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PROPAGATION OF CHAOS FOR THE KELLER-SEGEL EQUATION WITH A LOGARITHMIC CUT-OFF*

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Abstract. We consider a N-particle interacting system with the Newtonian potential aggregation and Brownian motions. Assuming that the initial data are independent and identically distributed (i.i.d.) with a common probability density function $\rho_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, (1+|x|)dx)$. We rigorously prove the propagation of chaos for this interacting system with a cut-off parameter $\varepsilon \sim (\ln N)^{-\frac{1}{d}}$: when $N \to \infty$, the empirical measure of the particle system converges in law to a probability measure and this measure possesses a density which is a weak solution to the mean-field Keller-Segel (KS) equation. More precisely, as $N \to \infty$, each particle path is approximated by a strong solution to a mean-field self-consistent stochastic differential equation (SDE). The global existence and uniqueness of strong solution to this SDE is proved and consequently we also prove the uniqueness of weak solution to the KS equation.

For $d = 2$, if $8\pi\nu > 1$, the propagation of chaos is valid globally in time. On the other hand, if $8\pi\nu < 1$, we show that the expectation of the collision time for the interacting particles system is bounded by $\frac{2\pi \mathrm{Var}\{\rho_0\}}{1-8\pi\nu}$. For $d \geq 3$, if $\|\rho_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}$ is bounded by a universal constant depending only on ν and d, then the propagation of chaos is also valid globally in time.

Key words. Newtonian potential aggregation, self-gravitating Brownian particles system, mean-field limit, L^∞ bound, log-Lipschitz continuity, uniqueness of weak solution, stability in Wasserstein metric.

Mathematics Subject Classification. 65M75, 35Q92, 35K55, 60H30.

1. Introduction.

The concept of the propagation of chaos was originated by Kac [26]. It is important for the kinetic theory that serves to relate the kinetic equations, such as the Fokker-Planck, Boltzmann and Vlasov equations, to the dynamics of N-particle systems of many indistinguishable individuals $\{(X_t^{i,N})_{t\geq 0}\}_{i=1}^N$ interacting with each other and following the same physical law. The propagation of chaos, also known as the molecular chaos, means that under the condition that the initial data $\{X_0^i\}_{i=1}^N$ are i.i.d. random variables with a common distribution function $f_0(x)$, this i.i.d. property is asymptotically preserved in time as $N \to \infty$. Following the framework of Sznitman [43], to prove the propagation of chaos, one needs to prove that the 2-particle marginal distribution $f_t^{(2),N}(x_1, x_2)$ narrowly converges to $f_t(x_1) \otimes f_t(x_2)$ for any $t \geq 0$, or equivalently the empirical measure $\frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$ converges in law to f_t, where f satisfies a mean-field partial differential equation (PDE) with the initial data $f_0(x)$ such as the Fokker-Planck equation.

Let $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\big)$ be a probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ ($\mathcal{F}$ is complete, $(\mathcal{F}_t)_{t\geq 0}$ is right continuous). We suppose that the space is endowed with N independent d-dimensional $\mathcal{F}_t$-Brownian motions $\{(B_t^i)_{t\geq 0}\}_{i=1}^N$. In this paper, we consider the interacting particle system of the following form:

$$X_t^i = X_0^i + \frac{1}{N-1}\sum_{j\neq i}^N \int_0^t F(X_s^i - X_s^j)ds + \sqrt{2\nu}B_t^i, \quad i = 1, \cdots, N \tag{1.1}$$

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where $(X_t^1, \cdots, X_t^N)_{t\geq 0}$ are the trajectories of N particles, $X_t^i \in \mathbb{R}^d$ for any $t \geq 0$, the initial data $\{X_0^i\}_{i=1}^N$ are the i.i.d. random variables with a common probability density function $\rho_0(x)$ and $\sqrt{2\nu}$ is a constant. The propagation of chaos for (1.1) with the smooth F has been rigorously proved by McKean in 70's and the mean-field equation is a class of nonlinear parabolic equations [32]. Furthermore, he also conjectured that for $d = 1$, $F(x) = \delta(x)$ (the Dirac distribution), the mean-field equation is the Burgers equation. This conjecture was proved in [8, 24, 42].

For $d = 2$, if the interacting force is taken by $F(x) = -\nabla^{\perp}\Phi(x)$ in (1.1), where the operator $\nabla^{\perp} = (\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})$ and $\Phi(x) = -\frac{1}{2\pi}\ln|x|$, then the mean-field limit equation becomes the incompressible Navier-Stokes equation. When $\sqrt{2\nu} = 0$, it is the incompressible Euler equation. In [31], Marchioro and Pulvirenti proved the mean-field limit for both the incompressible Navier-Stokes equation and Euler equation with some cut-off parameters $\varepsilon(N) \to 0$ as $N \to \infty$. Some of their techniques are adapted to solve our problem in this paper. The three dimensional Navier-Stokes equation and pathwise convergence rate with the stochastic vortex method have been studied in [18]. Osada also studied the propagation of chaos for the Navier-Stokes equation with the random vortex method in [35] without cut-off parameters. We refer to Chorin [13], Goodman [23] and Long [29] for the numerical aspect and convergence analysis for the random vortex method. More instances of the propagation of chaos have been studied in [5, 6, 19, 36, 37, 38]. Finally we refer readers to the long and informative article [43], in which Sznitman gives a comprehensive summary. We also refer to the recent important contribution [33].

Instead of taking curl of the Newtonian potential in the Navier-Stokes equation above, in this article we take the gradient of the Newton potential as the interacting attractive force $F(x) = -\nabla\Phi(x)$, $\forall\, x \in \mathbb{R}^d\backslash\{0\}$, $d \geq 2$ and $\sqrt{2\nu} > 0$ in (1.1), where the Newtonian potential is represented as

$$\Phi(x) = \begin{cases} \dfrac{1}{2\pi}\ln|x| & \text{if } d = 2, \\ -\dfrac{C_d}{|x|^{d-2}} & \text{if } d \geq 3, \end{cases} \tag{1.2}$$

where $C_d = \dfrac{1}{d(d-2)\alpha_d}$, $\alpha_d = \dfrac{\pi^{d/2}}{\Gamma(d/2+1)}$, i.e. α_d is the volume of d-dimensional unit ball. We rigorously derive the following mean-field KS equation:

$$\begin{cases} \partial_t\rho = \nu\triangle\rho - \nabla\cdot[\rho\nabla c], & x \in \mathbb{R}^d, \quad t > 0, \\ -\triangle c = \rho(t, x), \\ \rho(0, x) = \rho_0(x), \end{cases} \tag{1.3}$$

under the following assumption:

Assumption 1. *The initial density $\rho_0(x)$ satisfies*

1. $\rho_0(x) \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, (1+|x|)dx)$, $\int_{\mathbb{R}^d}\rho_0(x)dx = 1$;

2.

$$\|\rho_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} < \begin{cases} 8\pi\nu & \text{if } d = 2, \\ \dfrac{8\nu S_d}{d} & \text{if } d \geq 3, \end{cases} \tag{1.4}$$

where $S_d = \frac{d(d-2)}{4}2^{2/d}\pi^{1+1/d}\Gamma\left(\frac{d+1}{2}\right)^{-2/d}$, which is the best constant in the Sobolev inequality [28, pp.202].

In fact, the above assumption is sufficient for the existence of global weak solution to (1.3), see [1, 3, 4]. In the context of biological aggregation, the KS equation describes chemotaxis. $\rho(t,x)$ represents the bacteria density and $c(t,x)$ represents the chemical substance concentration.

Recently, the uniqueness of weak solution to the KS model (1.3) has been concerned by many scholars. Sugiyama [40] gave the uniqueness for 1-D Keller-Segel model by using the classical PDE theory. The optimal transport method [9] and the renormalizing argument [16] have been used to prove the uniqueness of weak solution to the KS model. In this paper, we introduce the following mean-field self-consistent stochastic process $(X_t)_{t\geq 0}$ underlying the KS equation:

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}^d} F(X_s - y)\rho(s,y)dyds + \sqrt{2\nu}B_t, \tag{1.5}$$

where we require $(X_t)_{t\geq 0}$ possessing a marginal density $(\rho_t)_{t\geq 0}$ for any $t \geq 0$ and the drift term is self-determined by $\int_{\mathbb{R}^d} F(X_s - y)\rho(s,y)dy$. By the Itô formula, we know that ρ is a weak solution to the KS equation.

We introduce the following notion of strong solution to the self-consistent SDE (1.5). We require $\rho \in L^\infty\big(0,T; L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, (1+|x|)dx)\big)$ for any $T > 0$ to insure that the self-consistent drift term

$$V(s,x) := \int_{\mathbb{R}^d} F(x-y)\rho(s,y)dy$$

is log-Lipschitz continuous, see Lemma 2.2 (*ii*). Then (1.5) becomes the following standard SDE:

$$X_t = X_0 + \int_0^t V(s,X_s)ds + \sqrt{2\nu}B_t, \tag{1.6}$$

and it is well known (utilizing Lemma 2.4) that the log-Lipschitz continuity of the drift term is enough for the existence and uniqueness of strong solution to this SDE. This kind of log-Lipschitz singularity also appeared in the 2D incompressible Euler equation and the uniqueness of weak solution was proved by Yudovich in [45]. Next, we give a precise definition of the strong solution to the self-consistent SDE (1.5).

DEFINITION 1. For any fixed $T > 0$, initial data X_0 and given probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ endowed with a d-dimensional $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion $(B_t)_{t\in[0,T]}$, if there is a stochastic process $(X_t)_{t\in[0,T]}$ adapted to $(\mathcal{F}_t)_{t\in[0,T]}$ and it has a time marginal density $\rho \in L^\infty\big(0,T; L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, (1+|x|)dx)\big)$ such that $\big(X_t, \rho_t\big)_{t\in[0,T]}$ satisfies (1.5) almost surely (a.s.) in the probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\big)$ for all $t \in [0,T]$, we say that $\big(X_t, \rho_t\big)_{t\geq 0}$ is a global strong solution to (1.5).

We will prove in Subsection 2.3 the following theorem about the uniqueness of weak solution (see the Definition 2) to (1.3) by utilizing the strong solution of (1.5) as a characteristic line.

THEOREM 1.1. *Assume the initial density $\rho_0(x)$ satisfies Assumption 1. Then for any fixed $T > 0$, we have*

(i) for any initial random variable X_0 with the density $\rho_0(x)$, there exists a unique global strong solution $\big(X_t, \rho_t\big)_{t\geq 0}$ to (1.5) and ρ is a weak solution to (1.3);

(ii) there exists a unique weak solution $\rho(t,x)$ in the class of

$$\rho \in L^\infty\big(0,T;L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d,(1+|x|)dx)\big)\cap L^2(0,T;H^1(\mathbb{R}^d))$$

to (1.3) *with the initial data ρ_0.*

Furthermore, with the help of (1.5), we also obtain the following theorem about the Dobrushin's type stability for (1.3) with respect to the initial data in the Wasserstein distance.

THEOREM 1.2. *For any fixed $T>0$, let $\rho^1,\rho^2 \in L^\infty\big(0,T;L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d,(1+|x|)dx)\big)\cap L^2(0,T;H^1(\mathbb{R}^d))$ be two weak solutions to* (1.3) *with the initial conditions $\rho_0^1(x),\rho_0^2(x)$ respectively and $\rho_0^1(x),\rho_0^2(x)$ satisfy Assumption 1. There exists two constants C (depending only on $\|\rho^1\|_{L^\infty(0,T;L^\infty\cap L^1(\mathbb{R}^d))}$ and $\|\rho^2\|_{L^\infty(0,T;L^\infty\cap L^1(\mathbb{R}^d))}$) and C_T (depending only on T) such that*

$$\sup_{t\in[0,T]}\mathcal{W}_1(\rho_t^1,\rho_t^2)\le C_T\max\Big\{\mathcal{W}_1(\rho_0^1,\ \rho_0^2),\{\mathcal{W}_1(\rho_0^1,\ \rho_0^2)\}^{\exp(-CT)}\Big\}.$$

where $\mathcal{W}_1$ is the Wasserstein distance, see Subsection 2.1.

Our last result deals with the propagation of chaos. Considering the interacting system (1.1) with $F(x)=-\nabla\Phi(x)=-\frac{C^*x}{|x|^d}$, $\forall\, x\in\mathbb{R}^d\backslash\{0\}$, $d\ge 2$, where Φ is given by (1.2), $C^*=\frac{\Gamma(d/2)}{2\pi^{d/2}}$. Then the first term on the right hand in (1.1) represents attractive force on X_t^i by all other particles. This system is also known as the self-gravitating Brownian particles [10, 11].

For $d=2$, it is well known that if $8\pi\nu<1$, the solution to the KS equation with the initial density ρ_0 concentrates before the time $\frac{2\pi\mathrm{Var}\{\rho_0\}}{1-8\pi\nu}$. In Subsection 3.1, when $8\pi\nu<1$, we show that the expectation of the collision time for (1.1) is also bounded by $\frac{2\pi\mathrm{Var}\{\rho_0\}}{1-8\pi\nu}$. Although we only prove the collision happens when $8\pi\nu<1$, the collision for (1.1) is generic. Recently, there is a deep result proved by Fournier and Jourdain [20, Proposition 4]: for any $N\ge 2$ and $T>0$, if $\{(X_t^{i,N})_{t\in[0,T]}\}_{i=1}^N$ is the solution to (1.1), then

$$\mathbb{P}(\exists s\in[0,T],\exists 1\le i<j\le N: X_s^{i,N}=X_s^{j,N})>0,$$

i.e. the singularity is visited and the particle system is not clearly well-defined. Therefore in order to obtain a global strong solution to the interacting particle system, we regularize the force term by a blob function $J(x)\in C^2(\mathbb{R}^d)$, $\operatorname{supp} J(x)\subset B(0,1)$, $J(x)\ge 0$ and $\int_{B(0,1)}J(x)dx=1$. Let $J_\varepsilon(x)=\frac{1}{\varepsilon^d}J(\frac{x}{\varepsilon})$, $\Phi_\varepsilon(x)=J_\varepsilon*\Phi(x)$ for $x\in\mathbb{R}^d$ and $F_\varepsilon(x)=-\nabla\Phi_\varepsilon(x)$. The regularized system is given by

$$X_t^{i,\varepsilon}=X_0^i+\frac{1}{N-1}\int_0^t\sum_{j\ne i}^N F_\varepsilon(X_s^{i,\varepsilon}-X_s^{j,\varepsilon})ds+\sqrt{2\nu}B_t^i,\quad i=1,\cdots,N. \tag{1.7}$$

This system has a unique global strong solution $\{(X_t^{i,\varepsilon})_{t\ge 0}\}_{i=1}^N$ by the standard theorem of SDE [34, Theorem 5.2.1].

