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006

ON VON KARMAN MODELING FOR TURBULENT FLOW NEAR A WALL*

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Abstract. Mixing-length models are often used by engineers in order to take into account turbulence phenomena in a flow. This kind of model is obtained by adding a turbulent viscosity to the laminar one in Navier-Stokes equations. When the flow is confined between two close walls, von Karman's model consists of adding a viscosity which depends on the rate of strain multiplied by the square of distance to the wall. In this short paper, we present a mathematical analysis of such modeling. In particular, we explain why von Karman's model is numerically ill-conditioned when using a finite element method with a small laminar viscosity. Details of analysis can be found in [1], [2].

Key words. Stokes Equations, Weighted Sobolev Spaces, Finite Element Method.

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1. Introduction. Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with a Lipschitz boundary $\partial\Omega$. We assume that Ω is occupied by a fluid with velocity $\boldsymbol{u}$ and pressure p, satisfying the incompressible Navier-Stokes equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} - \operatorname{div}(2\nu\epsilon(\boldsymbol{u})) + \boldsymbol{\nabla} p = \boldsymbol{F}, \tag{1}$$

$$\operatorname{div}(\boldsymbol{u}) = 0, \tag{2}$$

where ν is the kinematic viscosity, $\boldsymbol{F}$ is the external force and $\epsilon(\boldsymbol{u})$ is the strain tensor defined by

$$\epsilon(\boldsymbol{u})_{i,j} = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}_i}{\partial x_j} + \frac{\partial \boldsymbol{u}_j}{\partial x_i}\right), \quad 1 \le i,j \le 3. \tag{3}$$

The boundary conditions can be of several types such as the adherent condition $\boldsymbol{u} = 0$ on a part of $\partial\Omega$, and slip conditions $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ with $(\epsilon(\boldsymbol{u})\cdot\boldsymbol{n})\cdot\boldsymbol{t}_j = 0$, $j = 1,2$ on another part of $\partial\Omega$. Here $\boldsymbol{n}$ is the outward normal to $\partial\Omega$ and $\boldsymbol{t}_j$, $j = 1,2$ are two corresponding unit tangent vectors.

When the domain is very flat, for instance $\Omega = (0,1)\times(0,1)\times(0,\varepsilon)$ with $0 < \varepsilon \ll 1$, the viscosity of von Karman turbulent model depends on the square of the distance to the wall multiplied by $|\epsilon(\boldsymbol{u})|$. More precisely

$$\nu = \nu(\boldsymbol{x},\boldsymbol{u}) = \nu_0 + \beta_t\,|\epsilon(\boldsymbol{u})|\,d^2(\boldsymbol{x},\partial\Omega). \tag{4}$$

Here, β_t is a positive parameter, $d(\boldsymbol{x},\partial\Omega)$ is the distance from $\boldsymbol{x}\in\Omega$ to $\partial\Omega$, i.e. $d(\boldsymbol{x},\partial\Omega) = \inf_{y\in\partial\Omega}|\boldsymbol{x}-\boldsymbol{y}|$, $\nu_0 > 0$ is the constant kinematic laminar viscosity of the fluid and

$$|\epsilon(\boldsymbol{u})| = \left(\sum_{1\le i,j\le 3}\epsilon(\boldsymbol{u})_{i,j}^2\right)^{\frac{1}{2}}. \tag{5}$$

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We generalize this viscosity with ν to be

$$\nu = \nu_0 + \beta_t \left|\epsilon(\boldsymbol{u})\right| l^{2-\alpha} d^\alpha(\boldsymbol{x}, \partial\Omega), \tag{6}$$

where l is a characteristic length of the domain Ω and α is a constant parameter such that $0 \leq \alpha \leq 2$ ($\alpha = 2$ corresponds to von Karman modeling).
Our first consideration concerns the stationary Stokes problem related to (1)-(2) with a viscosity defined by (6).

