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# ON THE WAVEWISE ENTROPY INEQUALITIES FOR HIGH-RESOLUTION SCHEMES WITH SOURCE TERMS II: THE FULLY-DISCRETE CASE*

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**Abstract.** We extend the framework and the convergence criteria of wavewise entropy inequalities of H. Yang [35] to a class of fully-discrete high-resolution schemes for hyperbolic conservation laws with source terms. This approach is based on an extended theory of Yang [35] on wave tracking and wave analysis and the theory of Vol'pert [33] on BV solutions. For the Cauchy problem of convex conservation laws with source terms, we use one of the criteria to show the entropy convergence of the schemes with van Leer's flux limiter when the building block of the schemes is the Godunov or Engquish-Osher. The entropy convergence of the homogeneous counterparts of these schemes, originally introduced by Sweby [30], were established by the author [17].

**Key words.** Conservation laws with source terms, WEI framework, entropy convergence criteria.

**Mathematics Subject Classification.** 65M12 (35L60).

**1. Introduction.** The goal of this paper is to extend the framework and the convergence criteria of wavewise entropy inequalities, or WEIs, developed by Yang [35] to a large class of fully-discrete high-resolution schemes for initial value problems of hyperbolic conservation laws with source terms:

$$\begin{cases} w_t + f(w)_x = q(w), \\ w(x,0) = w_0(x), \end{cases} \tag{1.1}$$

where $f \in C^1(\mathbb{R})$, $q \in C^1(\mathbb{R})$, and $w_0 \in BV(\mathbb{R})$. Here $BV$ stands for the subspace of $L^1_{loc}$ consisting of functions $z$ with bounded total variation

$$TV(z) := \sup_{h \neq 0} \int_{\mathbb{R}} \frac{|z(x+h) - z(x)|}{|h|}\, dx. \tag{1.2}$$

The extension of Yang's framework in the semi-discrete case, from homogeneous to non-homogeneous, was accomplished in the earlier work by Yang and the author [36]. With the extended convergence criteria, we were able to show the entropy convergence of a class high-resolution schemes with Superbee and van Leer's flux limiters for non-homogeneous convex conservation laws [14, 16].

In this paper, we are interested in numerical solutions of the schemes that admit conservative form

$$u_j^{n+1} = H(u_{j-p}^n, \cdots, u_{j+p}^n; \lambda) = u_j^n - \lambda(g_{j+\frac{1}{2}}^n - g_{j-\frac{1}{2}}^n) + \tau q(u_j^n), \tag{1.3}$$

where $h$ and $\tau$ are, respectively, spatial and temporal step sizes, and $\lambda = \frac{\tau}{h}$; $u_j^n = u(x_j, t_n)$ are nodal values of the piecewise constant mesh function $u_h(x,t)$ approximating the solution $u(x,t)$. The numerical flux $g$ is given by

$$g_{j+\frac{1}{2}}^n = g_{j+\frac{1}{2}}[u_j^n; \lambda], \tag{1.4}$$

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where

$$g_{j+\frac{1}{2}}[v;\lambda] = g(v_{j-p+1}, v_{j-p+2}, \cdots, v_j, \cdots, v_{j+p}; \lambda), \tag{1.5}$$

for any data $\{v_j\}$. Throughout the paper, we simply write $g_{j+\frac{1}{2}}[v;\lambda]$ as $g_{j+\frac{1}{2}}[v]$ whenever there is no ambiguous. The function $g$ is Lipschitz continuous with respect to its first $2p$ arguments and is *consistent* with the conservation law in the sense that

$$g(u, u, \cdots, u, \lambda) \equiv f(u). \tag{1.6}$$

The homogeneous problems that correspond to (1.1) are

$$\begin{cases} w_t + f(w)_x = 0, \\ w(x,0) = w_0(x). \end{cases} \tag{1.7}$$

With the numerical flux given by (1.5), we call the corresponding schemes

$$\bar{u}_j^{n+1} = \bar{H}(u_{j-p}^n, \cdots, u_{j+p}^n; \lambda) = u_j^n - \lambda(g_{j+\frac{1}{2}}^n - g_{j-\frac{1}{2}}^n) \tag{1.8}$$

that are consistent with the problems (1.7) the *homogeneous counterparts* (HCPs) of the schemes (1.3). The schemes (1.8) are said to be *self-similar* if $\lambda$ is fixed, i.e. if $g$ is independent of step sizes. In this paper, we only consider conservative schemes with self-similar HCPs.

Let $T$ be a positive constant. A scheme of the form (1.3)-(1.5) for Cauchy problem (1.1) *converges* if, for every initial condition $w_0$ in BV and for each sequence of initial data $\{(u_j^n(0))^k\}_{k=1}^{\infty}$ with uniformly bounded variations that converges in $L^1_{loc}(\mathbb{R})$ to $w_0$, the corresponding sequence of (extended) numerical solutions $\{u^k\}$ generated by the scheme converges in $L^1_{loc}(\mathbb{R} \times [0,T))$ to the unique entropy solution $w$ of the problem (1.1) provided that the step sizes $h_k, \tau_k$ of $u^k$ vanish as $k \to \infty$.

The convergence analyses of numerical solutions, for homogeneous problems (1.7), in the early time were dominant by the method of cell entropy inequalities (CEIs), see, for example, [4, 9, 19, 22, 23, 24, 27] and the references therein. In the CEI approach, one tries to derive cell entropy inequalities for certain pairs of numerical entropy and entropy flux. Once these are obtained, the same arguments for Lax-Wendroff Theorem [20] ensure the entropy admissibility of the limit of the numerical solutions. Unfortunately, for a high-resolution scheme to satisfy numerical entropy inequalities at every mesh point is not an easy task. As a result, the convergence of many very effective methods, such as $\alpha$-, $\beta$-schemes constructed by Osher and Chakravarthy [1, 26], in their original setting, cannot be proved by this approach.

In the 1990s, different approaches for convergence analysis emerged. Among them, in this study, we focus our attention on Yang's method, since it has successfully enabled us to show the entropy convergence of the number of high-resolution schemes. In the papers [34, 35], Yang formed the concept of WEIs for a large class of total variation diminishing (TVD) schemes. Based on this concept Yang established several convergence criteria. In particular, for convex conservation laws, one of the criteria essentially states that, a WEI across the area of rarefaction where $u_j^n \leq u_{j+1}^n$ for all $x_j$ is sufficient for convergence to the entropy solution. Hence, in the convergence analysis, one may safely remove the shock area from scrutiny. Further, even in the rarefaction area, a much weaker condition than CEI is sufficient for convergence. Using this criterion, in fully- and semi-discrete version respectively, Yang proved the convergence of fully-discrete and semi-discrete MUSCL schemes and a class of high-resolution schemes based on flux limiters, for homogeneous problems with convex flux functions. For fully-discrete case, using Yang's convergence criterion, the author was able to show the entropy convergence of $\alpha$-, $\beta$-schemes introduced and studied by Osher and Chakravarthy [1, 26]. We also like to mention that for the Hermite type scheme, a cell entropy inequality was established by Jiang and Shu [10]. The proof is amazingly simple and the entropy convergence is implied for the one-dimensional scalar convex case.

In recent years, the numerical analysis of non-homogeneous problems (1.1) has attracted much attention. This includes studying numerical methods for the approximation of (1.1), see [2, 18, 21], for example; the error bounds related to the approximation of (1.1), see [28, 32] for example. However, the analytical tools in this area remain to be CEI, and hence, suffer to the aforementioned limitation.

In this paper, we extend Yang's entire WEI [35] framework to non-homogeneous conservation laws provided that the numerical flux satisfies the same conditions as in the homogeneous case. In particular, as an application of the extended criteria, we show that the fully-discrete Sweby's schemes based on van Leer's flux limiter remain convergent in the non-homogeneous case. The convergence of the HCPs of these schemes were established by the author [15].

This extended framework is developed for one-dimensional scalar convex conservation laws. The advantage is that in this case we can consider the entropy property of numerical solutions only for square entropy function based on DiPerna's results [6]. Although, WEI is effective in showing the entropy consistence for some high-order (in space) accurate schemes, it is very desirable to further extended this framework for uniformly high-order schemes (in space and in time), for example, SSP Runge-Kutta methods [5, 29]. We will attempt to show the second order SSP RK method has entropy convergence using the current WEI framework (may be with some modifications). The development will be reported in a future paper.

The paper is organized as follows. §2 consists of two parts. In the first part we review some properties of the discontinuities of BV weak solutions of conservation laws emphasizing entropy conditions which harbor the idea of WEI approach; and in the second part we show the total variation boundedness of the numerical solutions, which ensures existence of convergent subsequences of numerical solutions whose limits are weak solutions by the arguments of Lax-Wendroff [20]. The main results of the paper are in §3, where we give four extended WEI convergence criteria. These results are parallel to those of [35] for the HCPs of the schemes. We give proofs of the first two criteria since they reveal interesting effects of similarity transforms on the schemes with source terms. To prove the third criterion, we need to perform the wave separations, concentrations and splittings. These can be done similarly as in [34, 35, 36]. In §4 we use one of criteria to show the convergence of Sweby's schemes with van Leer's flux limiter for convex conservation laws with source terms.

In general, we only present proofs that are different from the corresponding ones in [15, 35]. However, in few cases we do provide proofs or sketch of proofs for the ones that are similar to their HCPs so that the paper is reasonably self-contained. Also, for better readability, we closely follow many notations defined in [15, 30, 35].

## 2. Preliminaries.

**2.1. Review of discontinuities of weak solutions.** The idea of WEI approach can be revealed from a simple observation: Let $U(w)$ be a convex entropy function, and $F(w)$ be its flux: $F' = U'f'$. In the area where the solution $w$ is smooth, the additional conservation law $U(w)_t + F(w)_x = U'(w)q(w)$ holds, and the entropy condition is automatically satisfied. Therefore, the entropy admissibility of a weak solution is solely determined by that of its discontinuities. The following is a closer examination of this observation.

For any two distinct numbers $w^-$ and $w^+$ in the domain of $f$, the function

$$W(x,t) = \begin{cases} w^- & \text{if } x < st, \\ w^+ & \text{if } x > st, \end{cases} \tag{2.1}$$

is a *traveling discontinuity*, provided that

$$s(w^+ - w^-) = f(w^+) - f(w^-) \tag{2.2}$$

holds. Clearly $W(x,t)$ is a weak solution of the homogeneous conservation law. Denote by $f[w; w^-, w^+]$ the linear function interpolating $f(w)$ at $w = w^-$ and $w = w^+$. Then $W$ is an *admissible traveling discontinuity* if

$$\operatorname{sgn}(w^+ - w^-)(f[w; w^-, w^+] - f(w)) \leq 0 \tag{2.3}$$

holds for all $w$ in between $w^-$ and $w^+$; otherwise, it is a *traveling expansion shock*.

