,15} = P_{15,17} = P_{15,18} = P_{15,20} = P_{15,22} = P_{15,23} = 0,$$
$$P_{15,25} = P_{15,26} = P_{15,28} = P_{16,16} = P_{16,18} = P_{16,19} = P_{16,20} = P_{16,21} = P_{16,22} = P_{16,24} = 0,$$
$$P_{16,27} = P_{16,28} = P_{17,17} = P_{17,19} = P_{17,21} = P_{17,23} = P_{17,24} = P_{17,25} = P_{17,26} = P_{17,28} = 0,$$
$$P_{18,18} = P_{18,19} = P_{18,20} = P_{18,22} = P_{18,23} = P_{18,25} = P_{18,27} = P_{18,28} = P_{19,19} = 0,$$
$$P_{19,21} = P_{19,23} = P_{19,24} = P_{19,25} = P_{19,27} = P_{19,28} = P_{20,20} = P_{20,21} = P_{20,22} = 0,$$
$$P_{20,23} = P_{20,26} = P_{20,27} = P_{20,28} = P_{21,21} = P_{21,23} = P_{21,24} = P_{21,26} = P_{21,27} = 0,$$
$$P_{21,28} = P_{22,22} = P_{22,24} = P_{22,25} = P_{22,26} = P_{22,27} = P_{22,28} = P_{23,23} = P_{23,25} = 0,$$
$$P_{23,26} = P_{23,27} = P_{23,28} = P_{24,24} = P_{24,25} = P_{24,26} = P_{24,27} = P_{24,28} = P_{25,25} = 0,$$
$$P_{25,26} = P_{25,27} = P_{25,28} = P_{26,26} = P_{26,27} = P_{26,28} = P_{27,27} = P_{27,28} = P_{28,28} = 0,$$

$$P_{1,28} = P_{2,28}P_{14,28} + P_{3,28}P_{12,28} + P_{4,28}P_{10,28} + P_{5,28}P_{8,28} + P_{6,28}P_{7,28},$$
$$P_{1,27} = P_{2,27}P_{14,28} + P_{3,27}P_{12,28} + P_{4,27}P_{10,28} + P_{5,27}P_{8,28} + P_{6,28}P_{7,27},$$
$$P_{1,26} = P_{2,26}P_{14,28} + P_{3,26}P_{12,28} + P_{4,25}P_{10,28} - P_{5,27}P_{7,28} + P_{5,28}P_{7,27},$$
$$P_{1,25} = P_{2,25}P_{14,28} + P_{3,25}P_{12,28} + P_{4,25}P_{8,28} + P_{4,27}P_{7,28} - P_{4,28}P_{7,27},$$
$$P_{1,24} = P_{2,24}P_{14,28} + P_{3,25}P_{10,28} - P_{3,26}P_{8,28} - P_{3,27}P_{7,28} + P_{3,28}P_{7,27},$$
$$P_{1,23} = P_{2,23}P_{14,28} + P_{3,23}P_{12,28} + P_{4,25}P_{6,28} - P_{4,27}P_{5,28} + P_{4,28}P_{5,27},$$
$$P_{1,22} = P_{2,24}P_{12,28} - P_{2,25}P_{10,28} + P_{2,26}P_{8,28} + P_{2,27}P_{7,28} - P_{2,28}P_{7,27},$$
$$P_{1,21} = P_{2,21}P_{14,28} + P_{3,23}P_{10,28} - P_{3,26}P_{6,28} + P_{3,27}P_{5,28} - P_{3,28}P_{5,27},$$
$$P_{1,20} = P_{2,21}P_{12,28} - P_{2,23}P_{10,28} + P_{2,26}P_{6,28} - P_{2,27}P_{5,28} + P_{2,28}P_{5,27},$$
$$P_{1,19} = P_{2,19}P_{14,28} + P_{3,23}P_{8,28} - P_{3,25}P_{10,28} - P_{3,27}P_{4,28} + P_{3,28}P_{4,27},$$
$$P_{1,18} = P_{2,19}P_{12,28} - P_{2,23}P_{8,28} + P_{2,25}P_{6,28} + P_{2,27}P_{4,28} - P_{2,28}P_{4,27},$$
$$P_{1,17} = P_{2,17}P_{14,28} - P_{3,23}P_{7,28} - P_{3,25}P_{5,28} - P_{3,26}P_{4,28} + P_{3,28}P_{4,25},$$
$$P_{1,16} = P_{2,19}P_{10,28} - P_{2,21}P_{8,28} + P_{2,24}P_{6,28} - P_{2,27}P_{3,28} + P_{2,28}P_{3,27},$$
$$P_{1,15} = P_{2,17}P_{12,28} + P_{2,23}P_{7,28} + P_{2,25}P_{5,28} + P_{2,26}P_{4,28} - P_{2,28}P_{4,25},$$
$$P_{1,14} = P_{2,14}P_{14,28} - P_{3,23}P_{7,27} - P_{3,25}P_{5,27} - P_{3,26}P_{4,27} + P_{3,27}P_{4,25},$$
$$P_{1,13} = P_{2,17}P_{10,28} + P_{2,21}P_{7,28} + P_{2,25}P_{5,28} + P_{2,26}P_{4,28} - P_{2,28}P_{4,25},$$
$$P_{1,12} = P_{2,14}P_{12,28} + P_{2,23}P_{7,27} + P_{2,25}P_{5,27} + P_{2,26}P_{4,27} - P_{2,27}P_{4,25},$$
$$P_{1,11} = P_{2,17}P_{8,28} + P_{2,19}P_{7,28} - P_{2,24}P_{4,28} - P_{2,25}P_{3,28} + P_{3,25}P_{2,28},$$
$$P_{1,10} = P_{2,14}P_{10,28} + P_{2,21}P_{7,27} + P_{2,24}P_{5,27} - P_{2,26}P_{3,27} + P_{2,27}P_{3,26},$$

