METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 26, No. 4, pp. 349–370, December 2019
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REALIZATIONS OF THE HOMOGENEOUS BESOV-TYPE SPACES*

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Abstract. Using the notion of realizations, we study the dilation commuting realizations of the homogeneous Besov-type spaces $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$, which are defined modulo polynomials of degree less than μ; the integer μ will be determined from the parameters n, s, p, q and τ.

Key words. Littlewood-Paley decomposition, distributions modulo polynomials, homogeneous Besov-type spaces.

Mathematics Subject Classification. 46E35.

1. Introduction and the main results. The homogeneous Besov-type spaces $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ are quasi-Banach spaces defined by functions in the space $\mathcal{S}'_\infty(\mathbb{R}^n)$ of tempered distributions modulo polynomials on $\mathbb{R}^n$. After $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ the inhomogeneous Besov-type spaces introduced by El Baraka in [9, 10], the homogeneous counterparts have been investigated in several papers, for instance by Liang et al. [12], Sawano et al. [16], Wu et al. [20], Yang et al. [22, 23, 24] and Yuan et al. [26].

Using the realizations, these spaces can be given in the space $\mathcal{S}'_\mu(\mathbb{R}^n)$ of tempered distributions modulo polynomials of degree less than μ, where the value of the positive integer μ is completely determined from the parameters n, s, p, q and τ.

The notion of realization of homogeneous spaces has been introduced by G. Bourdaud [3] in the case of the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$. Now, there are many papers in this subject for other function spaces than $\dot{B}^s_{p,q}(\mathbb{R}^n)$, as homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$ and homogeneous Sobolev spaces $\dot{W}^m_p(\mathbb{R}^n)$, see e.g., [5, 6, 14, 25]. There are also various works related to the realizations of certain homogeneous spaces as in e.g., Navier-Stokes, pseudodifferential operators, pointwise multipliers and wavelets, see e.g., [2, 8, 13, 19].

Our main result consists of the realization of $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$, where for this purpose, we first need to define the spaces of tempered distributions modulo polynomials, and for brevity, as all function spaces occurring in this work are defined on Euclidean space $\mathbb{R}^n$, we omit $\mathbb{R}^n$ in notations throughout the paper.

– $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the set of the integers, and $\mathbb{R}$ the set of the real numbers.

– For $s \in \mathbb{R}$, $[s]$ denotes the greatest integer less than or equal to s. For $x \in \mathbb{R}^n$, $E(x)$ denotes the vector $([x_1], \dots, [x_n]) \in \mathbb{Z}^n$. For $a \in \mathbb{R}$ we put $a_+ := \max(0, a)$. For $\alpha \in \mathbb{N}_0^n$ multi-indice we write $\frac{\partial^{|\alpha|} f}{\partial x^\alpha} = \partial^\alpha f = f^{(\alpha)}$ with $|\alpha| := \alpha_1 + \ldots + \alpha_n$.

– For $k \in \mathbb{N}$ we denote by $\mathcal{P}_k$ the set of all polynomials on $\mathbb{R}^n$ of degree less than k, we put $\mathcal{P}_0 = \{0\}$ and denote by $\mathcal{P}_\infty$ to the set of all polynomials on $\mathbb{R}^n$.

– For $k \in \mathbb{N}_0 \cup \{\infty\}$, the symbol $\mathcal{S}_k$ will be used for the set of functions φ in the Schwartz class $\mathcal{S}$ such that $\langle u, \varphi \rangle = 0$ for all $u \in \mathcal{P}_k$ (i.e., $\widehat{\varphi}^{(\alpha)}(0) = 0$ for all $|\alpha| < k$). The topological dual space of $\mathcal{S}_k$ is denoted by $\mathcal{S}'_k$. For all $f \in \mathcal{S}'$, we denote by $[f]_k$ the equivalence class of f modulo $\mathcal{P}_k$. The mapping which takes any $[f]_k$ to the

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restriction of f to $\mathcal{S}_k$ turns out to be an isomorphism from $\mathcal{S}'/\mathcal{P}_k$ onto $\mathcal{S}'_k$. For this reason, $\mathcal{S}'_k$ is called the space of *tempered distributions modulo* $\mathcal{P}_k$.

We second need to recall the notion of *realization* from the different works of G. Bourdaud [3]–[6], then we define the realized space.

DEFINITION 1. *Let $m \in \mathbb{N}_0 \cup \{\infty\}$ and $k \in \{0, \dots, m\}$. Let E be a vector subspace of $\mathcal{S}'_m$ endowed with a quasi-norm such that the continuous embedding $E \hookrightarrow \mathcal{S}'_m$ holds. A realization of E in $\mathcal{S}'_k$ is a continuous linear mapping $\sigma : E \to \mathcal{S}'_k$ such that $[\sigma(f)]_m = f$ for all $f \in E$. The image set $\sigma(E)$ is called the realized space of E with respect to σ.*

REMARK 1. A realization is entirely determined by the corresponding realized space, since for any $f \in E$, the element $\sigma(f)$ is *the unique representative* of f such that $\sigma(f) \in \sigma(E)$. We also note that if we require extra properties such that *translation or dilation invariance*, a realization of E in $\mathcal{S}_k$ for $k < m$ has some chances to be unique, in the case $k = m$ the identity is the unique realization, cf., see [6, props. 2.2, 2.4].

The set of realizations has a phenomenon of generation, in the sense that if a realization is known then it generates other realizations. We recall the following assertion and refer to [3, prop. 2] for the proof.

PROPOSITION 1. *Let $\sigma_0 : E \to \mathcal{S}'_k$ be a realization. For all finite family $(\mathcal{L}_\alpha)_{k \le |\alpha| \le m}$ of continuous linear functionals on E, the following formula defines a realization of E in $\mathcal{S}'_k$:*

$$\sigma(f)(x) := \sigma_0(f)(x) + \sum_{k \le |\alpha| \le m} \mathcal{L}_\alpha(f)\, x^\alpha .$$

Conversely, any realization of E is given in such a way.

On the other hand, the weak convergence of distributions to 0 at the infinity plays a major role in this work, we recall this fact.

DEFINITION 2. *A tempered distribution $f \in \mathcal{S}'$ tends to 0 at the infinity in the weak sense if $\lim_{\lambda \to 0} f(\lambda^{-1}(\cdot)) = 0$ in $\mathcal{S}'$. The set of all such distributions is denoted by $\widetilde{C}_0$.*

Here some examples of distributions vanish at the infinity in the weak sense: (i) functions in L_p for $1 \le p < \infty$, (ii) derivatives distributional of bounded functions, (iii) derivatives of the members of $\widetilde{C}_0$.

To give the main result we also need to introduce the Littlewood-Paley setting. We choose, once and for all, a standard cut-off function ρ. More precisely, we assume that ρ is a radial C^∞ function satisfying $0 \le \rho \le 1$, $\rho(\xi) = 1$ for $|\xi| \le 1$, $\rho(\xi) = 0$ for $|\xi| \ge 3/2$. We define $\gamma := \rho - \rho(2(\cdot))$. Then γ is supported by the compact annulus $1/2 \le |\xi| \le 3/2$ (the Tauberian condition), and the following identities hold:

$$\sum_{j \in \mathbb{Z}} \gamma(2^j \xi) = 1 \, (\forall \xi \in \mathbb{R}^n \setminus \{0\}), \quad \rho(2^{-k}\xi) + \sum_{j > k} \gamma(2^{-j}\xi) = 1 \, (\forall \xi \in \mathbb{R}^n, \forall k \in \mathbb{Z}).$$

We introduce the convolution operators $(Q_j)_{j \in \mathbb{Z}}$ by means of the following formula $\widehat{Q_j f} := \gamma(2^{-j}(\cdot))\widehat{f}$. It is clear that Q_j is defined on $\mathcal{S}'_\infty$ since $Q_j f = 0$ if, and only if, f is a polynomial on $\mathbb{R}^n$. In the following we say:

if $f \in \mathcal{S}'_\infty$ we set $Q_j f := Q_j f_1$ for all f_1 such that $[f_1]_\infty = f$.

The operators Q_j ($\forall j \in \mathbb{Z}$) take values in the space of analytical functions of exponential type, see the Paley-Wiener theorem e.g., in [11, thm. 1.7.7, p. 21]. Then we have the Littlewood-Paley decompositions of any tempered distribution or any smooth function: For every $f \in \mathcal{S}_\infty$ ($\mathcal{S}'_\infty$, respectively) it holds $f = \sum_{j\in\mathbb{Z}} Q_j f$ in $\mathcal{S}_\infty$ ($\mathcal{S}'_\infty$, respectively), and for every $f \in \mathcal{S}$ ($\mathcal{S}'$, respectively) and every $k \in \mathbb{Z}$ it holds $f = 2^{kn}\mathcal{F}^{-1}\rho(2^k(\cdot)) * f + \sum_{j>k} Q_j f$ in $\mathcal{S}$ ($\mathcal{S}'$, respectively).

To define the homogeneous Besov-type spaces, therefore we need some more notation (the dyadic cubes): For $k \in \mathbb{Z}$ and $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n$ we set

$$P_{k,\nu} := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \quad \nu_j \le 2^k x_j < \nu_j + 1,\ j = 1, \dots, n\}.$$

Definition 3. *Let $s, \tau \in \mathbb{R}$ and $p, q \in]0, +\infty]$. The homogeneous Besov-type space $\dot{B}^{s,\tau}_{p,q}$ is the set of $f \in \mathcal{S}'_\infty$ such that*

$$\|f\|_{\dot{B}^{s,\tau}_{p,q}} := \sup_{k\in\mathbb{Z}} \sup_{\nu\in\mathbb{Z}^n} 2^{kn\tau} \Big(\sum_{j\ge k} \Big(2^{sj} \|Q_j f\|_{L_p(P_{k,\nu})} \Big)^q \Big)^{1/q} < \infty. \tag{1}$$

As it is announced in the beginning, we turn to define the positive integer μ where its value will be justified and fixed throughout this paper: To any 5-tuple (n, s, p, q, τ) of real numbers we associate an integer $\mu \in \mathbb{N}_0$ defined as

$$\mu := \begin{cases} \big([s + n\tau - n/p] + 1\big)_+ & \text{if either} \quad s + n\tau - n/p \notin \mathbb{N}_0 \quad \text{or} \quad q > 1, \\ s + n\tau - n/p & \text{if} \quad s + n\tau - n/p \in \mathbb{N}_0 \ \text{and} \ 0 < q \le 1. \end{cases} \tag{2}$$

Then our main result is the following statement.

Theorem 1. *Let $s, \tau \in \mathbb{R}$ and $p, q \in]0, +\infty]$. Let $f \in \dot{B}^{s,\tau}_{p,q}$. Then the series $\sum_{j\in\mathbb{Z}} Q_j f$ converges in $\mathcal{S}'_\mu$. Let us define $\sigma_\mu(f)$ as the its sum which belongs to $\mathcal{S}'_\mu$. Then the mapping $\sigma_\mu : \dot{B}^{s,\tau}_{p,q} \to \mathcal{S}'_\mu$ defined in such a way is a translation and dilation commuting realization of $\dot{B}^{s,\tau}_{p,q}$ into $\mathcal{S}'_\mu$, and $\sigma_\mu(f)$ is the unique representative of f satisfying $\partial^\alpha \sigma_\mu(f) \in \widetilde{C}_0$ for all $|\alpha| = \mu$.*

An immediate consequence of Theorem 1 is a characterization of *the realized space* of the homogeneous Besov-type space without reference to Littlewood-Paley decompositions.