By the coupling method, as $N\to\infty$, we show that the N interacting particles $\{(X_t^{i,\varepsilon(N)})_{t\ge 0}\}_{i=1}^N$ respectively can be approximated by the processes $\{(X_t^i)_{t\ge 0}\}_{i=1}^N$, which are the strong solutions to

$$X_t^i=X_0^i+\int_0^t\int_{\mathbb{R}^d}F(X_s^i-y)\rho_s^i(y)dyds+\sqrt{2\nu}B_t^i,\quad i=1,\cdots,N, \tag{1.8}$$

where $(\rho_t^i)_{t\geq 0}$ is the time marginal density of $(X_t^i)_{t\geq 0}$; the initial data $\{X_0^i\}_{i=1}^N$ and Brownian motions $\{(B_t^i)_{t\geq 0}\}_{i=1}^N$ are same as these in the system (1.7). Recall that the self-consistent SDE (1.5) has a unique strong solution. Since the initial data and Brownian motions both are i.i.d., then the processes $\{(X_t^i)_{t\geq 0}\}_{i=1}^N$ are N copies of the strong solutions to (1.5) and $\rho^i(t,x) \equiv \rho(t,x)$ for $i = 1, \cdots, N$. By Theorem 1.1, ρ is the unique weak solution to the KS equation with the initial data ρ_0. The propagation of chaos result can be summarized as:

THEOREM 1.3. *Suppose ρ_0 satisfies Assumption 1 and $\{X_0^i\}_{i=1}^N$ are i.i.d. random variables with the common density ρ_0. Let $\{(X_t^{i,\varepsilon})_{t\geq 0}\}_{i=1}^N$ and $\{(X_t^i)_{t\geq 0}\}_{i=1}^N$ be the unique strong solutions to* (1.7) *and* (1.8) *respectively with the same i.i.d. initial data $\{X_0^i\}_{i=1}^N$ and Brownian motions $\{(B_t^i)_{t\geq 0}\}_{i=1}^N$. Then $\{(X_t^{i,\varepsilon})_{t\geq 0}\}_{i=1}^N$ are exchangeable, $\{(X_t^i)_{t\geq 0}\}_{i=1}^N$ are i.i.d. and there is a list of cut-off parameters $\varepsilon(N) \sim (\ln N)^{-\frac{1}{d}} \to 0$ as $N \to \infty$ such that for any $T > 0$ and all $1 \leq i \leq N$,*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|X_t^{i,\varepsilon(N)} - X_t^i\big|\Big] \to 0 \text{ as } N \to \infty.$$

Furthermore Corollary 3.1 and Corollary 3.2 show that for any $j \geq 1$, the j-particle marginal distribution $f_t^{(j),\varepsilon(N)}(x_1, \cdots, x_j)$ narrowly converges to $f(t,x)^{\otimes j}$ and $f(t,x)$ possesses a density $\rho(t,x)$ for any $t \geq 0$, where ρ is the unique weak solution to the mean-field KS equation with the initial data ρ_0.

We give a short review on results on the propagation of chaos for the KS equation. Stevens [39] derived the parabolic-parabolic KS equation from a large interacting particle system with birth and death processes. In [25], Haskovec and Schmeiser studied many-particle limit in the BBGKY hierarchy by using measure solutions of the KS system under the molecular chaos assumption. They also obtained some tightness and weak convergence results. However they pointed out that they could not prove the propagation of chaos due to lacking of uniqueness result for the limiting hierarchy. In [22], Godinho and Quininao considered the less singular force kernel, i.e. $F = \nabla\big(\frac{1}{\alpha-1}|x|^{1-\alpha}\big) = -\frac{x}{|x|^{\alpha+1}}$, $0 < \alpha < 1$, and they proved the propagation of chaos for the sub-critical KS equation.

This paper is organized as follows. The well-posedness for the KS equation and the self-consistent SDE are established in Section 2. In Section 3.1, if $8\pi\nu < 1$, we first show that the expectation of the collision time for the interacting particle system (1.1) is bounded by a uniform constant, and then we prove the propagation of chaos results. Finally, in the Appendix we provide a supplementary proof of Theorem 2.2.

2. Well-posedness for the self-consistent SDE. This section is devoted to prove the existence and uniqueness of strong solution to the self-consistent SDE (1.5). Notice that if the density $\rho(t,x) \in L^\infty\big(0,T; L^\infty \cap L^1(\mathbb{R}^d)\big)$, then $\int_{\mathbb{R}^d} F(X_s - y)\rho(s,y)dy$, as the drift term of (1.5), is Lipschitz continuous up to a logarithmic singularity (see Lemma 2.2), is also known as log-Lipschitz continuous [12]. Then we will adapt some techniques used in analysis of the incompressible Euler equation. In Subsection 2.1, we use the Osgood lemma [12, Lemma 5.2.1] to prove a Gronwall type inequality with a logarithmic singularity in Lemma 2.4. In Subsection 2.2, we regularize the self-consistent SDE (1.5) and give a uniform estimate for the density of the strong solution to this regularized problem. In Subsection 2.3, we prove the uniqueness of weak solution to the KS equation.

2.1. Preliminaries. We begin by introducing the definition of weak solution to the KS equation which we will deal with through this paper. Indeed, we ask for more regularities than needed for the definition and these regularities will be proved in Theorem 2.2.

Definition 2 (weak solution). Let the initial data $\rho_0(x) \in L^1_+ \cap L^{\frac{d}{2}}(\mathbb{R}^d)$ and $T > 0$. c is the chemical substance concentration associated with ρ and is given by $c(t,x) = -\Phi * \rho(t,x)$. We shall say that $\rho(t,x)$ is a weak solution to (1.3) with the initial data $\rho_0(x)$ if it satisfies:

1. Regularity:

$$\rho \in L^\infty\big(0,T; L^1_+ \cap L^{\frac{d}{2}}(\mathbb{R}^d)\big), \quad \rho^{\frac{d}{4}} \in L^2(0,T; H^1(\mathbb{R}^d)),$$
$$\text{and } \partial_t \rho \in L^q\big(0,T; W^{-1,p}_{loc}(\mathbb{R}^d)\big) \text{ for some } q,p \geq 1.$$

2. For all $\varphi \in C_0^\infty(\mathbb{R}^d)$ and $0 < t \leq T$, the following holds,

$$\int_{\mathbb{R}^d} \rho(t,x)\varphi(x)dx - \int_{\mathbb{R}^d} \rho_0(x)\varphi(x)dx - \nu \int_0^t \int_{\mathbb{R}^d} \rho(s,x)\Delta\varphi(x)dxds$$
$$= \int_0^t \int_{\mathbb{R}^d} \rho(s,x)\Big(\int_{\mathbb{R}^d} F(x-y)\rho(s,y)dy\Big) \cdot \nabla\varphi(x)dxds. \tag{2.1}$$

Remark 2.1. For $d \geq 3$, since $\rho^{\frac{d}{4}} \in L^2(0,T; H^1(\mathbb{R}^d))$, by the embedding theorems, $\rho \in L^{\frac{d}{2}}\big(0,T; L^{\frac{d^2}{2(d-2)}}(\mathbb{R}^d)\big)$. By the Hardy-Littlewood-Sobolev inequality, one also has

$$\begin{aligned} &\Big|\int_{\mathbb{R}^d} \rho(s,x)\Big(\int_{\mathbb{R}^d} F(x-y)\rho(s,y)dy\Big) \cdot \nabla\varphi(x)dx\Big| \\ &= \frac{C^*}{2}\Big| - \int_{\mathbb{R}^{2d}} \frac{(\nabla\varphi(x) - \nabla\varphi(y)) \cdot (x-y)}{|x-y|^2} \frac{\rho(s,x)\rho(s,y)}{|x-y|^{d-2}} dxdy\Big| \\ &\leq \frac{C^*}{2} \int_{\mathbb{R}^{2d}} \frac{\rho(s,x)\rho(s,y)}{|x-y|^{d-2}} dxdy \leq C(d)\|\rho\|^2_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}. \end{aligned} \tag{2.2}$$

Notice that $\frac{d^2}{2(d-2)} > \frac{2d}{d+2}$, thus the regularity of ρ is enough to make sense of each term in (2.1).

Now we give a lemma to collect some useful properties of the regularized force.

Lemma 2.1. *Suppose* $J(x) \in C^2(\mathbb{R}^d)$, *supp* $J(x) \subset B(0,1)$, $J(x) = J(|x|)$, $\int_{\mathbb{R}^d} J(x)dx = 1$ *and* $J(x) \geq 0$. *Let* $J_\varepsilon(x) = \frac{1}{\varepsilon^d} J(\frac{x}{\varepsilon})$ *and* $\Phi_\varepsilon(x) = J_\varepsilon * \Phi(x)$ *for* $x \in \mathbb{R}^d$, $F_\varepsilon(x) = -\nabla\Phi_\varepsilon(x)$. *Then* $F_\varepsilon(x) \in C^1(\mathbb{R}^d)$, $\nabla \cdot F_\varepsilon(x) = -J_\varepsilon(x)$ *and*

(i) $F_\varepsilon(0) = 0$ *and* $F_\varepsilon(x) = F(x)g(\frac{|x|}{\varepsilon})$ *for any* $x \neq 0$, *where* $g(r) = \frac{1}{C^*}\int_0^r J(s)s^{d-1}ds$, $C^* = \dfrac{\Gamma(d/2)}{2\pi^{d/2}}$, $d \geq 2$ *and* $g(r) = 1$ *for* $r \geq 1$;

(ii) $|F_\varepsilon(x)| \leq \min\{\frac{C|x|}{\varepsilon^d}, |F(x)|\}$ *and* $|\nabla F_\varepsilon(x)| \leq \frac{C}{\varepsilon^d}$.

Proof. Denote $r = |x|$ for any $x \in \mathbb{R}^d \backslash \{0\}$. Recall that

$$\triangle\Phi_\varepsilon(x) = J_\varepsilon(x) \quad \text{and} \quad \triangle_r = \frac{\partial_r(r^{d-1}\partial_r)}{r^{d-1}}.$$

Then

$$\partial_r(r^{d-1}\partial_r\Phi_\varepsilon(r)) = r^{d-1}J_\varepsilon(r). \tag{2.3}$$

Integrating (2.3), one has

$$r^{d-1}\partial_r\Phi_\varepsilon(r) = \int_0^r J_\varepsilon(s)s^{d-1}ds = \int_0^{\frac{r}{\varepsilon}} J(s)s^{d-1}ds.$$

Denote $g(\frac{r}{\varepsilon}) = \frac{1}{C^*}\int_0^{\frac{r}{\varepsilon}} J(s)s^{d-1}ds$, then

$$g(r) = 1 \text{ when } r \geq 1; \quad F_\varepsilon(x) = -\frac{x}{r}\partial_r\Phi_\varepsilon(r) = -\frac{C^*x}{r^d}g(\frac{r}{\varepsilon}) = F(x)g(\frac{r}{\varepsilon}),$$

i.e. (i) holds. By the definitions of J and g, one can easily find a positive constant C_1 such that

$$0 \leq g(r) \leq C_1\min\{1, r^d\}.$$

Then simple computation shows that there exists a constant $C > 0$ such that

$$|F_\varepsilon(x)| \leq \min\big\{C|x|^{1-d}g(\frac{|x|}{\varepsilon}),\, |F(x)|\big\} \leq \min\big\{C\frac{|x|}{\varepsilon^d},\, |F(x)|\big\}, \tag{2.4}$$

$$|\nabla F_\varepsilon(x)| \leq C\Big(|x|^{-d}(\frac{|x|}{\varepsilon})^d + |x|^{1-d}(\frac{|x|}{\varepsilon})^{d-1}\frac{1}{\varepsilon}\Big) \leq \frac{C}{\varepsilon^d}, \tag{2.5}$$

which finishes the proof of (ii). □

In this article we take a cut-off function $J(x) \geq 0$, $J(x) \in C_0^3(\mathbb{R}^d)$,

$$J(x) = \begin{cases} C(1+\cos\pi|x|)^2 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 1, \end{cases}$$

where C is a constant such that $C|\mathbb{S}^{d-1}|\int_0^1(1+\cos\pi r)^2 r^{d-1}dr = 1$, $|\mathbb{S}^{d-1}| = \frac{2\pi^{d/2}}{\Gamma(d/2)}$.

LEMMA 2.2. *For any function $\rho(x) \in L^\infty \cap L^1(\mathbb{R}^d)$, there exists a universal constant C (depending only on $\|\rho\|_{L^\infty\cap L^1}$) such that for all $\varepsilon \geq \varepsilon' \geq 0$, one has*
(i) $\int_{\mathbb{R}^d}|\rho(y)F_\varepsilon(x-y)|dy \leq C$.
(ii) $\int_{\mathbb{R}^d}|\rho(y)||F_\varepsilon(x-y) - F_\varepsilon(x'-y)|dy \leq C\,\omega\big(|x-x'|\big)$, *where*

$$\omega(r) = \begin{cases} 1 & \text{if } r \geq 1, \\ r(1-\ln r) & \text{if } 0 < r < 1. \end{cases} \tag{2.6}$$

(iii) $\int_{\mathbb{R}^d}|\rho(y)||F_\varepsilon(x-y) - F_{\varepsilon'}(x-y)|dy \leq C\,\varepsilon$.

Proof. For $d = 2$, (i) and (ii) have been proven by Kato [27, see Lemma 1.4.] or Marchioro and Pulvirenti [31]. Their proofs are also valid in the high dimensions. We omit the details here and prove (iii) below.

Since $F_\varepsilon(x) = F(x)$ for any $|x| \geq \varepsilon$ and $|F_\varepsilon(x)| \leq |F(x)|$ by Lemma 2.1, we have

$$\int_{\mathbb{R}^d} \big|F_\varepsilon(x-y) - F_{\varepsilon'}(x-y)\big|\rho(y)dy \leq C\|\rho\|_{L^\infty} \int_{\{y:|x-y|\leq\varepsilon\}} \frac{dy}{|x-y|^{d-1}}$$
$$\leq C\,\|\rho\|_{L^\infty}\,\varepsilon.$$

□

LEMMA 2.3. *Let X_i $(i = 1, 2)$ be two random variables with densities $\rho_i(x) \in L^\infty \cap L^1(\mathbb{R}^d)$ (X_1 and X_2 are not necessarily independent). For any $\varepsilon \geq \varepsilon' \geq 0$, define*

$$I := \int_{\mathbb{R}^d} F_\varepsilon(X_1 - y)\rho_1(y)dy - \int_{\mathbb{R}^d} F_{\varepsilon'}(X_2 - y)\rho_2(y)dy.$$

Then there exists a constant C (depending only on $\|\rho_1\|_{L^\infty\cap L^1}$ and $\|\rho_2\|_{L^\infty\cap L^1}$) such that

$$\mathbb{E}\big[|I|\big] \leq C\big(\varepsilon + \omega(\mathbb{E}[|X_1 - X_2|])\big), \tag{2.7}$$

where ω is given by (2.6).