$$
\begin{aligned}
P_{1,9} &= P_{2,17}P_{6,28} - P_{2,19}P_{5,28} - P_{2,21}P_{4,28} - P_{2,23}P_{3,28} + P_{2,28}P_{3,23},\\
P_{1,8} &= P_{2,14}P_{8,28} + P_{2,19}P_{7,27} - P_{2,14}P_{7,27} - P_{2,25}P_{3,27} + P_{2,27}P_{3,25},\\
P_{1,7} &= P_{2,14}P_{7,28} - P_{2,17}P_{7,27} + P_{2,24}P_{4,25} + P_{2,25}P_{3,26} - P_{2,26}P_{3,25},\\
P_{1,6} &= P_{2,14}P_{6,28} - P_{2,19}P_{5,27} - P_{2,21}P_{4,27} - P_{2,23}P_{3,27} + P_{2,27}P_{3,23},\\
P_{1,5} &= P_{2,14}P_{5,28} - P_{2,17}P_{5,27} - P_{2,21}P_{4,25} - P_{2,23}P_{3,26} + P_{2,26}P_{3,23},\\
P_{1,4} &= P_{2,14}P_{4,28} - P_{2,17}P_{4,27} + P_{2,19}P_{4,25} + P_{2,23}P_{3,25} - P_{2,25}P_{3,23},\\
P_{1,3} &= P_{2,14}P_{3,28} - P_{2,17}P_{3,27} - P_{2,19}P_{3,26} - P_{2,21}P_{3,25} + P_{2,24}P_{3,23},\\
P_{1,2} &= P_{2,14}P_{2,28} - P_{2,17}P_{2,27} + P_{2,19}P_{2,26} - P_{2,21}P_{2,25} + P_{2,23}P_{2,24},
\end{aligned}
$$