Through Vol'pert's BV solution theory [33], generic discontinuities of BV weak solutions are inherently related to the traveling discontinuities. Let $\mu(E)$ be the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$. We use $B_r(x_0)$ to denote the open ball centered at $x_0$ with the radius $r$. Let $a$ be an unit vector in $\mathbb{R}^n$, and $R_a(x_0)$ be the half space $\{(x - x_0) \bullet a > 0\}$ in $\mathbb{R}^n$. A point of *density (rarefaction)* for the set $E$ is a point $x$ for which

$$\lim_{r \to 0} \mu(E \bigcap B_r(x))/\mu(B_r(x)) = 1(0).$$

If $w(x)$ is a function defined on a set $E \subset \mathbb{R}^n$ and $x_0$ is not a point of rarefaction for $E$, then $L_E w(x_0)$ will denote the *approximate limit* of the function $w(x)$ at the point $x_0$ with respect to the set $E$, provided that for $\forall \varepsilon > 0$, $x_0$ is a point of rarefaction of the set

$$\{x : |w(x) - L_E w(x_0)| > \varepsilon, x \in E\}.$$

Definition 2.1. Let $w(x)$ be a function defined on $\mathbb{R}^n$.

($\alpha$) A point $x_0 \in \mathbb{R}^n$ is said to be *regular* if there exists a unit vector $a$ such that $l_a w(x_0)$ and $l_{-a} w(x_0)$ exist and are finite. Here, $l_a w(x_0) = L_{R_a(x_0)} w(x_0)$.

($\beta$) The point $x_0$ is said to be a *point of jump* of $w(x)$ if it is regular and $l_a w(x_0) \neq l_{-a} w(x_0)$. The set of the jump points of $w(x)$ is denoted by $\Gamma(w)$.

($\gamma$) If $x_0 \in \Gamma(w)$, then the value $a$ appearing in the definition ($\alpha$) is called *the normal* to $\Gamma(w)$ at the point $x_0$.

For $n = 2$, we apply the preceding concepts to a BV weak solution $w(x,t)$ of the conservation law, possibly with source terms. For any $(x_0, t_0) \in \Gamma(w)$, let $a$ be the normal to $\Gamma(w)$ at the point $(x_0, t_0)$ with positive spatial component, $w^+ = l_a w(x_0, t_0)$, and $w^- = l_{-a} w(x_0, t_0)$. We then call $W$, defined by (2.1)-(2.2), *the traveling discontinuity associated with $w$ at the jump point* $(x_0, t_0)$. Denote by $H_n$ the $n$-dimensional Hausdorff measure. Then the following basic result holds.

Lemma 2.2 (Vol'pert[33]). *A necessary and sufficient condition for a weak solution $w \in BV$ of $w_t + f(w)_x = q(w)$ to be an entropy solution is that* (2.3) *holds, for $H_1$-almost all points in* $\Gamma(w)$.

Roughly speaking, in the WEI approach, if a sequence of total variation bounded (TVB) numerical solutions approaches an entropy violating weak solutions, one may construct a sequence of numerical solutions with vanishing step sizes and vanishing source terms that converges to a traveling expansion shock and harbors an asymptotic traveling expansion shock, a concept that will be given in §3. Similarity transforms play a central role in the construction of such a sequence. Let $\mathcal{S}^{\varepsilon}_{x_0,t_0}$ be the similarity transform centered at a point $(x_0, t_0)$:

$$\mathcal{S}^{\varepsilon}_{x_0,t_0}((x,t)) = (x_0 + \varepsilon x, t_0 + \varepsilon t).$$

This induces a transform $T^{\varepsilon}_{x_0,t_0}$ in the set of the functions $\psi$ defined on a domain $\Omega \subset \mathbb{R}\times\mathbb{R}^+$: $T^{\varepsilon}_{x_0,t_0}\psi = \psi\circ\mathcal{S}^{\varepsilon}_{x_0,t_0}|_{\Omega}$, if $\mathcal{S}^{\varepsilon}_{x_0,t_0}\Omega \subset \Omega$, where $\phi|_{\Omega}$ denotes the restriction of $\phi$ to the set $\Omega$. Define $w_\varepsilon(x,t)$ by

$$w_\varepsilon(x,t) = (T^{\varepsilon}_{x_0,t_0}w)(x,t) = w \circ \mathcal{S}^{\varepsilon}_{x_0,t_0}((x,t)) = w(x_0 + \varepsilon x, t_0 + \varepsilon t).$$

Clearly, if $w(x,t)$ is a weak solution of $w_t + f(w)_x = q(w)$, then $w_\varepsilon(x,t)$ is the one of $w_t + f(w)_x = \varepsilon q(w)$. The following lemma (presented in [34] and still holds for non-homogeneous case) is one of the foundations of the WEI method. It shows that by successively zooming in around a jump point $(x_0,t_0)$ of a weak solution $w$, one can view it locally as a traveling discontinuity.

LEMMA 2.3 (Microscope Lemma). *Let $(x_0,t_0)$ be a jump point of a BV weak solution $w$ in the sense of Definition* 2.1. *If $\{\varepsilon_k\}_{k=1}^{\infty}$ is a sequence of positive numbers such that $\lim_{k\to\infty}\varepsilon_k = 0$, then the sequence $\{w_{\varepsilon_k}\}$ converges in $L^1_{loc}$ to the traveling discontinuity $W$ associated with the jump point $(x_0,t_0)$.*

**2.2. Preliminaries of the numerical schemes.** In this subsection, we present the conditions that will ensure the solutions of a scheme of the form (1.3)-(1.5) converge to a weak solution. Let $TV_u(t)$ be the total spatial variation of $u$. Then the TVD property of the HCP of the scheme (1.3)-(1.5) will guarantee the TVB property of (1.3)-(1.5). In the following, we assume that $\max|q'(u)| \le C$ for some constant $C$.

THEOREM 2.4. *Suppose the HCP of a scheme of the form* (1.3)-(1.5) *is TVD. Then we have $TV_u(t) \le TV_u(0)e^{CT}$, for $t \in [0,T]$.*

*Proof.* Let $(u_j^{n+1})^l$ be the solutions generated by the scheme (1.3)-(1.5), and $(\bar{u}_j^{n+1})^l$ be the solutions of its corresponding HCP (1.8). Then the TVD condition of HCP implies that

$$TV(\bar{u}_j^{n+1}) \le TV(u_j^n).$$

Now, by (1.3), we have

$$|\Delta_+ u_j^{n+1}| = |\Delta_+\bar{u}_j^{n+1} + \tau\Delta_+ q(u_j^n)| \le |\Delta_+\bar{u}_j^{n+1}| + \tau C|\Delta_+ u_j^n|.$$

Therefore, it follows that

$$TV_u(t) \le e^{CT}TV_u(0).$$

□

REMARK. The function $TV_{\bar{u}}(t)$ has a desirable invariant property under the similarity transform $T^{\varepsilon}_{x_0,t_0}$: Denote $(\hat{x},\hat{t}) = \mathcal{S}^{\varepsilon}_{x_0,t_0}((x,t))$ and $\hat{u} = T^{\varepsilon}_{x_0,t_0}(\bar{u})$. Then, for any $t_1, t_2 \in \mathbb{R}^+$ with $\hat{t}_1, \hat{t}_2 \geq 0$, we have $TV_{\hat{\bar{u}}}(t_2) - TV_{\hat{\bar{u}}}(t_1) = TV_{\bar{u}}(\hat{t}_2) - TV_{\bar{u}}(\hat{t}_1)$. This property is very important for the extension of the WEI framework.

With Theorem 2.4, using Helly's Theorem on the set of TVB functions and following the proof of the Lax–Wendroff Theorem [20], we obtain the following result.

THEOREM 2.5. *Suppose the HCP of a scheme of the form* (1.3)-(1.5) *is TVD, and* $\{(u_j^n)^l\}_{l=1}^{\infty}$ *is generated by* (1.3)-(1.5). *Suppose also that the step sizes* $\tau_l, h_l \to 0$ *as* $l \to \infty$, *and the initial conditions have uniformly bounded total variations. Then* $\{(u_j^n)^l\}_{l=1}^{\infty}$ *contains a subsequence* $\{(u_j^n)^{l_m}\}_{m=1}^{\infty}$ *which converges in* $L^1_{loc}(\mathbb{R} \times [0, T))$ *towards a weak solution of* (1.1) *as* $m \to \infty$.

**3. WEI criteria for convergence to the entropy solution.** For convenience, let $\Upsilon$ be the set of all sequences of numbers in $(0, 1)$ with zero limit. We use bold-faced letters to represent the sequences in $\Upsilon$, and use the corresponding light-faced ones with subscripts to represent the terms in such a sequence.

**3.1. General TVB schemes.** To show Theorem 3.1 that follows shortly, we begin by assuming that there is a sequence of numerical solutions $\{(u_j^n)^l\}_{l=1}^{\infty}$, generated by a TVB scheme of the form (1.3)-(1.5), converges to an entropy violating weak solution $w$. We also assume that the corresponding sequences of step sizes $\boldsymbol{\tau} \in \Upsilon$ and $\mathbf{h} \in \Upsilon$. By Lemma 2.2, there exists a jump point $(x_0, t_0)$ of $w$ with an associated traveling expansion shock $W$. Now for any $\boldsymbol{\varepsilon} \in \Upsilon$ applying similarity transforms $T^{\varepsilon_k}_{x_0,t_0}$ to $u^l$ for each $l$ and to $w$, we obtain $u^l_{\varepsilon_k}$ and $w_{\varepsilon_k}$ respectively. The numerical solution $u^l_{\varepsilon_k}$ satisfies the same scheme for $w_t + f(w)_x = \varepsilon_k q(w)$. For fixed $k$, $u^l_{\varepsilon_k} \to w_{\varepsilon_k}$ as $l \to \infty$ in $L^1_{loc}$. Applying Lemma 2.3 and using the same diagonal process as in [34], one may choose a sequence of increasing positive integers $\{l_k\}$ such that $\{u^{l_k}_{\varepsilon_k}\}_{k=1}^{\infty}$ converges in $L^1_{loc}$ to $W$. For simplicity, we denote $\{u^{l_k}_{\varepsilon_k}\}_{k=1}^{\infty}$ by $\{u^k\}_{k=1}^{\infty}$. Then $u^k$ is generated by the scheme

$$u_j^{n+1} = H_\varepsilon(u_{j-p}^n, \cdots, u_{j+p}^n; \lambda) = u_j^n - \lambda[g_{j+\frac{1}{2}}(u_j^n) - g_{j-\frac{1}{2}}(u_j^n)] + \varepsilon_k \tau_k q(u_j^n), \quad (3.1)$$

with $\boldsymbol{\varepsilon} \in \Upsilon$ . We call the scheme (3.1) *the* $\boldsymbol{\varepsilon}$*-scaled form* of the scheme (1.3)-(1.5), and we have obtained the first WEI convergence criterion.