Corollary 1. *Let σ_μ be the mapping defined in Theorem 1. Then the realized space $\sigma_\mu(\dot{B}^{s,\tau}_{p,q})$ coincides with the set of $f \in \mathcal{S}'_\mu$ such that $[f]_\mu \in \dot{B}^{s,\tau}_{p,q}$ and $f^{(\alpha)} \in \widetilde{C}_0$ for all $|\alpha| = \mu$; this set is denoted by $\dot{\widetilde{B}}^{s,\tau}_{p,q}$. The space $\dot{\widetilde{B}}^{s,\tau}_{p,q}$ is endowed with the quasi-seminorm $\|f\|_{\dot{\widetilde{B}}^{s,\tau}_{p,q}} := \|[f]_\mu\|_{\dot{B}^{s,\tau}_{p,q}}$.*

To prove Theorem 1, we will give in the first time an assertion which is a variant of the Nikol'skij representation method; see e.g., [15, p. 59], [18, p. 79], [21] for the case of inhomogeneous Besov spaces $B^s_{p,q}$. This assertion also presents *one of our contributions* in this work.

Theorem 2. *Let $s, \tau \in \mathbb{R}$ and $p, q \in]0, +\infty]$. Let a, b be real numbers such that $0 < a < b$. Let $(u_j)_{j\in\mathbb{Z}}$ be a sequence in $\mathcal{S}'$ such that*

- *$\widehat{u_j}$ is supported by the annulus $a2^j \le |\xi| \le b2^j$,*

- $A := \sup_{k\in\mathbb{Z}} \sup_{\nu\in\mathbb{Z}^n} 2^{kn\tau}\big(\sum_{j\geq k}\big(2^{sj}\|u_j\|_{L_p(P_{k,\nu})}\big)^q\big)^{1/q} < \infty$.

Then the series $\sum_{j\in\mathbb{Z}} u_j$ converges in $\mathcal{S}'_\mu$ to a limit u which belongs to $\mathcal{S}'_\mu$ and satisfies

$$\|[u]_\mu\|_{\dot{B}^{s,\tau}_{p,q}} \leq cA, \tag{3}$$

where c depends only on n, s, τ, p, q, a and b.

Remark 2. Since we work in $\mathcal{S}'_\mu$, Theorem 2 also holds if we replace, in its conclusion, the series $\sum_{j\in\mathbb{Z}} u_j$ by any series of type $\sum_{j\in\mathbb{Z}} u_j - v$ for every polynomial $v \in \mathcal{P}_\mu$; e.g., we will use in Section 4 below the particular cases $v(x) := \sum_{j\leq 0}\sum_{|\alpha|<\mu} u_j^{(\alpha)}(0)x^\alpha/\alpha!$ and $v(x) := \sum_{j\in\mathbb{Z}}\sum_{|\alpha|<\mu} u_j^{(\alpha)}(0)x^\alpha/\alpha!$. On the other hand, the value of the remarkable integer μ can be justified, this affirmation is due to G. Bourdaud in the case of the homogeneous Besov spaces $\dot{B}^s_{p,q}$ cf., [4, prop. 2.2.1] which is also taken for $\dot{B}^{s,\tau}_{p,q}$, see Proposition 4 and Remark 4 below.

The paper is organized as follows. In Section 2, we collect definitions and basic properties of $\dot{B}^{s,\tau}_{p,q}$. Section 3 is devoted to the proofs of main results where for technical reasons, we begin by proving Theorem 2 then Theorem 1. In Section 4, we discuss some remarks. In a final section (an appendix), we give some proofs.

Notations. We recall some usual notations. For all $k, N \in \mathbb{N}_0$ and all $f \in \mathcal{S}$, the standard seminorms are given by

$$\zeta_{N,k}(f) := \sup_{x\in\mathbb{R}^n} \sup_{|\alpha|\leq N} (1+|x|)^k\,|f^{(\alpha)}(x)|.$$

For a function $f \in L_1$ the Fourier transform and its inverse on $\mathbb{R}^n$ are defined by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\mathrm{d}x \quad \text{ and } \quad \mathcal{F}^{-1}f(x) := (2\pi)^{-n}\widehat{f}(-x).$$

The operators $\mathcal{F}$ and $\mathcal{F}^{-1}$ can be extended to the whole $\mathcal{S}'$ in the usual way. For $a \in \mathbb{R}^n$ and $\lambda > 0$, we denote by τ_a and h_λ the translation operator and the dilation operator, respectively, i.e., $\tau_a f := f(\cdot - a)$ and $h_\lambda f := f(\lambda^{-1}(\cdot))$. Constants c are strictly positives and depend only on the fixed parameters n, s, p, q and τ and probably on auxiliary functions, unless otherwise stated, its value may vary from line to line. Finally, we will use the symbol $\lesssim$; the notation $A \lesssim B$ means that $A \leq cB$.

2. The Besov-type spaces. In Definition 3 we have the following independence observation:

Remark 3. The spaces $\dot{B}^{s,\tau}_{p,q}$ are quasi-Banach independent of the choices of γ, i.e., if we take a function γ_1 (with the same properties of γ) positive, radial, C^∞ and supported by the annulus $a \leq |\xi| \leq b$ with $0 < 2a < b$ i.e., the Tauberian condition, and define $\|\cdot\|^{(\gamma_1)}_{\dot{B}^{s,\tau}_{p,q}}$ the resulting quasi-seminorm of $\dot{B}^{s,\tau}_{p,q}$ by replacing in (1) the quantity $Q_j f$ by $\big(2^{jn}h_{2^{-j}}\mathcal{F}^{-1}\gamma_1\big) * f$, then $\|\cdot\|^{(\gamma_1)}_{\dot{B}^{s,\tau}_{p,q}}$ is equivalent to (1), and there exist positive constants c_1, c_2 depending on $n, s, \tau, p, q,\ a, b,\ \mathcal{F}^{-1}\gamma$ and $\mathcal{F}^{-1}\gamma_1$ such that

$$c_1\|f\|_{\dot{B}^{s,\tau}_{p,q}} \leq \|f\|^{(\gamma_1)}_{\dot{B}^{s,\tau}_{p,q}} \leq c_2\|f\|_{\dot{B}^{s,\tau}_{p,q}} \qquad (\forall f \in \dot{B}^{s,\tau}_{p,q});$$

the proof of this fact can be found in [23, coro. 3.1].

We have also the chain of continuous embeddings $\mathcal{S}_\infty \hookrightarrow \dot{B}^{s,\tau}_{p,q} \hookrightarrow \mathcal{S}'_\infty$, see again [23, props. 3.1, 3.4], and $\|f\|_{\dot{B}^{s,\tau}_{p,q}} = 0$ if, and only if, f is a polynomial. Now, to explain why the spaces $\dot{B}^{s,\tau}_{p,q}$ are called homogeneous, we give the following assertion which will be proved in the final section:

PROPOSITION 2. *There exist two positive constants c_1 and c_2 such that the inequality*

$$c_1\|f\|_{\dot{B}^{s,\tau}_{p,q}} \le \lambda^{s+n\tau-n/p}\|h_\lambda f\|_{\dot{B}^{s,\tau}_{p,q}} \le c_2\|f\|_{\dot{B}^{s,\tau}_{p,q}} \tag{4}$$

holds for all $f \in \dot{B}^{s,\tau}_{p,q}$ and all $\lambda > 0$. In case $\lambda = 2^k$ ($k \in \mathbb{Z}$), the above inequality becomes an equality, i.e., $\|f\|_{\dot{B}^{s,\tau}_{p,q}} = 2^{k(s+n\tau-n/p)}\|h_{2^k} f\|_{\dot{B}^{s,\tau}_{p,q}}$.

In the following statement (that will be proved later on) there are some properties of $\dot{B}^{s,\tau}_{p,q}$ and a link with the "ordinary" homogeneous Besov spaces $\dot{B}^{s}_{p,q}$ which will be defined before:

DEFINITION 4. *Let $s \in \mathbb{R}$ and $p, q \in]0, +\infty]$. The homogeneous Besov space $\dot{B}^{s}_{p,q}$ is the set of $f \in \mathcal{S}'_\infty$ such that*

$$\|f\|_{\dot{B}^{s}_{p,q}} := \Big(\sum_{j\in\mathbb{Z}} \big(2^{sj}\|Q_j f\|_{L_p(\mathbb{R}^n)}\big)^q\Big)^{1/q} < \infty.$$

PROPOSITION 3. (i) *If $\tau = 0$, then $\dot{B}^{s,0}_{p,q} = \dot{B}^{s}_{p,q}$ holds in the sense of equivalent quasi-seminorms.*

(ii) *If $\tau < 0$, then the equality $\dot{B}^{s,\tau}_{p,q} = \mathcal{P}_\infty$ holds.*

(iii) *For all $f \in \dot{B}^{s,\tau}_{p,q}$, its first order derivatives $\partial_l f$ ($l = 1, \dots, n$) belong to $\dot{B}^{s-1,\tau}_{p,q}$ and $\|\partial_l f\|_{\dot{B}^{s-1,\tau}_{p,q}} \lesssim \|f\|_{\dot{B}^{s,\tau}_{p,q}}$.*

(iv) *The continuous embedding $\dot{B}^{s,\tau}_{p,q} \hookrightarrow \dot{B}^{s+n\tau-n/p}_{\infty,\infty}$ holds; this presents a link between $\dot{B}^{s}_{p,q}$ and $\dot{B}^{s,\tau}_{p,q}$.*

Noticing that the most properties of $\dot{B}^{s}_{p,q}$ (some of them are listed in above two propositions with the case of $\dot{B}^{s,\tau}_{p,q}$ and obtained by taking $\tau = 0$) can be found in [1, 7, 18].

3. Proofs of main results.

3.1. Preparation. We need the following two assertions; they also present another examples of functions in $\widetilde{C}_0$, where the first one is easy and the second is proved in [3] or [6, prop. 4.4].

LEMMA 1. *If a polynomial f belongs to $\widetilde{C}_0$, then $f = 0$, i.e., $\widetilde{C}_0 \cap \mathcal{P}_\infty = \{0\}$.*

LEMMA 2. *Any bounded function f, such that* $\operatorname{supp} \widehat{f}$ *is a compact set in $\mathbb{R}^n \setminus \{0\}$, belongs to $\widetilde{C}_0$.*

The proof of Theorem 2 is based on the following statement which is an estimate of functions in $\mathcal{S}$ and $\mathcal{S}_m$ and is proved in e.g., [14, prop. 2.5].

LEMMA 3. (i) *Let $k, m \in \mathbb{N}_0$. Then there exists a constant $c > 0$ such that*

$$|\varphi_j * f(x)| \le c2^{-jm}\zeta_{m,k}(f)\zeta_{m,k}(\varphi)\,(1 + |x|)^{-k}$$

for all $f \in \mathcal{S}$*, all* $\varphi \in \mathcal{S}_m$ *(with* $\varphi_j := 2^{nj} h_{2^{-j}} \varphi$*), all* $j \in \mathbb{N}_0$ *and all* $x \in \mathbb{R}^n$*.*

(ii) *Let* $k, m \in \mathbb{N}_0$*. Then there exists a constant* $c > 0$ *such that*

$$|\psi_j * f(x)| \leq c 2^{j(m+n)} \zeta_{m,k}(f) \zeta_{m,k}(\psi) \left(1 + 2^j |x|\right)^{-k}$$

for all $f \in \mathcal{S}_m$*, all* $\psi \in \mathcal{S}$ *(with* $\psi_j := 2^{nj} h_{2^{-j}} \psi$*), all* $j \in \mathbb{Z} \setminus \mathbb{N}$ *and all* $x \in \mathbb{R}^n$*.*

3.2. Proofs. We begin by the proof of Theorem 2 as mentioned in the Introduction.

Proof of Theorem 2. For the simplicity and clarity, we will subdivide the proof into several steps.

Step 1: convergence in $\mathcal{S}'_\mu$. We introduce a radial and positive function $\widetilde{\gamma} \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ such that $\widetilde{\gamma}(\xi) = 1$ if $a \leq |\xi| \leq b$, and we define $\widetilde{Q}_k := \widetilde{\gamma}(2^{-k} D)$; i.e., $\widehat{\widetilde{Q}_k f} = \widetilde{\gamma}(2^{-k}(\cdot)) \widehat{f}$ for all $k \in \mathbb{Z}$. We have $\widetilde{Q}_k u_k = u_k$.