$$
\begin{aligned}
P_{1,1} = 2(&P_{2,14}P_{2,28}P_{14,28} - P_{2,17}P_{2,27}P_{14,28} + P_{2,19}P_{2,26}P_{14,28} - P_{2,21}P_{2,25}P_{14,28}\\
&+P_{2,23}P_{2,24}P_{14,28} + P_{2,14}P_{3,28}P_{12,28} - P_{2,17}P_{3,27}P_{12,28} + P_{2,19}P_{3,26}P_{12,28}\\
&-P_{2,21}P_{3,25}P_{12,28} + P_{2,24}P_{3,23}P_{12,28} + P_{2,1}P_{4,28}P_{10,28} - P_{2,17}P_{4,27}P_{10,28}\\
&+P_{2,19}P_{4,25}P_{10,28} + P_{2,23}P_{3,25}P_{10,28} - P_{2,25}P_{3,23}P_{10,28} + P_{2,14}P_{5,28}P_{8,28}\\
&-P_{2,17}P_{5,27}P_{8,28} - P_{2,21}P_{4,25}P_{8,28} - P_{2,23}P_{3,26}P_{8,28} + P_{2,26}P_{3,3,23}P_{8,28}\\
&+P_{2,14}P_{6,28}P_{7,28} - P_{2,19}P_{5,27}P_{7,28} - P_{2,21}P_{4,27}P_{7,28} - P_{2,23}P_{3,27}P_{7,28}\\
&+P_{2,27}P_{3,23}P_{7,28} - P_{2,17}P_{6,28}P_{7,27} + P_{2,24}P_{4,25}P_{6,28} + P_{2,25}P_{3,26}P_{6,28}\\
&-P_{2,26}P_{3,25}P_{6,28} + P_{2,19}P_{5,28}P_{7,27} - P_{2,24}P_{4,27}P_{5,28} - P_{2,25}P_{3,27}P_{5,28}\\
&+P_{2,27}P_{3,25}P_{5,28} + P_{2,21}P_{4,28}P_{7,27} + P_{2,23}P_{3,28}P_{7,27} - P_{2,28}P_{3,23}P_{7,27}\\
&+P_{2,24}P_{4,28}P_{5,27} - P_{2,26}P_{3,27}P_{4,28} + P_{2,27}P_{3,26}P_{4,28} + P_{2,25}P_{3,28}P_{5,27}\\
&-P_{2,28}P_{3,25}P_{5,27} + P_{2,26}P_{3,28}P_{4,27} - P_{2,27}P_{3,28}P_{4,25} - P_{2,28}P_{3,26}P_{4,27}\\
&+P_{2,28}P_{3,27}P_{4,25})
\end{aligned}
$$

in coordinates $(X_1, \dots, X_{28}, Z, P_1, \dots, P_{28}, P_{1,1}, \dots, P_{28,28})$ of $L(J)$.

5. Goursat Equation (B_ℓ)**.** Now, utilizing the Second Reduction Theorem , we will construct the model equation (B_ℓ) from the same standard differential system (X, D) constructed in §3 and §4, which is the local model corresponding to the Gourst gradation of type X_ℓ. First $(W; C^*, N)$ is constructed as follows; $W = W(X)$ is the collection of hyperplanes v in each tangent space $T_x(X)$ at $x \in X$ which contains the fibre $\partial D(x)$ of the derived system ∂D of D, which is same as $R(X)$

$$
W(X) = \bigcup_{x \in X} W_x \subset J(X, 3s+1),
$$
$$
W_x = \{v \in \mathrm{Gr}(T_x(X), 3s+1) \mid v \supset \partial D(x)\} \cong \mathbb{P}^1.
$$

Moreover C^* is the canonical system obtained by the Grassmaniann construction and N is the lift of ∂D. Precisely, C^* and N are given by

$$
C^*(v) = \nu_*^{-1}(v) \supset N(v) = \nu_*^{-1}(\partial D(x)),
$$

for each $v \in W(X)$ and $x = \nu(v)$, where $\nu : W(X) \to X$ is the projection.

We introduce a fibre coordinate λ by $\varpi = \varpi_1 + \lambda \varpi_2$, where

$$
C^* = \{\, \varpi = 0 \,\} \qquad \text{and} \qquad \partial D = \{\varpi_1 = \varpi_2 = 0\}.
$$

Here $(w_1, w_2, z_1, \dots, z_s, y_1, \dots, y_s, x_1, \dots, x_s, \lambda)$ constitutes a coordinate system on $W(X)$. Then we have

$$
d\varpi = (dy_1 + \lambda dx_1) \wedge \omega_1 + \cdots + (dy_s + \lambda dx_s) \wedge \omega_s + d\lambda \wedge \varpi_2,
$$

$$\mathrm{Ch}\,(C^*) = \{\ \varpi = \varpi_2 = \omega_1 = \cdots = \omega_s = dy_1 + \lambda dx_1 = \cdots = dy_s + \lambda dx_s = d\lambda = 0\ \},$$
$$N = \{\ \varpi = \varpi_2 = 0\ \},$$
$$\mathrm{Ch}\,(N) = \{\ \varpi_1 = \varpi_2 = \omega_1 = \cdots = \omega_s = dy_1 = \cdots = dy_s = dx_1 = \cdots = dx_s = 0\ \}.$$