THEOREM 3.1. *A TVB scheme* (1.3)-(1.5) *for the Cauchy problem* (1.1) *converges if there exists no sequence of solutions* $\{u^k\}_{k=1}^{\infty}$ *generated by the* $\boldsymbol{\varepsilon}$*-scaled form* (3.1) *of the scheme with* $\boldsymbol{\varepsilon} \in \Upsilon$ *that converges in* $L^1_{loc}(\mathbb{R} \times [0, T))$ *to a traveling expansion shock.*

**3.2. Schemes with TVD HCP.** Clearly, similarity transform $T^{\varepsilon}_{x_0,t_0}$ preserves the total variation in space. This property enables us to obtain stronger and more practical convergence criteria than Theorem 3.1. For this purpose, we consider the solutions $\bar{u}$ generated by (1.8), the HCP of the scheme (1.3)-(1.5). Let $TV_{\bar{u}}(t)$ be the total spatial variation of $\bar{u}$ at the time $t$. We denote $TTV_{\bar{u}}(t_1, t_2)$ be the total temporal variation of $TV_{\bar{u}}(t)$ from $t_1$ to $t_2$. Let $W(x, t)$ be a traveling discontinuity defined by (2.1) with the two states $w^-$ and $w^+$, where $w^-$ and $w^+$ are two distinct real constants in the domain of $f$. Throughout the remaining part of the paper, we assume that $q'(u) \geq 0$, and we make a convention: the phrase "$\boldsymbol{\varepsilon}$-scaled form" automatically implies that $\boldsymbol{\tau}, \mathbf{h}, \boldsymbol{\varepsilon} \in \boldsymbol{\Upsilon}$.

DEFINITION 3.2. We call a sequence of numerical solutions $\{u^k\}_{k=1}^{\infty}$ generated by an $\boldsymbol{\varepsilon}$-scaled form (3.1) of the scheme (1.3)-(1.5) *a TV-stable sequence of a numerical* *traveling discontinuity with the limit* $W$, if there exist $\boldsymbol{\varepsilon'} \in \Upsilon$ and positive constants $C_0$ and $C$ such that

(i) $u^k \to W$ in $L^1_{loc}(\mathbb{R} \times [0, 1))$,

(ii) $TV_{u^k}(t) < C_0$ for all $t$ and $k$, and

(iii) $TTV^k_{\bar{u}}(0, 1) < C\varepsilon'_k$ for each $k$.

When $W$ is a traveling expansion shock, we call $\{u^k\}_{k=1}^{\infty}$ *a TV-stable sequence of a numerical traveling expansion shock.*

THEOREM 3.3. *A scheme of the form* (1.3)-(1.5) *with TVD HCP for the Cauchy problem* (1.1) *converges if no* $\boldsymbol{\varepsilon}$*-scaled form* (3.1) *of the scheme is able to generate a TV-stable sequence of a numerical traveling expansion shock.*

*Proof.* We assume that the convergence of a scheme given by (1.3)-(1.5) fails. By Theorem 2.4 and Theorem 3.1 there exists an $\boldsymbol{\varepsilon'}$-scaled form (3.1) of the scheme which is capable of generating a sequence of functions $\{u^\nu\}_{\nu=1}^{\infty}$ that converges in $L^1_{loc}(\mathbb{R} \times [0, T))$ to a traveling expansion shock $W$ of the form (2.1). Moreover, $TV_{u^\nu}(t) < C_0$ for all $\nu$ and some constant $C_0$. Our goal is to find a sequence $\{\hat{u}^k\}_{k=1}^{\infty}$ generated by an $\boldsymbol{\varepsilon}$-scaled form (3.1) of the scheme such that $\hat{u}^k \to W$ in $L^1_{loc}$, $TV_{\hat{u}^k}(t) \le C_0$ and $TTV_{\hat{\bar{u}}^k}(0, 1) \le C\varepsilon'_k$. To this end, since $TV_{\bar{u}^\nu}$ is monotone decreasing, we have $TTV_{\bar{u}^\nu}(t_1, t_2) = TV_{\bar{u}^\nu}(t_1) - TV_{\bar{u}^\nu}(t_2)$ for $t_1, t_2 \in [0, 1]$, and for any positive integers $n$ and $\nu$, there is an integer $m(n, \nu)$ such that $0 \le m(n, \nu) \le n - 1$ and

$$TTV_{\bar{u}^\nu}(m(n, \nu)/n, (m(n, \nu) + 1)/n) \le \frac{1}{n} TTV_{\bar{u}^\nu}(0, 1) \le \frac{C'_0}{n}.$$

Let $t_{n,\nu} = m(n, \nu)/n$, and $x_{n,\nu} = st_{n,\nu}$. For each $k$, one can first choose a sufficiently large $n = n_k$ so that $C'_0/n_k < \varepsilon'_k$. Then, since $u^\nu \to W$ in $L^1_{loc}(\mathbb{R} \times [0, T))$, one can choose a sufficiently large $\nu = \nu_k$ so that

$$\int_0^1 \int_{st-1}^{st+1} |u^{\nu_k}(x, t) - W(x, t)| dxdt < \varepsilon'_k / n_k^2,$$

$\tau_k := n_k \tau'_{\nu_k} < \frac{1}{k}$, and $h_k := n_k h'_{\nu_k} < \frac{1}{k}$. For simplicity we set $\hat{x}_k = x_{n_k,\nu_k}$, $\hat{t}_k = t_{n_k,\nu_k}$, and $\hat{u}^k(x, t) = T^{1/n_k}_{\hat{x}_k, \hat{t}_k} u^{\nu_k}(x, t)$. We then have

$$\int_0^1 \int_{st-n_k}^{st+n_k} |\hat{u}^k(x, t) - W(x, t)| dxdt < \varepsilon'_k,$$

since $T^c_{s\alpha,\alpha} W(x, t) = W(x, t)$ for any positive constants $\alpha$ and $c$. Therefore $\hat{u}^k \to W$ in $L^1_{loc}(\mathbb{R} \times [0, 1))$. Next, since $\tau_k, h_k < \frac{1}{k}$, and the source term of $\hat{u}^k$ is $\varepsilon_k \tau_k q(\hat{u}^k)$, where $\varepsilon_k := \frac{\varepsilon'_{\nu_k}}{n_k} \to 0$ as $k \to \infty$, $\{\hat{u}^k\}_{k=1}^{\infty}$ is generated by an $\boldsymbol{\varepsilon}$-scaled form of the scheme. Moreover, $TV_{\hat{u}^k}(t) \le C_0$ since similarity transforms preserve the spatial variation. Finally, by the remark, we have

$$TTV_{\hat{\bar{u}}^k}(0, 1) = TTV_{\bar{u}^{\nu_k}}(m(n_k, \nu_k)/n_k, (m(n_k, \nu_k) + 1)/n_k) \le \frac{C'_0}{n_k} < \varepsilon'_k.$$

Therefore, $\{\hat{u}^k\}_{k=1}^{\infty}$ is a TV-stable sequence of numerical traveling expansion shock. The Theorem is proved. ∎

**3.3. Extremum Traceable schemes, general flux $f$.** To connect the numerical flux with the exact flux, we make the following assumption which is needed to develop more practical convergence criteria.

ASSUMPTION 3.4. *The numerical fluxes $g^n_{j+\frac{1}{2}}$ satisfy*

$$g^n_{j+\frac{1}{2}} \geq f(u^n_j) \geq g^n_{j-\frac{1}{2}} \quad \textit{if} \;\; u^n_j - u^n_{j\pm1} \geq 0,$$

*and*

$$g^n_{j+\frac{1}{2}} \leq f(u_j) \leq g^n_{j-\frac{1}{2}} \quad \textit{if} \;\; u^n_j - u^n_{j\pm1} \leq 0.$$

In order to track the waves (discontinuities) of the sequences of numerical solutions, first, as in [34, 35], we use the following notion of paths to be the boundaries of the transition areas of the discontinuities of the numerical solutions.

DEFINITION 3.5 (Definition 2.5 [35]). A grid point valued function $x_{I_n} = I_n h + c$, $0 \leq t_n \leq t_N = T$, is said to be an *$\varepsilon$-path of the first type with respect to a numerical solution $u$* in $[0, T]$ if for $\varepsilon > 0$ the following conditions hold:

(i) The relation $u^n_j = u^n_{I_n}$ holds if $j$ is between $I_n$ and $I_{n+1}$ and $j \neq I_{n+1}$.

(ii) The inequality holds:

$$\min(\mathrm{sgn}(\Delta_+ u^n_{I_{n+1}})\Delta_- u^n_{I_{n+1}}, |\Delta_+ u^n_{I_{n+1}}|) < \varepsilon.$$

(iii) The total variation of the numerical solution along the path is bounded, for some constant $C$, by $C\varepsilon$:

$$\sum_{n=0}^{N-1} |u^{n+1}_{I_{n+1}} - u^n_{I_{n+1}}| + |u^n_{I_{n+1}} - u^n_{I_n}| < C\varepsilon.$$

DEFINITION 3.6 (Definition 2.6 [35]). A grid point valued function $x_{I_n} = I_n h + c$, $0 \leq t_n \leq t_N = T$, is said to be an *$\varepsilon$-path of the second type with respect to a numerical solution $u$* in $[0, T]$ if for $\varepsilon > 0$ the following conditions hold:

(i) The integer valued function $I_n$ is a monotone for $0 \leq n \leq N$. Moreover, $|I_{n+1} - I_n| \leq 1$ for $0 \leq n \leq N-1$.

(ii) There is a constant $A$ such that for $0 \leq n \leq N-1$, $|u^n_j - A| < \varepsilon$ holds if $x_j$ is in the stencil of the scheme at $(x_{I_{n+1}+1}, t_n)$ or $(x_{I_{n+1}-1}, t_n)$.

Along an $\varepsilon$-path of either type, the numerical flux of an $\boldsymbol{\varepsilon}$-scared form and the exact flux have the following relationship.