Proof. A direct computation shows that

$$|I| \leq \int_{\mathbb{R}^d} |F_\varepsilon(X_1-y) - F_\varepsilon(X_2-y)|\rho_1(y)dy + \int_{\mathbb{R}^d} |F_\varepsilon(X_2-y) - F_{\varepsilon'}(X_2-y)|\rho_1(y)dy$$
$$+ \int_{\mathbb{R}^d} |F_{\varepsilon'}(X_2-y)\rho_1(y) - F_{\varepsilon'}(X_2-y)\rho_2(y)|dy =: I_1 + I_2 + I_3. \tag{2.8}$$

By Lemma 2.2 (ii) and (iii), there exists a constant C (depending only on $\|\rho_1\|_{L^\infty\cap L^1}$) such that

$$I_1 \leq C\,\omega\big(|X_1 - X_2|\big), \tag{2.9}$$

$$I_2 \leq C\,\varepsilon. \tag{2.10}$$

Suppose $(Y_1; Y_2)$ is an independent copy of $(X_1; X_2)$. By Lemma 2.2 (ii), there exists a constant C (depending only on $\|\rho_2\|_{L^\infty\cap L^1}$) such that

$$\begin{aligned}
\mathbb{E}[I_3] &= \mathbb{E}[\int_{\mathbb{R}^d} |F_{\varepsilon'}(X_2-y)\rho_1(y) - F_{\varepsilon'}(X_2-y)\rho_2(y)|dy] \\
&= \mathbb{E}_x\mathbb{E}_y\big[|F_{\varepsilon'}(X_2 - Y_1) - F_{\varepsilon'}(X_2 - Y_2)|\big] \\
&= \mathbb{E}_y\big[\int_{\mathbb{R}^d} |F_{\varepsilon'}(x - Y_1) - F_{\varepsilon'}(x - Y_2)|\rho_2(x)dx\big] \\
&\leq C\,\mathbb{E}[\omega(|Y_1 - Y_2|)] = C\,\mathbb{E}[\omega(|X_1 - X_2|)].
\end{aligned} \tag{2.11}$$

Taking the expectation of (2.8) and combining (2.9), (2.10), (2.11) and the concavity of $\omega(r)$, we obtain that

$$\begin{aligned}
\mathbb{E}[|I|] &\leq C\big(\varepsilon + \mathbb{E}[\omega(|X_1 - X_2|)]\big) \\
&\leq C\big(\varepsilon + \omega\big(\mathbb{E}[|X_1 - X_2|]\big)\big),
\end{aligned} \tag{2.12}$$

where C is a constant depending only on $\|\rho_1\|_{L^\infty\cap L^1}$ and $\|\rho_2\|_{L^\infty\cap L^1}$. □

The following lemma is a Gronwall type inequality with a logarithmic singularity, which is an application of the Osgood lemma.

LEMMA 2.4. *Assume that a sequence of nonnegative continuous functions* $\{\alpha_\varepsilon(t)\}_{\varepsilon>0}$ *satisfy*

$$\alpha_\varepsilon(t) \le C\int_0^t \alpha_\varepsilon(s)[1-\ln \alpha_\varepsilon(s)]ds + C\varepsilon T \quad \textit{for all } t\in[0,T],$$

where C is a constant. Then there exists two constants C_T (depending only on T) and $\varepsilon_0(T)>0$ such that if $\varepsilon<\varepsilon_0(T)$, then

$$\sup_{t\in[0,T]} \alpha_\varepsilon(t) \le C_T\varepsilon^{\exp(-CT)} < 1. \tag{2.13}$$

Proof. We divide into the following two cases:

Case (i): If $\alpha_\varepsilon(t)<1$ for all $t\in[0,T]$, we claim (2.13) holds. Indeed the proof follows directly from the Osgood lemma, where we take $\gamma=C$, $\mu(x)=x(1-\ln x)$, $x<1$ and $a=C\varepsilon T$ in the cited reference. Then

$$-\mathrm{M}(\alpha_\varepsilon(t))+\mathrm{M}(a)\le Ct, \quad \textit{where} \quad \mathrm{M}(x)=\int_x^1\frac{dr}{\mu(r)}.$$

Simple computation shows that

$$\begin{aligned}\alpha_\varepsilon(t) &\le \exp\big(1-(1-\ln Ct\varepsilon)\exp(-Ct)\big)\\ &=\exp(1-\exp(-Ct))(CT\varepsilon)^{\exp(-Ct)} \le C_T\varepsilon^{\exp(-CT)}.\end{aligned} \tag{2.14}$$

Case (ii): There exists $T_1<T$ such that $\alpha_\varepsilon(t)<1$ when $t\in[0,T_1)$ and $\alpha_\varepsilon(T_1)=1$. Choosing $\varepsilon_0(T)$ such that $C_T{\varepsilon_0}^{\exp(-CT)}=1$, if $\varepsilon<\varepsilon_0(T)$, then we show that the case (ii) can not happen.

From (2.14), we obtain that for all $t\in[0,T_1)$,

$$\alpha_\varepsilon(t)\le C_T\varepsilon^{\exp(-CT)}. \tag{2.15}$$

Using the continuity of $\alpha_\varepsilon(t)$ in (2.15), one has if $\varepsilon<\varepsilon_0(T)$, then

$$1=\alpha_\varepsilon(T_1)\le C_T\varepsilon^{\exp(-CT)}<C_T{\varepsilon_0}^{\exp(-CT)}=1,$$

which is a contradiction. □

Now we introduce a topology of the 1-Wasserstein space which will be used for proving the well-posedness of weak solution to the KS equation. Consider the following space

$$\mathcal{P}_1(\mathbb{R}^d)=\Big\{f|\ f \text{ is a probability measure on } \mathbb{R}^d \text{ and } \int_{\mathbb{R}^d}|x|df(x)<+\infty\Big\}.$$

We denote the Kantorovich-Rubinstein distance in $\mathcal{P}_1(\mathbb{R}^d)$ as follows

$$\mathcal{W}_1(f,\ g)=\inf_{\pi\in\Lambda(f,\ g)}\Big\{\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|d\pi(x,y)\Big\},$$

where $\Lambda(f,\ g)$ is the set of joint probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals f and g. If $f,\ g$ have densities $\rho_1,\ \rho_2$ respectively, we also denote the distance as $\mathcal{W}_1(\rho_1,\ \rho_2)$. In [44, Theorem 6.18], it has been proven that $\mathcal{P}_1(\mathbb{R}^d)$ endowed with this distance is a complete metric space. And by [44, Theorem 6.9], the following proposition holds.

PROPOSITION 2.1. *For a sequence of* $\{f_k\}_{k=1}^{\infty}$ *and* f *in* $\mathcal{P}_1(\mathbb{R}^d)$*, the convergence of* $\{f_k\}_{k=1}^{\infty}$ *to* f *in the 1-Wasserstein distance implies the narrow convergence of* $\{f_k\}_{k=1}^{\infty}$*, i.e.*

$$\mathcal{W}_1(f_k, f) \xrightarrow{k\to\infty} 0 \Rightarrow \int \varphi df_k(x) \xrightarrow{k\to\infty} \int \varphi df(x) \quad \textit{for any } \varphi \in C_b(\mathbb{R}^d),$$

where $C_b(\mathbb{R}^d)$ *is the space of continuous and bounded functions.*

In this paper we use the following time dependent space $L^\infty\big(0, T; \mathcal{P}_1(\mathbb{R}^d)\big)$:

$$\big\{f(t,x)|\ f(t,\cdot) \text{ is a probability measure on } \mathbb{R}^d \\ \text{for any time } t \text{ and } \sup_{t\in[0,T]} \int_{\mathbb{R}^d} |x| df(t,x) < +\infty\big\}.$$

endowed with metric

$$\mathcal{M}_T(f_t^1, f_t^2) = \sup_{t\in[0,T]} \mathcal{W}_1(f_t^1, f_t^2).$$

And the following proposition is well known, c.f. [6].

PROPOSITION 2.2. $L^\infty\big(0, T; \mathcal{P}_1(\mathbb{R}^d)\big)$ *is a complete metric space.*

At last, we introduce the Kantorovitch-Rubinstein or Wasserstein metric on the set $\mathcal{P}_1(\mathcal{C})$ of probability measures on $\mathcal{C} := C([0,T], \mathbb{R}^d)$ with bounded first moment, defined by

$$\mathcal{D}_T(m_1, m_2) = \inf_{m\in\Lambda(m_1,m_2)} \Big\{ \int_{\mathcal{C}\times\mathcal{C}} \sup_{t\in[0,T]} |x_t - y_t| dm(x,y) \Big\}, \tag{2.16}$$

where $(x_t)_{0\le t\le T}$ and $(y_t)_{0\le t\le T}$ are two canonical processes on $\mathcal{C}$. The formula (2.16) defines a complete metric on the set $\mathcal{P}_1(\mathcal{C})$ and gives a topology of $\mathcal{P}_1(\mathcal{C})$, see [43].

2.2. Regularization for the self-consistent SDE and the uniform estimates in $L^\infty(\mathbb{R}^d)$. First, we state a result on the global existence and uniqueness of strong solution to the following regularized self-consistent SDE:

$$X_t^\varepsilon = X_0 + \int_0^t \int_{\mathbb{R}^d} F_\varepsilon(X_s^\varepsilon - y)\rho_\varepsilon(s,y) dy ds + \sqrt{2\nu} B_t, \tag{2.17}$$

where we require $(X_t^\varepsilon)_{t\ge 0}$ possessing a marginal density $(\rho_\varepsilon(t,x))_{t\ge0}$ and the initial density $\rho_0(x)$ satisfies $\int_{\mathbb{R}^d} \rho_0(x)dx = 1$ and $\rho_0(x) \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, (1+|x|)dx)$. The PDE associated to (2.17) is the following regularized KS equation:

$$\begin{cases} \partial_t \rho_\varepsilon = \nu \triangle \rho_\varepsilon - \nabla\cdot[\rho_\varepsilon \nabla c_\varepsilon], \quad x \in \mathbb{R}^d, \quad t > 0, \\ -\triangle c_\varepsilon = J_\varepsilon * \rho_\varepsilon(t,x), \\ \rho_\varepsilon(0,x) = \rho_0(x), \end{cases} \tag{2.18}$$

which has a unique global weak solution ρ_ε in the class of $L^\infty\big(0,T;L^2(\mathbb{R}^d)\big)\cap L^2\big(0,T;H^1(\mathbb{R}^d)\big)$, and $\int_{\mathbb{R}^d}\rho_\varepsilon(t,x)dx\equiv 1$.

Now we give the results about the existence and uniqueness (in the strong sense) of (2.17).

THEOREM 2.1. *Suppose $\rho_0(x)\in L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d,(1+|x|)dx)$ and X_0 is a random variable with the density $\rho_0(x)$. Then for any fixed $T>0$, $\varepsilon>0$,* (2.17) *has a unique strong solution $(X_t^\varepsilon,\rho_\varepsilon(t,y))_{t\geq 0}$ with the initial data X_0 and $\rho_\varepsilon(t,y)$ is the unique weak solution to* (2.18).

Proof. We first solve for ρ_ε from (2.18) directly. Then plugging ρ_ε into (2.17) and it becomes a linear SDE. Hence we can solve for a solution $(X_t^\varepsilon)_{t\geq 0}$ from this linear SDE. By the Itô formula, we know that the marginal density of $(X_t^\varepsilon)_{t\geq 0}$ is exactly ρ_ε. This strategy has been used in [6] and now we give the detailed proof.

Suppose $\rho_\varepsilon(t,x)$ is the unique weak solution to (2.18). Let $\tilde{V}_\varepsilon(t,x)=\int_{\mathbb{R}^d}F_\varepsilon(x-y)\rho_\varepsilon(t,y)dy$. Since $\tilde{V}_\varepsilon(t,x)$ is bounded and Lipschitz continuous, the following linear SDE

$$X_t^\varepsilon = X_0+\int_0^t \tilde{V}_\varepsilon\big(s,X_s^\varepsilon\big)ds+\sqrt{2\nu}B_t$$

has a unique strong solution $(X_t^\varepsilon)_{t\geq 0}$ and it admits a time marginal density denoted by $\tilde{\rho}_\varepsilon(t,x)$ (see [41, Theorem 9.1.9]). For any $\varphi(x)\in C_b^2(\mathbb{R}^d)$, The Itô formula states that

$$\varphi(X_t^\varepsilon)=\varphi(X_0)+\int_0^t\nabla\varphi(X_s^\varepsilon)\tilde{V}_\varepsilon\big(s,X_s^\varepsilon\big)ds \\ +\sqrt{2\nu}\int_0^t\nabla\varphi(X_s^\varepsilon)dB_s+\nu\int_0^t\triangle\varphi(X_s^\varepsilon)ds. \tag{2.19}$$

Taking expectation of (2.19), $\tilde{\rho}_\varepsilon$ is a weak solution to the following linear Fokker-Planck equation:

$$\begin{cases} \partial_t\tilde{\rho}_\varepsilon(t,x)=\nu\triangle\tilde{\rho}_\varepsilon(t,x)-\nabla\cdot[\tilde{V}_\varepsilon(t,x)\tilde{\rho}_\varepsilon(t,x)], & x\in\mathbb{R}^d, \quad t>0,\\ \tilde{\rho}_\varepsilon(0,x)=\rho_0(x). \end{cases} \tag{2.20}$$

Since ρ_ε is also a weak solution to (2.20) and the weak solution in the class of $L^\infty\big(0,T;L^2(\mathbb{R}^d)\big)\cap L^2\big(0,T;H^1(\mathbb{R}^d)\big)$ to (2.20) is unique, then

$$\tilde{\rho}_\varepsilon=\rho_\varepsilon,$$

which means that $(X_t^\varepsilon,\rho_\varepsilon(t,y))_{t\geq 0}$ is a strong solution to (2.17).

The uniqueness of strong solution to (2.17) comes from the uniqueness of weak solution to (2.18). In fact, suppose $(X_t^{\varepsilon,1})_{t\geq 0}$ and $(X_t^{\varepsilon,2})_{t\geq 0}$ are two strong solutions to (2.17) with the same initial data. Let ρ_ε^1 and ρ_ε^2 be the densities of those two solutions respectively. By the Itô formula, one knows that ρ_ε^1 and ρ_ε^2 both are weak solutions to (2.18) with the same initial data $\rho_0(x)$. Since the weak solutions to (2.18) is unique, one has

$$\rho_\varepsilon^1=\rho_\varepsilon^2. \tag{2.21}$$

Hence by standard argument, since $F_\varepsilon*\rho_\varepsilon^1=F_\varepsilon*\rho_\varepsilon^2$ is smooth, uniqueness holds (in the strong sense) in (2.17). ◻

Next we present a uniform estimate on $\|\rho_\varepsilon\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}$ and other related estimates for ρ_ε, which is crucial for deriving the mean-field equation (see [31]).

THEOREM 2.2. *Suppose $\rho_\varepsilon(t,x)$ is the unique weak solution to* (2.18) *with the initial condition $\rho_0(x)$ satisfying Assumption 1. Then for any fixed $T>0$, there exists a constant C (depending only on T, $\|\rho_0\|_{L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d,(1+|x|)dx)}$ and data in* (1.4)*) such that*
i) $\|\rho_\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{R}^d))}=1,\ \|\rho_\varepsilon\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\le C$ *and* $\int_{\mathbb{R}^d}|x|\rho_\varepsilon(t,x)dx\le C$.
ii) $\int_0^T\|\nabla\rho_\varepsilon\|^2_{L^2(\mathbb{R}^d)}dt\le C$ *and* $\int_0^T\|\partial_t\rho_\varepsilon\|^2_{H^{-1}(\mathbb{R}^d)}dt\le C$.

The proof will be given in the Appendix.

2.3. Global existence and uniqueness of strong solution to the self-consistent SDE and weak solution to the KS equation. In this subsection, with *a-priori* estimates of weak solution to the regularized KS equation in Theorem 2.2, we show that the KS equation (1.3) has a unique weak solution and the corresponding self-consistent SDE (1.5) also has a unique strong solution.