Suppose for a moment that μ exists and is given by (2). Let $f \in \mathcal{S}_\mu$.

Substep 1.1: the case either $s + n\tau - n/p \notin \mathbb{N}_0$ *or* $q > 1$. Here $\mu := \big([s + n\tau - n/p] + 1\big)_+$, i.e., $\mu > s + n\tau - n/p$. The assumption on $\widetilde{\gamma}$ yields $\langle \widetilde{Q}_k u_k, f \rangle = \langle u_k, \widetilde{Q}_k f \rangle$. Then it follows

$$\sum_{k \in \mathbb{Z}} |\langle u_k, f \rangle| \leq \sum_{k \in \mathbb{Z}} \sum_{\nu \in \mathbb{Z}^n} \|u_k\|_{L_\infty(P_{k,\nu})} \|\widetilde{Q}_k f\|_{L_1(P_{k,\nu})}. \tag{5}$$

Now, we are going to estimate $\|u_k\|_{L_\infty(P_{k,\nu})}$, where we will prove:

$$\|u_k\|_{L_\infty(P_{k,\nu})} \lesssim 2^{k(n/p - s - n\tau)} A \qquad (\forall k \in \mathbb{Z},\ \forall \nu \in \mathbb{Z}^n). \tag{6}$$

We first observe that

$$\begin{aligned} \|u_k\|_{L_p(P_{k,\nu})} &\lesssim 2^{-ks} \Big(\sum_{j \geq k} 2^{sjq} \|u_j\|^q_{L_p(P_{k,\nu})} \Big)^{1/q} \\ &\lesssim 2^{-k(s+n\tau)} A \qquad (\forall k \in \mathbb{Z},\ \forall \nu \in \mathbb{Z}^n), \end{aligned} \tag{7}$$

and we continue by considering the cases $p \geq 1$ and $p < 1$ separately, for technical reasons.

- *The case* $p \geq 1$. By Hölder's inequality and (7) we have

$$\begin{aligned} |u_k(x)| = |\widetilde{Q}_k u_k(x)| &\lesssim 2^{kn} \sum_{\eta \in \mathbb{Z}^n} \int_{P_{k,\eta}} (2 + 2^k |x - y|)^{-n-1} |u_k(y)| \, \mathrm{d}y \\ &\lesssim 2^{kn} \sum_{\eta \in \mathbb{Z}^n} \|u_k\|_{L_p(P_{k,\eta})} \Big(\int_{P_{k,\eta}} (2 + 2^k |x-y|)^{-(n+1)p'} \, \mathrm{d}y \Big)^{1/p'} \quad (p' := p/(p-1)) \\ &\lesssim 2^{k(n-s-n\tau)} A \sum_{\eta \in \mathbb{Z}^n} \Big(\int_{P_{k,\eta}} (2 + 2^k |x-y|)^{-(n+1)p'} \, \mathrm{d}y \Big)^{1/p'}. \end{aligned} \tag{8}$$

For $x \in P_{k,\nu}$ and $y \in P_{k,\eta}$ we have $1 + |\nu - \eta| \leq \sqrt{n}(2 + 2^k |x - y|)$, then

$$\sum_{\eta \in \mathbb{Z}^n} \Big(\int_{P_{k,\eta}} (2 + 2^k |x-y|)^{-(n+1)p'} \, \mathrm{d}y \Big)^{1/p'} \lesssim 2^{-kn/p'} \sum_{\eta \in \mathbb{Z}^n} (1 + |\nu - \eta|)^{-n-1}.$$

By translation invariance with respect to the summation on η, the last inequality is bounded by $c\,2^{-kn/p'}$. Inserting this estimate into (8). By taking the supremum on $x \in P_{k,\nu}$ in the first term of (8), we then get (6).

- *The case* $0 < p < 1$. We write

$$|u_k(x)| = |\widetilde{Q}_k u_k(x)| \lesssim 2^{kn} \sum_{\eta\in\mathbb{Z}^n} \int_{P_{k,\eta}} (2 + 2^k|x-y|)^{-n-1} |u_k(y)|\,\mathrm{d}y$$

$$\lesssim 2^{kn} \sup_{\omega\in\mathbb{Z}^n} \|u_k\|^{1-p}_{L_\infty(P_{k,\omega})} \sum_{\eta\in\mathbb{Z}^n} \int_{P_{k,\eta}} (2 + 2^k|x-y|)^{-n-1} |u_k(y)|^p\,\mathrm{d}y. \quad (9)$$

As above, $x \in P_{k,\nu}$ and $y \in P_{k,\eta}$ imply $1 + |\nu-\eta| \le \sqrt{n}(2 + 2^k|x-y|)$, and by (7) we also have that the right-hand side of (9) (with $x \in P_{k,\nu}$) is bounded by

$$c2^{kn} 2^{-(s+n\tau)kp} A^p \sup_{\omega\in\mathbb{Z}^n} \|u_k\|^{1-p}_{L_\infty(P_{k,\omega})} \sum_{\eta\in\mathbb{Z}^n} (1 + |\nu-\eta|)^{-n-1}.$$

Again, by translation invariance with respect to the summation on η, we obtain

$$\|u_k\|_{L_\infty(P_{k,\nu})} \lesssim 2^{(n/p-s-n\tau)kp} A^p \sup_{\omega\in\mathbb{Z}^n} \|u_k\|^{1-p}_{L_\infty(P_{k,\omega})} \qquad (\forall k \in \mathbb{Z},\ \forall \nu \in \mathbb{Z}^n);$$

take the supremum over all $\nu \in \mathbb{Z}$ in the left-hand side, the estimate (6) holds again.

We now turn to (5). By Lemma 3 we have, since $f \in \mathcal{S}$ (recall that $f \in \mathcal{S}_\mu \subset \mathcal{S}$),

$$\|\widetilde{Q}_k f\|_{L_1(P_{k,\nu})} \lesssim 2^{-kN_1} \zeta_{N_1,n+1}(f) \int_{P_{k,\nu}} (1 + |y|)^{-n-1}\,\mathrm{d}y \qquad (\text{with} \quad k > 0) \quad (10)$$

for some positive integer N_1. Since $f \in \mathcal{S}_\mu$, and by the change of variables $x := 2^k y$ in $2^{kn} \int_{P_{k,\nu}} (1 + 2^k|y|)^{-n-1} \mathrm{d}y$, we get

$$\|\widetilde{Q}_k f\|_{L_1(P_{k,\nu})} \lesssim 2^{\mu k} \zeta_{\mu,n+1}(f) \int_{P_{0,\nu}} (1 + |x|)^{-n-1}\,\mathrm{d}x \qquad (\text{with} \quad k \le 0). \quad (11)$$

Now, inserting (6), (10) and (11) into (5), i.e.,

$$\sum_{k\in\mathbb{Z}} |\langle u_k, f\rangle| \le c(f) A \Big(\int_{\mathbb{R}^n} (1 + |x|)^{-n-1}\,\mathrm{d}x \Big) \sum_{k\in\mathbb{Z}} 2^{(n/p-s-n\tau)k} \min(2^{\mu k}, 2^{-kN_1}),$$

choosing N_1 such that $N_1 + s + n\tau - n/p > 0$ and using the fact that $\mu - s - n\tau + n/p > 0$, we obtain the desired conclusion.

Substep 1.2: the case $\mu := s + n\tau - n/p \in \mathbb{N}_0$ *and* $0 < q \le 1$. We first instead of (5) use the following splitting

$$\sum_{k\in\mathbb{Z}} |\langle u_k, f\rangle| \le \sum_{k>0} \sum_{\nu\in\mathbb{Z}^n} \|u_k\|_{L_\infty(P_{k,\nu})} \|\widetilde{Q}_k f\|_{L_1(P_{k,\nu})} + \sum_{k\le 0} |\langle u_k, f\rangle|\,.$$

Using (6) and (10), the estimate of the first part (i.e., $\sum_{k>0} \ldots$) can be done as in Substep 1.1, and will be omitted. Now, for the second one (i.e., $\sum_{k\le 0} \ldots$), we will distinguish between the cases $p \geq 1$ and $p < 1$.

- *The case 1.2.1:* $p \geq 1$. We first use the Hölder inequality (recall $p' := p/(p-1)$)

$$\sum_{k\leq 0} |\langle u_k, f\rangle| \leq \sum_{k\leq 0} \sum_{\nu\in\mathbb{Z}^n} \|u_k\|_{L_p(P_{k,\nu})} \|\widetilde{Q}_k f\|_{L_{p'}(P_{k,\nu})}. \tag{12}$$

Since the assumption $f \in \mathcal{S}_\mu$, we proceed as in (10)–(11) by taking $L_{p'}(P_{k,\nu})$ instead of $L_1(P_{k,\nu})$, we obtain on the one hand

$$\sum_{k\leq 0} \|u_k\|_{L_p(P_{k,\nu})} \|\widetilde{Q}_k f\|_{L_{p'}(P_{k,\nu})}$$
$$\leq c\zeta_{\mu,n+1}(f)\Big(\sum_{k\leq 0} 2^{(s+n\tau)k} \|u_k\|_{L_p(P_{k,\nu})}\Big)\Big(\int_{P_{0,\nu}} (1+|x|)^{-(n+1)p'}\,\mathrm{d}x\Big)^{1/p'}, \tag{13}$$

where the term in the last integral $\int_{P_{0,\nu}} \dots$ is obtained by a change of variables $x := 2^k y$, see again (11) and in particular the sentence just before this formula.

On the other hand, we will use the following property of the support of $\widehat{u}_k$: for all couples $(j,k) \in \mathbb{Z}^2$ it holds:

$$\text{if } |j-k| \geq \log_2(b/a) \text{ then } \{\xi : a2^j \leq |\xi| \leq b2^j\} \cap \{\xi : a2^k \leq |\xi| \leq b2^k\} = \emptyset. \tag{14}$$

Thus $\sum_{k\in\mathbb{Z}} \widehat{u_k}(\xi)$, for all $\xi \in \mathbb{R}^n \setminus \{0\}$, contains at most $2m+1$ non-vanishing terms (where m is the non-negative integer near of $\log_2(b/a)$) corresponding to the compact annulus $2^{k-l} \leq |\xi| \leq 2^{k+l}$ ($l = -m, \dots, m$). Consequently, $k \in \Lambda$ with $\operatorname{Card}\Lambda = 2m+1$ where $\operatorname{Card}\Lambda$ denotes the cardinal number of a finite set $\Lambda \subsetneqq \mathbb{Z}$.