LEMMA 3.7 (see Lemma 2.7 [35] for the result of the homogeneous case). *Let $\{(u^n_j)^k\}^{\infty}_{k=1}$ be a sequence of solutions generated by an $\boldsymbol{\varepsilon}$-scaled form (3.1) of the scheme (1.3)-(1.5) that satisfies Assumption 3.4. For each $k$, let $x_{I^k_n} = I^k_n h_k + c_k$ be an $\varepsilon_k$-path of either type in $[0, T]$, where $T = N^k\tau > 0$ and $\varepsilon_k \leq \varepsilon$. We then have*

$$\sum_{n=0}^{N^k-1} |(g^n_{I^k_{n+1}\pm\frac{1}{2}})^k - f((u^n_{I^k_{n+1}})^k)|\tau < C\varepsilon, \tag{3.2}$$

*where $C$ depends on $T$, the Lipschitz coefficients of $g$ and* $\max|q(u)|$.

DEFINITION 3.8 (see Definition 3.7[36]). Let $\{u^k\}_{k=1}^{\infty}$ be a sequence of solutions generated by an $\boldsymbol{\varepsilon}$-scaled form (3.1) of the scheme (1.3)-(1.5) that satisfies Assumption 3.4. We call a sequence of pairs of $\varepsilon_k$-paths of either type, $\{x^k_{I^k_n}, x^k_{J^k_n}, 0 \le n\tau_k \le N^k\tau_k = T^k\}_{k=1}^{\infty}$, where $x^k_{I^k_n} = I^k_n h_k + c_k$, $x^k_{J^k_n} = J^k_n h_k + c_k$ , and $T^k \ge 1 \ge T^k - \tau_k$, an *asymptotic traveling wave* (*ATW*) *of* $\{u^k\}$ if $x^k_{I^k_n} < x^k_{J^k_n}$ for each $k$, and if there are linear function $x(t) = st + r$ and two distinct constants $L$ and $R$ such that:

(i) If we set $\xi^k_L = x^k_{I^k_n}$ and $\xi^k_R = x^k_{J^k_n}$ for $n\tau_k \le t \le (n+1)\tau_k$ and $0 \le n \le N^k - 1$, then both $\xi^k_L$ and $\xi^k_R$ converge to $x(t)$ uniformly on the $t$-interval $[0, 1]$ as $k \to \infty$;

(ii) $|(u^n_{I^n_k})^k - L| < \varepsilon_k$ and $|(u^n_{J^n_k})^k - R| < \varepsilon_k$, for $0 \le n \le N^k$;

(iii) In the case $s = 0$, for each $k$, if either path of the pair $\{x^k_{I^k_n}, x^k_{J^k_n}\}$, say $x^k_{I^k_n}$, is of the second type, then $x^k_{I^k_n}$ is a constant.

We call $x(t)$ *the limit path* of the ATW, $L$ and $R$ the *two states* of the ATW.

Next, in order to study entropy properties of an ATW, as Osher in [23], for any convex entropy $U(w)$ and its flux $F(w)$: $F' = U'f'$, we have the equality $U(u^{n+1}_j) - U(u^n_j) = U'(\eta^n_j)(u^{n+1}_j - u^n_j)$ for some $\eta^n_j$ in between $u^n_j$ and $u^{n+1}_j$, and we adopt the numerical entropy flux

$$G^n_{j-\frac{1}{2}} \stackrel{\text{def}}{=} F(\eta^n_j) + U'(\eta^n_j)[g_{j-\frac{1}{2}} - f(\eta^n_j)].$$

Applying Lemma 3.7 for the conservation laws of the form: $U(w)_t + F(w)_x = \varepsilon_k U'(w)q(w)$ with $\boldsymbol{\varepsilon} \in \Upsilon$, then $G^n_{j-\frac{1}{2}}$ satisfies the following.

COROLLARY 3.9. *If* $\{u^k\}_{k=1}^{\infty}$ *satisfies the conditions of Lemma* 3.7*, then*

$$\sum_{n=0}^{N^k-1} |(G^n_{I^k_{n+1}-\frac{1}{2}})^k - F((u^n_{I^k_{n+1}})^k)|\,\tau < C\varepsilon.$$

In the following, for the given step sizes $h$ and $\tau$, we denote $\Delta^t_{\pm} v(x,t) := \pm(v(x, t\pm\tau) - v(x,t))$, $\Delta^x_{\pm} v(x,t) := \pm(v(x\pm h, t) - v(x,t))$, $D^t_{\pm} := (1/\tau)\Delta^t_{\pm}$, and $D^x_{\pm} := (1/h)\Delta^x_{\pm}$. Adapting Osher's proof in [23], we obtain the equality (3.3) for the numerical solutions $\{u^k\}_{k=1}^{\infty}$ generated by an $\boldsymbol{\varepsilon}$-scaled form (3.1) of the scheme (1.3)-(1.5).

$$\begin{aligned} &h_k[D^t_+ U((u^n_j)^k) + D^x_+(G^n_{j-\frac{1}{2}})^k - \varepsilon_k U'((u^n_j)^k)q((u^n_j)^k)] \\ &= \int_{(\eta^n_j)^k}^{(\eta^n_{j+1})^k} U''(w)((g^n_{j+\frac{1}{2}})^k - f(w))dw. \end{aligned} \tag{3.3}$$

Let $\phi(x,t)$ be a smooth function with compact support in the domain $-\infty < x < \infty$, $0 < t < 1$. Set $(\phi^n_j)^k = \phi(x^k_j, t^k_n)$ and define

$$\widehat{\Phi}^k \stackrel{\text{def}}{=} \tau_k \sum_{n=0}^{N^k-1} \sum_{j=I^k_{n+1}}^{J^k_{n+1}-1} h_k[D^t_+ U((u^n_j)^k) + D^x_+(G^n_{j-\frac{1}{2}})^k - \varepsilon_k U'((u^n_j)^k)q((u^n_j)^k)](\phi^n_j)^k, \tag{3.4}$$

and

$$\overline{\Phi}^k \stackrel{\text{def}}{=} \tau_k \sum_{n=0}^{N^k-1} \phi(st^k_n + r, t^k_n) \sum_{j=I^k_{n+1}}^{J^k_{n+1}-1} \int_{(\eta^n_j)^k}^{(\eta^n_{j+1})^k} U''(w)((g^n_{j+\frac{1}{2}})^k - f(w))dw. \tag{3.5}$$

We then have the following important result.

LEMMA 3.10 (see Lemma 2.11 [35] for the result of the HCP of the scheme (1.3)-(1.5)). *Suppose* $\{u^k\}_{k=1}^{\infty}$ *satisfies the conditions of Lemma* 3.7. *Let* $\{I_n^k h_k + c_k, J_n^k h_k + c_k\}$ *be an ATW of* $\{u^k\}_{k=1}^{\infty}$ *with the limit path* $x(t) = st + r$ *and the two states* $L$ *and* $R$. *We then have*

$$\lim_{k\to\infty} \overline{\Phi}^k = \lim_{k\to\infty} \widehat{\Phi}^k = [F(R) - F(L) - s(U(R) - U(L))] \int_{x=st+r} \phi(x,t)dt. \tag{3.6}$$

To derive the corollary below, we choose $U(w) = w$, and using (3.4) we have $\widehat{\Phi}^k = 0$. Hence $f(R) - f(L) = s(R - L)$ by (3.6). The second equality of the corollary follows from the integration by parts.

COROLLARY 3.11. *With the conditions of Lemma* 3.10, *we have the following discrete Rankine-Hugoniot condition:*

$$f(R) - f(L) = s(R - L). \tag{3.7}$$

*Moreover*

$$\lim_{k\to\infty} \overline{\Phi}^k = \lim_{k\to\infty} \widehat{\Phi}^k = \int_L^R U''(w)(f[w; L, R] - f(w))dw \int_{x=st+r} \phi(x,t)dt. \tag{3.8}$$

To develop the extremum tracking procedure for numerical solutions, we need the following two definitions. Denote the set of the grid points by $X = \{(x_j, t_n) : -\infty \leq j \leq \infty, 0 \leq n \leq \infty\}$ and we consider a numerical solution $u$ on $X$. A finite set of successive grid of points $\{x_q, \cdots, x_r\}$ with $r \geq q$ is said to be the *stencil of spatial maximum,* or simply *an* $\overline{E}$*-stencil* of $u$ at the time $t_n$, provided $u_q^n = \cdots = u_r^n, u_{q-1}^n < u_q^n$ and $u_{r+1}^n < u_r^n$. Notions of *an* $\underline{E}$*-stencils* for minimal and $E$*-stencils* for general extrema are defined similarly.

DEFINITION 3.12 (Definition 2.13 [35]). A nonempty subset of $X$ denoted by $\overline{E}_{t_n,t_m}, n \leq m$, is called *a ridge of the numerical solution* $u$ *from* $t_n$ *to* $t_m$ if

(i) for all $\nu$, $n \leq \nu \leq m$, the set

$$P_{\overline{E}}(\nu) := \{x_j : (x_j, t_\nu) \in \overline{E}_{t_n,t_m}\} = \{x_{q^\nu}, \cdots, x_{r^\nu}\}$$

is not empty and is an $\overline{E}$-stencil of $u$ at $t_\nu$;

(ii) for all $\nu$, $n \leq \nu \leq m - 1$,

$$P_{\overline{E}}(\nu) \cup P_{\overline{E}}(\nu + 1) = \{x_j : \min(q^\nu, q^{\nu+1}) \leq j \leq \max(r^\nu, r^{\nu+1})\}.$$

The set $P_{\overline{E}}(\nu)$ is called *the x-projection of* $\overline{E}_{t_n,t_m}$ at $t_\nu$. The value of $u$ along the ridge is denoted by $V_{\overline{E}}(\nu) : V_{\overline{E}}(\nu) = u_j^\nu$ for $q^\nu \leq j \leq r^\nu$.

If, for all $\nu$, $n \leq \nu \leq m$, the $\overline{E}$-stencil in the item (i) of the definition is replaced by an $\underline{E}$-stencil, then the set is called a *trough* of $u$ from $t_n$ to $t_m$ and is denoted by $\underline{E}_{t_n,t_m}$. The related notions $P_{\underline{E}}(\nu)$ and $V_{\underline{E}}(\nu)$ are defined similarly. Ridges and troughs are also called *extremum paths.* When we do not distinguish between ridges and troughs, we use $E_{t_n,t_m}$, $P_E(\nu)$, and $V_E(\nu)$ for either type. We write

$$E^1_{t_n,t_m} < (\leq) E^2_{t_n,t_m}, \text{ if } \max P_{E^1}(\nu) < (\leq) \max P_{E^2}(\nu) \text{ for } n \leq \nu \leq m.$$

DEFINITION 3.13 (Definition 2.14 [35]). A scheme is said to be *extremum traceable* if there exists a positive constant $c \geq 1$ such that for each numerical solution $u$ of the scheme and each integer $N > 0$, there exists a finite or infinite collection of extremum paths $\{E^l_{t_0,t_N}\}_{l=l_1}^{l_2}$ with the following properties:

(i) $\{P_{E^l}(N)\}_{l=l_1}^{l_2}$ is precisely the set of $E$-stencils of $u^n_j$ at the time $t_N$ arranged in ascending spatial coordinates.