Proof of Theorem 1.1. Let $\rho_\varepsilon(t,x)$ be the unique weak solution to the regularized KS equation (2.18) with the initial condition $\rho_0(x)$. Using the uniform estimates and by the standard argument (see a proof in Appendix), we have

Claim 1: there exists a subsequence ρ_ε (without relabeling) such that for any ball B_R,

$$\rho_\varepsilon\to\rho \text{ in } L^2\big(0,T;L^2(B_R)\big) \text{ as } \varepsilon\to 0, \tag{2.22}$$

and $\rho(t,x)$ is a weak solution to (1.3) with the following regularities:
i) $\rho\in L^\infty\big(0,T;L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d,(1+|x|)dx)\big)$,
ii) $\rho\in L^2\big(0,T;H^1(\mathbb{R}^d)\big)$ and $\partial_t\rho\in L^2\big(0,T;H^{-1}(\mathbb{R}^d)\big)$.

Combining $\int_{\mathbb{R}^d}\rho_\varepsilon dx=1$ and $\int_{\mathbb{R}^d}|x|\rho_\varepsilon dx\le C$, we also obtain that

$$\int_{\mathbb{R}^d}\rho(t,x)dx=1. \tag{2.23}$$

We split the proof into the following three steps:

Step 1 We split into three sub-steps to prove the existence of strong solution to (1.5). Let $(X_t^\varepsilon)_{t\ge0}$ be the strong solution to (2.17). Step 1.1 proves that $\{(X_t^\varepsilon)_{t\ge0}\}_{\varepsilon>0}$ is a Cauchy sequence and denotes by $(X_t)_{t\ge0}$ the limit point of $\{(X_t^\varepsilon)_{t\ge0}\}_{\varepsilon>0}$. Step 1.2 shows that $(X_t)_{t\ge0}$ has a time marginal density $\rho(t,x)$ and it is a weak solution to (1.3). Step 1.3 shows that $(X_t,\rho_t)_{t\ge0}$ is a strong solution to (1.5).

Step 1.1 For $\varepsilon>\varepsilon^{'}>0$, considering equation (2.17) and denoting by X_t^ε, $X_t^{\varepsilon^{'}}$ two strong solutions to (2.17) starting from the same initial data X_0 and Brownian motion. Let f_ε, $f_{\varepsilon'}$ be the laws of stochastic processes $(X_t^\varepsilon)_{t\ge0}$ and $(X_t^{\varepsilon^{'}})_{t\ge0}$ respectively. We also have $df_\varepsilon=\rho_\varepsilon(t,x)dx$, $df_{\varepsilon'}=\rho_{\varepsilon'}(t,x)dx$ and ρ_ε, $\rho_{\varepsilon'}$ are two weak solutions to (2.18) with the same initial condition ρ_0. We show that there exists a constant C_T and $\varepsilon_0(T)>0$ such that if $\varepsilon<\varepsilon_0(T)$,

$$\sup_{t\in[0,T]}\mathcal{W}_1(f_\varepsilon,f_{\varepsilon'})\le\mathbb{E}\big[\sup_{t\in[0,T]}|X_t^\varepsilon-X_t^{\varepsilon^{'}}|\big]\le C_T\varepsilon^{e^{-CT}}. \tag{2.24}$$

Considering (2.17) and subtracting one equation from the other one, one has

$$\sup_{\tau\in[0,t]}|X_\tau^\varepsilon-X_\tau^{\varepsilon^{'}}|\le\int_0^t\Big|\int_{\mathbb{R}^d}F_\varepsilon(X_s^\varepsilon-y)\rho_\varepsilon(s,y)dy-\int_{\mathbb{R}^d}F_{\varepsilon'}(X_s^{\varepsilon^{'}}-y)\rho_{\varepsilon'}(s,y)dy\Big|ds.$$

By Lemma 2.3, the fact that ω is nondecreasing, and the uniform estimates of ρ_ε in Theorem 2.2, there exists a constant C such that

$$
\begin{aligned}
&\mathbb{E}\big[\sup_{\tau\in[0,t]}|X_\tau^\varepsilon - X_\tau^{\varepsilon'}|\big]\\
&\le \int_0^t \mathbb{E}\big[\big|\int_{\mathbb{R}^d}\big(F_\varepsilon(X_s^\varepsilon - y)\rho_\varepsilon(s,y) - F_{\varepsilon'}(X_s^{\varepsilon'} - y)\rho_{\varepsilon'}(s,y)\big)dy\big|\big]ds\\
&\le C\int_0^t \Big(\varepsilon + \omega\big(\mathbb{E}\big[|X_s^\varepsilon - X_s^{\varepsilon'}|\big]\big)\Big)ds\\
&\le C\int_0^t \omega\big(\mathbb{E}\big[\sup_{\tau\in[0,s]}|X_\tau^\varepsilon - X_\tau^{\varepsilon'}|\big]\big)ds + C\varepsilon t. \qquad (2.25)
\end{aligned}
$$

By Lemma 2.4, there exists a constant C_T and $\varepsilon_0(T) > 0$ such that if $\varepsilon < \varepsilon_0(T)$, (2.24) holds. And then there exists a unique $C([0,T],\mathbb{R}^d)$-valued stochastic process $(X_t)_{t\in[0,T]}$ such that

$$
\mathbb{E}\big[\sup_{t\in[0,T]}|X_t^\varepsilon - X_t|\big] \le C_T\varepsilon^{\exp(-CT)}. \qquad (2.26)
$$

Furthermore, since $L^\infty\big(0,T;\mathcal{P}_1(\mathbb{R}^d)\big)$ is a complete metric space by Proposition 2.2, and combining (2.24), there exists a unique $f(t,x)\in L^\infty\big(0,T;\mathcal{P}_1(\mathbb{R}^d)\big)$ such that

$$
\mathcal{M}_T(f_\varepsilon, f) \le C_T\varepsilon^{\exp(-CT)}, \qquad (2.27)
$$

and $\mathcal{L}\{X_t\} = f$.

Step 1.2 we prove that

$$
df(t,x) = \rho(t,x)dx. \qquad (2.28)
$$

where ρ is a weak solution to (1.3).

By Proposition 2.1 and (2.27), one has

$$
\int_{\mathbb{R}^d}\varphi(x)df_\varepsilon(t,x) \to \int_{\mathbb{R}^d}\varphi(x)df(t,x) \text{ as } \varepsilon\to 0, \qquad (2.29)
$$

for any $\varphi(x)\in C_b(\mathbb{R}^d)$.

By (2.22), for all $\varphi(x)\in C_0^\infty(\mathbb{R}^d)$, one has

$$
\int_{\mathbb{R}^d}\varphi(x)\rho_\varepsilon(t,x)dx \to \int_{\mathbb{R}^d}\varphi(x)\rho(t,x)dx \text{ as } \varepsilon\to 0, \qquad (2.30)
$$

where ρ is a weak solution to (1.3). Recall (2.23), i.e. $\int_{\mathbb{R}^d}\rho(t,x)dx = 1$, since $df_\varepsilon(t,x) = \rho_\varepsilon(t,x)dx$, combining (2.29) and (2.30) yields (2.28).

Step 1.3 $(X_t,\rho_t)_{t\ge 0}$ is a strong solution to (1.5).

By Lemma 2.3, we have

$$
\mathbb{E}\big[\big|\int_{\mathbb{R}^d}F_\varepsilon(X_s^\varepsilon - y)\rho_\varepsilon(s,y)dy - \int_{\mathbb{R}^d}F(X_s - y)\rho(s,y)dy\big|\big] \le C\omega(\mathbb{E}[|X_s^\varepsilon - X_s|]) + C\varepsilon.
$$

Then by (2.26), we have

$$
\begin{aligned}
&\mathbb{E}\big[\big|\int_0^t\int_{\mathbb{R}^d}F_\varepsilon(X_s^\varepsilon - y)\rho_\varepsilon(s,y)dyds - \int_0^t\int_{\mathbb{R}^d}F(X_s - y)\rho(s,y)dyds\big|\big]\\
&\le CT\omega(\mathbb{E}[\sup_{t\in[0,T]}|X_t^\varepsilon - X_t|]) + C\varepsilon T \to 0 \text{ as } \varepsilon\to 0.
\end{aligned}
$$

Thus there exists a subsequence of $\int_0^t \int_{\mathbb{R}^d} F_\varepsilon(X_s^\varepsilon - y)\rho_\varepsilon(s,y)dyds$ (without relabeling) such that

$$\int_0^t \int_{\mathbb{R}^d} F_\varepsilon(X_s^\varepsilon - y)\rho_\varepsilon(s,y)dyds \to \int_0^t \int_{\mathbb{R}^d} F(X_s - y)\rho(s,y)dyds \text{ a.s.} \quad \text{as } \varepsilon \to 0.$$

Taking $\varepsilon \to 0$ in (2.17), we conclude that for all $t \in [0,T]$,

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}^d} F(X_s - y)\rho(s,y)dyds + \sqrt{2\nu}B_t \text{ a.s.}$$

i.e. $(X_t, \rho_t)_{t\geq 0}$ is a strong solution to (1.5).

Step 2 Uniqueness of the strong solution to (1.5).

Assume $(X_t, \rho_t)_{t\geq 0}, (\bar{X}_t, \bar{\rho}_t)_{t\geq 0}$ are two strong solutions to (1.5) with the same initial data and Brownian motion. Then

$$\bar{X}_t - X_t = \int_0^t \Big(\int_{\mathbb{R}^d} F(\bar{X}_s - y)\bar{\rho}(s,y)dy - \int_{\mathbb{R}^d} F(X_s - y)\rho(s,y)dy \Big) ds. \tag{2.31}$$

Taking expectation of (2.31) and using Lemma 2.3, one has

$$\begin{aligned} \mathbb{E}[\sup_{t\in[0,T]} |\bar{X}_t - X_t|] &\leq \int_0^T \mathbb{E}\big[\big|\int_{\mathbb{R}^d} F(\bar{X}_s - y)\bar{\rho}(s,y)dy - \int_{\mathbb{R}^d} F(X_s - y)\rho(s,y)dy\big|\big] ds \\ &\leq C \int_0^T \omega(\mathbb{E}[\sup_{s\in[0,t]} |\bar{X}_s - X_s|])dt. \end{aligned} \tag{2.32}$$

By $\mathbb{E}[|\bar{X}_0 - X_0|] = 0$ and the Osgood lemma, we obtain that $\mathbb{E}[\sup_{t\in[0,T]} |\bar{X}_t - X_t|] \equiv 0$ and

$$\mathcal{M}_T(\bar{\rho}, \rho) \leq \mathbb{E}[\sup_{t\in[0,T]} |\bar{X}_t - X_t|] = 0. \tag{2.33}$$

Therefore $\bar{\rho} = \rho$ and $X_t = \bar{X}_t$ a.s. for all $t \geq 0$.

Step 3 Uniqueness of the weak solution to (1.3).

Suppose ρ, $\bar{\rho}$ are two weak solutions to (1.3) with the same initial data ρ_0. For any fixed random variable X_0 with the density ρ_0, by the following Proposition 2.3 (i), there exists two processes $(X_t)_{t\geq 0}$ and $(\bar{X}_t)_{t\geq 0}$ such that $(X_t, \rho_t)_{t\geq 0}$ and $(\bar{X}_t, \bar{\rho}_t)_{t\geq 0}$ both are strong solutions to (1.5) with the same initial data (X_0, ρ_0). Thus (2.33) holds, which gives the uniqueness of (1.3).

PROPOSITION 2.3. *Assume the initial density $\rho_0(x)$ satisfies Assumption 1. The relationship between the weak solution to* (1.3) *and the strong solution to* (1.5) *can be expressed:*

(i) If $\rho(t,x)$ is a weak solution to (1.3) *with the initial data $\rho_0(x)$, then for any random variable X_0 with the density ρ_0, there is a unique process $(X_t)_{t\geq 0}$ with the density ρ such that $(X_t, \rho_t)_{t\geq 0}$ is a strong solution to* (1.5) *with the initial data (X_0, ρ_0).*

(ii) If $(X_t, \rho_t)_{t\geq 0}$ is a strong solution to (1.5) *with the initial data (X_0, ρ_0), then ρ is a weak solution to* (1.3) *with the initial data ρ_0.*

Proof. To prove (i), we first prove the uniqueness of weak solution in the class of $L^\infty\big(0,T;L^2(\mathbb{R}^d)\big)\cap L^2\big(0,T;H^1(\mathbb{R}^d)\big)$ to the following linear Fokker-Planck equation:

$$\begin{cases} \partial_t\rho = \nu\triangle\rho - \nabla\cdot[V_g(t,x)\rho],\ x\in\mathbb{R}^d,\ t>0, \\ \rho(0,x)=\rho_0(x). \end{cases} \tag{2.34}$$

where $V_g(t,x)=\int_{\mathbb{R}^d}F(x-y)g(t,y)dy,\ g(t,x)\in L^\infty\big(0,T;L^\infty\cap L^1(\mathbb{R}^d)\big)$ is a given function.

Suppose ρ_1, ρ_2 are two weak solutions to (2.34) with the same initial condition, subtracting one equation from another and taking test function as $\rho_1-\rho_2$, since $V(t,x)$ is bounded (see Lemma 2.2 (i)), one has

$$\begin{aligned}\frac{d}{dt}\frac{\|\rho_1-\rho_2\|_2^2}{2} &= \int_{\mathbb{R}^d} V(t,x)(\rho_1-\rho_2)\cdot\nabla(\rho_1-\rho_2)dx - \nu\|\nabla(\rho_1-\rho_2)\|_2^2 \\ &\le C_\nu\|\rho_1-\rho_2\|_2^2 + \frac{\nu}{2}\|\nabla(\rho_1-\rho_2)\|_2^2 - \nu\|\nabla(\rho_1-\rho_2)\|_2^2 \le C_\nu\|\rho_1-\rho_2\|_2^2.\end{aligned}$$

Then combining the zero initial condition, we obtain $\|\rho_1-\rho_2\|_2\equiv 0$.

Next let $V_\rho(t,x)=\int_{\mathbb{R}^d}F(x-y)\rho(t,y)dy$, where ρ is a weak solution to (1.3). Since V_ρ is log-Lipschitz continuous, repeating the proof of the existence and uniqueness for (1.5), one can prove that the following linear SDE

$$X_t = X_0 + \int_0^t V_\rho\big(s,X_s\big)ds + \sqrt{2\nu}B_t, \tag{2.35}$$

has a unique strong solution $(X_t)_{t\ge0}$ with a marginal density $\tilde\rho$ and $\tilde\rho$ is a weak solution to (2.34) associated to V_ρ. Notice that ρ is also a weak solution to (2.34) associated to V_ρ. By the uniqueness of weak solution to (2.34), we obtain $\tilde\rho=\rho$, i.e. $(X_t,\rho_t)_{t\ge0}$ is a strong solution to (1.5).