We turn to (12). Then in its left-hand side, we have $\sum_{k\leq 0} \dots = \sum_{k\leq 0,\, k\in\Lambda} \dots$. The set Λ is constituted by *consecutive elements*, say $\Lambda := \{J, J+1, \dots, J+2m\}$ where $J < 0$. We continue, since

$$k \geq J \;\Rightarrow\; P_{k,\nu} \subset P_{J,E(2^{J-k}\nu)} \cup P_{J,E(2^{J-k}\nu)+w_0},$$

where $w_0 = (1,1,\dots,1) \in \mathbb{Z}^n$. Then

$$\|u_k\|_{L_p(P_{k,\nu})} \leq \|u_k\|_{L_p(P_{J,E(2^{J-k}\nu)})} + \|u_k\|_{L_p(P_{J,E(2^{J-k}\nu)+w_0})}, \tag{15}$$

and it holds

$$\sum_{k\leq 0} 2^{(s+n\tau)k} \|u_k\|_{L_p(P_{k,\nu})} = \sum_{k\leq 0,\, k\in\Lambda} 2^{(s+n\tau)k} \|u_k\|_{L_p(P_{k,\nu})}$$
$$\leq \max(2^{2mn\tau}, 1)\, 2^{Jn\tau} \sum_{k=J}^{J+2m} 2^{sk}\big(\|u_k\|_{L_p(P_{J,E(2^{J-k}\nu)})} + \|u_k\|_{L_p(P_{J,E(2^{J-k}\nu)+w_0})}\big)$$
$$\leq 2\max(2^{2mn\tau}, 1) \sum_{k=J}^{J+2m} \Big(2^{Jn\tau} 2^{sk} \sup_{\eta\in\mathbb{Z}^n} \|u_k\|_{L_p(P_{J,\eta})}\Big)$$
$$\leq 2\max(2^{2mn\tau}, 1) \sum_{k=J}^{J+2m} \Big(\sup_{j\in\mathbb{Z}} \sup_{\eta\in\mathbb{Z}^n} 2^{n\tau j} \sum_{l\geq j} 2^{sl} \|u_l\|_{L_p(P_{j,\eta})}\Big)$$
$$\lesssim \sup_{j\in\mathbb{Z}} \sup_{\eta\in\mathbb{Z}^n} 2^{n\tau j} \Big(\sum_{l\geq j} \big(2^{sl} \|u_l\|_{L_p(P_{j,\eta})}\big)^q\Big)^{1/q} \quad (\text{since } 0 < q \leq 1)$$
$$\lesssim A \qquad (\forall \nu \in \mathbb{Z}^n). \tag{16}$$

Finally by replacing (16) into (13), and by taking into account that $x \in P_{0,\nu}$ implies $1+|\nu| \leq \sqrt{n}(2+|x|)$, we get, from the second member of (12),

$$\begin{aligned}\sum_{k\leq 0} |\langle u_k, f\rangle| &\leq cA\zeta_{\mu,n+1}(f) \sum_{\nu\in\mathbb{Z}^n} \Big(\int_{P_{0,\nu}} (1+|x|)^{-(n+1)p'}\,\mathrm{d}x\Big)^{1/p'}\\ &\leq cA\zeta_{\mu,n+1}(f) \sum_{\nu\in\mathbb{Z}^n} (1+|\nu|)^{-n-1} \lesssim A\zeta_{\mu,n+1}(f).\end{aligned}$$

• *The case 1.2.2:* $0 < p < 1$. We introduce a real number $0 < d < p < 1$ which will be chosen later on. The function $\widehat{u_k} * \widehat{\widetilde{Q}_k f}$ is supported by the ball $|\xi| \leq (b+3/2)2^k$, then the Bernstein inequality yields

$$\begin{aligned}\sum_{k\leq 0} |\langle u_k, f\rangle| &\lesssim \sum_{k\leq 0} 2^{kn(1/d-1)} \Big(\int_{\mathbb{R}^n} |u_k(x)\widetilde{Q}_k f(x)|^d\,\mathrm{d}x\Big)^{1/d}\\ &\lesssim \sum_{k\leq 0} 2^{kn(1/d-1)} \Big(\sum_{\nu\in\mathbb{Z}^n} \int_{P_{k,\nu}} |u_k(x)\widetilde{Q}_k f(x)|^d\,\mathrm{d}x\Big)^{1/d}.\end{aligned}$$

By using the Minkowski inequality w.r.t. $\ell_{1/d}(\ell_1(\mathbb{Z}^n))$, the third term in the last inequality is bounded by

$$c\Big(\sum_{\nu\in\mathbb{Z}^n} \Big\{\sum_{k\leq 0} \Big(2^{kn(1-d)} \int_{P_{k,\nu}} |u_k(x)\widetilde{Q}_k f(x)|^d\,\mathrm{d}x\Big)^{1/d}\Big\}^d\Big)^{1/d}. \tag{17}$$

By the Hölder inequality with exponents p/d and $w := p/(p-d)$, it holds

$$\Big(\int_{P_{k,\nu}} |u_k(x)\widetilde{Q}_k f(x)|^d\,\mathrm{d}x\Big)^{1/d} \leq \|u_k\|_{L_p(P_{k,\nu})} \|\widetilde{Q}_k f\|_{L_{dw}(P_{k,\nu})}, \tag{18}$$

on the one hand. On the other, by Lemma 3 since $k \leq 0$ we obtain, for a positive integer N and a change of variables $y := 2^k x$,

$$\begin{aligned}\|\widetilde{Q}_k f\|_{L_{dw}(P_{k,\nu})} &\leq c2^{k(\mu+n)}\zeta_{\mu,N}(f)\Big(\int_{P_{k,\nu}} (1+|2^k x|)^{-Ndw}\,\mathrm{d}x\Big)^{1/(dw)}\\ &= c2^{k(\mu+n-n/d+n/p)}\zeta_{\mu,N}(f)\Big(\int_{P_{0,\nu}} (1+|y|)^{-Ndw}\,\mathrm{d}y\Big)^{1/(dw)}. \quad (19)\end{aligned}$$

Inserting now (19) and (18) into (17), it follows

$$\begin{aligned}\sum_{k\leq 0} |\langle u_k, f\rangle| \lesssim \zeta_{\mu,N}(f)\Big\{ \sum_{\nu\in\mathbb{Z}^n} \Big(\sum_{k\leq 0} 2^{k(s+n\tau)} \|u_k\|_{L_p(P_{k,\nu})}\Big)^d\\ \times \Big(\int_{P_{0,\nu}} (1+|y|)^{-Ndw}\,\mathrm{d}y\Big)^{1/w}\Big\}^{1/d}.\end{aligned}$$

But we can here apply (16) since (15) is also valid in the following sense:

$$\|u_k\|_{L_p(P_{k,\nu})} \leq 2^{1/p-1}\big(\|u_k\|_{L_p(P_{J,E(2^{J-k}\nu)})} + \|u_k\|_{L_p(P_{J,E(2^{J-k}\nu)+w_0})}\big); \tag{20}$$

therefore, we find the bound

$$cA\zeta_{\mu,N}(f)\Big\{\sum_{\nu\in\mathbb{Z}^n}\Big(\int_{P_{0,\nu}}(1+|y|)^{-Ndw}\,\mathrm{d}y\Big)^{1/w}\Big\}^{1/d}.$$

Since $1+|\nu|\le(1+\sqrt{2n})(1+|y|)$ if $y\in P_{0,\nu}$, it follows

$$\sum_{k\le 0}|\langle u_k,f\rangle|\lesssim A\zeta_{\mu,N}(f)\Big(\sum_{\nu\in\mathbb{Z}^n}(1+|\nu|)^{-Nd}\Big)^{1/d}.$$

As $\prod_{i=1}^n(1+|\nu_i|)\le(1+|\nu|)^n$, it suffices to choose $N\in\mathbb{N}$ sufficiently large such that $N>n/p$ and d satisfying $n/N<d<p$.

Step 2: proof of (3). Now $\sum_{j\in\mathbb{Z}}u_j =: u$ in $\mathcal{S}'_\mu$ is well defined, and we are going to estimate $\|[u]_\mu\|_{\dot{B}^{s,\tau}_{p,q}}$.

Since $\widehat{u_j}$ is supported by the annulus $a2^j\le|\xi|\le b2^j$, there exist two integers m_1 and m_2, depending only on a and b, such that $Q_k(u_j)=0$ if $j\le k+m_1$ or $j\ge k+m_2$ (m_1 and m_2 are the integers near to $\log_2(1/2b)$ and $\log_2(3/2a)$, respectively, with $m_1<m_2$).

We will estimate $\|Q_k u\|_{L_p(P_{l,\nu})}$. We put $d:=\min(1,p)$ and use the Minkowski inequality, then for all $\nu\in\mathbb{Z}^n$ and all couples $(k,l)\in\mathbb{Z}^2$ we obtain

$$\begin{aligned}\|Q_k u\|_{L_p(P_{l,\nu})}&\le\Big(\sum_{k+m_1<j<k+m_2}\|Q_k(u_j)\|^d_{L_p(P_{l,\nu})}\Big)^{1/d}\\&\le\Big(\sum_{k+m_1<j<k+m_2}2^{sjq}\|Q_k(u_j)\|^q_{L_p(P_{l,\nu})}\Big)^{1/q}\Big(\sum_{k+m_1<\ell<k+m_2}2^{-sd\ell}\Big)^{1/d}.\end{aligned}$$

Thus

$$\|Q_k u\|_{L_p(P_{l,\nu})}\lesssim 2^{-sk}\Big(\sum_{k+m_1<j<k+m_2}2^{sjq}\|Q_k(u_j)\|^q_{L_p(P_{l,\nu})}\Big)^{1/q}. \tag{21}$$

We now continue by considering the cases $p\ge1$ and $p<1$ separately.

Substep 2.1: the case $p\ge 1$. By using the Hölder inequality (recall $p':=p/(p-1)$), we obtain

$$\begin{aligned}\|Q_k(u_j)\|^p_{L_p(P_{l,\nu})}&=\int_{P_{l,\nu}}\Big(\sum_{w\in\mathbb{Z}^n}\int_{P_{l,w}}|u_j(y)|\,|2^{kn}\mathcal{F}^{-1}\gamma(2^k(x-y))|\mathrm{d}y\Big)^p\mathrm{d}x\\ \lesssim\int_{P_{l,\nu}}\Big(&\sum_{w\in\mathbb{Z}^n}\Big\{\int_{P_{l,w}}|u_j(y)|^p\mathrm{d}y\Big\}^{1/p}\Big\{\int_{P_{l,w}}|2^{kn}\mathcal{F}^{-1}\gamma(2^k(x-y))|^{p'}\mathrm{d}y\Big\}^{1/p'}\Big)^p\mathrm{d}x,\end{aligned} \tag{22}$$

we continue by the Minkowski inequality, we get the bound

$$c\left(\sum_{w\in\mathbb{Z}^n}\|u_j\|_{L_p(P_{l,w})}\Big(\int_{P_{l,\nu}}\Big\{\int_{P_{l,w}}|2^{kn}\mathcal{F}^{-1}\gamma(2^k(x-y))|^{p'}\,\mathrm{d}y\Big\}^{p/p'}\mathrm{d}x\Big)^{1/p}\right)^p.$$

Since $x\in P_{l,\nu}$ and $y\in P_{l,w}$ imply that

$$1+|w-\nu|\le\sqrt{n}(2+2^l|x-y|)\le\sqrt{n}(2+2^k|x-y|)\quad\text{with}\quad k\ge l,$$

then

$$\int_{P_{l,\nu}} \Big\{ \int_{P_{l,w}} |2^{kn}\mathcal{F}^{-1}\gamma(2^k(x-y))|^{p'}\,\mathrm{d}y \Big\}^{p/p'} \mathrm{d}x \lesssim 2^{(k-l)np}(1+|w-\nu|)^{-(n+1)p}\,.$$

We turn to (22), it holds

$$\|Q_k(u_j)\|_{L_p(P_{l,\nu})} \lesssim 2^{(k-l)n} \sum_{w\in\mathbb{Z}^n} \|u_j\|_{L_p(P_{l,w})}(1+|w-\nu|)^{-(n+1)}\,. \tag{23}$$

On the one hand, as in (14) the series $\sum_{k\in\mathbb{Z}} \widehat{Q_k u}(\xi)$, for all $\xi \in \mathbb{R}^n \setminus \{0\}$, contains at most three non-vanishing terms corresponding to the compact annulus $2^{k-r-1} \le |\xi| \le 3\cdot 2^{k+r-1}$ $(r=-1,0,1)$, consequently, $k\in\Lambda$ with $\operatorname{Card}\Lambda = 3$. Then we turn to (21) and obtain

$$\Big(\sum_{k\ge l} 2^{skq}\|Q_k u\|^q_{L_p(P_{l,\nu})}\Big)^{1/q} \lesssim \Big(\sum_{k\ge l,\,k\in\Lambda}\ \sum_{j=k+m_1}^{k+m_2} 2^{sjq}\|Q_k(u_j)\|^q_{L_p(P_{l,\nu})}\Big)^{1/q}. \tag{24}$$