(ii) If $E^l_{t_0,t_N}$ is a ridge (trough), then $V_{E^l}(n)$ is a non increasing (non decreasing) function of $n$.

(iii) Let $P_{E^l}(n) = \{x_{q^l(n)}, \cdots, x_{r^l(n)}\}$ for $1 \leq n \leq N$. If $P_{E^l}(n) \cap P_{E^l}(n+1) = \emptyset$, then

$$|\, u^n_{q^l(n+1)} - u^n_{r^l(n)} \,| \leq c \,|\, V_{E^l}(n+1) - V_{E^l}(n) \,| \quad \text{when} \quad q^l(n+1) > r^l(n),$$

$$|\, u^n_{r^l(n+1)} - u^n_{q^l(n)} \,| \leq c \,|\, V_{E^l}(n+1) - V_{E^l}(n) \,| \quad \text{when} \quad q^l(n) > r^l(n+1).$$

(iv) If $l_2 > l_1$, then $E^l_{t_0,t_N} < E^{l+1}_{t_0,t_N}$ for $l_1 \leq l \leq l_2 - 1$.

THEOREM 3.14. *If an $\boldsymbol{\varepsilon}$-scaled form (3.1) of the scheme (1.3)-(1.5) can be written in an increment form*

$$u^{n+1}_j = u^n_j - C_{j-\frac{1}{2}} \Delta u_{j-\frac{1}{2}} + D_{j+\frac{1}{2}} \Delta u_{j+\frac{1}{2}} + \varepsilon_k \tau_k q(u^n_j), \tag{3.9}$$

*then, for sufficient small $\varepsilon$ with $\varepsilon_k < \varepsilon$, the sufficient conditions for (3.1) to be extremum traceable are the following inequalities:*

$$0 \leq C_{j+\frac{1}{2}},\; 0 \leq D_{j+\frac{1}{2}},\; 0 \leq C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \leq 1, \;\textit{for all } j; \tag{3.10}$$

*there is a positive constant $\mu$ such that, if $u^n_j$ is a space extremum, then*

$$\max\{C_{j\pm\frac{1}{2}}, C_{j\pm\frac{3}{2}}, D_{j\pm\frac{1}{2}}, D_{j+\frac{3}{2}}\} \leq \frac{\mu}{4} < \frac{1}{4}. \tag{3.11}$$

We omit the proof, since it can be easily derived from the proof of Theorem 2.3 in [17]. Recall, Yang's Lemma 2.15 [35], an extremum traceable scheme is TVD, and following the results of Theorem 3.14 and Theorem 2.3 [17], we have

COROLLARY 3.15. *If an $\boldsymbol{\varepsilon}$-scaled form (3.9) satisfies the conditions (3.10)-(3.11), then the HCP of the scheme (1.3)-(1.5) is extremum traceable and hence TVD.*

LEMMA 3.16 (Lemma 2.17 [35]). *Suppose $u$ is given by an extremum traceable scheme. Suppose also that $TTV_u(t_0,t_N) \leq \varepsilon/(2c-1)$, where $N$ is a positive integer and $c$ is the constant in Definition 3.13. Let $E_{t_0,t_N}$ be an extremum path of $u^n_j$ that satisfies the properties (ii) and (iii) of Definition 3.13. Then exists an $\varepsilon$-path $x_{I_n}$ of the first type for $0 \leq n \leq N$ such that $x_{I_n} \in P_E(n)$.*

The important concepts of *asymptotic traveling discontinuity* (*ATD*) and *asymptotic traveling expansion shock* (*ATES*) will be introduced shortly. Loosely speaking, the essential structure of a traveling wave of a numerical solution is the moving transition from a left limit to a different right limit. When using an ATW to track the wave, the transition region is bounded either by extremum paths, i.e., ridge (trough) lines, which can be approximated by an $\varepsilon$-path of the first type, or by rim lines which can be approximated by an $\varepsilon$-path of the second type. If an ATW contains several large jumps and these jumps can be split into essentially monotone waves, then the ATW becomes an asymptotic traveling discontinuity (ATD).

DEFINITION 3.17. An ATW $\{x^k_{I^k_n}, x^k_{J^k_n}; 0 \le n\tau_k \le N^k\tau_k = T^k\}_{k=1}^{\infty}$ of $\{u^k\}_{k=1}^{\infty}$ with the left state $L$ and the right state $R$ is called an *ATD of* $\{u^k\}_{k=1}^{\infty}$ if the following three properties hold.

(i) Each of the two paths is either an $\varepsilon_k$-path of the second type or an $\varepsilon_k$-path of the first type that is also an extremum path as in Lemma 3.16. Moreover, if any of the paths is a ridge (trough), then it is on the side of the larger (smaller) state of $L$ and $R$.

(ii) For each $k$, if $I^k_n \le q < r \le J^k_n$ and $0 \le n \le N^k$, then

$$-((u^n_r)^k - (u^n_q)^k)\mathrm{sgn}(R - L) \le \varepsilon_k.$$

(iii) For each $k$, if $0 \le n \le N^k$,and $I^k_n \le j < J^k_n$ and $((u^n_{j+1})^k - (u^n_j)^k)(R-L) < 0$, then

$$(u^n_j)^k, (u^n_{j+1})^k \in \mathcal{N}_{\varepsilon_k}(\{w : \ f(w) = f[w; L, R]\})$$

where $\mathcal{N}_\delta(S)$ denotes the $\delta$-neighborhood of a set $S$. An ATD of $\{u^k\}_{k=1}^{\infty}$ is called an *ATES* of $\{u^k\}_{k=1}^{\infty}$ if the entropy condition (2.3) with $w^- = L$ and $w^+ = R$ fails. In this case we also say that $\{u^k\}_{k=1}^{\infty}$ *harbors the ATES* $\{x^k_{I^k_n}, x^k_{J^k_n}\}_{k=1}^{\infty}$.

Now we are ready to state the third WEI criterion for the convergence.

THEOREM 3.18. *A scheme of the form (1.3)-(1.5) satisfying Assumption 3.4, with TVD HCP and extremum traceable $\boldsymbol{\varepsilon}$-scaled form (3.1), converges if no $\boldsymbol{\varepsilon}$-scaled form (3.1) is able to create a sequence of solutions $\{u^k\}_{k=1}^{\infty}$ that harbors an ATES.*

*Sketch of the proof.* (see the proof of Theorem 2.19 [35] for the HCP of the scheme) We argue by contradiction. If the convergence fails, then by Theorem 3.3 there exists a TV-stable sequence $\{u^k\}$ of a numerical traveling expansion shock generated by an $\boldsymbol{\varepsilon}$-scaled form (3.1) of the scheme. Since $TV_{u^k}(t) < C_0$ for all $k$ and $t \in [0, 1]$ and since $\{u^k\}$ converges to $W$ in $L^1_{loc}(\mathbb{R} \times [0, 1))$, in the compact domain $\Omega = \{(x, t) : (x, t) \in [st - 1, st + 1] \times [0, 1]\}$ there may exist at most uniformly bounded number of large oscillations which asymptotically either travel away from the line $x = st$ as infinitesimally thin spikes in the graph of the numerical solutions, or move along the line. Also since the $\boldsymbol{\varepsilon}$-scaled form (3.1) is extremum traceable, i.e., non-oscillatory, we can use approximate extremum paths to track these oscillations. Because the sequence $\{u^k\}$ is TV-stable, the amplitudes of these oscillations are essentially stationary, and it contains a subsequence, in which the approximate paths becomes $\varepsilon$-paths as described in Definitions 3.5 and 3.6. Using similarity transforms and selecting subsequences, we may push those oscillations which do not travel along the line $x = st$ out of the interested domain $\Omega$ (this effect is called wave separation). Hence, all the strong oscillations which remain in $\Omega$ travel along the line (this effect is called wave concentration). Finally it can be shown that these oscillations consists of finite number of strong ATWs which dominated the entropy estimate and the oscillations of small amplitude whose contributions to the entropy estimate are negligible, and at least one of the strong ATWs must be an ATES (this analysis is called wave splitting). The entire proof can be directly translated from that of Theorem 3.13 in [34]. ☐

**3.4. Extremum traceable schemes, convex flux $f$.** For any collection of data $\{v_j\}$, denote $\tilde{v}_j = H_\varepsilon(v_{j-p}, \cdots, v_{j+p}; \lambda)$ (see (3.1)), $\check{v}_j = \frac{v_j + \tilde{v}_j}{2}$ , and $f[w; L, R]$ be the linear function interpolating $f(w)$ at $w = L$ and $w = R$. In this subsection, we assume that $f''(w) \geq 0$.

Definition 3.19 (see Definition 2.20 [35]). We call *an ordered pair of numbers* $\{L, R\}$ *a rarefying pair* if $L < R$ and $f[w; L, R] > f(w)$ when $L < w < R$. We call *a collection of data* $\Gamma = \{v_j\}_{j=I-p}^{J+p}$ *an $\varepsilon$-rarefying collection* of the $\boldsymbol{\varepsilon}$-scaled form (3.1) to the rarefying pair $\{L, R\}$ if, for $\varepsilon > 0$,

(i) $L = v_I \leq v_{I+1} \leq \cdots \leq v_J = R$;

(ii) $\tilde{v}_I \leq \tilde{v}_{I+1} \leq \cdots \leq \tilde{v}_J$, $|L - \tilde{v}_I| < \varepsilon$, $|R - \tilde{v}_J| < \varepsilon$;

(iii) either $v_{I-1} \geq v_I$ or $v_I = v_{I+1}$; and either $v_{J+1} \leq v_J$ or $v_{J-1} = v_J$.

The conditions (i) and (ii) imply that $\check{v}_I \leq \check{v}_{I+1} \leq \cdots \leq \check{v}_J$, $|L - \check{v}_I| < \varepsilon/2$, and $|R - \check{v}_J| < \varepsilon/2$. We define the piecewise constant function $g_\Gamma$ associated with the $\varepsilon$-rarefying collection $\Gamma$ as follows:

$$g_\Gamma(w) = g_{j+\frac{1}{2}}[v] \quad \text{for } w \in (\check{v}_j, \check{v}_{j+1}), \quad I \leq j \leq J-1.$$

Definition 3.20. We call an $\varepsilon$-rarefying collection $\Gamma = \{v_j\}_{j=I-2}^{J+2}$ of the $\boldsymbol{\varepsilon}$-scaled form (3.1) to the pair $\{L, R\}$ *an $\varepsilon$-normal collection*, if it satisfies the following relation:

$$L = v_{I-2} = v_{I-1} = v_I = v_{I+1} \leq \cdots \leq v_{J-1} = v_J = v_{J+1} = v_{J+2} = R. \tag{3.12}$$

Next, we present the last WEI criterion of convergence, which states that the WEI across the area of the rarefaction is sufficient for convergence.