Combining the uniqueness of strong solution to (1.5), we finish the proof of (i). The proof of (ii) is directly from the Itô formula and by using the energy estimates for the regularity of ρ defined in Definition 2. □

2.4. Dobrushin's type stability of the KS equation.

Proof of Theorem 1.2. For any fixed $T>0$, let ρ^1,ρ^2 be two weak solutions to (1.3) with the initial conditions $\rho_0^1(x)$ and $\rho_0^2(x)$ respectively. Following the idea of Dobrushin, we take $\pi_0(x,y)\in\Lambda(\rho_0^1,\rho_0^2)$ as the optimal joint probability measure with marginals ρ_0^1dx and ρ_0^2dx, i.e.

$$\mathcal{W}_1(\rho_0^1,\ \rho_0^2) = \int_{\mathbb{R}^{2d}}|x-y|\pi_0(dx,dy). \tag{2.36}$$

Then taking two initial random variables X_0^1 and X_0^2 with joint distribution $\pi_0(x,y)$, one has

$$\mathcal{W}_1(\rho_0^1,\ \rho_0^2) = \mathbb{E}[|X_0^1-X_0^2|]. \tag{2.37}$$

By Proposition 2.3, we know that the following two self-consistent SDEs:

$$X_t^i = X_0^i + \int_0^t\int_{\mathbb{R}^d}F(X_s^i-y)\rho^i(s,y)dyds + \sqrt{2\nu}B_t,\quad i=1,2, \tag{2.38}$$

have unique strong solutions $(X_t^i, \rho_t^i)_{t\geq 0}$, where $(\rho_t^i)_{t\geq 0}$ is the marginal density of $(X_t^i)_{t\geq 0}$. From Lemma 2.3, there exists a constant C (depending only on $\|\rho^1\|_{L^\infty(0,T;L^\infty\cap L^1(\mathbb{R}^d))}$ and $\|\rho^2\|_{L^\infty(0,T;L^\infty\cap L^1(\mathbb{R}^d))}$) such that

$$\begin{aligned}\mathbb{E}[|X_t^1 - X_t^2|] &\leq \mathbb{E}[|X_0^1 - X_0^2|] + \int_0^t \mathbb{E}[|\int_{\mathbb{R}^d} F(X_s^1 - y)\rho^1 dy - \int_{\mathbb{R}^d} F(X_s^2 - y)\rho^2 dy|] ds \\ &\leq \mathbb{E}[|X_0^1 - X_0^2|] + C\int_0^t \omega(\mathbb{E}[|X_s^1 - X_s^2|]) ds. \end{aligned} \tag{2.39}$$

Notice that $\omega(r) = r(1-\ln r)$ for $0 < r < 1$ by (2.6). Applying Lemma 2.4 to (2.39), there exists a small enough constant $C_0(T)$ such that if $\mathbb{E}[|X_0^1 - X_0^2|] < C_0(T)$, then $\mathbb{E}[|X_t^1 - X_t^2|] < 1$ for any $t \in [0,T]$. Indeed, one has that

$$\mathbb{E}[|X_t^1 - X_t^2|] \leq C_T\{\mathbb{E}[|X_0^1 - X_0^2|]\}^{\exp(-CT)} < 1 \; for\ any\ t \in [0,T]. \tag{2.40}$$

Combining (2.37), one has

$$\mathbb{E}[|X_t^1 - X_t^2|] \leq C_T\{\mathcal{W}_1(\rho_0^1,\ \rho_0^2)\}^{\exp(-CT)}. \tag{2.41}$$

Otherwise, if $\mathbb{E}[|X_0^1 - X_0^2|] \geq C_0(T)$, then there exists two cases:

(i) for any $t \in [0,T]$, $\mathbb{E}[|X_t^1 - X_t^2|] \geq C_0(T)$;

(ii) there exists a $t_0 \in (0,T]$ such that $\mathbb{E}[|X_t^1 - X_t^2|] \geq C_0(T)$ for $t \in [0,t_0)$ and $\mathbb{E}[|X_{t_0}^1 - X_{t_0}^2|] < C_0(T)$.

For the case (i), by the definition of $\omega(r)$, one obtains that there exists a constant C_1 (depending only on $C_0(T)$) such that $\omega(r) \leq C_1 r$ for $r \geq C_0(T)$. Thus combining (2.39), we get

$$\mathbb{E}[|X_t^1 - X_t^2|] \leq \mathbb{E}[|X_0^1 - X_0^2|] + C_1 C\int_0^t \mathbb{E}[|X_s^1 - X_s^2|] ds. \tag{2.42}$$

By the Gronwall inequality, for any $t \in [0,T]$, one has

$$\mathbb{E}[|X_t^1 - X_t^2|] \leq \mathbb{E}[|X_0^1 - X_0^2|] \exp\big(C_1 CT\big) = \mathcal{W}_1(\rho_0^1,\ \rho_0^2) \exp\big(C_1 CT\big). \tag{2.43}$$

For the case (ii), the estimate for the interval $t \in [0,t_0)$ is reduced to the case (i) and one has

$$\mathbb{E}[|X_t^1 - X_t^2|] \leq \mathcal{W}_1(\rho_0^1,\ \rho_0^2) \exp\big(C_1 CT\big). \tag{2.44}$$

For the interval $t \in [t_0,T]$, choosing t_0 as a new initial time and repeating the proof of (2.41) gives the following inequality

$$\mathbb{E}[|X_t^1 - X_t^2|] \leq C_T\{\mathbb{E}[|X_{t_0}^1 - X_{t_0}^2|]\}^{\exp(-C(t-t_0))}. \tag{2.45}$$

By (2.44) and the continuity of $\mathbb{E}[|X_t^1 - X_t^2|]$, one has $\mathbb{E}[|X_{t_0}^1 - X_{t_0}^2|] \leq \mathcal{W}_1(\rho_0^1,\ \rho_0^2) \exp\big(C_1 CT\big)$. Therefore combining (2.44) and (2.45), we obtain

$$\begin{aligned}&\mathbb{E}[|X_t^1 - X_t^2|] \\ &\leq C_T \max\Big\{\mathcal{W}_1(\rho_0^1,\ \rho_0^2), \{\mathcal{W}_1(\rho_0^1,\ \rho_0^2)\}^{\exp(-CT)}\Big\} \text{ for any } t \in [0,T].\end{aligned} \tag{2.46}$$

Let $\pi_t(x,y)$ be the joint distribution of X_t^1 and X_t^2. Clearly one has $\pi_t(x,y) \in \Lambda(\rho_t^1, \rho_t^2)$. Hence $\sup_{t\in[0,T]} \mathcal{W}_1(\rho_t^1, \rho_t^2) \leq \sup_{t\in[0,T]} \mathbb{E}[|X_t^1 - X_t^2|]$. Combining (2.41), (2.43) and (2.46) finishes the proof immediately. □

3. Propagation of chaos. This section is divided into three subsections. We first prove a result on the collision between particles. Then we prove Theorem 1.3. Finally, we show the propagation of chaos for the KS equation by applying Theorem 1.3 .

3.1. Collision between particles. For $d = 2$, it is well known that if $8\pi\nu < 1$, the solution to the KS equation with the initial density ρ_0 concentrates before the time

$$T^c := \frac{2\pi \mathrm{Var}\{\rho_0\}}{1 - 8\pi\nu}.$$

In this subsection, we show that the expectation of the collision time for the interacting particle system (1.1) is also bounded by this constant.

THEOREM 3.1. *Assume $d = 2$. Given N i.i.d. random variables $\{X_0^i\}_{i=1}^N$ with the common density ρ_0. Let $\{(X_t^i)_{t\geq 0}\}_{i=1}^N$ be the strong solution to* (1.1) *with the initial data $\{X_0^i\}_{i=1}^N$. If $8\pi\nu < 1$, fix $T > T^c$ and define a stopping time by*

$$\tau_\varepsilon = \inf\{t \geq 0 : \min_{i\neq j} |x_t^i - x_t^j| \leq \varepsilon\} \wedge T. \tag{3.1}$$

Let $\tau = \lim\limits_{\varepsilon\to 0} \tau_\varepsilon$, we have

$$\mathbb{E}(\tau) \leq T^c. \tag{3.2}$$

Proof. From the definition (3.1), there exists a unique strong solution $\{(X_t^i)_{t\geq 0}\}_{i=1}^N$ to the interacting particle system (1.1) up to time $\tau_{\frac{\varepsilon}{2}}$ with the initial data $\{X_0^i\}_{i=1}^N$. Since $F_\varepsilon(x) = F(x)$ for any $|x| \geq \varepsilon$ by Lemma 2.1, then $\{(X_t^i)_{t\geq 0}\}_{i=1}^N$ equals to $\{(X_t^{i,\varepsilon})_{t\geq 0}\}_{i=1}^N$ (the strong solution of (1.7)) for all $t \leq \tau_\varepsilon$, i.e.

$$X_{\tau_\varepsilon}^i = X_0^i - \frac{1}{N-1}\sum_{j\neq i}^N \frac{1}{2\pi}\int_0^{\tau_\varepsilon} \frac{X_s^i - X_s^j}{|X_s^i - X_s^j|^2} ds + \sqrt{2\nu} B_{\tau_\varepsilon}^i, \quad i = 1, \cdots, N.$$

Summing all of the equations, one has

$$\sum_{i=1}^N X_{\tau_\varepsilon}^i = \sum_{i=1}^N X_0^i + \sum_{i=1}^N \sqrt{2\nu} B_{\tau_\varepsilon}^i. \tag{3.3}$$

Taking expectation of (3.3), by the exchangeability of $\{(X_t^i)_{t\geq 0}\}_{i=1}^N$ and $\mathbb{E}[B_{\tau_\varepsilon}^i] = 0$ [21, pp.28, Theorem 1], we have

$$\mathbb{E}[X_{\tau_\varepsilon}^i] = \mathbb{E}[X_0^i] =: X_0.$$

Since the system (1.1) has a unique strong solution until the explosion time $\tau := \inf\{t > 0 : \min\limits_{i\neq j} |x_t^i - x_t^j| = 0\}$, and $F_\varepsilon(x) = F(x)$ for any $|x| \geq \varepsilon$ by (i) of Lemma 2.1, we know $X_t^i \equiv X_t^{i,\varepsilon}$ for any $1 \leq i \leq N$ and $t \leq \tau_\varepsilon$, where $(X_t^{i,\varepsilon})_{t\geq 0}$ is the unique global strong solution to (1.7).

Next, we will estimate $\mathbb{E}[\tau_\varepsilon]$ by computing the variance of process $(X_t^{i,\varepsilon})_{t\geq 0}$ at the stopping time τ_ε. By the Itô formula and (1.7), for $i = 1, \cdots, N$, one has

$$\begin{aligned} &d[|X_t^{i,\varepsilon} - X_0|^2] \\ &= 2(X_t^{i,\varepsilon} - X_0)\cdot\Big(\frac{1}{N-1}\sum_{j\neq i}^N F_\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon})dt + \sqrt{2\nu}dB_t^i\Big) + 4\nu dt. \end{aligned} \tag{3.4}$$

Since

$$F_\varepsilon(-x) = F(-x)g(\frac{|x|}{\varepsilon}) = -F(x)g(\frac{|x|}{\varepsilon}) = -F_\varepsilon(x),$$

summing all of (3.4) and integrating, one has

$$\sum_{i=1}^N |X_t^{i,\varepsilon} - X_0|^2 = \sum_{i=1}^N |X_0^i - X_0|^2 + \frac{2}{N-1}\sum_{\substack{i,j=1\\ i\neq j}}^N \int_0^t X_s^{i,\varepsilon}\cdot F_\varepsilon(X_s^{i,\varepsilon} - X_s^{j,\varepsilon})ds$$
$$+\sum_{i=1}^N 2\sqrt{2\nu}\int_0^t (X_s^{i,\varepsilon} - X_0)\cdot dB_s^i + 4\nu Nt. \tag{3.5}$$

Since $x\cdot F_\varepsilon(x) = x\cdot F(x)g(\frac{|x|}{\varepsilon}) \le 0$, we have

$$\sum_{\substack{i,j=1\\ i\neq j}}^N X_s^{i,\varepsilon}\cdot F_\varepsilon(X_s^{i,\varepsilon} - X_s^{j,\varepsilon}) = \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^N (X_s^{i,\varepsilon} - X_s^{j,\varepsilon})\cdot F_\varepsilon(X_s^{i,\varepsilon} - X_s^{j,\varepsilon}) \le 0.$$

Then taking expectation of (3.5), by the exchangeability of $\{(X_t^{i,\varepsilon})_{t\ge 0}\}_{i=1}^N$, one has

$$\mathbb{E}\big[|X_t^{i,\varepsilon} - X_0|^2\big] \le \mathbb{E}\big[|X_0^i - X_0|^2\big] + 4\nu t + \frac{2\sqrt{2\nu}}{N}\big(\mathbb{E}[\sum_{i=1}^N\int_0^t (X_s^{i,\varepsilon} - X_0)\cdot dB_s^i]^2\big)^{\frac{1}{2}}$$
$$\le \mathbb{E}\big[|X_0^i - X_0|^2\big] + 4\nu t + 2\sqrt{2\nu}\big(\int_0^t \mathbb{E}\big[|X_s^{i,\varepsilon} - X_0|^2\big]ds\big)^{\frac{1}{2}},$$

the last inequality comes from the Itô isometry. Hence

$$\int_0^T \mathbb{E}\big[|X_t^{i,\varepsilon} - X_0|^2\big]dt < +\infty. \tag{3.6}$$

Therefore applying [21, pp.28, Theorem 1] deduces that

$$\mathbb{E}\big[\int_0^{\tau_\varepsilon} (X_t^{i,\varepsilon} - X_0)\cdot dB_t^i\big] = 0. \tag{3.7}$$

Since $F_\varepsilon(x) = F(x)$ for any $|x| \ge \varepsilon$, by the definition of τ_ε and the fact that $\{(X_t^{i,\varepsilon})_{t\ge 0}\}_{i=1}^N$ is the unique solution to (1.1) on $[0,\tau_\varepsilon]$, we have $(X_s^{i,\varepsilon} - X_s^{j,\varepsilon})\cdot F_\varepsilon(X_s^{i,\varepsilon} - X_s^{j,\varepsilon}) = -\frac{1}{2\pi}$ for all $s\in[0,\tau_\varepsilon]$. Then

$$\frac{2}{N-1}\sum_{\substack{i,j=1\\ i\neq j}}^N X_s^{i,\varepsilon}\cdot F_\varepsilon(X_s^{i,\varepsilon} - X_s^{j,\varepsilon}) = -\frac{N}{2\pi} \quad \text{for all } s\in[0,\tau_\varepsilon]. \tag{3.8}$$

Taking expectation of (3.5), by the exchangeability of $\{(X_t^{i,\varepsilon})_{t\ge 0}\}_{i=1}^N$ and choosing t as τ_ε in (3.5), one has

$$\mathbb{E}[|X_{\tau_\varepsilon}^i - X_0|^2] = \mathbb{E}[|X_0^i - X_0|^2] + (4\nu - \frac{1}{2\pi})\mathbb{E}[\tau_\varepsilon]$$
$$= \mathrm{Var}\{X_0^1\} + (4\nu - \frac{1}{2\pi})\mathbb{E}[\tau_\varepsilon]. \tag{3.9}$$

By the positivity of left hand, we obtain

$$\mathbb{E}[\tau_\varepsilon] \le \frac{2\pi \mathrm{Var}\{X_0^1\}}{1-8\pi\nu} = T^c.$$

By the monotone convergence theorem, we achieve (3.2). □

3.2. Proof of Theorem 1.3. In order to find out the relationship between the N paths of (1.7) and paths of (1.8), we construct the following regularized self-consistent SDEs:

$$\bar{X}_t^{i,\varepsilon} = X_0^i + \int_0^t \int_{\mathbb{R}^d} F_\varepsilon(\bar{X}_s^{i,\varepsilon} - y) df_s^{i,\varepsilon}(y) ds + \sqrt{2\nu} B_t^i, \quad i = 1, \cdots, N, \tag{3.10}$$

where $(f_t^{i,\varepsilon})_{t\ge 0}$ is the marginal density of $\{(\bar{X}_t^{i,\varepsilon})_{t\ge 0}\}$ and the initial data $\{X_0^i\}_{i=1}^N$ and Brownian motions $\{(B_t^i)_{t\ge 0}\}_{i=1}^N$ are the same as those of (1.7). In Section 2, Theorem 2.1 stated the existence and uniqueness of strong solution to (2.17), which implies that the processes $\{(\bar{X}_t^{i,\varepsilon})_{t\ge 0}\}_{i=1}^N$ are N copies of the strong solutions to (2.17) since the initial data and Brownian motions both are i.i.d..