We set $\Lambda := \{J, J+1, J+2\}$ (Λ is constituted by consecutive elements), and using both (22) and (23) with $l := J$, then since we have $1 \le 2^{(k-J)n} \le 2^{2n}$, the right-hand side of (24) is bounded by

$$\begin{aligned} &c\Big(\sum_{k=J}^{J+2}\ \sum_{j=k+m_1}^{k+m_2} 2^{jqs}\Big\{ \sum_{w\in\mathbb{Z}^n} \|u_j\|_{L_p(P_{J,w})}(1+|w-\nu|)^{-(n+1)}\Big\}^q\Big)^{1/q} \lesssim \\ &\qquad \lesssim \sup_{\eta\in\mathbb{Z}^n} \Big(\sum_{j\ge J+m_1} 2^{jqs}\|u_j\|^q_{L_p(P_{J,\eta})} \sum_{k=J}^{J+2} 1\Big)^{1/q} \sum_{w\in\mathbb{Z}^n}(1+|w-\nu|)^{-(n+1)} \lesssim \\ &\qquad \lesssim \sup_{\eta\in\mathbb{Z}^n} \Big(\sum_{j\ge J+m_1} 2^{jqs}\|u_j\|^q_{L_p(P_{J,\eta})}\Big)^{1/q}. \end{aligned} \tag{25}$$

On the other hand, by the elementary inequality

$$[2^{m_1}\eta_j] \le 2^{J+m_1}x_j < [2^{m_1}\eta_j] + [2^{m_1}] + 2 \qquad (x\in P_{J,\eta},\ j=1,\dots,n)$$

we obtain

$$P_{J,\eta} \subset \bigcup_{r=0}^{[2^{m_1}]+1} P_{J+m_1,E(2^{m_1}\eta)+rw_0} \tag{26}$$

where $w_0 = (1,1,\dots,1)\in\mathbb{Z}^n$. Then we continue the estimation of (25), we find

$$\begin{aligned} &\Big(\sum_{j\ge J+m_1} 2^{sjq}\|u_j\|^q_{L_p(P_{J,\eta})}\Big)^{1/q} \\ &\qquad \le \Big(\sum_{j\ge J+m_1} 2^{sjq}\Big\{ \sum_{r=0}^{[2^{m_1}]+1} \|u_j\|_{L_p(P_{J+m_1,E(2^{m_1}\eta)+rw_0})}\Big\}^q\Big)^{1/q} \\ &\qquad \lesssim \sum_{r=0}^{[2^{m_1}]+1} \Big(\sum_{j\ge J+m_1} 2^{sjq}\|u_j\|^q_{L_p(P_{J+m_1,E(2^{m_1}\eta)+rw_0})}\Big)^{1/q} \lesssim 2^{-n\tau J}A. \end{aligned}$$

Hence $\big(\sum_{k\geq J} 2^{skq}\|Q_k u\|^q_{L_p(P_{J,\nu})}\big)^{1/q}$ is bounded by $c2^{-Jn\tau}A$, and the desired result follows.

Substep 2.2: the case $0<p<1$. Using (6), we begin by the following estimate

$$\begin{aligned}\|Q_k(u_j)\|^p_{L_p(P_{l,\nu})} &\leq \int_{P_{l,\nu}}\Big(\sum_{w\in\mathbb{Z}^n}\|u_j\|_{L_\infty(P_{j,w})}\int_{P_{j,w}}2^{kn}|\mathcal{F}^{-1}\gamma(2^k(x-y))|\,\mathrm{d}y\Big)^p\,\mathrm{d}x\\ &\lesssim 2^{jp(n/p-s-n\tau)}A^p\int_{P_{l,\nu}}\Big(\sum_{w\in\mathbb{Z}^n}\int_{P_{j,w}}2^{kn}|\mathcal{F}^{-1}\gamma(2^k(x-y))|\,\mathrm{d}y\Big)^p\,\mathrm{d}x.\end{aligned}$$

Then we have

$$\begin{aligned}&\int_{P_{l,\nu}}\Big(\sum_{w\in\mathbb{Z}^n}\int_{P_{j,w}}2^{kn}|\mathcal{F}^{-1}\gamma(2^k(x-y))|\,\mathrm{d}y\Big)^p\,\mathrm{d}x =\\ &\qquad=\int_{P_{l,\nu}}\Big(\int_{\mathbb{R}^n}2^{kn}|\mathcal{F}^{-1}\gamma(2^k(x-y))|\,\mathrm{d}y\Big)^p\,\mathrm{d}x \;=\; 2^{-ln}\|\mathcal{F}^{-1}\gamma\|^p_{L_1(\mathbb{R}^n)}.\end{aligned}$$

Hence

$$\|Q_k(u_j)\|_{L_p(P_{l,\nu})}\lesssim 2^{j(n/p-s-n\tau)}2^{-ln/p}A \qquad (\forall k,l,j\in\mathbb{Z},\ \forall\nu\in\mathbb{Z}^n). \tag{27}$$

As in the previous substep that it has been observed that the series $\sum_{k\in\mathbb{Z}}\widehat{Q_k u}(\xi)$ is given for $k\in\Lambda$ with $\operatorname{Card}\Lambda=3$ and $\Lambda:=\{J,J+1,J+2\}$. Then from (21) we obtain

$$\Big(\sum_{k\geq l}2^{skq}\|Q_k u\|^q_{L_p(P_{l,\nu})}\Big)^{1/q}\lesssim\Big(\sum_{k\geq l,\,k\in\Lambda}\ \sum_{j=k+m_1}^{k+m_2}2^{sjq}\|Q_k(u_j)\|^q_{L_p(P_{l,\nu})}\Big)^{1/q}. \tag{28}$$

Using both (27) and (28) with $l:=J$, then the right-hand side of (28) is bounded by

$$\begin{aligned}&cA\,2^{-Jn/p}\Big(\sum_{k=J}^{J+2}\sum_{j=k+m_1}^{k+m_2}2^{jq(n/p-n\tau)}\Big)^{1/q}\lesssim\\ &\qquad\lesssim A\,2^{-Jn/p}\Big(\sum_{k=J}^{J+2}2^{kq(n/p-n\tau)}\Big)^{1/q}\lesssim 2^{-Jn\tau}A.\end{aligned}$$

Hence agian we obtain that $\big(\sum_{k\geq J}2^{skq}\|Q_k u\|^q_{L_p(P_{J,\nu})}\big)^{1/q}$ is bounded by $c2^{-Jn\tau}A$. The proof is complete. □

Proof of Theorem 1. Let $f\in\dot{B}^{s,\tau}_{p,q}$.

Step 1. The convergence of the series $\sum_{j\in\mathbb{Z}}Q_j f$ in $\mathcal{S}'_\mu$ follows directly from Theorem 2, see also [22, lem. 2.4]. We put $\sigma_\mu(f):=\sum_{j\in\mathbb{Z}}Q_j f$ in $\mathcal{S}'_\mu$.

Step 2. We prove $\partial^\alpha\sigma_\mu(f)\in\widetilde{C}_0$ for $|\alpha|=\mu$. Owing to Proposition 3(iv), the proof in the case $s+n\tau-n/p\notin\mathbb{N}_0$ or $q>1$ is similar to that in [6, p. 483, step 3] or [14, p. 170] since $[\partial^\alpha\sigma_\mu(f)]_\infty\in\dot{B}^{s-\mu,\tau}_{p,q}\hookrightarrow\dot{B}^{s+n\tau-n/p-\mu}_{\infty,\infty}$ ($|\alpha|=\mu$) and $s+n\tau-n/p-\mu<0$. We now see the case $\mu=s+n\tau-n/p\in\mathbb{N}_0$ and $0<q\leq1$. By again Proposition 3(iii)–(iv) we have $[\partial^\alpha\sigma_\mu(f)]_\infty\in\dot{B}^{0}_{\infty,\infty}$ (with $|\alpha|=\mu$) which implies that $\|Q_j f^{(\alpha)}\|_{L_\infty(\mathbb{R}^n)}\lesssim 2^{-j\mu}\|[\sigma_\mu(f)]_\infty\|_{\dot{B}^{s,\tau}_{p,q}}$ for all $j\in\mathbb{Z}$. We put

$$f_m:=\sum_{|j|\leq m}Q_j f^{(\alpha)} \qquad (\alpha\in\mathbb{N}_0^n \text{ is fixed such that } |\alpha|=\mu \text{ and } m=1,2,\ldots),$$

then for all $\lambda > 0$ and all $g \in \mathcal{S}$ we write

$$\langle h_\lambda \partial^\alpha \sigma_\mu(f), g\rangle = \big\langle h_\lambda\big(\partial^\alpha \sigma_\mu(f) - f_m\big), g\big\rangle + \langle h_\lambda f_m, g\rangle. \tag{29}$$

Since $\|f_m\|_{L_\infty(\mathbb{R}^n)} < \infty$ and $\widehat{f}_m$ is supported in $\mathbb{R}^n \setminus \{0\}$ by the annulus $2^{-m-1} \le |\xi| \le 3 \cdot 2^{m-1}$, by Lemma 2 it holds that $\langle h_\lambda f_m, g\rangle$ tends to 0 with $\lambda \to 0$.

To treat the second term in (29), we introduce a positive integer r such that $2^{-r-1} < \lambda \le 2^{-r}$. The function $\mathcal{F}\big(h_\lambda(Q_j f^{(\alpha)})\big)$ is supported by the annulus $2^{j+r-1} \le |\xi| \le 3 \cdot 2^{j+r}$. Then

$$\mathcal{F}\big(Q_{k+r} h_\lambda(Q_j f^{(\alpha)})\big) = 0 \quad \text{if} \quad k - j \ge 3 \quad \text{or} \quad k - j \le -2\,.$$

Hence

$$\big\langle h_\lambda(Q_j f^{(\alpha)}), g\big\rangle = \sum_{k\in\mathbb{Z}} \big\langle Q_{k+r} h_\lambda(Q_j f^{(\alpha)}), g\big\rangle = \sum_{l=-2}^{3} \big\langle h_\lambda(Q_j f^{(\alpha)}), Q_{j+r+l}\, g\big\rangle;$$

here we used the fact that γ is a radial function on $\mathbb{R}^n$. Finally we obtain

$$\big\langle h_\lambda\big(\partial^\alpha \sigma_\mu(f) - f_m\big), g\big\rangle = \sum_{|j|>m} \sum_{l=-2}^{3} \big\langle h_\lambda(Q_j f^{(\alpha)}), Q_{j+r+l}\, g\big\rangle. \tag{30}$$

As in (26) it holds

$$\begin{aligned} \big|\big\langle h_\lambda\big(\partial^\alpha \sigma_\mu(f) - f_m\big), g\big\rangle\big| &\le \sum_{\nu\in\mathbb{Z}^n} \sum_{|j|>m} \sum_{l=-2}^{3} \|h_\lambda(Q_j f^{(\alpha)})\|_{L_\infty(P_{j,\nu})} \|Q_{j+r+l}\, g\|_{L_1(P_{j,\nu})} \\ &\lesssim \sum_{r=0}^{1+[1/\lambda]} \sum_{\nu\in\mathbb{Z}^n} \sum_{|j|>m} \sum_{l=-2}^{3} \|Q_j f^{(\alpha)}\|_{L_\infty(P_{j,E(\nu/\lambda)+rw_0})} \|Q_{j+r+l}\, g\|_{L_1(P_{j,\nu})}\,, \end{aligned} \tag{31}$$

where $w_0 := (1, 1, \dots, 1) \in \mathbb{Z}^n$, on the one hand. On the other, in (30) we have the equality $\sum_{|j|>m} \cdots = \sum_{|j|>m, j\in\Lambda} \cdots$ with $\operatorname{Card}\Lambda = 3$, (noticing that $\Lambda := \{J, J+1, J+2\}$, cf., see the sentence just before formula (28)). Then, using (6) with u_k, $P_{k,\nu}$ and s replaced by $Q_j f^{(\alpha)}$, $P_{j,\, E(\nu/\lambda)+rw_0}$ and $s - \mu$, respectively, it holds that

$$\|Q_j f^{(\alpha)}\|_{L_\infty(P_{j,E(\nu/\lambda)+rw_0})} \lesssim 2^{j(n/p - s + |\alpha| - n\tau)} \|f^{(\alpha)}\|_{\dot{B}^{s-\mu,\tau}_{p,q}} \;=\; c\|f^{(\alpha)}\|_{\dot{B}^{s-\mu,\tau}_{p,q}}$$

where the positive constant c is independent of j, ν and λ (recall $|\alpha| = \mu$). Hence

$$\begin{aligned} \big|\big\langle h_\lambda\big(\partial^\alpha \sigma_\mu(f) - f_m\big), g\big\rangle\big| &\le c(2 + [1/\lambda])\|f^{(\alpha)}\|_{\dot{B}^{s-\mu,\tau}_{p,q}} \sum_{l=-2}^{3} \sum_{j=J}^{J+2} \sum_{\nu\in\mathbb{Z}^n} \|Q_{j+r+l}\, g\|_{L_1(P_{j,\nu})} \\ &= c(2 + [1/\lambda])\|f^{(\alpha)}\|_{\dot{B}^{s-\mu,\tau}_{p,q}} \sum_{l=-2}^{3} \sum_{j=J}^{J+2} \|Q_{j+r+l}\, g\|_{L_1(\mathbb{R}^n)}\,. \end{aligned}$$