Theorem 3.21. *A scheme (1.3)-(1.5) satisfying Assumption 3.4, with TVD HCP and extremum traceable $\boldsymbol{\varepsilon}$-scaled form (3.1), converges for convex conservation laws (1.1) if, for any rarefying pair $\{L, R\}$ and $\varepsilon$-rarefying collections $\{v_j\}_{j=-p}^{n+p}$ of the $\boldsymbol{\varepsilon}$-scaled form (3.1) to the pair, there is a constant $\delta > 0$ such that the quadrature inequality*

$$\delta < \int_L^R f[w; L, R]dw - \int_{\check{v}_I}^{\check{v}_J} g_\Gamma(w)dw \tag{3.13}$$

*holds, provided that $\varepsilon$ is sufficiently small.*

The proof of the theorem is similar to that of Theorem 2.21 in [35]. Nonetheless, we provide a sketch of the proof here for convenience.

*Sketch of the proof.* Assume a scheme satisfies the conditions of the theorem does not converge. Then by Theorem 3.18, there is a sequence of numerical solutions $\{u^k\}$ generated by an extremum traceable $\boldsymbol{\varepsilon}$-scaled form (3.1) that harbors an ATES $\{x^k_{I^k_n}, x^k_{J^k_n}\}$. Since $f$ is convex, the two states $L$ and $R$ of the ATES form a rarefying pair $\{L, R\}$ and, by DiPerna's results [6] for scalar convex conservation laws, it suffices to consider the entropy property of the wave with the entropy function $U(w) = w^2/2$. Similarly, as the proof of the HCP in [35], for any $\varepsilon > 0$, sufficiently large $k$, and $0 \leq n \leq N^k$, the inequality

$$\Big|\sum_{j=I^k_{n+1}}^{J^k_{n+1}-1} \int_{(\eta^n_j)^k}^{(\eta^n_{j+1})^k} ((g^n_{j+\frac{1}{2}})^k - f(w))dw - \int_{\check{v}_I}^{\check{v}_J} (g_\Gamma(w) - f(w))dw\,\Big| < \varepsilon \tag{3.14}$$

holds for some $\varepsilon$-rarefying collection $\Gamma = \{v_j\}_{j=I-p}^{J+p}$ of the $\boldsymbol{\varepsilon}$-scaled form (3.1) to the pair $\{L, R\}$, where $(\eta_j^n)^k = ((u_j^n)^k + (u_j^{n+1})^k)/2$. Now, let $\delta$ be the constant in (3.13) for the rarefying pair $\{L, R\}$. Then there is a constant $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, every $\varepsilon$-rarefying collection $\Gamma = \{v_j\}_{j=I-p}^{J+p}$ of the $\boldsymbol{\varepsilon}$-scaled form (3.1) to the pair satisfies

$$\begin{aligned}\int_{\check{v}_I}^{\check{v}_J} (g_\Gamma(w) - f(w))dw &\le \int_{\check{v}_I}^{\check{v}_J} g_\Gamma(w) - \int_L^R f(w)dw + C\varepsilon \\ &< \int_L^R (f[w; L, R] - f(w))dw - \delta + C\varepsilon,\end{aligned} \tag{3.15}$$

where $C = \max\{|f(w)| : |w - L| < \varepsilon/2 \text{ or } |w - R| < \varepsilon/2\}$. Now let $k$ be sufficiently large and let $\Gamma = \{v_j\}_{j=I-p}^{J+p}$ be an $\varepsilon$-rarefying collection of (3.1) that satisfies (3.14). Since it also satisfies (3.15), we have

$$\sum_{j=I_{n+1}^k}^{J_{n+1}^k - 1} \int_{(\eta_j^n)^k}^{(\eta_{j+1}^n)^k} ((g_{j+\frac{1}{2}}^n)^k - f(w))dw < \int_L^R (f[w; L, R] - f(w))dw - \delta + (C+1)\varepsilon \tag{3.16}$$

for $0 \le n \le N^k$. With the special entropy function $U(w) = w^2/2$, we apply the inequality to $\overline{\Phi}^k$ that is defined by (3.5) and we obtain

$$\overline{\Phi}^k \le \tau_k \sum_{n=0}^{N^k - 1} \phi(st_n^k + r, t_n^k)\Big[\int_L^R (f[w; L, R] - f(w))dw - \delta + (C+1)\varepsilon\Big], \tag{3.17}$$

where $\phi$ is a nonnegative smooth test function. Taking the limit on both sides while applying Corollary 3.11 to the left side, we arrive at

$$\int_L^R (f[w; L, R] - f(w))dw \le \int_L^R (f[w; L, R] - f(w))dw - \delta + (C+1)\varepsilon,$$

where we have eliminated the common positive factor $\int_{x=st+r} \phi(x,t)dt$. This is a contradiction for sufficiently small $\varepsilon$. $\square$

**4. Application.** For the homogeneous conservation laws (1.7), the convergence of high-resolution schemes with van Leer's flux limiter, introduced by Sweby [30], was established by the author [17]. In this section we illustrate that the aforementioned schemes with source terms will converge as well, by applying the last extended WEI convergence criterion. First, to introduce the schemes, we need some shorthand notations. Let the flux differences be

$$f_{j+\frac{1}{2}}^+ = f(u_{j+1}^n) - g_{j+\frac{1}{2}}^E, \tag{4.1}$$

and

$$f_{j+\frac{1}{2}}^- = g_{j+\frac{1}{2}}^E - f(u_j^n), \tag{4.2}$$

where $g_{j+\frac{1}{2}}^E = g^E(u_j^n, u_{j+1}^n)$ is the flux of an $E$-scheme [25] that satisfies

$$\operatorname{sgn}(u_{j+1}^n - u_j^n)[g_{j+\frac{1}{2}}^E - f(u)] \le 0, \tag{4.3}$$

for all $u$ in between $u_j^n$ and $u_{j+1}^n$.

Following Sweby [30], we use

$$\nu^+_{j+\frac{1}{2}} = \frac{\lambda f^+_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}, \qquad \nu^-_{j+\frac{1}{2}} = \frac{\lambda f^-_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}, \tag{4.4}$$

to define a series of local CFL numbers, where, by convention, $\Delta u^n_{j+\frac{1}{2}} = \Delta_+ u^n_j = \Delta_- u^n_{j+1} = u^n_{j+1} - u^n_j$. We also assume that the local CFL numbers satisfy $|\nu^\pm_{j+\frac{1}{2}}| \le 1$ for all $j \in \mathbb{Z}$, and set

$$\alpha^+_{j+\frac{1}{2}} = \frac{1}{2}(1 - \nu^+_{j+\frac{1}{2}}), \qquad \alpha^-_{j+\frac{1}{2}} = \frac{1}{2}(1 + \nu^-_{j+\frac{1}{2}}); \tag{4.5}$$

and

$$r^+_j = \frac{\alpha^+_{j-\frac{1}{2}} f^+_{j-\frac{1}{2}}}{\alpha^+_{j+\frac{1}{2}} f^+_{j+\frac{1}{2}}}, \qquad r^-_j = \frac{\alpha^-_{j+\frac{1}{2}} f^-_{j+\frac{1}{2}}}{\alpha^-_{j-\frac{1}{2}} f^-_{j-\frac{1}{2}}}. \tag{4.6}$$

Using these notations, Sweby's schemes with flux limiter, in the non-homogeneous case, are given by

$$u^{n+1}_j = u^n_j - \lambda\,(\,g_{j+\frac{1}{2}} - g_{j-\frac{1}{2}}\,) + \tau q(u^n_j), \tag{4.7}$$

where

$$g_{j+\frac{1}{2}} = g^E_{j+\frac{1}{2}} + \varphi(r^+_j)\alpha^+_{j+\frac{1}{2}} f^+_{j+\frac{1}{2}} - \varphi(r^-_{j+1})\alpha^-_{j+\frac{1}{2}} f^-_{j+\frac{1}{2}}, \tag{4.8}$$

and $\varphi$ is a flux limiter, which is Lipschitz continuous function and its graph lies in the second order TVD region of the HCP of (4.7)-(4.8)) [30]:

$$\{(r, \varphi_\Phi(r)) : \varphi_\Phi(r) = \max(0, \min(\Phi r, 1), \min(r, \Phi)), 1 \le \Phi \le 2, r \in \mathbb{R}\}. \tag{4.9}$$

To study the entropy convergence of the schemes (4.7)-(4.8), we consider a special limiter: $\varphi = \varphi_{VL}$, i.e. the van Leer's flux limiter [30]:

$$\varphi_{VL}(r) = \begin{cases} 0 & r \le 0, \\ \dfrac{2r}{1+r} & r > 0, \end{cases} \tag{4.10}$$

which is one of the famous limiters that resides in the region (4.9) and $\varphi_{VL}$ approaches to 2, the upper boundary of the region, as $r \to \infty$. For the remainder of the paper, we also assume that the building block of the schemes is Godunov [8] or Engquist-Osher [7] scheme. Now, we recall that for Godunov scheme, the flux is given by

$$g^{God}(u_j, u_{j+1}) = \begin{cases} \min_{u_j \le w \le u_{j+1}} f(w) & \text{when} \quad u_j \le u_{j+1}, \\ \max_{u_j \ge w \ge u_{j+1}} f(w) & \text{when} \quad u_j \ge u_{j+1}, \end{cases} \tag{4.11}$$

and for Engquist-Osher scheme, the flux is given by

$$g^{EO}(u_j, u_{j+1}) = \int_0^{u_j} \max(f'(w), 0)dw + \int_0^{u_{j+1}} \min(f'(w), 0)dw + f(0). \tag{4.12}$$

The following lemma can be shown in the same way as we did for its HCP [15], since Assumption 3.4 does not involve the source terms.

LEMMA 4.1. *A scheme of the form* (4.7)-(4.8) *with van Leer's flux limiter satisfies Assumption* 3.4.