2. Analysis of a stationary Stokes problem. Let us consider the equations

$$-\operatorname{div}(2\nu\epsilon(\boldsymbol{u})) + \boldsymbol{\nabla} p = \boldsymbol{F}, \text{ in } \Omega, \tag{7}$$
$$\operatorname{div}(\boldsymbol{u}) = 0, \text{ in } \Omega, \tag{8}$$

with the following boundary condition

$$\boldsymbol{u} = 0 \text{ on } \partial\Omega. \tag{9}$$

The details of analysis of problem defined by (7)-(9) can be found in [1] and [2]. In the following, we present results leading to our main theorem.

First we start by multiplying Equation (7) by a vectorial test function $\boldsymbol{w}$ vanishing on the boundary $\partial\Omega$, and we proceed with an integration by part. We also multiply (8) by a test function q and we formally obtain the two following relations:

$$\int_\Omega 2\nu\epsilon(\boldsymbol{u}) * \epsilon(\boldsymbol{w}) d\boldsymbol{x} - \int_\Omega p \operatorname{div}(\boldsymbol{w}) d\boldsymbol{x} = \int_\Omega \boldsymbol{F} \cdot \boldsymbol{w} d\boldsymbol{x}, \tag{10}$$
$$\int_\Omega \operatorname{div}(\boldsymbol{u}) q d\boldsymbol{x} = 0, \tag{11}$$

where $*$ denotes the tensorial product: $\epsilon(\boldsymbol{u}) * \epsilon(\boldsymbol{w}) = \sum_{1\leq i,j\leq 3} \epsilon(\boldsymbol{u})_{i,j} \cdot \epsilon(\boldsymbol{w})_{i,j}$.

Let us remark that if ν is given by (6), then we have two kinds of integrals in (10) : $\int_\Omega \epsilon(\boldsymbol{u}) * \epsilon(\boldsymbol{w}) dx$ and $\int_\Omega d^\alpha(\boldsymbol{x}, \partial\Omega)) \left|\epsilon(\boldsymbol{u})\right| \epsilon(\boldsymbol{u}) * \epsilon(\boldsymbol{w}) dx$. In order to give meaning to these two integrals, it is sufficient to take $\boldsymbol{u}$ and $\boldsymbol{w}$ in the Sobolev spaces $H_0^1(\Omega)^3$ for the first integral and $(W_{d^\alpha}^{1,3}(\Omega))^3$ for the second one, where $W_{d^\alpha}^{1,3}(\Omega)$ denotes the weighted Sobolev space given by

$$W_{d^\alpha}^{1,3}(\Omega) = \{w \in L_{d^\alpha}^3(\Omega) \text{ and } \boldsymbol{\nabla} w \in (L_{d^\alpha}^3(\Omega))^3\}, \tag{12}$$
$$\text{with } L_{d^\alpha}^3(\Omega) = \left\{ w : \int \left|w(\boldsymbol{x})\right|^3 d^\alpha(\boldsymbol{x}, \partial\Omega) d\boldsymbol{x} < \infty \right\}. \tag{13}$$

Note that for $\alpha = 0$, $W_{d^0}^{1,3}(\Omega) = W^{1,3}(\Omega) \subset H^1(\Omega)$ and equation (10) can be posed in $W_0^{1,3}(\Omega)^3$. More precisely, when $\alpha = 0$ and for $\boldsymbol{F} \in L^{3/2}(\Omega)^3$, we can look for $\boldsymbol{u} \in W_0^{1,3}(\Omega)^3$ and $p \in L^{3/2}(\Omega)$ satisfying (10) and (11) for every $\boldsymbol{w} \in W_0^{1,3}(\Omega)^3$ and $q \in L^{3/2}(\Omega)$. In [3] and [4] one can find a proof of existence and uniqueness of solution to this problem, with p defined up to a constant.

For $\alpha < 2$ and close to 2, the situation is not so obvious. The space $W_{d^\alpha}^{1,3}(\Omega)$ is not embedded in $H^1(\Omega)$ (see [5]), and we have to take a Banach space of type $H_0^1(\Omega)^3 \cap (W_{d^\alpha}^{1,3}(\Omega))^3$ in order to analyse equation (10).