By Young's inequality we have $\|Q_j g\|_{L_1(\mathbb{R}^n)} \lesssim \|g\|_{L_1(\mathbb{R}^n)}$ for all $j \in \mathbb{Z}$, then

$$\big|\big\langle h_\lambda\big(\partial^\alpha \sigma_\mu(f) - f_m\big), g\big\rangle\big| \lesssim (2 + [1/\lambda])\|f^{(\alpha)}\|_{\dot{B}^{s-\mu,\tau}_{p,q}} \|g\|_{L_1(\mathbb{R}^n)}\,,$$

which yields

$$\lim_{m\to\infty}\left|\langle h_\lambda\big(\partial^\alpha\sigma_\mu(f)-f_m\big),g\rangle\right|=0,$$

see again (30) and (31). Then from the expression (29), for a fixed arbitrarily $\varepsilon>0$, there exists a positive integer m_0, such that for all $m\geq m_0$ it holds

$$\left|\langle h_\lambda\partial^\alpha\sigma_\mu(f),g\rangle\right|\leq\varepsilon+|\langle h_\lambda f_m,g\rangle|,$$

which gives $\lim_{\lambda\to 0}\langle h_\lambda\partial^\alpha\sigma_\mu(f),g\rangle=0$, and this proves that $\partial^\alpha\sigma_\mu(f)\in\widetilde{C}_0$ if $|\alpha|=\mu$ in all cases.

We note that if we put $\widetilde{\sigma}(f):=\sum_{j\in\mathbb{Z}}Q_jf+v$ with any nonzero polynomial $v\in\mathcal{P}_\mu$, then $\widetilde{\sigma}$ are realizations of $\dot{B}^{s,\tau}_{p,q}$ into $\mathcal{S}'_\mu$, and it is clear that $\partial^\alpha\widetilde{\sigma}(f)\in\widetilde{C}_0$ if $|\alpha|=\mu$ and $\widetilde{\sigma}(f)=\sigma_\mu(f)$ in $\mathcal{S}'_\mu$, i.e., $[\widetilde{\sigma}(f)]_\mu=[\sigma_\mu(f)]_\mu$.

Step 3. Since $\tau_a\circ Q_j=Q_j\circ\tau_a$ for all $j\in\mathbb{Z}$ and all $a\in\mathbb{R}^n$, the mapping $f\to\sigma_\mu(f):=\sum_{j\in\mathbb{Z}}Q_jf$ commutes itself with τ_a. For the commutation with dilations, we first observe that from the equality

$$\langle h_\lambda\big(h_\beta f\big)^{(\alpha)},\varphi\rangle=\beta^{n-|\alpha|}\langle h_\lambda f^{(\alpha)},h_{1/\beta}\varphi\rangle\qquad(\forall\lambda>0,\forall\beta>0,\forall\varphi\in\mathcal{S}),$$

we have, if $f^{(\alpha)}\in\widetilde{C}_0$ then $\big(h_\beta f\big)^{(\alpha)}\in\widetilde{C}_0$. It holds that the realized space $\dot{\widetilde{B}}^{s,\tau}_{p,q}$ is dilation invariant. Second, we then obtain $\sigma_\mu(f)-f=:v_1$ with $v_1\in\mathcal{P}_\mu$. Also, for all $\lambda>0$ we have $\sigma_\mu(h_\lambda f)-h_\lambda f=:v_2$ with $v_2\in\mathcal{P}_\mu$. Then this yields

$$\sigma_\mu(h_\lambda f)-h_\lambda\sigma_\mu(f)=v_2-h_\lambda v_1;$$

as $v_2-h_\lambda v_1\in\mathcal{P}_\mu$, then $\sigma_\mu(h_\lambda f)=h_\lambda\sigma_\mu(f)$ in $\mathcal{S}'_\mu$, i.e., $[\sigma_\mu(h_\lambda f)]_\mu=[h_\lambda\sigma_\mu(f)]_\mu$. The mapping σ_μ defined from $\dot{B}^{s,\tau}_{p,q}$ into $\mathcal{S}'_\mu$ commutes with dilations.

Step 4. The uniqueness of $\sigma_\mu(f)$, the representative of f, follows immediately by Lemma 1. The proof is complete. □

Proof of Corollary 1. We prove $\sigma_\mu(\dot{B}^{s,\tau}_{p,q})=\dot{\widetilde{B}}^{s,\tau}_{p,q}$. Indeed, by definition the embedding in the one sense holds. Let now $g\in\dot{\widetilde{B}}^{s,\tau}_{p,q}$. By Theorem 1 we have $\partial^\alpha\big(g-\sigma_\mu([g]_\mu)\big)\in\widetilde{C}_0$ for all $|\alpha|=\mu$, i.e., $g-\sigma_\mu([g]_\mu)\in\mathcal{P}_\mu$, then g and $\sigma_\mu([g]_\mu)$ coincide in $\mathcal{S}'_\mu$, and the converse embedding holds too. □

4. Concluding Remarks. We are interested in the integer μ in the sense of Remark 2.

Proposition 4. *Let $s\in\mathbb{R}$, $p,q\in]0,+\infty]$ and $\tau\geq 0$ be given such that $\mu\geq 1$. Let a,b be real numbers such that $0<a<b$. Then there exists a sequence $(u_j)_{j\in\mathbb{Z}}$ in $\mathcal{S}'$ satisfying*

(i) *$\widehat{u_j}$ is supported by the annulus $a2^j\leq|\xi|\leq b2^j$,*

(ii) $A:=\sup_{k\in\mathbb{Z}}\sup_{\nu\in\mathbb{Z}^n}2^{kn\tau}\big(\sum_{j\geq k}\big(2^{sj}\|u_j\|_{L_p(P_{k,\nu})}\big)^q\big)^{1/q}<\infty$,

and such that the series $\sum_{j\in\mathbb{Z}}u_j$ diverges in $\mathcal{S}'_{\mu-1}$.

Proof. Only some changes are needed w.r.t. the proof given in [4, prop. 2.2.1]. We briefly outline this. We put $m:=\mu-1$. Let $\varphi\in\mathcal{D}$ be such that $\int_{\mathbb{R}^n}\varphi(x)\mathrm{d}x=1$

(this implies $\partial_{x_1}^m \varphi \in \mathcal{S}_m$), then we split $\big\langle \sum_{j\in\mathbb{Z}} u_j, \partial_{x_1}^m \varphi \big\rangle$ into $I_1 + I_2$ where

$$I_1 := (-1)^m \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \Big(\partial_{x_1}^m u_j(x) - \partial_{x_1}^m u_j(0)\Big)\varphi(x)\mathrm{d}x,$$
$$I_2 := (-1)^m \sum_{j\in\mathbb{Z}} \partial_{x_1}^m u_j(0).$$

It suffices to choose $(u_j)_{j\in\mathbb{Z}}$ such that (i) and (ii) are satisfied, and in more

$$|I_2| = \sum_{j\in\mathbb{Z}} \partial_{x_1}^m u_j(0) = \infty \quad \text{and} \quad |I_1| \le c \sum_{j\in\mathbb{Z}} \|\nabla \partial_{x_1}^m u_j\|_\infty < \infty,$$

which are possible, where it suffices to change s by $s + n\tau$ in all occurrences in the proof of [4, prop. 2.2.1]. □

Remark 4. If there exists a sequence $(u_j)_{j\in\mathbb{Z}}$ in $\mathcal{S}'$ satisfying (ii) of Proposition 4 with $\tau < 0$, then $u_j = 0$ for all $j \in \mathbb{Z}$, see below the proof of Proposition 3(ii) by replacing $Q_j f$ by u_j.

Remark 5. In [3, 6] Bourdaud has given the construction of realizations from the homogeneous Besov spaces $\dot{B}^s_{p,q}$ into the tempered distributions space $\mathcal{S}'$. This construction is also hold for $\dot{B}^{s,\tau}_{p,q}$. Namely, for all $f \in \dot{B}^{s,\tau}_{p,q}$ we have

$$\sigma_{\mu,1}(f) := \textstyle\sum_{k\in\mathbb{Z}} Q_k f, \quad \text{if either} \quad s + n\tau < n/p \quad \text{or} \quad s + n\tau = n/p \text{ and } q \le 1, \tag{32}$$

$$\sigma_{\mu,2}(f) := \textstyle\sum_{k\in\mathbb{Z}} \Big(Q_k f - \sum_{|\alpha|<\mu} (Q_k f)^{(\alpha)}(0)\, x^\alpha/\alpha!\Big),$$
$$\text{if either } s + n\tau - n/p \in \mathbb{R}^+ \setminus \mathbb{N}_0 \text{ or } s + n\tau - n/p \in \mathbb{N} \text{ and } q \le 1, \tag{33}$$

$$\sigma_{\mu,3}(f) := \textstyle\sum_{k>0} Q_k f + \sum_{k\le 0} \Big(Q_k f - \sum_{|\alpha|<\mu} (Q_k f)^{(\alpha)}(0)\, x^\alpha/\alpha!\Big),$$
$$\text{if } s + n\tau - n/p \in \mathbb{N}_0 \text{ and } q > 1, \tag{34}$$

where these series converge in $\mathcal{S}'$ and satisfy $\partial^\alpha \sigma_{\mu,i}(f) \in \widetilde{C}_0$ $(i = 1, 2, 3)$ for all $|\alpha| = \mu$.

In that case, $\dot{\widetilde{B}}^{s,\tau}_{p,q}$ can be characterized as a subset of $\mathcal{S}'$ by other properties; e.g., in the case $\mu \ge 1$ we have the following two statements and refer to [6] for this direction.

Proposition 5. *Let $s, \tau \in \mathbb{R}$ and $p, q \in]0, +\infty]$ satisfy either $s + n\tau - n/p \in \mathbb{R}^+ \setminus \mathbb{N}_0$ or $s + n\tau - n/p \in \mathbb{N}$ and $0 < q \le 1$. Then every element f of $\dot{\widetilde{B}}^{s,\tau}_{p,q}$ satisfies $f \in \mathcal{S}'$, $[f]_\mu \in \dot{B}^{s,\tau}_{p,q}$ and the following properties:*
(i) f is a function of class $C^{\mu-1}$,
(ii) $f^{(\alpha)}(0) = 0$ for all $|\alpha| \le \mu - 1$,
(iii) $f^{(\alpha)} \in \widetilde{C}_0$ for all $|\alpha| = \mu$.