The proof of Theorem 1.3 by the coupling method can be realized by two steps: (i) (2.26) gives the connection between (1.8) and (3.10), (ii) the following proposition shows the connection between (1.7) and (3.10).

PROPOSITION 3.1. *Suppose* $\{(X_t^{i,\varepsilon})_{t\ge 0}\}_{i=1}^N$ *and* $\{(\bar{X}_t^{i,\varepsilon})_{t\ge 0}\}_{i=1}^N$ *are the unique strong solution to* (1.7) *and* (3.10) *respectively, with the same i.i.d. initial data* $\{X_0^i\}_{i=1}^N$ *and Brownian motions* $\{(B_t^i)_{t\ge 0}\}_{i=1}^N$. *Then for any* $\varepsilon > 0, 1 \le i \le N$ *and* $T > 0$, *one has*

$$\mathbb{E}\big[\sup_{t\in[0,T]} |X_t^{i,\varepsilon} - \bar{X}_t^{i,\varepsilon}|\big] \le \frac{C_T}{\sqrt{N-1}\varepsilon^{(d-1)}} \exp\big(\frac{C_T}{\varepsilon^d}\big) \tag{3.11}$$

where C_T *is a constant depending only on* T *and* d.

Proof. Following the spirit of [43], since $\big|\nabla F_\varepsilon(x)\big| \le \frac{C}{\varepsilon^d}$ by Lemma 2.1, one has

$$\begin{aligned}\big|X_t^{i,\varepsilon} - \bar{X}_t^{i,\varepsilon}\big| \le \Big| \int_0^t \frac{1}{N-1} \sum_{j\ne i}^N \big(F_\varepsilon(X_s^{i,\varepsilon} - X_s^{j,\varepsilon}) - F_\varepsilon(\bar{X}_s^{i,\varepsilon} - \bar{X}_s^{j,\varepsilon}) \\ + F_\varepsilon(\bar{X}_s^{i,\varepsilon} - \bar{X}_s^{j,\varepsilon}) - \int_{\mathbb{R}^d} F_\varepsilon(\bar{X}_s^{i,\varepsilon} - y) df_s^{i,\varepsilon}(y)\big) ds\Big| \\ \le \frac{1}{N-1}\int_0^t \sum_{j\ne i}^N \big(\frac{C}{\varepsilon^d}|X_s^{i,\varepsilon} - \bar{X}_s^{i,\varepsilon}| + \frac{C}{\varepsilon^d}|\bar{X}_s^{j,\varepsilon} - X_s^{j,\varepsilon}|\big) + \Big|\sum_{j\ne i}^N A_j^i(s)\Big| ds\end{aligned} \tag{3.12}$$

where

$$A_j^i(s) = F_\varepsilon(\bar{X}_s^{i,\varepsilon} - \bar{X}_s^{j,\varepsilon}) - \int_{\mathbb{R}^d} F_\varepsilon(\bar{X}_s^{i,\varepsilon} - y) df_s^{i,\varepsilon}(y). \tag{3.13}$$

From (3.12), one has

$$
\begin{aligned}
&\sup_{s\in[0,t]}|X_s^{i,\varepsilon}-\bar X_s^{i,\varepsilon}|\\
&\le \frac{1}{N-1}\int_0^t\Big\{\sum_{j\neq i}^N\big(\frac{C}{\varepsilon^d}|X_s^{i,\varepsilon}-\bar X_s^{i,\varepsilon}|+\frac{C}{\varepsilon^d}|\bar X_s^{j,\varepsilon}-X_s^{j,\varepsilon}|\big)+|\sum_{j\neq i}^N A_j^i(s)|\Big\}ds\\
&\le \frac{1}{N-1}\int_0^t\sum_{j\neq i}^N\big(\frac{C}{\varepsilon^d}\sup_{\tau\in[0,s]}|X_\tau^{i,\varepsilon}-\bar X_\tau^{i,\varepsilon}|+\frac{C}{\varepsilon^d}\sup_{\tau\in[0,s]}|\bar X_\tau^{j,\varepsilon}-X_\tau^{j,\varepsilon}|\big)ds\\
&\quad+\frac{1}{N-1}\int_0^t|\sum_{j\neq i}^N A_j^i(s)|ds. \qquad (3.14)
\end{aligned}
$$

Denote by $m_{N+1}(\omega^1,\cdots,\omega^N,y)\in\mathcal{P}(\mathcal{C}^{N+1})$ ($\mathcal{C}=C([0,T],\mathbb{R}^d)$) the joint distribution of $\big(X_t^{1,\varepsilon},\cdots,X_t^{N,\varepsilon},\bar X_t^{i,\varepsilon}\big)_{t\ge0}$ and $m_3(\omega^i,\omega^j,y)=\int_{\mathcal{C}^{N-2}}m_{N+1}(d\omega^1,\cdots,\omega^i,\cdots,\omega^j,\cdots,d\omega^N,y)$ for any $1\le i\neq j\le N$. Since $\{(X_t^{i,\varepsilon})_{t\ge0}\}_{i=1}^N$ are exchangeable stochastic processes and $\{(\bar X_t^{i,\varepsilon})_{t\ge0}\}_{i=1}^N$ are i.i.d. stochastic processes, then $m_3(\omega^i,\omega^j,y)=m_3(\omega^j,\omega^i,y)$ for any $1\le i\neq j\le N$ and then we obtain the following exchangeability qualities: for any $t\in[0,T]$,

$$
\begin{aligned}
&\mathbb{E}[\sup_{s\in[0,t]}|X_s^{i,\varepsilon}-\bar X_s^{i,\varepsilon}|]=\int_{\mathcal{C}^3}\sup_{s\in[0,t]}|\omega_s^i-y_s|dm_3(\omega^i,\omega^j,y)\\
&=\int_{\mathcal{C}^3}\sup_{s\in[0,t]}|\omega_s^i-y_s|dm_3(\omega^j,\omega^i,y)=\int_{\mathcal{C}^3}\sup_{s\in[0,t]}|\omega_s^j-y_s|dm_3(\omega^i,\omega^j,y)\\
&=\mathbb{E}[\sup_{s\in[0,t]}|X_s^{j,\varepsilon}-\bar X_s^{j,\varepsilon}|]. \qquad (3.15)
\end{aligned}
$$

Hence taking expectation of (3.14), one has

$$
\begin{aligned}
&\mathbb{E}\big[\sup_{s\in[0,t]}|X_s^{i,\varepsilon}-\bar X_s^{i,\varepsilon}|\big]\\
&\le\int_0^t\mathbb{E}\big[|\frac{1}{N-1}\sum_{j\neq i}^N A_j^i(s)|\big]ds+\frac{2C}{\varepsilon^d}\int_0^t\mathbb{E}\big[\sup_{\tau\in[0,s]}|X_\tau^{i,\varepsilon}-\bar X_\tau^{i,\varepsilon}|\big]ds. \qquad (3.16)
\end{aligned}
$$

Applying the Gronwall's lemma deduces that

$$
\begin{aligned}
\mathbb{E}\big[\sup_{s\in[0,t]}\big|X_s^{i,\varepsilon}-\bar X_s^{i,\varepsilon}\big|\big]&\le\exp\big(\frac{2Ct}{\varepsilon^d}\big)\int_0^t\mathbb{E}\big[|\frac{1}{N-1}\sum_{j\neq i}^N A_j^i(s)|\big]ds\\
&\le\exp\big(\frac{2CT}{\varepsilon^d}\big)\int_0^T\Big\{\mathbb{E}\big[|\frac{1}{N-1}\sum_{j\neq i}^N A_j^i(s)|^2\big]\Big\}^{\frac12}ds. \qquad (3.17)
\end{aligned}
$$

Because $\{(\bar X_t^{i,\varepsilon})_{t\ge0}\}_{i=1}^N$ are i.i.d. random variables, when $j\neq k$, one has

$$
\mathbb{E}\big[A_j^i(s)A_k^i(s)\big]=0.
$$

Hence

$$
\mathbb{E}\big[|\frac{1}{N-1}\sum_{j\neq i}^N A_j^i(s)|^2\big]=\frac{1}{(N-1)^2}\mathbb{E}\big[\sum_{\substack{j,k=1\\ j,k\neq i}}^N A_j^i(s)A_k^i(s)\big]\le\frac{\mathbb{E}\big[(A_2^1(s))^2\big]}{N-1}. \qquad (3.18)
$$

For all $\varepsilon > 0$, $\big|F_\varepsilon(x)\big| \le \min\{\frac{C|x|}{\varepsilon^d}, \frac{C}{|x|^{d-1}}\}$ from Lemma 2.1, we have

$$\begin{aligned}
&\mathbb{E}\big[(A_2^1(s))^2\big] \\
&= \mathbb{E}\Big[\big(F_\varepsilon(\bar{X}_s^{1,\varepsilon} - \bar{X}_s^{2,\varepsilon}) - \int_{\mathbb{R}^d} F_\varepsilon(\bar{X}_s^{1,\varepsilon} - y) df_s^{1,\varepsilon}(y)\big)^2\Big] \\
&\le 2\mathbb{E}\big[F_\varepsilon^2(\bar{X}_s^{1,\varepsilon} - \bar{X}_s^{2,\varepsilon}) + \big(\int_{\mathbb{R}^d} F_\varepsilon(\bar{X}_s^{1,\varepsilon} - y) df_s^{1,\varepsilon}(y)\big)^2\big] \le 4\mathbb{E}\big[F_\varepsilon^2(\bar{X}_s^{1,\varepsilon} - \bar{X}_s^{2,\varepsilon})\big] \\
&\le C\int_{|x-y|\le\varepsilon} \frac{|x-y|^2}{\varepsilon^{2d}} df_s^{1,\varepsilon}(x) df_s^{2,\varepsilon}(y) + C\int_{|x-y|>\varepsilon} \frac{1}{|x-y|^{2(d-1)}} df_s^{1,\varepsilon}(x) df_s^{2,\varepsilon}(y) \\
&\le \frac{C}{\varepsilon^{2(d-1)}}. \qquad (3.19)
\end{aligned}$$

Combining (3.17), (3.18) and (3.19) together yields (3.11). □

Proof of Theorem 1.3. Let $\big\{(X_t^i)_{t\ge0}\big\}_{i=1}^N$ and $\big\{(\bar{X}_t^{i,\varepsilon})_{t\ge0}\big\}_{i=1}^N$ be the unique strong solution to (1.8) and (3.10) respectively, with the same i.i.d. initial data $\{X_0^i\}_{i=1}^N$ and Brownian motions $\{(B_t^i)_{t\ge0}\}_{i=1}^N$. Similarly with (2.26), one can obtain that there exists a constant C_T and $\varepsilon_0(T) > 0$ such that if $\varepsilon < \varepsilon_0(T)$, then for any $1 \le i \le N$ and $T > 0$,

$$\mathbb{E}\big[\sup_{t\in[0,T]} |\bar{X}_t^{i,\varepsilon} - X_t^i|\big] \le C_T\varepsilon^{\exp(-CT)}. \qquad (3.20)$$

Combining (3.11) and (3.20) together, one has

$$\begin{aligned}
\mathbb{E}\big[\sup_{t\in[0,T]} |X_t^{i,\varepsilon} - X_t^i|\big] &\le \mathbb{E}\big[\sup_{t\in[0,T]} |X_t^{i,\varepsilon} - \bar{X}_t^{i,\varepsilon}|\big] + \mathbb{E}\big[\sup_{t\in[0,T]} |\bar{X}_t^{i,\varepsilon} - X_t^i|\big] \\
&\le \frac{C_T}{\sqrt{N-1}\varepsilon^{(d-1)}} \exp\big(\frac{C_T}{\varepsilon^d}\big) + C_T\varepsilon^{\exp(-CT)}. \qquad (3.21)
\end{aligned}$$

We choose $\varepsilon = \varepsilon(N) = \lambda(\ln N)^{-\frac{1}{d}} \to 0$ as $N \to \infty$ in (3.21), where λ is a large enough positive constant. And then

$$\begin{aligned}
\mathbb{E}\big[\sup_{t\in[0,T]} |X_t^{i,\varepsilon(N)} - X_t^i|\big] \le \frac{C_T N^{\frac{C_T}{\lambda^d}} (\ln N)^{\frac{d-1}{d}}}{\lambda^{d-1}\sqrt{N-1}} + C_T\varepsilon^{\exp(-CT)} \\
\to 0 \text{ as } N \to \infty, \qquad (3.22)
\end{aligned}$$

which ends the proof of Theorem 1.3. □

3.3. Propagation of chaos.

COROLLARY 3.1. *Let $\{(X_t^{i,\varepsilon})_{t\ge0}\}_{i=1}^N$ and $\{(X_t^i)_{t\ge0}\}_{i=1}^N$ be two processes defined in Theorem 1.3. Denote by $F_t(x_1, \cdots, x_N)$ the joint marginal distribution of $\big(X_t^{1,\varepsilon}, \cdots, X_t^{N,\varepsilon}\big)_{t\ge0}$, $f_t^{(j),\varepsilon} = \int_{\mathbb{R}^{(N-j)d}} F_t(\cdot, dx_{j+1}, \cdots, dx_N)$ be the j-th marginal distribution of $F_t(x_1, \cdots, x_N)$, and $(f_t)_{t\ge0}$ be the common time marginal distribution of $\big\{(X_t^i)_{t\ge0}\big\}_{i=1}^N$. Then there is a list of cut-off parameters $\varepsilon(N) \sim (\ln N)^{-\frac{1}{d}} \to 0$ as $N \to \infty$ such that for any $j \ge 1$,*

$$\mathcal{M}_T(f_t^{(j),\varepsilon(N)}, f_t^{\otimes j}) \to 0 \text{ as } N \to \infty.$$

Proof. Denote by $\tilde{F}_t(x_1, \cdots, x_N, \hat{x}_1, \cdots, \hat{x}_j)$ the joint marginal distribution of $\big(X_t^{1,\varepsilon}, \cdots, X_t^{N,\varepsilon}, X_t^1, \cdots, X_t^j\big)_{t\geq 0}$, then one has the following facts

$$\begin{aligned} &f_t^{\otimes j} = \int_{\mathbb{R}^{Nd}} \tilde{F}_t(dx_1, \cdots, dx_N, \cdot), \\ &f_t^{(j),\varepsilon} = \int_{\mathbb{R}^{Nd}} \tilde{F}_t(\cdot, dx_{j+1}, \cdots, dx_N, d\hat{x}_1, \cdots, d\hat{x}_j), \\ &\int_{\mathbb{R}^{(N-j)d}} \tilde{F}_t(\cdot, dx_{j+1}, \cdots, dx_N, \cdot) \in \Lambda(f_t^{(j),\varepsilon}, f_t^{\otimes j}). \end{aligned}$$

Applying (3.21) and the exchangeability of $\{(X_t^{i,\varepsilon})_{t\geq 0}\}_{i=1}^N$ and $\{(X_t^i)_{t\geq 0}\}_{i=1}^N$, we obtain

$$\begin{aligned} &\mathcal{M}_T(f_t^{(j),\varepsilon(N)}, f_t^{\otimes j}) \\ &\leq \sup_{t\in[0,T]} \int_{\mathbb{R}^{2jd}} \big(|x_1 - \hat{x}_1| + \cdots + |x_j - \hat{x}_j|\big) \int_{\mathbb{R}^{(N-j)d}} \tilde{F}_t(dx_1, \cdots, dx_N, d\hat{x}_1, \cdots, d\hat{x}_j) \\ &= j \sup_{t\in[0,T]} \int_{\mathbb{R}^{(N+j)d}} |x_1 - \hat{x}_1| \tilde{F}_t(dx_1, \cdots, dx_N, d\hat{x}_1, , \cdots, d\hat{x}_j) \\ &\leq j\mathbb{E}_{x_1,\cdots,x_N,\hat{x}_1,\cdots,\hat{x}_j}\big[\sup_{t\in[0,T]} |X_t^{1,\varepsilon(N)} - X_t^1|\big] \to 0 \text{ as } N\to\infty. \end{aligned} \tag{3.23}$$

∎

For another perspective, similar with [43], we have another propagation of chaos result.