Before considering problem (10) $-$ (11), let us recall some results on weighted Sobolev spaces found in [5] and used in [1], [2]. For a 3×3 tensor $\boldsymbol{\kappa}$ depending on

$\boldsymbol{x} \in \Omega$, we adopt the notations:

$$\|\boldsymbol{\kappa}\|_{(L^2(\Omega))^{3\times 3}} = \sum_{1\le i,j\le 3} \left(\int_\Omega |\boldsymbol{\kappa}_{i,j}(\boldsymbol{x})|^2 \, d\boldsymbol{x} \right)^{1/2},$$

$$\|\boldsymbol{\kappa}\|_{(L^3_{d^\alpha}(\Omega))^{3\times 3}} = \sum_{1\le i,j\le 3} \left(\int_\Omega |\boldsymbol{\kappa}_{i,j}(\boldsymbol{x})|^3 \, d^\alpha(\boldsymbol{x}, \partial\Omega) d\boldsymbol{x} \right)^{1/3}.$$

- There exists $\alpha_0 \in]0,2[$ such that for $\alpha \in [0,\alpha_0[$, $W^{1,3}_{d^\alpha}(\Omega) \subset H^1(\Omega)$, but for $\alpha > \alpha_0$, $W^{1,3}_{d^\alpha}(\Omega) \cap H^1(\Omega) \neq W^{1,3}_{d^\alpha}(\Omega)$.
- For $0 \le \alpha < 2$, there is a trace operator $\Gamma : W^{1,3}_{d^\alpha}(\Omega) \to L^3(\partial\Omega)$ and $W^{1,3}_{d^\alpha,0}(\Omega) = \{w \in W^{1,3}_{d^\alpha}(\Omega) : \Gamma(w) = 0\}$.
- For $0 \le \alpha < 2$, we have a Korn inequality in $W^{1,3}_{d^\alpha,0}(\Omega)$: there exists a positive constant β such that $\beta \|\boldsymbol{\nabla w}\|_{(L^3_{d^\alpha}(\Omega))^{3\times 3}} \le \|\epsilon(\boldsymbol{w})\|_{(L^3_{d^\alpha}(\Omega))^{3\times 3}}$ for every $\boldsymbol{w} \in W^{1,3}_{d^\alpha,0}(\Omega)^3$ (see [6]).
- For $\alpha \geqq 2$, a vanished trace of a function of $W^{1,3}_{d^\alpha}(\Omega)$ has no meaning and the problem of existence of a Korn inequality is open!

Starting from these considerations, we now consider only the case when $\alpha \in [0,2[$. Let us define the Banach spaces

$$\boldsymbol{V} = H^1_0(\Omega)^3 \cap W^{1,3}_{d^\alpha,0}(\Omega)^3, \tag{14}$$

$$\boldsymbol{V}_{\mathrm{div}} = \{\boldsymbol{w} \in \boldsymbol{V} : \mathrm{div}(\boldsymbol{w}) = 0\}, \tag{15}$$

provided with the norm $\|\boldsymbol{w}\|_{\boldsymbol{V}} = \|\boldsymbol{\nabla w}\|_{(L^2(\Omega))^{3\times 3}} + \|\boldsymbol{\nabla w}\|_{(L^3_{d^\alpha}(\Omega))^{3\times 3}}$.

By taking $\boldsymbol{w} \in \boldsymbol{V}_{\mathrm{div}}$ in (10), we have to look for $\boldsymbol{u} \in \boldsymbol{V}_{\mathrm{div}}$ satisfying

$$\int_\Omega 2\nu\epsilon(\boldsymbol{u}) * \epsilon(\boldsymbol{w}) d\boldsymbol{x} = \int_\Omega \boldsymbol{F} \cdot \boldsymbol{w} d\boldsymbol{x} \quad \forall \boldsymbol{w} \in \boldsymbol{V}_{\mathrm{div}}. \tag{16}$$

In order to give a meaning to the right side of equalities (10) or (16), we assume $\boldsymbol{F} \in L^2(\Omega)^3 \oplus L^{3/2}_{d^{-\alpha/2}}(\Omega)^3$.