Proof. We first note that $\mu \ge 1$. Let g be a function in $\dot{B}^{s,\tau}_{p,q}$. Clearly $[\sigma_{\mu,2}(g)]_\mu = [\sigma_\mu(g)]_\mu$, where $\sigma_{\mu,2}$ and σ_μ are the realizations defined in (33) and

Theorem 1, respectively. Then the convergence of $\sigma_{\mu,2}(g)$ in $\mathcal{S}'_\mu$ results from Remark 2 and we have $\|[\sigma_{\mu,2}(g)]_\mu\|_{\dot{B}^{s,\tau}_{p,q}} \lesssim \|g\|_{\dot{B}^{s,\tau}_{p,q}}$. The function

$$f := \sigma_{\mu,2}(g)$$

easily satisfies (i)–(ii), this can be done by the same method used in the proofs of [6, prop. 4.8] and [14, thm. 4.5]; however by Theorem 1, the item (iii) is also satisfied for f since $\partial^\alpha \sigma_{\mu,2}(g) = \partial^\alpha \sigma_\mu(g)$ if $|\alpha| = \mu$. □

Remark 6. If $s, \tau \in \mathbb{R}$ and $p, q \in]0, +\infty]$ satisfy $s + n\tau - n/p \in \mathbb{R}^+ \setminus \mathbb{N}_0$, then every element f of $\dot{\widetilde{B}}{}^{s,\tau}_{p,q}$ satisfies

$$|f^{(\alpha)}(x)| \leq c|x|^{s+n\tau-n/p-\mu+1} \|[f]_\mu\|_{\dot{B}^{s,\tau}_{p,q}} \qquad (\forall x \in \mathbb{R}^n,\ \forall |\alpha| = \mu - 1)$$

where the positive constant c is independent of f and x. This follows by [14, prop. 5.1/(5.2)] and the embedding $\dot{B}^{s,\tau}_{p,q} \hookrightarrow \dot{B}^{s+n\tau-n/p}_{\infty,\infty}$.

Proposition 6. *Let $s, \tau \in \mathbb{R}$ and $p, q \in]0, +\infty]$. If $\mu \geq 1$ then $\dot{\widetilde{B}}{}^{s,\tau}_{p,q}$ is the set of distributions f satisfying $[f]_\mu \in \dot{B}^{s,\tau}_{p,q}$ and*
(i) *$f^{(\alpha)} \in \widetilde{C}_0$ for all $|\alpha| = \mu$,*
(ii) *$\langle f^{(\alpha)}, \gamma\rangle = 0$ for all $|\alpha| = \mu - 1$,*
(iii) *$f \in \mathcal{S}'_{\mu-1}$.*

Proof. That is the same as in [6, lem. 4.13/proof], just it suffices to take the realization from $\dot{B}^{s,\tau}_{p,q}$ into $\mathcal{S}'_{\mu-1}$ defined by

$$\sigma(f) := \widetilde{\sigma}_\mu(f) - c \sum_{|\beta|=\mu-1} \left\langle \partial^\beta\left(\widetilde{\sigma}_\mu(f)\right), \gamma\right\rangle \frac{x^\beta}{\beta!}$$

where $1/c := \int_{\mathbb{R}^n} \gamma(\xi)\mathrm{d}\xi$, and $\widetilde{\sigma}_\mu : \dot{B}^{s,\tau}_{p,q} \to \mathcal{S}'$ is the realization defined by $\widetilde{\sigma}_\mu := \sigma_{\mu,1}$ or $\sigma_{\mu,2}$ or $\sigma_{\mu,3}$ given in (32) or (33) or (34), respectively. □

Remark 7. Using the assertions (i) and (ii) of Proposition 6, in [6, 4.4.2] it has been proved the nonexistence of dilation commuting realization of the Besov spaces $\dot{B}^s_{p,q}$ into $\mathcal{S}'_{\mu-1}$ in case $s - n/p \in \mathbb{N}$ and $q > 1$ (here $\tau = 0$ and $\mu := s - n/p + 1$). We have the same characterization in the Besov-type spaces $\dot{B}^{s,\tau}_{p,q}$ when $s + n\tau - n/p \in \mathbb{N}$, $q > 1$ and $\tau > 0$ (recall that $\dot{B}^{s,\tau}_{p,q}$ is reduced to $\mathcal{P}_\infty$ if $\tau < 0$, see Proposition 3(ii)). Namely:
(i) If $\sigma : \dot{B}^{s,\tau}_{p,q} \to \mathcal{S}'_m$ $(m \in \mathbb{N}_0)$ is a dilation commuting realization, then it holds

$$\left|\left\langle h_\lambda\left(\partial^\alpha \sigma(f)\right), g\right\rangle\right| \leq c\lambda^{n/p-s-n\tau+|\alpha|}\zeta_{N,k}(g)\|[\sigma(f)]_m\|_{\dot{B}^{s,\tau}_{p,q}}, \tag{35}$$

for all $\lambda > 0$, all $\alpha \in \mathbb{N}_0^n$, all $f \in \dot{B}^{s,\tau}_{p,q}$ and all $g \in \mathcal{S}_m$, the positive constant c does not depend on λ, α, N, k, f and g. Indeed, assume that σ be given; since we have

$$\langle \sigma^{(\alpha)}(h_\lambda f), g\rangle = \lambda^{-|\alpha|}\langle h_\lambda(\sigma^{(\alpha)}(f)), g\rangle \qquad (\forall g \in \mathcal{S}_m),$$

then (35) is an easy computation of the equality $\sigma^{(\alpha)}(h_\lambda f) = \lambda^{-|\alpha|}h_\lambda(\sigma^{(\alpha)}(f))$ in $\mathcal{S}'_m$, separately continuous functions property cf., [17, coro, sect. 34.2, p. 354] and the inequality

$$\|[\sigma^{(\alpha)}(h_\lambda f)]_{m+|\alpha|}\|_{\dot{B}^{s-|\alpha|,\tau}_{p,q}} \lesssim \|[\sigma(h_\lambda f)]_m\|_{\dot{B}^{s,\tau}_{p,q}} \lesssim \lambda^{n/p-s-n\tau}\|[\sigma(f)]_m\|_{\dot{B}^{s,\tau}_{p,q}},$$

recall that $\partial^\alpha \sigma : \dot{B}^{s-|\alpha|,\tau}_{p,q} \to \mathcal{S}'_{m+|\alpha|}$.
(ii) Assume that $s+n\tau-n/p \in \mathbb{N}$ and $q>1$ (then $\mu = s+n\tau-n/p+1$). If we put $m := \mu-1$ the inequality (35) becomes

$$\big|\big\langle h_\lambda\big(\partial^\alpha\sigma(f)\big), g\big\rangle\big| \le c\zeta_{N,k}(g)\|[\sigma(f)]_{\mu-1}\|_{\dot{B}^{s,\tau}_{p,q}}, \quad (\forall|\alpha| = \mu-1, \forall\lambda>0). \tag{36}$$

However in [6, pp. 486-487] it has been proved the existence of functions f_0 (see again [6, (4.8)] for their expressions) such that $[f_0]_\mu \in \dot{B}^s_{p,q}$ and the inequality (36), with $f := f_0$, $g := \gamma$ and $\tau = 0$, cannot be hold. This is also true in $\dot{B}^{s,\tau}_{p,q}$ for all $\tau>0$ by the same functions f_0, since $[f_0]_\mu \in \dot{B}^{s,\tau}_{p,q}$ can be easily obtained by both Proposition 2 and properties of f_0; we omit details. Then there are nonexistence of dilation commuting realizations of $\dot{B}^{s,\tau}_{p,q}$ into $\mathcal{S}'_{\mu-1}$ in this situation.

5. Annexe.

Proof of Proposition 2. Step 1. We first prove (4) with $\lambda := 2^m$, $m \in \mathbb{Z}$. We consider $f \in \dot{B}^{s,\tau}_{p,q}$ and put $f_m := h_{2^{-m}} f$. By using the identity

$$Q_j f_m = Q_{j-m} f(2^m(\cdot)), \tag{37}$$

we have

$$\|Q_j f_m\|_{L_p(P_{k+m,\nu})} = 2^{-nm/p}\|Q_{j-m} f\|_{L_p(P_{k,\nu})} \qquad (\forall k \in \mathbb{Z}, \forall \nu \in \mathbb{Z}^n). \tag{38}$$

Then we write

$$\begin{aligned}
&\Big(2^{n\tau(k+m)q} \sum_{j\ge k+m} 2^{sjq}\|Q_j f_m\|^q_{L_p(P_{k+m,\nu})}\Big)^{1/q}\\
&= 2^{(n\tau-n/p)m}\Big(2^{n\tau kq} \sum_{j\ge k+m} 2^{sjq}\|Q_{j-m} f\|^q_{L_p(P_{k,\nu})}\Big)^{1/q}\\
&= 2^{(s+n\tau-n/p)m}\Big(2^{n\tau kq} \sum_{l\ge k} 2^{slq}\|Q_l f\|^q_{L_p(P_{k,\nu})}\Big)^{1/q} \le 2^{(s+n\tau-n/p)m}\|f\|_{\dot{B}^{s,\tau}_{p,q}}.
\end{aligned} \tag{39}$$

Taking the supremum over all $k \in \mathbb{Z}$ and all $\nu \in \mathbb{Z}^n$ in the first term of (39) we deduce

$$\|f_m\|_{\dot{B}^{s,\tau}_{p,q}} \le 2^{(s+n\tau-n/p)m}\|f\|_{\dot{B}^{s,\tau}_{p,q}}.$$

For the converse inequality we proceed as in (37)–(39). Indeed, by the equality $Q_j f(2^m(\cdot)) = Q_{j+m} f_m$ we get $\|Q_j f\|_{L_p(P_{k-m,\nu})} = 2^{nm/p}\|Q_{j+m} f_m\|_{L_p(P_{k,\nu})}$ ($\forall k \in \mathbb{Z}$, $\forall \nu \in \mathbb{Z}^n$), and

$$\begin{aligned}
&\Big(2^{n\tau(k-m)q} \sum_{j\ge k-m} 2^{sjq}\|Q_j f\|^q_{L_p(P_{k-m,\nu})}\Big)^{1/q}\\
&= 2^{(n/p-n\tau)m}\Big(2^{n\tau kq} \sum_{j\ge k-m} 2^{sjq}\|Q_{j+m} f_m\|^q_{L_p(P_{k,\nu})}\Big)^{1/q}\\
&= 2^{(n/p-n\tau-s)m}\Big(2^{n\tau kq} \sum_{l\ge k} 2^{slq}\|Q_l f_m\|^q_{L_p(P_{k,\nu})}\Big)^{1/q} \le 2^{(n/p-n\tau-s)m}\|f_m\|_{\dot{B}^{s,\tau}_{p,q}}.
\end{aligned}$$

Taking again the supremum over all $k \in \mathbb{Z}$ and all $\nu \in \mathbb{Z}^n$, we obtain

$$\|f\|_{\dot{B}^{s,\tau}_{p,q}} \leq 2^{-(s+n\tau-n/p)m} \|f_m\|_{\dot{B}^{s,\tau}_{p,q}}.$$

Step 2. We consider $\lambda > 0$. Let $m \in \mathbb{Z}$ be such that $2^m \leq \lambda < 2^{m+1}$. We also consider a function $f \in \dot{B}^{s,\tau}_{p,q}$ and put $f_{m,\lambda} := f\big(2^{-m}\lambda(\cdot)\big)$. By Step 1, it holds that

$$\|f(\lambda(\cdot))\|_{\dot{B}^{s,\tau}_{p,q}} = 2^{(s+n\tau-n/p)m} \|f_{m,\lambda}\|_{\dot{B}^{s,\tau}_{p,q}}.$$

Then it suffices to prove $c_1\|f\|_{\dot{B}^{s,\tau}_{p,q}} \leq \|f_{m,\lambda}\|_{\dot{B}^{s,\tau}_{p,q}} \leq c_2\|f\|_{\dot{B}^{s,\tau}_{p,q}}$ with positive constants c_1 and c_2 independent of m, λ and f. By Remark 3 we introduce the equivalent quasi-seminorm in $\dot{B}^{s,\tau}_{p,q}$ defined by the function $\gamma_1 := \gamma\big(2^m\lambda^{-1}(\cdot)\big)$ which has a support in the annulus $1/2 \leq |\xi| \leq 3$ (recall that $1 \leq 2^{-m}\lambda < 2$). On the one hand, by a simple change of variables we have

$$\big(2^{jn}h_{2^{-j}}\mathcal{F}^{-1}\gamma_1\big) * f_{m,\lambda} = Q_j f(2^{-m}\lambda(\cdot))\,. \tag{40}$$

