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TWO COMPUTATIONS CONCERNING THE ISOVECTORS OF THE BACKWARD HEAT EQUATION WITH QUADRATIC POTENTIAL*

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Abstract. We determine the isovectors of the backward heat equation with quadratic potential term in the space variable. This generalizes the calculations of Lescot-Zambrini (cf.[4]). These results first appeared in the first author's PhD thesis (Rouen, 2013).

Key words. Lie algebra of isovectors, Vector fields, Quadratic potential, Backward heat equation and Schrödinger equation.

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1. Backward Equation.

The fundamental equation of Euclidean Quantum Mechanics of Zambrini is the backward heat equation with potential $V(t,q)$:

$$\begin{cases} \theta^2 \frac{\partial \eta_u^V}{\partial t} &= -\frac{\theta^4}{2}\frac{\partial^2 \eta_u^V}{\partial q^2} + V(t,q)\eta_u^V \qquad (C_1^{(V)}) \\ \eta_u^V(0,q) &= u(q) \end{cases}$$

where t represents the time variable, q the space variable, and θ is a real parameter, strictly positive (in physics, $\theta = \sqrt{\hbar}$).

This equation is not well–posed in general, but existence and uniqueness of a solution are insured whenever the initial condition u belongs to the set of **analytic vectors** for the operator appearing on the right–hand side of the equation (see *e.g.* [1], Lemma 4, p. 429).

We shall denote by $\eta_u(t,q) = \eta_u^0(t,q)$ the solution of the backward heat equation with null potential:

$$\begin{cases} \theta^2 \frac{\partial \eta_u}{\partial t} &= -\frac{\theta^4}{2}\frac{\partial^2 \eta_u}{\partial q^2} \qquad (C_1^{(0)}) \\ \eta_u(0,q) &= \eta_u^0(0,q) = u(q). \end{cases}$$

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1.1. Generalization of results due to Lescot-Zambrini (cf.[4]).

Theorem 1.1. *Let $a(t), b(t)$ and $c(t)$ be continuous functions and $V(t,q) = a(t)q^2 + b(t)q + c(t)$.*

For all initial conditions u, the solution $\eta_u^V(t,q)$ of:

$$\theta^2 \frac{\partial \eta_u^V}{\partial t} = -\frac{\theta^4}{2}\frac{\partial^2 \eta_u^V}{\partial q^2} + V(t,q)\eta_u^V \qquad (C_1^{(V)})$$

such that $\eta_u^V(0,q) = u(q)$ is given by:

$$\eta_u^V(t,q) = \varphi_1(t,q)\ \eta_u\left(\varphi_2(t,q), \varphi_3(t,q)\right), \tag{1.1}$$

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where: $\eta_u(t,q) = \eta_u^0(t,q)$ *and* $\varphi_1(t,q)$, $\varphi_2(t,q)$, $\varphi_3(t,q)$ *only depend on* $a(t)$, $b(t)$ *and* $c(t)$ *(via formulas (1.7), (1.8), (1.9), (1.11), (1.12), (1.13) and (1.14)).*

Proof. We shall make formula (1.1) explicit with the following initial conditions:

$$\begin{cases} \varphi_1(0,q) = 1 \\ \varphi_2(0,q) = 0 \\ \varphi_3(0,q) = q. \end{cases}$$

In this case:

$$\eta_u^V(0,q) = \eta_u(0,q) = u(q).$$

We now differentiate formula (1.1) with respect to t and q; η_u and its derivatives will be taken in $(\varphi_2(t,q), \varphi_3(t,q))$.

$$\frac{\partial \eta_u^V}{\partial t} = \frac{\partial \varphi_1}{\partial t}\eta_u + \varphi_1(\frac{\partial \eta_u}{\partial t}\frac{\partial \varphi_2}{\partial t} + \frac{\partial \eta_u}{\partial q}\frac{\partial \varphi_3}{\partial t}),$$

$$\frac{\partial \eta_u^V}{\partial q} = \frac{\partial \varphi_1}{\partial q}\eta_u + \varphi_1(\frac{\partial \eta_u}{\partial t}\frac{\partial \varphi_2}{\partial q} + \frac{\partial \eta_u}{\partial q}\frac{\partial \varphi_3}{\partial q}),$$