Now we define the functional $\boldsymbol{J}$ on $\boldsymbol{V}_{\mathrm{div}}$ by

$$\boldsymbol{J}(\boldsymbol{v}) = \int_\Omega [2A(\boldsymbol{x}, |\epsilon(\boldsymbol{v})|) - \boldsymbol{F} \cdot \boldsymbol{v}] d\boldsymbol{x}, \tag{17}$$

where

$$A(\boldsymbol{x}, s) = \nu_0 \frac{s^2}{2} + \beta_t \frac{s^3}{3} l^{2-\alpha} d^\alpha(\boldsymbol{x}, \partial\Omega). \tag{18}$$

It is easy to show that $\boldsymbol{J}$ is Gâteau-differentiable on $\boldsymbol{V}_{\mathrm{div}}$, and its derivative at $\boldsymbol{u}$ in direction $\boldsymbol{w}$ is given by

$$D\boldsymbol{J}(\boldsymbol{u})(\boldsymbol{w}) = \int_\Omega 2\nu\epsilon(\boldsymbol{u}) * \epsilon(\boldsymbol{w}) d\boldsymbol{x} - \int_\Omega \boldsymbol{F} \cdot \boldsymbol{w} d\boldsymbol{x}, \tag{19}$$

with ν depending on $\boldsymbol{u}$ via (6). It follows that if $\boldsymbol{u} \in \boldsymbol{V}_{\mathrm{div}}$ is solution of (16), then $D\boldsymbol{J}(\boldsymbol{u})(\boldsymbol{w}) = 0$ for every $\boldsymbol{w} \in \boldsymbol{V}_{\mathrm{div}}$. In [2], it is proven that the functional $\boldsymbol{J}$ is continuous, coercive and strictly convex on $\boldsymbol{V}_{\mathrm{div}}$. Then, there exists a unique $\boldsymbol{u}$ which minimizes $\boldsymbol{J}$ on $\boldsymbol{V}_{\text{div}}$ and $D\boldsymbol{J}(\boldsymbol{u}) = 0$. Moreover, $\boldsymbol{u}$ is the unique solution of Problem (16). The main result of this analysis is

Proposition 1. *When* $0 \leq \alpha < 2$, *and* ν *is defined by* (6), *there exists a unique solution of Problem* (16) *(see [2]).*

Concerning the pressure p which appears in Problem $(10) - (11)$, it is well known that an inf – sup condition allows to obtain its existence and uniqueness by knowing the velocity $\boldsymbol{u} \in \boldsymbol{V}_{\text{div}}$. In [2], it is shown that the dual space of $L^3_{d^\alpha}(\Omega)$ can be identified to $L^{3/2}_{d^{-\alpha/2}}(\Omega)$ by means of an integral on Ω. Since $\text{div}(\boldsymbol{w}) \in L^2(\Omega) \cap L^3_{d^\alpha}(\Omega)$, the pressure p will be considered in the space

$$Q = L^2_0(\Omega) \oplus L^{3/2}_{d^{-\alpha/2},0}(\Omega),$$

in order to give meaning to the second integral of (10). The index 0 in $L^2_0(\Omega)$ and $L^{3/2}_{d^{-\alpha/2},0}(\Omega)$ signifies that the mean values of functions in these spaces are zero.

When α is close to 2 (recall that $\alpha = 2$ in von Karman modeling), the space $L^2(\Omega) \cap L^{3/2}_{d^{-\alpha/2}}(\Omega)$ is not reduced to the null space, and we are not able to verify the inequality $\inf_{q \in Q} \sup_{\boldsymbol{w} \in \boldsymbol{V}} \frac{\int_\Omega q \, \text{div}(\boldsymbol{w}) dx}{\|q\|_Q \|\boldsymbol{w}\|_{\boldsymbol{V}}} > 0$. As a consequence, even if we are looking for a pressure p defined up to a constant in Problem $(10) - (11)$, we are not able to directly verify its existence and uniqueness.