On the other hand, for all $\nu \in \mathbb{Z}^n$ and all $k \in \mathbb{Z}$ we have

$$x \in P_{k,\nu} \;\Rightarrow\; 2^{-m}\lambda x \in P_{k,E(2^{-m}\lambda\nu)} \cup P_{k,E(2^{-m}\lambda\nu)+w_0} \cup P_{k,E(2^{-m}\lambda\nu)+2w_0}, \tag{41}$$

where $w_0 := (1,1,\dots,1) \in \mathbb{Z}^n$, and

$$\|Q_j f(2^{-m}\lambda(\cdot))\|_{L_p(P_{k,\nu})} \leq \max(1,\, 2^{2(1/p-1)})\Big(\|Q_jf\|_{L_p(P_{k,E(2^{-m}\lambda\nu)})} + \\ + \|Q_jf\|_{L_p(P_{k,E(2^{-m}\lambda\nu)+w_0})} + \|Q_jf\|_{L_p(P_{k,E(2^{-m}\lambda\nu)+2w_0})}\Big) \qquad (\forall j \in \mathbb{Z},\, \forall \nu \in \mathbb{Z}^n).$$

Then we obtain

$$\begin{aligned}\|f_{m,\lambda}\|_{\dot{B}^{s,\tau}_{p,q}} &= c_1 \sup_{k\in\mathbb{Z}} \sup_{\nu\in\mathbb{Z}^n} 2^{n\tau k}\Big(\sum_{j\geq k} 2^{sjq}\|\big(2^{jn}h_{2^{-j}}\mathcal{F}^{-1}\gamma_1\big) * f_{m,\lambda}\|^q_{L_p(P_{k,\nu})}\Big)^{1/q} \\ &\leq c_2 \sup_{k\in\mathbb{Z}} \sup_{\nu\in\mathbb{Z}^n} 2^{n\tau k}\Big(\sum_{j\geq k} 2^{sjq}\Big\{\sum_{l=0}^{2}\|Q_jf\|_{L_p(P_{k,E(2^{-m}\lambda\nu)+lw_0})}\Big\}^q\Big)^{1/q} \\ &\leq c_3 \sup_{k\in\mathbb{Z}} \sup_{w\in\mathbb{Z}^n} 2^{n\tau k}\Big(\sum_{j\geq k} 2^{sjq}\|Q_jf\|^q_{L_p(P_{k,w})}\Big)^{1/q} \;=\; c_3\|f\|_{\dot{B}^{s,\tau}_{p,q}}.\end{aligned}$$

For the converse inequality, we proceed as in (40)–(41) and it is clear that the roles of $f_{m,\lambda}$ and f may be exchanged. Indeed, we write $Q_jf = \big(2^{jn}h_{2^{-j}}\mathcal{F}^{-1}\gamma_1\big) * f_{m,\lambda}(2^m\lambda^{-1}(\cdot))$, and

$$x \in P_{k,\nu} \;\Rightarrow\; 2^m\lambda^{-1}x \in P_{k,E(2^m\lambda^{-1}\nu)} \cup P_{k,E(2^m\lambda^{-1})+w_0}.$$

We continue as above, and we deduce that $\|f\|_{\dot{B}^{s,\tau}_{p,q}} \leq c\|f_{m,\lambda}\|_{\dot{B}^{s,\tau}_{p,q}}$ holds as desired. □

REMARK 8. For Proposition 2 we can see [26, rem. 8.1, p. 254].

Proof of Proposition 3. Step 1: proof of (i). Obviously we have $\|f\|_{\dot{B}^{s,0}_{p,q}} \leq \|f\|_{\dot{B}^s_{p,q}}$ for all $f \in \dot{B}^s_{p,q}$. For the converse inequality, we assume that $q < \infty$ (the case $q = \infty$ can be done completely similar) and we will apply twice the Fatou's lemma. We consider the union of the two dyadic cubes $P_{-k,\nu_0} \cup P_{-k,\nu_1} =: \widetilde{P}_k$ with $\nu_0 :=$

$(0, 0, \dots, 0)$ and $\nu_1 := (-1, -1, \dots, -1)$, $((\nu_0, \nu_1) \in (\mathbb{Z}^n)^2)$, i.e., if $x \in \widetilde{P}_k$ then $-2^k \leq x_j < 2^k$ for $j = 1, \dots, n$. Let $f \in \dot{B}^{s,0}_{p,q}$. Then the inequality

$$\|Q_j f\|_{L_p(P_{l,\nu})} \lesssim 2^{-js} \|f\|_{\dot{B}^{s,0}_{p,q}} \qquad (\forall l \in \mathbb{Z},\, \forall j \geq l,\, \forall \nu \in \mathbb{Z}^n)$$

yields that $\big|Q_j f(\cdot)\big|^p \mathbf{1}_{\widetilde{P}_k}(\cdot)$ with $-j \leq k$ are positive and Lebesgue measurable functions on $\mathbb{R}^n$, where $\mathbf{1}_{\widetilde{P}_k}$ denotes the characteristic function of the set $\widetilde{P}_k$. Then on the one hand, we have

$$\begin{aligned}\|Q_j f\|^q_{L_p(\mathbb{R}^n)} &= \Big(\int_{\mathbb{R}^n} \liminf_{k \to \infty} \big|Q_j f(x)\big|^p \mathbf{1}_{\widetilde{P}_k}(x)\,\mathrm{d}x \Big)^{q/p} \\ &\leq \Big(\liminf_{k \to \infty} \int_{\mathbb{R}^n} \big|Q_j f(x)\big|^p \mathbf{1}_{\widetilde{P}_k}(x)\,\mathrm{d}x \Big)^{q/p} \\ &\leq \liminf_{k \to \infty} \|Q_j f\|^q_{L_p(\widetilde{P}_k)} \qquad (\forall j \in \mathbb{Z}),\end{aligned}$$

(the last term is obtained by using the following easy assertion: if $g_n \geq 0$ are measurable functions and $\beta \geq 0$ then $\big(\liminf_{n\to\infty} g_n(x)\big)^\beta \leq \liminf_{n\to\infty} g_n^\beta(x)$), and

$$\begin{aligned}\sum_{j \geq l} 2^{jsq} \|Q_j f\|^q_{L_p(\mathbb{R}^n)} &\leq \sum_{j \geq l} \liminf_{k \to \infty} 2^{jsq} \|Q_j f\|^q_{L_p(\widetilde{P}_k)} \\ &\leq \liminf_{k \to \infty} \sum_{j \geq l} 2^{jsq} \|Q_j f\|^q_{L_p(\widetilde{P}_k)}.\end{aligned} \tag{42}$$

But in the last term of (42) and since k is large, there exists a positive integer k_0 such that for all $k \geq \max(k_0, l)$ we have $\sum_{j \geq l} \cdots \leq \sum_{j \geq -k} \cdots$, and by making $l \to -\infty$ in the left-hand side of (42) we find

$$\|f\|^q_{\dot{B}^s_{p,q}} \leq \liminf_{k \to \infty} \sum_{j \geq -k} 2^{jsq} \|Q_j f\|^q_{L_p(\widetilde{P}_k)}. \tag{43}$$

On the other hand, since $P_{-k,\nu_0} \cap P_{-k,\nu_1} = \emptyset$, then it holds

$$\sum_{j \geq -k} 2^{jsq} \|Q_j f\|^q_{L_p(\widetilde{P}_k)} \lesssim \sum_{j \geq -k} 2^{jsq} \|Q_j f\|^q_{L_p(P_{-k,\nu_0})} + \sum_{j \geq -k} 2^{jsq} \|Q_j f\|^q_{L_p(P_{-k,\nu_1})}.$$

Now, taking the supremum over, both, all $\nu \in \mathbb{Z}^n$ and all $k \in \mathbb{Z}$ in the right-hand side of the last inequality and inserting this into (43), the wanted estimate will be obtained.

Step 2: proof of (ii). We assume that $\tau < 0$ and prove $\dot{B}^{s,\tau}_{p,q} \subset \mathcal{P}_\infty$. We restrict ourselves to $q < \infty$ since the case $q = \infty$ can be done in the same manner. We consider $f \in \dot{B}^{s,\tau}_{p,q}$. For all $\nu \in \mathbb{Z}^n$ and all $k \in \mathbb{Z} \setminus \mathbb{N}$ we have $P_{0,\nu} \subset P_{k,E(2^k\nu)} \cup P_{k,E(2^k\nu)+w_0}$, where $w_0 = (1, 1, \dots, 1) \in \mathbb{Z}^n$. Then in the first time we get

$$\|Q_j f\|_{L_p(P_{0,\nu})} \leq c\big(\|Q_j f\|_{L_p(P_{k,E(2^k\nu)})} + \|Q_j f\|_{L_p(P_{k,E(2^k\nu)+w_0})}\big) \quad (\forall j \in \mathbb{Z},\, k \leq 0), \tag{44}$$

where $c = \max(1,\, 2^{1/p-1})$, see e.g., (15) and (20). In the second, by assumption on f, we obtain

$$2^{n\tau k} \Big(\sum_{j \geq k} 2^{sjq} \|Q_j f\|^q_{L_p(P_{k,E(2^k\nu)})} \Big)^{1/q} \leq \|f\|_{\dot{B}^{s,\tau}_{p,q}} \qquad (\forall k \in \mathbb{Z}),$$

and the same inequality holds with $P_{k,E(2^k\nu)+w_0}$ instead of $P_{k,E(2^k\nu)}$. Then using (44), we drive

$$\|Q_j f\|_{L_p(P_{0,\nu})} \lesssim 2^{-sjq} 2^{-n\tau k} \|f\|_{\dot{B}^{s,\tau}_{p,q}} \qquad (\forall k \leq 0,\, \forall j \geq k,\, \forall \nu \in \mathbb{Z}^n).$$

Letting $k \to -\infty$ we conclude $Q_j f = 0$ a.e. for any $P_{0,\nu}$ and all $j \in \mathbb{Z}$, and since ν is arbitrary then $Q_j f = 0$ a.e. on $\mathbb{R}^n$, which implies that $f \in \mathcal{P}_\infty$.

Step 3: proof of (iii). We introduce a function $\psi \in \mathcal{S}_\infty$ such that $\widehat{\psi}(\xi) := \xi_1 \gamma(\xi)$, and we set $\psi_j := 2^{jn} h_{2^{-j}} \psi$, $(j \in \mathbb{Z})$. By Remark 3, we can replace in Definition 3 the operator Q_j by ψ_j. Now, since $Q_j(\partial_1 f) = 2^j \psi_j * f$, the desired result follows.

Step 4: proof of (iv). By (6) with $Q_j f$ instead of u_k, we get

$$\begin{aligned} |Q_j f(x)| &\lesssim \sum_{\nu \in \mathbb{Z}^n} 2^{jn} \int_{P_{j,\nu}} \left| \mathcal{F}^{-1} \widetilde{\gamma}\big(2^j(x-y)\big)\, Q_j f(y) \right| \mathrm{d}y \\ &\lesssim \Big(2^{jn} \int_{\mathbb{R}^n} \left| \mathcal{F}^{-1} \widetilde{\gamma}(2^j z) \right| \mathrm{d}z \Big) \sup_{\omega \in \mathbb{Z}^n} \|Q_j f\|_{L_\infty(P_{j,\omega})} \lesssim 2^{(n/p - s - n\tau)j} \|f\|_{\dot{B}^{s,\tau}_{p,q}} \end{aligned}$$

for all $x \in \mathbb{R}^n$ and all $j \in \mathbb{Z}$. The result follows. □

Remark 9. Of course, the statements of Proposition 3 are essentially known. For example (iii) can be found in [20], and the assertion (iv) is proved in [24] with the restriction $\tau > 1/p$ and $q < \infty$ or $\tau \geq 1/p$ and $q = \infty$. See also e.g., [23, 26].

Acknowledgement. We would like to thank the referees for their valuable comments and suggestions which led to an improvement of the paper.

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