Hence $(W(X); C^*, N)$ is an IG-manifold of corank 1 (see §2.2 [15]). Then we can utilize the integration of ϖ in §4 (and §3). In particular, by the calculation in (4.1) (and (3.3)), we obtain

$$\begin{cases} \varpi = dZ - \displaystyle\sum_{i=1}^{s+1} P_i dX_i, \\ \varpi_2 = dP_1 + \displaystyle\sum_{k=1}^{s} a_k dP_{k+1} + \hat{g} dX_1 - \sum_{k=1}^{s} \hat{g}_k dX_{k+1}, \end{cases} \tag{5.1}$$

where $(X_1, \ldots, X_{s+1}, Z, P_1, \ldots, P_{s+1})$ constitute a coordinate system of $J = W/\mathrm{Ch}\,(C^*)$ and $(X_1, \ldots, X_{s+1}, Z, P_1, \ldots, P_{s+1}, a_1, \ldots, a_s)$ constitute a coordinate system of W. Here we note, in the case of BD_ℓ type,

$$f_1 = \sum_{k=2}^{p+1} x_k y_k, \qquad f_k = x_k y_1 + x_1 y_k \quad (k = 2, \ldots, p+1),$$

so that

$$\hat{g}_1 = \frac{1}{2} \sum_{k=2}^{p+1} a_k^2, \qquad \hat{g}_k = a_1 a_k \quad (k = 2, \ldots, p+1), \qquad \text{and} \qquad \hat{g} = \frac{1}{2} \sum_{k=2}^{p+1} a_1 a_k^2.$$

Now our model equation is constructted as follows; Let $(R(X); D_X^1, D_X^2) = (R(W); D_W^1, D_W^2)$ be the Lagrange Grassmann bundle over $(W; C^*, N)$.

$$R(W) = \bigcup_{x \in W} R_w, \qquad R_w = \{\hat{v} \subset N(w) \mid d\varpi\ |_{\hat{v}} = 0, \quad \hat{v} \text{ is maximal}\},$$

where $C^* = \{\varpi = 0\}$. Moreover D_W^1 and D_W^2 are defined by

$$D_W^1(\hat{v}) = \tau_*^{-1}(C^*(w)) \supset D_W^2(\hat{v}) = \tau_*^{-1}(\hat{v}),$$

for $\hat{v} \in R(W), w = q(\hat{v})$ and $\tau : R(W) \to W$ is the projection. Namely we collect maximal isotropic subspaces of $(N(w), d\varpi)$. Infact $v = q_*(\hat{v})$ is a legendrian subspace of (J, C) such that $v \subset \iota(w) = q_*(N(w))$, where $q : W \to J = W/\mathrm{Ch}\,(C^*)$ is the projection and $\iota : W \to I^1(J)$ is the canonical immersion (see Theorem 2.1 [15]). Thus we define a map $\zeta : R(W) \to L(J)$ by $\zeta(\hat{v}) = q_*(\hat{v})$. Then we have

$$\zeta(R_w) = \{v \in L(J) \mid v \subset \bar{w} \subset C(u)\} \cong L(\bar{w}/\bar{w}^\perp) \cong U(s)/O(s),$$

where $u = q(w), \bar{w} = \iota(w)$ and $L(\bar{w}/\bar{w}^\perp)$ denotes the Lagrange Grassmann manifold of the symplectic vector space $\bar{w}/\bar{w}^\perp$of dimension $2s$. Hence our equation $\zeta(R(W))$ is the collection of legendrian subspaces $v = q_*(\hat{v})$ such that $v \subset \bar{w} = q_*(N(w))$, i.e. $\hat{v} \subset N(w)$, for $w \in W$. Since $(X, \partial D)$ is a regular differential system of type $\mathfrak{c}^1(s, 2)$ and of Cartan rank s, we can check that $(R(X); D_X^1, D_X^2)$ is a PD manifold of second order on an open dense subset $\hat{R}$ of $R(X) = R(W)$ (see Proposition 7.3 [15] for detail) and $\zeta : R(W) \to L(J)$ is an immersion on $\hat{R}$.

Now, substituting $dP_i - \sum_{j=1}^{s+1} P_{ij}dX_j$ $(i = 1, \ldots, s+1)$ into (5.1) , we obtain the parametric description of Goursat equation (B_ℓ)

$$P_{11} + \sum_{k=1}^{s} a_k P_{k+1,1} + \hat{g} = 0, \quad P_{1,j+1} + \sum_{k=1}^{s} a_k P_{k+1,j+1} - \hat{g}_j = 0 \quad (j = 1, \ldots, s). \tag{5.2}$$