$$\begin{aligned}\frac{\partial^2 \eta_u^V}{\partial q^2} = {} & \frac{\partial^2 \varphi_1}{\partial q^2}\eta_u + 2\,\frac{\partial \varphi_1}{\partial q}\left(\frac{\partial \eta_u}{\partial t}\frac{\partial \varphi_2}{\partial q} + \frac{\partial \eta_u}{\partial q}\frac{\partial \varphi_3}{\partial q}\right) \\ & + \varphi_1\left[\frac{\partial^2 \eta_u}{\partial t^2}(\frac{\partial \varphi_2}{\partial q})^2 + \frac{\partial^2 \eta_u}{\partial q^2}(\frac{\partial \varphi_3}{\partial q})^2 + 2\,\frac{\partial^2 \eta_u}{\partial t \partial q}\frac{\partial \varphi_2}{\partial q}\frac{\partial \varphi_3}{\partial q} + \frac{\partial \eta_u}{\partial t}\frac{\partial^2 \varphi_2}{\partial q^2} + \frac{\partial \eta_u}{\partial q}\frac{\partial^2 \varphi_3}{\partial q^2}\right]\end{aligned}$$

and

$$V(t,q)\eta_u^V = a(t)q^2\ \varphi_1\ \eta_u + b(t)q\ \varphi_1\ \eta_u + c(t)\ \varphi_1\ \eta_u.$$

So, it's enough to have:

$$\theta^2\varphi_1\frac{\partial \varphi_2}{\partial t} - \theta^2\varphi_1(\frac{\partial \varphi_3}{\partial q})^2 + \frac{\theta^4}{2}(2\,\frac{\partial \varphi_1}{\partial q}\frac{\partial \varphi_2}{\partial q} + \varphi_1\frac{\partial^2 \varphi_2}{\partial q^2}) = 0 \tag{1.2}$$

$$\frac{\theta^4}{2}\varphi_1(\frac{\partial \varphi_2}{\partial q})^2 = 0 \tag{1.3}$$

$$\theta^2\varphi_1\frac{\partial \varphi_3}{\partial t} + \frac{\theta^4}{2}(2\frac{\partial \varphi_1}{\partial q}\frac{\partial \varphi_3}{\partial q} + \varphi_1\frac{\partial^2 \varphi_3}{\partial q^2}) = 0 \tag{1.4}$$

$$\theta^4\frac{\partial \varphi_2}{\partial q}\frac{\partial \varphi_3}{\partial q} = 0 \tag{1.5}$$

$$\theta^2\frac{\partial \varphi_1}{\partial t} + \frac{\theta^4}{2}\frac{\partial^2 \varphi_1}{\partial q^2} - \varphi_1\left(a(t)q^2 + b(t)q + c(t)\right) = 0\,. \tag{1.6}$$

As $\varphi_1(0,q)=1$, the equation (1.3) gives us $(\frac{\partial\varphi_2}{\partial q})^2=0$, then

$$\varphi_2=\varphi_2(t)$$

and (1.5) is then automatically satisfied.

So, the equation (1.2) implies $\varphi_1(t,q)(\frac{\partial\varphi_2}{\partial t}-(\frac{\partial\varphi_3}{\partial q})^2)=0$, then for all (t,q):

$$\frac{\partial\varphi_2}{\partial t}=(\frac{\partial\varphi_3}{\partial q})^2$$
$$\frac{\partial\varphi_3}{\partial q}=A(t)$$

$$\varphi_3(t,q)=A(t)q+B(t) \tag{1.7}$$

and

$$\varphi_2(t)=\int_0^t A^2(s)\,ds\,, \tag{1.8}$$

then the equation (1.4) gives:

$$\varphi_1\frac{\partial\varphi_3}{\partial t}+\theta^2\frac{\partial\varphi_1}{\partial q}\frac{\partial\varphi_3}{\partial q}=0$$
$$\varphi_1\left(\dot A(t)q+\dot B(t)\right)+\theta^2A(t)\frac{\partial\varphi_1}{\partial q}=0$$

$$\varphi_1(t,q)=k(t)\,e^{-\frac{1}{\theta^2A(t)}\left(\frac{\dot A(t)}{2}q^2+\dot B(t)q\right)}\,, \tag{1.9}$$

therefore:

$$\frac{\partial\varphi_1}{\partial t}=\left\{\dot k(t)+\frac{k(t)}{\theta^2A(t)}\left(\frac{1}{2}(\frac{\dot A^2(t)}{A(t)}-\ddot A(t))q^2+(\frac{\dot A(t)}{A(t)}\dot B(t)-\ddot B(t))q\right)\right\}$$
$$\times e^{-\frac{1}{\theta^2A(t)}\left(\frac{\dot A(t)}{2}q^2+\dot B(t)q\right)}\,,$$

$$\frac{\partial\varphi_1}{\partial q}=\frac{-1}{\theta^2}\frac{\varphi_1}{A(t)}(\dot A(t)q+\dot B(t))$$

and

$$\frac{\partial^2\varphi_1}{\partial q^2}=\frac{1}{\theta^4}\frac{\varphi_1}{A^2(t)}\left(\dot A^2(t)q^2+2\dot A(t)\dot B(t)q-\theta^2\dot A(t)A(t)+\dot B^2(t)\right)\,,$$

then the equation (1.6) becomes:

$$k(t)\left(-\frac{1}{2}\frac{\ddot A(t)}{A(t)}+\frac{\dot A^2(t)}{A^2(t)}-a(t)\right)q^2+k(t)\left(-\frac{\ddot B(t)}{A(t)}+2\dot B(t)\frac{\dot A(t)}{A^2(t)}-b(t)\right)q$$
$$+\theta^2\dot k(t)-k(t)c(t)+\frac{k(t)}{2}\frac{\dot B^2(t)}{A^2(t)}-\theta^2\frac{k(t)}{2}\frac{\dot A(t)}{A(t)}=0.$$