It is easy to see that, by following Sweby's derivation [30], the schemes (4.7)-(4.8) can be written in an increment form.

$$u_j^{n+1} = u_j^n - C_{j-\frac{1}{2}}\Delta u_{j-\frac{1}{2}} + D_{j+\frac{1}{2}}\Delta u_{j+\frac{1}{2}} + \tau q(u_j^n), \tag{4.13}$$

with

$$C_{j+\frac{1}{2}} = \nu^+_{j+\frac{1}{2}}\{1 + \alpha^+_{j+\frac{1}{2}}[\frac{\varphi(r^+_{j+1})}{r^+_{j+1}} - \varphi(r^+_j)]\}, \tag{4.14}$$

and

$$D_{j+\frac{1}{2}} = -\nu^-_{j+\frac{1}{2}}\{1 + \alpha^-_{j+\frac{1}{2}}[\frac{\varphi(r^-_j)}{r^-_j} - \varphi(r^-_{j+1})]\}. \tag{4.15}$$

If we denote the solutions of the HCP of (4.13)-(4.15) by $\bar{u}_j^{n+1}$, i.e. $\bar{u}_j^{n+1} = u_j^n - C_{j-\frac{1}{2}}\Delta u_{j-\frac{1}{2}} + D_{j+\frac{1}{2}}\Delta u_{j+\frac{1}{2}}$, we then can write an $\boldsymbol{\varepsilon}$-scaled form of (4.13)-(4.15) as

$$u_j^{n+1} = \bar{u}_j^{n+1} + \varepsilon_k\tau q(u_j^n). \tag{4.16}$$

LEMMA 4.2 (see Lemma 2.5 [17] for the result of the homogeneous case). *An $\boldsymbol{\varepsilon}$-scaled form (4.16) of the scheme (4.13)-(4.15) with the building block of Godunov or Engquist-Osher scheme is extremum traceable, provided that $\varepsilon_k < \varepsilon$ for sufficiently small $\varepsilon$, and*

$$\nu^+_{j+\frac{1}{2}} - \nu^-_{j+\frac{1}{2}} \leq \frac{2}{2+\Phi} \quad \textit{for all} \ \ j, \tag{4.17}$$

*where $\Phi$ is given by (4.9); and when $u_j^n$ is an extremum, there is a constant $\mu$, $0 \leq \mu < 1$, such that $\lambda K' = \lambda \max_{u^n_{j-2}\leq w\leq u^n_{j+2}} |\, f'(w)\,| \leq \frac{\mu}{6}$.*

With Lemma 4.1 and Theorem 3.14, we can simplify Theorem 3.21 for the scheme (4.13)-(4.15) as follows.

LEMMA 4.3 (see Lemma 3.5 [17] for the result of the homogeneous case). *A scheme of the form (4.13)-(4.15) satisfying the conditions (3.10)-(3.11) converges for convex conservation laws, provided that for each rarefying pair $\{L, R\}$ there is a constant $\delta > 0$ such that the inequality (3.13) holds for all $\varepsilon$-normal corrections of the $\boldsymbol{\varepsilon}$-scaled form (4.16) to the pair $\{L, R\}$ for sufficiently small $\varepsilon$ with $\varepsilon_k < \varepsilon$.*

*Proof.* Let $\Lambda = \{\kappa_{P-2}, \cdots, \kappa_{Q+2}\}$ be an arbitrary $\varepsilon$-rarefying collection of the $\boldsymbol{\varepsilon}$-scaled form (4.16) to the pair $\{L, R\}$. Without loss of generality, we assume $|\varepsilon_k\tau q| < \varepsilon$ for all $k$. Let

$$S' = \int_{\check{\kappa}_P}^{\check{\kappa}_Q} g_\Lambda(w)\,dw = \sum_{j=P}^{Q-1}(\check{\kappa}_{j+1} - \check{\kappa}_j)\, g_{j+\frac{1}{2}}[\kappa]. \tag{4.18}$$

By (i) and (iii) of Definition 3.19, either $\kappa_P$ or $\kappa_{P+1}$ is a minimum. In either case, Assumption 3.4 and the condition (ii) of Definition 3.19 imply that

$$\begin{aligned} \varepsilon > |L - \tilde{\kappa}_P| &= |\tilde{\kappa}_P - \kappa_P| \\ &\geq \lambda |g_{P+\frac{1}{2}}[\kappa] - g_{P-\frac{1}{2}}[\kappa]| - |\varepsilon_k \tau q| \geq \lambda |g_{P\pm\frac{1}{2}}[\kappa] - f(L)| - |\varepsilon_k \tau q|, \end{aligned} \tag{4.19}$$

or

$$\lambda |g_{P\pm\frac{1}{2}}[\kappa] - f(L)| \leq \varepsilon + |\varepsilon_k \tau q| < 2\varepsilon. \tag{4.20}$$

Similarly, we have

$$\varepsilon > |R - \tilde{\kappa}_Q| \geq \lambda |g_{Q\pm\frac{1}{2}}[\kappa] - f(R)| - |\varepsilon_k \tau q|, \tag{4.21}$$

or

$$\lambda |g_{Q\pm\frac{1}{2}}[\kappa] - f(R)| \leq \varepsilon + |\varepsilon_k \tau q| < 2\varepsilon. \tag{4.22}$$

Next, we construct an $\varepsilon$-normal collection $\Gamma = \{v_j\}_{j=I-2}^{J+2}$ as follows. First, let $I = P-1$ and $J = Q+1$ and we also set $v_{I-2} = v_{I-1} = v_I = L$, $v_J = v_{J+1} = v_{J+2} = R$, and $v_j = \kappa_j$ for $I+1 \leq j \leq J-1$. Then, we can easily obtain the following relations

$$g_{I\pm\frac{1}{2}}[v] = f(L), \qquad g_{J\pm\frac{1}{2}}[v] = f(R), \tag{4.23}$$

$$\tilde{v}_I = L + \varepsilon_k \tau q(L), \qquad \tilde{v}_J = R + \varepsilon_k \tau q(R),$$

and

$$\tilde{v}_I \leq \tilde{v}_{I+1} \leq \cdots \leq \tilde{v}_J.$$

Therefore $\Gamma = \{v_j\}_{j=I-2}^{J+2}$ indeed is an $\varepsilon$-normal collection. Now, let $G$ be the Lipschitz constant of the numerical flux $g$, and $K = \max\{|f(L)|, |f(R)|\} + 2G(R-L)$. Denote

$$S = \int_{\check{v}_I}^{\check{v}_J} g_\Gamma(w)\, dw = \sum_{j=I}^{J-1} (\check{v}_{j+1} - \check{v}_j) g_{j+\frac{1}{2}}[v], \tag{4.24}$$

then a-priori estimate $|S - S'| \leq 4K\varepsilon$ holds. Let $\delta'$ be a constant such that for all $\varepsilon$-normal collections of the scheme (4.16) to the pair $\{L, R\}$ the inequality (3.13) holds for $\delta = \delta'$. Thus, for $\delta = \delta'$, the inequality (3.13) also holds for the $\varepsilon$-normal collection $\Gamma = \{v_j\}_{j=I-2}^{J+2}$. Therefore, for $\delta = \frac{\delta'}{2}$, the inequality (3.13) holds for all $\varepsilon$-rarefying collection of the scheme to the pair $\{L, R\}$ provided that $\varepsilon \leq \frac{\delta}{4K}$. It remains to show the a-priori estimate. Notice that $g_{j+\frac{1}{2}}[\kappa] = g_{j+\frac{1}{2}}[v]$, for $P \leq j \leq Q-1$, and therefore, $\check{\kappa}_j$ for $P+1 \leq j \leq Q-1$ are independent of $\kappa_i$ for $i < P$ or $i > Q$. Thus, $\check{\kappa}_j = \check{v}_j$ for $P+1 \leq j \leq Q-1$, and we have

$$\begin{aligned} |S - S'| \leq{}& |\check{v}_{I+1} - \check{v}_I||g_{I+\frac{1}{2}}[v]| + |\check{v}_{I+1} - \check{\kappa}_{I+1}||g_{I+\frac{3}{2}}[v]| \\ &+ |\check{v}_J - \check{v}_{J-1}||g_{J-\frac{1}{2}}[v]| + |\check{v}_{J-1} - \check{\kappa}_{J-1}||g_{J-\frac{3}{2}}[v]|. \end{aligned} \tag{4.25}$$

The relationship of $\Lambda$ and $\Gamma$ and the inequalities (4.20) and (4.22) yield:

$$|\check{v}_{I+1} - \check{\kappa}_{I+1}| \leq \frac{\varepsilon}{2} + \frac{|\varepsilon_k \tau q|}{2} < \varepsilon, \quad |\check{v}_{J-1} - \check{\kappa}_{J-1}| \leq \frac{\varepsilon}{2} + \frac{|\varepsilon_k \tau q|}{2} < \varepsilon, \tag{4.26}$$

$$|\check{v}_{I+1}-\check{v}_I| = |\frac{\lambda}{2}(g_{I+\frac{3}{2}}-f(L))| = |\frac{\lambda}{2}(g_{P+\frac{1}{2}}-f(L))| < \varepsilon, \tag{4.27}$$

and similarly we have

$$|\check{v}_J-\check{v}_{J-1}| < \varepsilon. \tag{4.28}$$

Finally, $|S-S'| < 4K\varepsilon$ follows from the inequalities (4.25)-(4.28). □

For an $\varepsilon$-normal collection $\Gamma = \{v_j\}_{j=I-2}^{J+2}$, we denote the vertex $(v_j, f(v_j))$ by $V_j$ and the area of convex polygon $V_{j_1}V_{j_2}\cdots V_{j_r}$ by $S_{j_1,\dots,j_r}$. Let $\sigma_\Gamma = \max_{I-2\le j\le J+2}|\nu^{\pm}_{j\pm\frac{1}{2}}|$, and let

$$\alpha_j = \begin{cases} 0.5 & \text{if } \Delta_+v_{j-2} = \Delta_+v_{j+1} = 0, \\ 1 & \text{otherwise.} \end{cases}$$

First, we need the following lemma, which was Yang's result [35].

Lemma 4.4 (Lemma 3.7 [35]). *For $I < i < J-1$, we have*

$$S_{I,I+1,\dots,J} - \sum_{j=I+1}^{J-1} S_{j-1,j,j+1} \ge S_{I,i,i+1,J} - (S_{I,i,i+1} + S_{i,i+1,J}).$$

For the $\boldsymbol{\varepsilon}$-scaled form (4.16), we will have a very important estimate (4.29). The proof is the similar to the one given for its HCP by the author [17]. The derivations of the estimations are the same, except for (4.16) we will have extra finite terms of the form $\varepsilon_k\tau q'$ (may be multiplied by a bounded quantity), which all vanish to zero as $k\to\infty$. This means that if $\varepsilon$ is small enough and $\varepsilon_k\tau q' < \varepsilon$, then the contributions of these terms are negligible, and therefore the inequality (4.29) still holds for the $\boldsymbol{\varepsilon}$-scaled form (4.16). For this reason, we omit the proof.