In fact we can describe the immersion $\zeta : R(W) \to L(J)$ in coordinates as follows (see §4.1 [15] for detail); Let $(X_1, X_{i+1}, Z, P_1, P_{i+1}, A^{i+1}, B^{i+1}, S)(1 \leqq i \leqq s)$ be the coordinate system of $I^1(J)$ induced from a canonical coordinate $(X_1, \ldots, X_{s+1}, Z, P_1, \ldots, P_{s+1})$ of J (see §2.1 [15]). Then, by (5.1), $\iota(W)$ is given by

$$A^{i+1} = -a_i, \qquad B^{i+1} = \hat{g}_i, \qquad S = -\hat{g} + \sum_{i=1}^{s} A^{i+1}B^{i+1} = -4\hat{g}. \tag{5.3}$$

Moreover, following the argument in §4.1 [15], we can choose the cooodinate system $(X_1, X_{i+1}, Z,\ P_1, P_{i+1}, a_i, P^*_{i+1,j+1})$ $(1 \leqq i, j \leqq s)$ of $R(W)$ so that $\zeta : R(W) \to L(J)$ is given by

$$\zeta^* P_{i+1,j+1} = P^*_{i+1,j+1}, \qquad \zeta^* P_{1,i+1} = \iota^* B^{i+1} + \sum_{j=1}^{s} P^*_{i+1,j+1} \iota^* A^{j+1},$$

$$\zeta^* P_{11} = \iota^* S + \sum_{i=1}^{s}\sum_{j=1}^{s} P^*_{i+1,j+1} \iota^* A^{i+1} \iota^* A^{j+1},$$

where $(X_1, X_{i+1}, Z, P_1, P_{i+1}, P_{11}, P_{1,j+1}, P_{i+1,j+1})(1 \leqq i, j \leqq s)$ is the coordinate system of $L(J)$ induced from a canonical coordinate system $(X_1, X_{i+1}, Z, P_1, P_{i+1})(1 \leqq i \leqq s)$ of J.

Then, substituting (5.3) into the above, we obtain

$$\zeta^* P_{1,i+1} = \hat{g}_i - \sum_{j=1}^{s} \zeta^* P_{i+1,j+1} a_j,$$

$$\zeta^* P_{11} = -4\hat{g} + \sum_{i=1}^{s}\sum_{j=1}^{s} \zeta^* P_{i+1,j+1} a_i a_j = -4\hat{g} - \sum_{i=1}^{s} (\zeta^* P_{1,i+1} - \hat{g}_i) a_i = -\sum_{i=1}^{s} \zeta^* P_{1,i+1} a_i - \hat{g}.$$

This gives us (5.2).

REFERENCES

[1] W. M. Boothby, *Homogeneous complex contact manifolds*, Proc. Symp. Pure Math., Amer. Math. Soc., 3 (1961), pp. 144–154.

[2] N. Bourbaki, *Groupes et algèbles de Lie, Chapitre 4,5 et 6*, Hermann, Paris (1968).

[3] E. Cartan, *Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre*, Ann. Ec. Normale, 27 (1910), pp. 109–192.

[4] ——, *Sur les systèmes en involution d'équations aux dérivées partielles du second ordre à une fonction inconnue de trois variables indépendantes*, Bull. Soc. Math. France, 39 (1911), pp. 352–443.

[5] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York (1972).

[6] N. Tanaka, *On differential systems, graded Lie algebras and pseudo-groups*, J. Math. Kyoto Univ., 10 (1970), pp. 1–82.

[7] ______ , *On the equivalence problems associated with simple graded Lie algebras*, Hokkaido Math. J., 8 (1979), pp. 23–84.

[8] D. The, *Exceptionally simple PDE*, Differential Geometry and its Applications, 56 (2018), pp. 13–41.

[9] K. Yamaguchi, *Contact geometry of higher order*, Japanese J. of Math., 8 (1982), pp. 109–176.

[10] ______ , *Geometrization of Jet bundles*, Hokkaido Math. J., 12 (1983), pp. 27–40.

[11] ______ , *Typical classes in involutive systems of second order*, Japanese J. Math., 11 (1985), pp. 265–291.

[12] ______ , *Differential systems associated with simple graded Lie algebras*, Adv. Studies in Pure Math., 22 (1993), pp. 413–494.

[13] ______ , *G_2-Geometry of Overdetermined Systems of Second Order*, Trends in Math. (Analysis and Geometry in Several Complex Variables) (1999), Birkhäuser, Boston, pp. 289–314.

[14] ______ , *Contact Geometry of Second Order I*, Differential Equations: Geometry, Symmetries and Integrability: The Abel Symposium 2008, Abel Symposia, 5 (2009), pp. 335–386.

[15] ______ , *Contact Geometry of Second Order II*, Adv. Studies in Pure Math., 82 (2019), pp. 99–195.