As $k(t) \neq 0$, the equation (1.6) is equivalent to the following system:

$$\begin{cases} -\dfrac{1}{2}\dfrac{\ddot{A}(t)}{A(t)} + \dfrac{\dot{A}^2(t)}{A^2(t)} - a(t) = 0 \\ -\dfrac{\ddot{B}(t)}{A(t)} + 2\dot{B}(t)\,\dfrac{\dot{A}(t)}{A^2(t)} - b(t) = 0 \\ \theta^2\dot{k}(t) - k(t)c(t) + \dfrac{k(t)}{2}\dfrac{\dot{B}^2(t)}{A^2(t)} - \theta^2\dfrac{k(t)}{2}\dfrac{\dot{A}(t)}{A(t)} = 0 \end{cases} \tag{1.10}$$

knowing that:

$$\begin{cases} A(0) & = 1 \\ B(0) & = 0 \\ k(0) & = 1 \\ \dot{A}(0) & = 0 \\ \dot{B}(0) & = 0. \end{cases}$$

This is due to initial conditions at time t on the functions $\varphi_1(t,q)$, $\varphi_2(t,q)$, and $\varphi_3(t,q)$. Let

$$A(t) = e^{\int_0^t \rho(s)ds}\,, \tag{1.11}$$

then

$$\rho(t) = \frac{\dot{A}(t)}{A(t)}$$

and

$$\dot{\rho}(t) + \rho^2(t) = \frac{\ddot{A}(t)}{A(t)}\,,$$

therefore the first equation of (1.10) is equivalent to a Riccati equation:

$$\begin{cases} \rho^2(t) - \dot{\rho}(t) & = 2a(t) \\ \rho(0) = \frac{\dot{A}(0)}{A(0)} & = 0\,. \end{cases} \tag{1.12}$$

The second equation of (1.10) gives: $2\,\frac{\dot{A}(t)}{A(t)}\dot{B}(t) - \ddot{B}(t) = A(t)b(t)$, then

$$\begin{aligned} \left(A^{-2}(t)\dot{B}(t)\right)' &= -2A^{-3}(t)\dot{A}(t)\dot{B}(t) + A^{-2}(t)\ddot{B}(t) \\ &= -A^{-2}(t)\left(2\,\frac{\dot{A}(t)}{A(t)}\dot{B}(t) - \ddot{B}(t)\right) \\ &= -A^{-2}(t)\left(A(t)b(t)\right) \\ &= -\frac{b(t)}{A(t)}\,, \end{aligned}$$

therefore $B(t)$ is the solution of:

$$\begin{cases} \dot{B}(t) & = -A^2(t)\displaystyle\int_0^t \frac{b(s)}{A(s)}\,ds \\ B(0) & = 0\,. \end{cases} \tag{1.13}$$

Let $\Delta(t) = \int_0^t \frac{b(s)}{A(s)}\, ds$ and define k by:

$$\begin{cases} \dot{k}(t) &= \frac{k(t)}{\theta^2}\left(c(t) - \frac{1}{2}A^2(t)\Delta^2(t) + \frac{\theta^2}{2}\rho(t)\right) \\ k(0) &= 1, \end{cases} \tag{1.14}$$

this makes the first equation of (1.7) satisfied. □

REMARK 1.2. If $a(t), b(t)$ and $c(t)$ were constants, then we could apply the method of Rosencrans (cf. [6]).

1.2. Particular cases.

i) $\mathbf{V(t,q) = a(t)q^2}$ **(semiclassical case).**

Then

$$B(t) = 0\,, \Delta(t) = 0\,,$$

$$\begin{aligned} k(t) &= k(0)\, e^{\frac{1}{\theta^2}\int_0^t \left(c(s) - \frac{1}{2}A^2(s)\Delta^2(s) + \frac{\theta^2}{2}\rho(s)\right) ds} \\ &= e^{\frac{1}{2}\int_0^t \frac{\dot{A}(s)}{A(s)}\, ds} \\ &= \sqrt{A(t)} \end{aligned}$$

and

$$\eta_u^V(t,q) = \sqrt{A(t)}\, e^{-\frac{q^2}{2\theta^2}\rho(t)}\, \eta_u\left(\varphi_2(t), A(t)q\right)$$

is the solution of $(C_1^{(V)})$ such that $\eta_u^V(0,q) = \eta_u(0,q) = u(q)$.