Lemma 4.5 (see Lemma 3.6 [17] for the result of the homogeneous case). *Let $\Gamma = \{v_j\}_{j=I-2}^{J+2}$ be an $\varepsilon$-normal collection of $\boldsymbol{\varepsilon}$-scaled form (4.16) to a rarefying pair $\{L,R\}$. Then the numerical solutions of the $\boldsymbol{\varepsilon}$-scaled form (4.16) for convex conservation laws satisfy, for sufficiently small $\varepsilon$ and $\sigma_\Gamma$, the following inequality*

$$\int_L^R (f[w;L,R]-g_\Gamma)dw \ge S_{I,I+1,\dots,J} - \sum_{j=I+1}^{J-1}\alpha_j S_{j-1,j,j+1}. \tag{4.29}$$

Finally, with the Lemmas 4.3, 4.4 and 4.5, we can show the entropy convergence of the scheme (4.13)-(4.15).

Theorem 4.6 (see Theorem 3.8 [17] for the result of the homogeneous case). *The scheme (4.13)-(4.15) satisfying the conditions of (3.10)-(3.11) converges for the convex problem (1.1), provided that $\varphi = \varphi_{VL}$ is van Leer's flux limiter (4.10), $g^E(\cdot,\cdot)$ is either the Godunov flux (4.11) or Engquist-Osher flux (4.12) and, $\sigma$ and $\varepsilon$ are sufficiently small.*

*Proof.* For each $\varepsilon$-normal collection $\Gamma = \{v_i\}_{i=I-2}^{J+2}$ to a rarefying pair $\{L,R\}$, we have $\check{v}_I = L + \frac{\varepsilon_k\tau}{2}q(L)$ and $\check{v}_J = R + \frac{\varepsilon_k\tau}{2}q(R)$. Recall that $q'(w)\ge 0$ and

$$g_\Gamma(w) = g_{j+\frac{1}{2}}[v] \quad \text{for } w\in(\check{v}_j,\check{v}_{j+1}), \quad I\le j\le J-1.$$

*Case 1.* If $q(L) \geq 0$, than we have $q(R) \geq 0$ as well. Let $c$ be a constant such that $|g_\Gamma(w)| \leq c$, for $w \in (R, \check{v}_J)$ and we set $g_\Gamma(w) = -c$, when $w \in (L, \check{v}_I)$. We obtain

$$\int_{\check{v}_I}^{\check{v}_J} g_\Gamma(w)dw = \{-\int_L^{\check{v}_I} + \int_L^R + \int_R^{\check{v}_J}\}g_\Gamma(w)dw \leq c\varepsilon_k\tau\frac{q(L)+q(R)}{2} + \int_L^R g_\Gamma(w)dw$$

*Case 2.* If $q(L) \leq 0$, and $q(R) \geq 0$, we let $c$ be a constant such that $|g_\Gamma(w)| \leq c$, for $w \in (R, \check{v}_J) \cup (\check{v}_I, L)$. Now we have

$$\int_{\check{v}_I}^{\check{v}_J} g_\Gamma(w)dw = \{\int_{\check{v}_I}^L + \int_L^R + \int_R^{\check{v}_J}\}g_\Gamma(w)dw \leq c\varepsilon_k\tau\frac{-q(L)+q(R)}{2} + \int_L^R g_\Gamma(w)dw$$

*Case 3.* If $q(L) \leq 0$, and $q(R) \leq 0$, we let $c$ be a constant such that $|g_\Gamma(w)| \leq c$, for $w \in (\check{v}_I, L)$ and set $g_\Gamma(w) = -c$, when $w \in (\check{v}_J, R)$. We obtain

$$\int_{\check{v}_I}^{\check{v}_J} g_\Gamma(w)dw = \{\int_{\check{v}_I}^L + \int_L^R + \int_R^{\check{v}_J}\}g_\Gamma(w)dw \leq c\varepsilon_k\tau\frac{-q(L)-q(R)}{2} + \int_L^R g_\Gamma(w)dw.$$

In all cases, without loss of generality, for any given $\varepsilon > 0$ we let $c\varepsilon_k\tau\frac{|q(L)|+|q(R)|}{2} < \varepsilon$ for all $k$. Thus,

$$\int_{\check{v}_I}^{\check{v}_J} g_\Gamma(w)dw \leq \int_L^R g_\Gamma(w)dw + \varepsilon. \tag{4.30}$$

Now, we set

$$d_1(\Gamma) = \max_{I \leq i \leq J} \min(v_i - L, R - v_i).$$

Since $J - I$ is finite, $d_1(\Gamma) = \min(v_j - L, R - v_j)$ for some $j$ between $I$ and $J$. We then let

$$d_2(\Gamma) = \max_{I \leq i \leq J, i \neq j} \min(v_i - L, R - v_i).$$

We also have $d_2(\Gamma) = \min(v_k - L, R - v_k)$ for some $k \neq j$ between $I$ and $J$. Clearly, we can choose $j$ and $k$ so that $|j - k| = 1$.

To complete the proof, we argue by contradiction. Hence, we assume that for certain convex $f$, the scheme of the form (4.13)-(4.15) does not converge. By Lemma 4.3 and (4.30), there is a rarefying pair $\{L, R\}$ such that for each $\delta > 0$, $\delta' = \frac{1}{2}\delta$, and $\varepsilon = \frac{1}{2}\delta$, there is an $\varepsilon$-normal collection $\Gamma = \{v_j\}_{j=I-2}^{J+2}$ of the $\boldsymbol{\varepsilon}$-scared form (4.16) to the pair that satisfies

$$\int_L^R \{f[w; L, R] - g_\Gamma(w)\}dw \leq \delta' + \varepsilon = \delta.$$

It follows that there is a sequence of $\varepsilon$-normal collections $\{\Gamma_\nu\}_{\nu=1}^\infty$, where $\Gamma_\nu = \{v_j^\nu\}_{j=I^\nu-2}^{J^\nu+2}$ such that

$$\lim_{\nu\to\infty} \int_L^R \{f[w; L, R] - g_{\Gamma_\nu}(w)\} \leq 0. \tag{4.31}$$

The following three cases exhaust all possibilities.

*Case* 1. $\limsup_{\nu\to\infty} d_2(\Gamma_\nu) > 0$. Set $\rho = \frac{1}{2}\limsup_{\nu\to\infty} d_2(\Gamma_\nu)$. Then, there is a subsequence of the $\varepsilon$-normal collections, still denoted by $\{\Gamma_\nu\}_{\nu=1}^{\infty}$, and a corresponding sequence of integers $\{i(\nu)\}_{\nu=1}^{\infty}$ such that

$$L + \rho \le v_{i(\nu)}^{\nu} \le v_{i(\nu)+1}^{\nu} \le R - \rho,$$

and $\sup_\nu \sigma_{\Gamma_\nu} \le \sigma$. For simplicity, we fix a $\nu$ and drop it from the notation. Set $\gamma = f[\frac{L+R}{2}; L, R] - f(\frac{L+R}{2})$. It is a positive constant since $\{L, R\}$ is a rarefying pair. Applying Lemmas 4.4 and 4.5, we have

$$\begin{aligned} \int_L^R \{f[w; L, R] - g_{\Gamma_\nu}(w)\}dw &\ge S_{I,i,i+1,J} - (S_{I,i,i+1} + S_{i,i+1,J}) \qquad (4.32)\\ &= \frac{1}{2}\{(v_i - v_I)(f[v_{i+1}; L, R] - f(v_{i+1})) + (v_J - v_{i+1})(f[v_i; L, R] - f(v_i))\}\\ &> \eta, \end{aligned}$$

if $\eta = 2\rho^2\gamma/(R - L)$. This contradicts (4.31).

*Case* 2. $\limsup_{\nu\to\infty} d_1(\Gamma_\nu) > \limsup_{\nu\to\infty} d_2(\Gamma_\nu) = 0$. Set $\rho = \frac{1}{2}\limsup_{\nu\to\infty} d_1(\Gamma_\nu)$. Then, there is a subsequence of the $\varepsilon$-normal collections, still denoted by $\{\Gamma_\nu\}_{\nu=1}^{\infty}$, and a corresponding sequence of integers $\{i^\nu\}_{\nu=1}^{\infty}$ such that $\lim_{\nu\to\infty} v_{i^\nu-1}^{\nu} = L, \lim_{\nu\to\infty} v_{i^\nu+1}^{\nu} = R$, and $\lim_{\nu\to\infty} v_{i^\nu}^{\nu} = v \in [L + \rho, R - \rho]$. We then have

$$\int_L^R (f[w; L, R] - g_{\Gamma_\nu}(w))dw \to \int_L^R (f[w; L, R] - g_\Gamma(w))dw,$$

where $\Gamma$ is the following $\varepsilon$-normal collection: $I = 0, J = 4, v_{-2} = v_{-1} = v_0 = v_1 = L, v_2 = v$, and $v_3 = v_4 = v_5 = v_6 = R$. By Lemma 4.5, we have

$$\int_L^R (f[w; L, R] - g_\Gamma(w))dw \ge S_{1,2,3} - \alpha_2 S_{1,2,3} = \frac{1}{2} S_{1,2,3} > 0$$

for $\alpha_2 = \frac{1}{2}$ since $\Delta_+ v_0 = \Delta_+ v_3 = 0$. This contradicts (4.31).

*Case* 3. $\limsup_{\nu\to\infty} d_1(\Gamma_\nu) = 0$. Then, there exists a sequence of integers $\{i^\nu\}$ with $I^\nu + 1 \le i^\nu < J^\nu - 1$ such that $\lim_{\nu\to\infty} v_{i^\nu}^{\nu} = L, \lim_{\nu\to\infty} v_{i^\nu+1}^{\nu} = R$. We then have

$$\int_L^R (f[w; L, R] - g_{\Gamma_\nu}(w))dw \to \int_L^R (f[w; L, R] - g_\Gamma(w))dw,$$

where $\Gamma$ is the following $\varepsilon$-normal collection: $I = 0, J = 3, v_{-2} = v_{-1} = v_0 = v_1 = L, v_2 = v_3 = v_4 = v_5 = R$. In this case, the numerical flux $g_\Gamma(w)$ becomes $E$-flux $g^E(L, R)$. Hence, we have

$$\int_L^R (f[w; L, R] - g_\Gamma(w))dw \ge \int_L^R (f[w; L, R] - f(w))dw.$$

The right-hand side of the inequality is a positive constant since $\{L, R\}$ is a rarefying pair. This contradicts (4.31) again. We have thus completed the proof of Theorem 4.6. $\square$

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