Corollary 3.2. *Let $\{(X_t^{i,\varepsilon})_{t\geq 0}\}_{i=1}^N$ and $\{(X_t^i)_{t\geq 0}\}_{i=1}^N$ be defined in Theorem 1.3, which can be considered as the canonical process on $\mathcal{C} = C([0,T], \mathbb{R}^d)$. Denote $m_N(\omega_1, \cdots, \omega_N) \in \mathcal{P}(\mathcal{C}^N)$ as the joint probability measure of $\big(X_t^{1,\varepsilon}, \cdots, X_t^{N,\varepsilon}\big)_{t\geq 0}$ and $m \in \mathcal{P}(\mathcal{C})$ as the common probability measure of $\big\{(X_t^i)_{t\geq 0}\big\}_{i=1}^N$. Then there is a list of cut-off parameters $\varepsilon(N) \sim (\ln N)^{-\frac{1}{d}} \to 0$ as $N\to\infty$ such that for any $j \geq 1$, $(X_t^{1,\varepsilon(N)}, \cdots, X_t^{j,\varepsilon(N)})_{t\geq 0}$ converges in law to the j-independent random variables $(X_t^1, \cdots, X_t^j)_{t\geq 0}$, which is equivalent to that the empirical measure $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,\varepsilon(N)}}$ (regarded as a $\mathcal{P}_1(\mathcal{C})$-valued random variable) converges in probability to m as $N\to\infty$.*

Proof. The proof is similar to that of Corollary 3.1. Denote by $m^{(j),\varepsilon} = \int_{\mathcal{C}^{(N-j)d}} m_N(\omega_1, \cdots, \omega_j, d\omega_{j+1}, \cdots, d\omega_N)$ the j-th marginal probability measure of m_N, one has

$$\begin{aligned} \mathcal{D}_T(m^{(j),\varepsilon(N)}, m^{\otimes j}) &\leq \mathbb{E}\big[\sum_{i=1}^j \sup_{t\in[0,T]} |X_t^{i,\varepsilon(N)} - X_t^i|\big] \\ &= j\mathbb{E}\big[\sup_{t\in[0,T]} |X_t^{1,\varepsilon(N)} - X_t^1|\big] \to 0 \text{ as } N\to\infty. \end{aligned}$$

It has been proven by Sznitman [43, Proposition 2.2.] that $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,\varepsilon(N)}}$ converges in law to m. Since m is a constant random variable, then the convergence in probability follows. ∎

4. Appendix.

Proof of Claim 1 in Section 2.3. Based on the uniform estimates in Theorem 2.2, there exists a constant C which is independent of ε such that

$$\int_0^T \| \nabla\rho_\varepsilon \|^2_{L^2(\mathbb{R}^d)} \, dt \le C \text{ and } \int_0^T \| \partial_t\rho_\varepsilon \|^2_{H^{-1}(\mathbb{R}^d)} \, dt \le C,$$

then the following compact embedding holds: for any ball B_R centered at 0 with radius R,

$$H^1(B_R) \hookrightarrow\hookrightarrow L^2(B_R) \hookrightarrow H^{-1}(B_R).$$

By the Lions-Aubin lemma and combining with the regularity, one arrives at

$$\rho_\varepsilon \text{ is compact in } L^2(0,T;L^2(B_R)).$$

Consequently, there exists a subsequence ρ_ε without relabeling such that

$$\rho_\varepsilon \to \rho \text{ in } L^2(0,T;L^2(B_R)) \text{ as } \varepsilon \to 0. \tag{4.1}$$

By the uniform estimates in Lemma 2.2, the regularity of ρ follows:
i) $\rho \in L^\infty\big(0,T;L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d,(1+|x|)dx)\big)$,
ii) $\rho \in L^2\big(0,T;H^1(\mathbb{R}^d)\big)$ and $\partial_t\rho \in L^2\big(0,T;H^{-1}(\mathbb{R}^d)\big)$.
Now, we prove that $\rho(t,x)$ is exactly a weak solution to (1.3). For any test function $\varphi(x) \in C_0^\infty(\mathbb{R}^d)$, it satisfies the following equation

$$\begin{aligned}&\int_{\mathbb{R}^d} \rho_\varepsilon(t,\cdot)\varphi(x)dx - \int_{\mathbb{R}^d} \rho_0\varphi(x)dx - \nu\int_0^t\int_{\mathbb{R}^d} \rho_\varepsilon(t,x)\Delta\varphi dxds \\ &= \int_0^t\int_{\mathbb{R}^d} \rho_\varepsilon(t,x)\Big(\int_{\mathbb{R}^d} F_\varepsilon(x-y)\rho_\varepsilon(t,y)dy\Big)\cdot\nabla\varphi(x)dxds.\end{aligned} \tag{4.2}$$

By the weak convergence of $\rho_\varepsilon(t,x)$, the linear parts converge as follows

$$\int_{\mathbb{R}^d} \varphi(x)\rho_\varepsilon(t,x)dx \to \int_{\mathbb{R}^d} \varphi(x)\rho(t,x)dx, \text{ as } \varepsilon \to 0. \tag{4.3}$$

$$\int_0^t\int_{\mathbb{R}^d} \rho_\varepsilon(t,x)\Delta\varphi dxds \to \int_0^t\int_{\mathbb{R}^d} \rho(t,x)\Delta\varphi dxds, \text{ as } \varepsilon \to 0. \tag{4.4}$$

The nonlinear part is divided as follows

$$\begin{aligned}&\Big|\int_{\mathbb{R}^d} \Big[\rho_\varepsilon(t,x)\int_{\mathbb{R}^d} F_\varepsilon(x-y)\rho_\varepsilon(t,y)dy - \rho(t,x)\int_{\mathbb{R}^d} F(x-y)\rho(t,y)dy\Big]\cdot\nabla\varphi(x)dx\Big| \\ &\le \Big|\int_{\mathbb{R}^d} \rho_\varepsilon(t,x)\Big[\int_{\mathbb{R}^d} F_\varepsilon(x-y)\rho_\varepsilon(t,y)dy - \int_{\mathbb{R}^d} F(x-y)\rho(t,y)dy\Big]\cdot\nabla\varphi(x)dx\Big| \\ &\quad + \Big|\int_{\mathbb{R}^d} \big[\rho_\varepsilon(t,x)-\rho(t,x)\big]\int_{\mathbb{R}^d} F(x-y)\rho(t,y)dy\cdot\nabla\varphi(x)dx\Big|.\end{aligned} \tag{4.5}$$

By Lemma 2.2, $V(t,x) = \int_{\mathbb{R}^d} |F(x-y)\rho(t,y)|dy \le C(\rho_0)$ for any $x \in \mathbb{R}^d$ and $V(t,x)$ is continuous in space. Hence $V(t,x)\cdot\nabla\varphi(x) \in C_0(\mathbb{R}^d)$, and then the second term goes to zero, i.e.

$$\Big|\int_{\mathbb{R}^d} \big[\rho_\varepsilon(t,x)-\rho(t,x)\big]\int_{\mathbb{R}^d} F(x-y)\rho(t,y)dy\cdot\nabla\varphi(x)dx\Big| \to 0 \text{ as } \varepsilon \to 0. \tag{4.6}$$

With the strong convergence (4.1), the first term is estimated by

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^d}\rho_\varepsilon(t,x)\Big[\int_{\mathbb{R}^d}F_\varepsilon(x-y)\rho_\varepsilon(t,y)dy-\int_{\mathbb{R}^d}F(x-y)\rho(t,y)dy\Big]\cdot\nabla\varphi(x)dx\Big|\\
&\le C\int_{\mathbb{R}^d}\rho_\varepsilon(t,x)\Big|\int_{\mathbb{R}^d}\big[F_\varepsilon(x-y)\rho_\varepsilon(t,y)-F(x-y)\rho_\varepsilon(t,y)\big]dy\Big|dx\\
&\quad+\Big|\int_{\mathbb{R}^d}\rho_\varepsilon(t,x)\int_{\mathbb{R}^d}\big[F(x-y)\rho_\varepsilon(t,y)-F(x-y)\rho(t,y)\big]dy\cdot\nabla\varphi(x)dx\Big|\\
&\le C(\rho_0)\varepsilon+\Big|\int_{\mathbb{R}^d}\big[\rho(t,y)-\rho_\varepsilon(t,y)\big]\int_{\mathbb{R}^d}\rho_\varepsilon(t,x)F(x-y)\cdot\nabla\varphi(x)dxdy\Big|\\
&\le C(\rho_0)\varepsilon+\Big|\int_{|y|\le R}\big[\rho(t,y)-\rho_\varepsilon(t,y)\big]\int_{\mathbb{R}^d}\rho_\varepsilon(t,x)F(x-y)\cdot\nabla\varphi(x)dxdy\Big|\\
&\quad+\Big|\int_{|y|>R}\big[\rho(t,y)-\rho_\varepsilon(t,y)\big]\int_{\mathbb{R}^d}\rho_\varepsilon(t,x)F(x-y)\cdot\nabla\varphi(x)dxdy\Big|.
\end{aligned}\tag{4.7}
$$

Since $\int_{\mathbb{R}^d}|\rho_\varepsilon(t,x)F(x-y)\cdot\nabla\varphi(x)|dx\le C$, combining with (4.1), one has

$$
\begin{aligned}
&\int_0^t\Big|\int_{|y|\le R}\big[\rho(t,y)-\rho_\varepsilon(t,y)\big]\int_{\mathbb{R}^d}\rho_\varepsilon(t,x)F(x-y)\\
&\qquad\qquad\cdot\nabla\varphi(x)dxdy\Big|ds\to 0 \text{ as } \varepsilon\to 0.
\end{aligned}\tag{4.8}
$$

Since $\rho(t,y),\rho_\varepsilon(t,y)\in L^1(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d}|\rho_\varepsilon(t,x)F\big(x-y\big)\cdot\nabla\varphi(x)|dx\le C$ uniformly with $\varepsilon\ge 0$,

$$
\begin{aligned}
&\int_0^t\Big|\int_{|y|>R}\big[\rho(t,y)-\rho_\varepsilon(t,y)\big]\int_{\mathbb{R}^d}\rho_\varepsilon(t,x)F(x-y)\\
&\qquad\qquad\cdot\nabla\varphi(x)dxdy\Big|ds\to 0 \text{ as } R\to\infty.
\end{aligned}\tag{4.9}
$$

Combining (4.7), (4.8) and (4.9) leads to

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^d}\rho_\varepsilon(t,x)\Big[\int_{\mathbb{R}^d}F_\varepsilon(x-y)\rho_\varepsilon(t,y)dy\Big]\cdot\nabla\varphi(x)dxds\\
&\to\int_0^t\int_{\mathbb{R}^d}\rho(t,x)\Big[\int_{\mathbb{R}^d}F(x-y)\rho(t,y)dy\Big]\cdot\nabla\varphi(x)dxds \text{ as } \varepsilon\to 0.
\end{aligned}\tag{4.10}
$$

Owing to (4.3), (4.4) and (4.10), passing to the limit $\varepsilon\to 0$ in (4.2), we obtain that for any $t\in(0,T]$

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\rho(t,\cdot)\varphi(x)dx-\int_{\mathbb{R}^d}\rho_0\varphi(x)dx-\nu\int_0^t\int_{\mathbb{R}^d}\rho(t,x)\Delta\varphi dxds\\
&=\int_0^t\int_{\mathbb{R}^d}\rho(t,x)\Big(\int_{\mathbb{R}^d}F(x-y)\rho(t,y)dy\Big)\cdot\nabla\varphi(x)dxds,
\end{aligned}
$$

which finishes the proof.

Proof of Theorem 2.2. First, multiplying (2.18) with $p\rho_\varepsilon^{p-1}$, $p\ge 2$, one has

$$
\frac{d}{dt}\int_{\mathbb{R}^d}\rho_\varepsilon^p dx+\frac{4(p-1)\nu}{p}\int_{\mathbb{R}^d}\big|\nabla(\rho_\varepsilon^{\frac{p}{2}})\big|^2dx\le(p-1)\int_{\mathbb{R}^d}\rho_\varepsilon^{p+1}dx.\tag{4.11}
$$

For $d \geq 3$, using the Gagliardo-Nirenberg-Sobolev inequality (see [7], [14] and [17]), for any $p \geq \frac{d}{2} - 1$, one has

$$(p-1)\int_{\mathbb{R}^d} {\rho_\varepsilon}^{p+1} dx \leq (p-1){S_d}^{-1}\int_{\mathbb{R}^d} \left|\nabla({\rho_\varepsilon}^{\frac{p}{2}})\right|^2 dx \Big(\int_{\mathbb{R}^d} {\rho_\varepsilon}^{\frac{d}{2}} dx\Big)^{\frac{2}{d}},$$

where $S_d = \frac{d(d-2)}{4} 2^{2/d}\pi^{1+1/d}\Gamma\left(\frac{d+1}{2}\right)^{-2/d}$. Then taking $p = \frac{d}{2}$ in (4.11), one has

$$\frac{d}{dt}\int_{\mathbb{R}^d} \rho_\varepsilon^{\frac{d}{2}} dx \leq (\frac{d}{2}-1)\int_{\mathbb{R}^d} \left|\nabla(\rho_\varepsilon^{\frac{d}{4}})\right|^2 dx \Big({S_d}^{-1}\|\rho_\varepsilon\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} - \frac{8\nu}{d}\Big),$$

which means that if $\|\rho_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} < \frac{8\nu S_d}{d}$, we have

$$\|\rho_\varepsilon\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} < \|\rho_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} < \frac{8\nu S_d}{d}.$$

In [2, Theorem 4.2.], the uniform bound of $\|\rho_\varepsilon\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}$ has been proven under the above condition.

For $d = 2$, under the sharp condition of $\int_{\mathbb{R}^2} \rho_0 dx < 8\pi\nu$, the estimate of $\|\rho_\varepsilon\|_{L^\infty(0,T;L^3(\mathbb{R}^2))}$ was given by [3, 15] with a bounded second moment. Similarly, with a bound of the first moment, there exists a constant C_T (depending only on T) such that

$$\sup_{t\in[0,T]} \|\rho_\varepsilon\|_{L^3}^3 \leq C_T. \tag{4.12}$$

Then, we establish the uniform bound of $\|\rho_\varepsilon\|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))}$ under this sharp condition and divide the proof into two steps by using an iteration method.