However, the existence of a pressure satisfying $(10) - (11)$ with the velocity $\boldsymbol{u}$ given above, can be deducted from two inf – sup conditions. The first inf – sup is the pairing $H^1_0(\Omega)^3 - L^2_0(\Omega)$ and the other one is $W^{1,3}_{d^\alpha,0}(\Omega)^3 - L^{3/2}_{d^{-\alpha/2},0}(\Omega)$, (see [2]). Due to these two inf-sup conditions, we can prove the existence and uniqueness of $\overline{p}_1 \in L^2_0(\Omega)$ and $\overline{p}_2 \in L^{3/2}_{d^{-\alpha/2},0}(\Omega)$ satisfying:

$$\int_\Omega \overline{p}_1 \, \text{div}(\boldsymbol{w}) \, dx = \int_\Omega 2\nu_0 \boldsymbol{\epsilon}(\boldsymbol{u}) * \boldsymbol{\epsilon}(\boldsymbol{w}) dx - \int_\Omega \boldsymbol{F} \cdot \boldsymbol{w} dx, \quad \forall \boldsymbol{w} \in H^1_0(\Omega)^3,$$

$$\int_\Omega \overline{p}_2 \, \text{div}(\boldsymbol{w}) \, dx = \int_\Omega 2\beta_t l^{2-\alpha} d^\alpha(x, \partial\Omega) \, |\boldsymbol{\epsilon}(\boldsymbol{u})| \, \boldsymbol{\epsilon}(\boldsymbol{u}) * \boldsymbol{\epsilon}(\boldsymbol{w}) dx, \ \forall \boldsymbol{w} \in W^{1,3}_{d^\alpha,0}(\Omega)^3.$$

By setting $p = \overline{p}_1 + \overline{p}_2$, then $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ satisfies relations (10)-(11) for every $(\boldsymbol{w}, q) \in \boldsymbol{V} \times Q$.

These considerations lead to:

Proposition 2 (see [2]). *When* $0 \leq \alpha < 2$, *and* ν *is defined by* (6), *there exists a solution* $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ *satisfying* $(10) - (11)$ *for every* $(\boldsymbol{w}, q) \in \boldsymbol{V} \times Q$. *Moreover, if* $(\boldsymbol{u}_1, p_1)$ *and* $(\boldsymbol{u}_2, p_2)$ *are two solutions, then* $\boldsymbol{u}_1 = \boldsymbol{u}_2$. *When* $\alpha \in [0, \alpha_0[$, *i.e when* $W^{1,3}_{d^\alpha}(\Omega) \subset H^1(\Omega)$, *then* $p_1 = p_2$.

3. Conclusions and remarks.

- When $\nu_0 = 0$ in (6), then $H^1_0(\Omega)^3$ can be dropped in the definition of $\boldsymbol{V}$ in order to set Problem $(10) - (11)$. It follows that for $0 \leq \alpha < 2$, the functional J can be minimized on $X_{\text{div}} = \{\boldsymbol{v} \in W^{1,3}_{d^\alpha,0}(\Omega)^3 : \text{div}(\boldsymbol{v}) = 0\}$. In the case $\nu_0 = 0$, for every $\alpha \in [0, 2[$, Problem $(10) - (11)$ possesses a unique solution $(\boldsymbol{u}, p) \in W^{1,3}_{d^\alpha,0}(\Omega)^3 \times L^{3/2}_{d^{-\alpha/2},0}(\Omega)$.

- For $\alpha = 2$, there is no trace of $\boldsymbol{u}$ on $\partial\Omega$ when $\boldsymbol{u} \in W^{1,3}_{d^2}(\Omega)^3$. In this case it is not possible to take $\nu_0 = 0$ when $\boldsymbol{u} = 0$ is imposed on the boundary $\partial\Omega$. Moreover Korn's inequality is probably wrong in $W^{1,3}_{d^2}(\Omega)^3$ (Kalamajska's conjecture [6]). The analysis of von Karman model ($\alpha = 2$) remains open.
- A direct consequence of previous remarks is: when using a finite element approximation on von Karman model ($\alpha = 2$) and when the laminar viscosity ν_0 is "small" with respect to the numerical viscosity of the method, **the obtained results can strongly depend on the mesh!** In this case, it is necessary to take a very thin meshing close to the walls in order to remedy to this situation. Numerical examples are given in [1] and [2].

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