Step 1 Define $q_k = 2^k + 2$, $k \geq 0$. We have already obtained the uniform estimate for ρ_ε in $L^{q_0}(\mathbb{R}^2)$ with $q_0 = 3$ in (4.12). Now taking $p = q_k$, $k = 1, 2, \cdots$, in (4.11), the inequality becomes

$$\frac{d}{dt}\int_{\mathbb{R}^2} \rho_\varepsilon^{q_k} dx \leq -\frac{4(q_k-1)\nu}{q_k}\int_{\mathbb{R}^2} \left|\nabla(\rho_\varepsilon^{\frac{q_k}{2}})\right|^2 dx + (q_k-1)\int_{\mathbb{R}^2} \rho_\varepsilon^{q_k+1} dx. \tag{4.13}$$

In this step, we derive the following inequality:

$$\frac{d}{dt}\int_{\mathbb{R}^2} \rho_\varepsilon^{q_k} dx \leq -\int_{\mathbb{R}^2} \rho_\varepsilon^{q_k} dx + C2^{2k}\Big\{\big(\int_{\mathbb{R}^2} \rho_\varepsilon^{q_{k-1}} dx\big)^\gamma + \big(\int_{\mathbb{R}^2} \rho_\varepsilon^{q_{k-1}} dx\big)^2\Big\}, \tag{4.14}$$

where $\gamma = \frac{q_k}{q_{k-1}} \leq 2$.

Ladyzhenskaya's inequality reads

$$\int_{\mathbb{R}^2} \rho_\varepsilon^{2q_k} dx \leq 2\big(\int_{\mathbb{R}^2} \rho_\varepsilon^{q_k} dx\big)\big(\int_{\mathbb{R}^2} \left|\nabla(\rho_\varepsilon^{\frac{q_k}{2}})\right|^2 dx\big). \tag{4.15}$$

Using the interpolation inequality yields

$$\int_{\mathbb{R}^2} \rho_\varepsilon^{q_k+1} dx \leq \big(\int_{\mathbb{R}^2} \rho_\varepsilon^{q_{k-1}} dx\big)^{\frac{(q_k+1)(1-\theta)}{q_{k-1}}} \big(\int_{\mathbb{R}^2} \rho_\varepsilon^{2q_k} dx\big)^{\frac{(q_k+1)\theta}{2q_k}}, \tag{4.16}$$

$$\int_{\mathbb{R}^2} \rho_\varepsilon^{q_k} dx \leq \big(\int_{\mathbb{R}^2} \rho_\varepsilon^{q_{k-1}} dx\big)^{\frac{q_k(1-\beta)}{q_{k-1}}} \big(\int_{\mathbb{R}^2} \rho_\varepsilon^{2q_k} dx\big)^{\frac{\beta}{2}}, \tag{4.17}$$

where $\theta=\frac{2q_k(q_k+1-q_{k-1})}{(q_k+1)(2q_k-q_{k-1})}$, $\beta=\frac{2(q_k-q_{k-1})}{2q_k-q_{k-1}}$.
Plugging (4.17) into (4.15), one has

$$\int_{\mathbb{R}^2}\rho_\varepsilon^{2q_k}dx\le 2^{\frac{2}{2-\beta}}\Big(\int_{\mathbb{R}^2}\rho_\varepsilon^{q_{k-1}}dx\Big)^{\frac{2q_k(1-\beta)}{q_{k-1}(2-\beta)}}\Big(\int_{\mathbb{R}^2}\big|\nabla(\rho_\varepsilon^{\frac{q_k}{2}})\big|^2dx\Big)^{\frac{2}{2-\beta}}. \tag{4.18}$$

Plugging (4.18) into (4.16), one has

$$\begin{aligned}\int_{\mathbb{R}^2}\rho_\varepsilon^{q_k+1}dx&\le 2^{\frac{(q_k+1)\theta}{q_k(2-\beta)}}\Big(\int_{\mathbb{R}^2}\rho_\varepsilon^{q_{k-1}}dx\Big)^{\frac{(q_k+1)(2-\theta-\beta)}{q_{k-1}(2-\beta)}}\Big(\int_{\mathbb{R}^2}\big|\nabla(\rho_\varepsilon^{\frac{q_k}{2}})\big|^2dx\Big)^{\frac{(q_k+1)\theta}{q_k(2-\beta)}}\\&\le 4\Big(\int_{\mathbb{R}^2}\rho_\varepsilon^{q_{k-1}}dx\Big)\Big(\int_{\mathbb{R}^2}\big|\nabla(\rho_\varepsilon^{\frac{q_k}{2}})\big|^2dx\Big)^{\frac12}\\&\le 2a^{-1}\Big(\int_{\mathbb{R}^2}\rho_\varepsilon^{q_{k-1}}dx\Big)^2+2a\int_{\mathbb{R}^2}\big|\nabla(\rho_\varepsilon^{\frac{q_k}{2}})\big|^2dx,\end{aligned} \tag{4.19}$$

for any $a>0$. Let $a=\frac{\nu}{q_k}$ and take (4.19) into (4.13),

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2}\rho_\varepsilon^{q_k}dx&\le-\frac{2(q_k-1)\nu}{q_k}\int_{\mathbb{R}^2}\big|\nabla(\rho_\varepsilon^{\frac{q_k}{2}})\big|^2dx+\frac{2{q_k}^2}{\nu}\Big(\int_{\mathbb{R}^2}\rho_\varepsilon^{q_{k-1}}dx\Big)^2\\&\le-\nu\int_{\mathbb{R}^2}\big|\nabla(\rho_\varepsilon^{\frac{q_k}{2}})\big|^2dx+\frac{2{q_k}^2}{\nu}\Big(\int_{\mathbb{R}^2}\rho_\varepsilon^{q_{k-1}}dx\Big)^2.\end{aligned} \tag{4.20}$$

Plugging (4.18) into (4.17) induces that

$$\begin{aligned}\int_{\mathbb{R}^2}\rho_\varepsilon^{q_k}dx&\le 2^{\frac{\beta}{2-\beta}}\Big(\int_{\mathbb{R}^2}\rho_\varepsilon^{q_{k-1}}dx\Big)^{\frac{2q_k(1-\beta)}{q_{k-1}(2-\beta)}}\Big(\int_{\mathbb{R}^2}\big|\nabla(\rho_\varepsilon^{\frac{q_k}{2}})\big|^2dx\Big)^{\frac{\beta}{2-\beta}}\\&\le\delta' b^{-\frac{1}{\delta'}}\Big(2\int_{\mathbb{R}^2}\rho_\varepsilon^{q_{k-1}}dx\Big)^{\frac{2q_k(1-\beta)}{q_{k-1}(2-\beta)\delta'}}+\delta b^{\frac1\delta}\Big(\int_{\mathbb{R}^2}\big|\nabla(\rho_\varepsilon^{\frac{q_k}{2}})\big|^2dx\Big)^{\frac{\beta}{(2-\beta)\delta}},\end{aligned}$$

for any $b>0$, where $\delta+\delta'=1$.

Let $\delta b^{\frac1\delta}=\nu$ and $\delta=\frac{\beta}{2-\beta}=\frac{2^{k-1}}{2^k+2}\in[\frac18,\frac12]$, the above inequality becomes

$$\begin{aligned}\int_{\mathbb{R}^2}\rho_\varepsilon^{q_k}dx&\le(1-\delta)\Big(\frac{\nu}{\delta}\Big)^{-\frac{\delta}{1-\delta}}\Big(2\int_{\mathbb{R}^2}\rho_\varepsilon^{q_{k-1}}dx\Big)^{\gamma}+\nu\int_{\mathbb{R}^2}\big|\nabla(\rho_\varepsilon^{\frac{q_k}{2}})\big|^2dx\\&\le\lambda\Big(\int_{\mathbb{R}^2}\rho_\varepsilon^{q_{k-1}}dx\Big)^{\gamma}+\nu\int_{\mathbb{R}^2}\big|\nabla(\rho_\varepsilon^{\frac{q_k}{2}})\big|^2dx,\end{aligned} \tag{4.21}$$

where $\gamma=\frac{2q_k(1-\beta)}{q_{k-1}(2-\beta)(1-\delta)}=\frac{q_k}{q_{k-1}}\le 2$ and $\lambda=2\max\{(1-\delta)(\frac{\nu}{\delta})^{-\frac{\delta}{1-\delta}}\}$.
Plugging (4.21) into (4.20), we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2}\rho_\varepsilon^{q_k}dx&\le-\int_{\mathbb{R}^2}\rho_\varepsilon^{q_k}dx+\lambda\Big(\int_{\mathbb{R}^2}\rho_\varepsilon^{q_{k-1}}dx\Big)^{\gamma}+\frac{2{q_k}^2}{\nu}\Big(\int_{\mathbb{R}^2}\rho_\varepsilon^{q_{k-1}}dx\Big)^2\\&\le-\int_{\mathbb{R}^2}\rho_\varepsilon^{q_k}dx+C2^{2k}\Big\{\Big(\int_{\mathbb{R}^2}\rho_\varepsilon^{q_{k-1}}dx\Big)^{\gamma}+\Big(\int_{\mathbb{R}^2}\rho_\varepsilon^{q_{k-1}}dx\Big)^2\Big\},\end{aligned}$$

which gives the inequality (4.14).

Step 2 Denote $x_k(t):=\int_{\mathbb{R}^2}\rho_\varepsilon^{q_k}dx$, the inequality (4.14) can be recast as an ordinary differential inequality problem:

$$x_k'(t)\le-x_k(t)+C2^{2k}\Big(x_{k-1}^{\gamma}(t)+x_{k-1}^2(t)\Big),$$

where $0 < \gamma \le 2$ and C is a constant independent of k. From the above inequality, one has

$$\begin{aligned} x_k(t) &\le x_k(0)e^{-t} + C2^{2k} \max\big\{1, \sup_{t\in[0,T]} x_{k-1}^2(t)\big\}(1-e^{-t}) \\ &\le C2^{2k} \max\big\{1, \sup_{t\in[0,T]} x_{k-1}^2(t), x_k(0)\big\}. \end{aligned} \tag{4.22}$$

Plugging $x_k(0) = \int_{\mathbb{R}^2} \rho_0^{q_k} dx \le \|\rho_0\|_{L^1} \|\rho_0\|_{L^\infty}^{q_k-1} \le D^{q_k}$, $D = \max\big\{1, \|\rho_0\|_{L^1(\mathbb{R}^2)}, \|\rho_0\|_{L^\infty(\mathbb{R}^2)}\big\}$, into (4.22), we achieve

$$\begin{aligned} x_k(t) &\le C2^{2k} \max\big\{\sup_{t\in[0,T]} x_{k-1}^2(t), D^{2^k+2}\big\} \\ &\le C2^{2k}\big(C2^{2(k-1)}\big)^2\big(C2^{2(k-2)}\big)^{2^2}\cdots\big(C2^2\big)^{2^{k-1}} \max\big\{\sup_{t\in[0,T]} x_0^{2^k}(t), D^{2^{k+1}}\big\} \\ &= C^{2^k-1}2^{2^{k+2}-4k-4} \max\big\{\sup_{t\in[0,T]} x_0^{2^k}(t), D^{2^{k+1}}\big\}. \end{aligned} \tag{4.23}$$

Take the power $\frac{1}{2^k+2}$ to both sides of (4.23), since $\|\rho_\varepsilon\|_{L^{2^k+2}(\mathbb{R}^2)} = x_k^{\frac{1}{2^k+2}}(t)$, then the estimate is obtained by passing to the limit $k \to \infty$,

$$\|\rho_\varepsilon\|_{L^\infty(0,T;\ L^\infty(\mathbb{R}^2))} \le C \max\big\{\sup_{t\in[0,T]} x_0(t), D^2\big\}. \tag{4.24}$$

Since $x_0(t) = \int_{\mathbb{R}^2} \rho_\varepsilon^3 dx$, combining (4.12) and (4.24) together, we finish the proof of the uniform estimate of $\|\rho_\varepsilon\|_{L^\infty(0,T;\ L^\infty(\mathbb{R}^2))}$.

Now we prove (ii). Taking $p = 2$ in (4.11), one has

$$\frac{d}{dt}\int_{\mathbb{R}^d} \rho_\varepsilon^2 dx + 2\nu \int_{\mathbb{R}^d} \big|\nabla\rho_\varepsilon\big|^2 dx \le \int_{\mathbb{R}^d} \rho_\varepsilon^3 dx \le C. \tag{4.25}$$

Integrating (4.25) from 0 to T yields

$$\int_{\mathbb{R}^d} \rho_\varepsilon^2(T,x)dx + 2\nu \int_0^T \int_{\mathbb{R}^d} \big|\nabla\rho_\varepsilon\big|^2 dxdt \le CT + \int_{\mathbb{R}^d} \rho_0^2(x)dx,$$

then $\int_0^T \int_{\mathbb{R}^d} \big|\nabla\rho_\varepsilon\big|^2 dxdt \le CT + \int_{\mathbb{R}^d} \rho_0^2(x)dx$, i.e.

$$\int_0^T \|\nabla\rho_\varepsilon\|_{L^2(\mathbb{R}^d)}^2 dt \le CT + \int_{\mathbb{R}^d} \rho_0^2(x)dx. \tag{4.26}$$

Multiplying (2.18) with a test function $\varphi(x) \in C_0^\infty(\mathbb{R}^d)$, then integrating in space, one has

$$\int_{\mathbb{R}^d} \partial_t\rho_\varepsilon\varphi dx = \int_{\mathbb{R}^d} \Big(\nabla c_\varepsilon \rho_\varepsilon - \nu\nabla\rho_\varepsilon\Big)\nabla\varphi dx. \tag{4.27}$$

From Lemma 2.2, $|\nabla c_\varepsilon| \le \int_{\mathbb{R}^d} \big|\rho_\varepsilon(t,y)F_\varepsilon(x-y)\big|dy \le C$, it follows

$$\Big|\int_{\mathbb{R}^d} \partial_t\rho_\varepsilon\varphi dx\Big| \le \|\nabla\varphi\|_{L^2}\big(C + \nu\|\nabla\rho_\varepsilon\|_{L^2(\mathbb{R}^d)}\big). \tag{4.28}$$

Hence this directly derives that

$$\begin{aligned} \|\partial_t \rho_\varepsilon\|_{H^{-1}(\mathbb{R}^d)} &\le \sup_{\varphi \in C_0^\infty(\mathbb{R}^d)} \frac{\left| \int_{\mathbb{R}^d} \partial_t \rho_\varepsilon \varphi dx \right|}{\|\varphi\|_{H^1(\mathbb{R}^d)}} \\ &\le C + \nu \|\nabla \rho_\varepsilon\|_{L^2(\mathbb{R}^d)} \text{ for any } t \in [0,T]. \end{aligned} \tag{4.29}$$

By (4.26), integrating (4.29) in time space, one achieves

$$\int_0^T \|\partial_t \rho_\varepsilon\|_{H^{-1}(\mathbb{R}^d)}^2 dt \le 2CT + 2\nu \int_0^T \|\nabla \rho_\varepsilon\|_{L^2(\mathbb{R}^d)}^2 dt, \tag{4.30}$$

thus the proof of Theorem 2.2 is completed. □

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