In this case, as in the next one, the formulas really involve only $A(t)$.

i – a) $\mathbf{V(t,q) = \frac{\omega^2}{2}q^2}$.

Then

$$\rho(t) = -\omega\, \tanh(\omega t)\,, A(t) = \frac{1}{\cosh(\omega t)}\,, \varphi_2(t,q) = \frac{1}{\omega}\tanh(\omega t)$$

and according to **i)**

$$\eta_u^V(t,q) = \frac{1}{\sqrt{\cosh(\omega t)}}\, e^{\frac{\omega}{2\theta^2}\tanh(\omega t) q^2}\, \eta_u\left(\frac{1}{\omega}\tanh(\omega t), \frac{q}{\cosh(\omega t)}\right)$$

is the solution of $(C_1^{(V)})$.

This result is due to Zambrini (cf.[7], p.227, (51)).

The existence of such a formula was mentioned in [5], p. 96, and the similar result for the Schrödinger equation with quadratic potential (with $\cos(\omega t)$ taking the place of $\cosh(\omega t)$) appeared in [5], p. 83.

ii) $\mathbf{V(t,q) = \lambda(t)q}$.

Then $\rho(t) = 0$, $A(t) = 1$, $\dot{B}(t) = -\int_0^t \lambda(s)\, ds$, $B(t) = \int_0^t \dot{B}(u)\, du = -\int_0^t (\int_0^u \lambda(s)\, ds)\, du$

and

$$\eta_u^V(t,q) = k(t)\, e^{-\frac{1}{\theta^2}\dot{B}(t)q}\, \eta_u(t, q + B(t))$$

is the solution of $(C_1^{(V)})$, such that $\eta_u^V(0,q) = \eta_u(0,q) = u(q)$.

ii − a) $\mathbf{V(t,q) = \lambda q}$.

Then $A(t) = 1, \Delta(t) = \lambda t, \dot{B}(t) = -\lambda t, B(t) = -\frac{\lambda}{2}t^2, k(t) = e^{-\frac{\lambda^2}{6\theta^2}t^3}$ and

$$\eta_u^V(t,q) = e^{-\frac{1}{\theta^2}(\frac{\lambda^2}{6}t^3 - \lambda t q)}\, \eta_u(t, q - \frac{\lambda t^2}{2})$$

is the solution of $(C_1^{(V)})$.

We hereby recover a result of Lescot and Zambrini (cf.[4], p.219, (1)).

2. Isovectors. Let $\mathcal{G}_V$ defined by:

$$\mathcal{G}_V = \{N \,|\, \mathcal{L}_N(I) \subset I\}$$

and $\mathcal{H}_V$ by:

$$\mathcal{H}_V = \{N \in \mathcal{G}_V \,|\, \frac{\partial N^S}{\partial S} = 0\}\,.$$

These definitions are taken from Lescot-Zambrini (cf.[4]).

We remind the reader that a $\mathcal{G}_V$ is a Lie algebra for the usual Lie bracket of vectors fields, and $\mathcal{H}_V$ is a Lie subalgebra of $\mathcal{G}_V$ (cf.[4], p.210-211).

According to Lescot-Zambrini (cf.[4], p.214-215, 3.9'- 3.29), for all $N \in \mathcal{H}_V$, we can write:

$$N^q := \frac{1}{2}\dot{T}_N(t)q + \ell_N(t)$$

$$N^t := T_N(t)$$

$$N^S := h(q,t,S)\,,$$

for $\sigma(t), l(t)$ and $T_N(t)$ satisfying:

$$-\frac{1}{4}\dddot{T}_N\, q^2 - \ddot{\ell}q + \dot{\sigma} + T_N\frac{\partial V}{\partial t} + \frac{1}{2}\dot{T}_N q\frac{\partial V}{\partial q} + \ell\frac{\partial V}{\partial q} + \dot{T}_N V - \frac{\theta^2}{4}\ddot{T}_N = 0\,. \quad (2.1)$$

Let us set $l_N = l$.

Proposition 2.1. *Let $\mathcal{K}_V$ defined by:*

$$\mathcal{K}_V = \{N \in \mathcal{H}_V / \ell_N = 0\},$$

then $\mathcal{K}_V$ is a Lie subalgebra of $\mathcal{H}_V$.

Proof. Let $N_1, N_2 \in \mathcal{K}_V$; one has

$$\begin{aligned}
[N_1,N_2]^q &= [N_1,N_2](q)\\
&= N_1(N_2^q) - N_2(N_1^q)\\
&= N_1^q \frac{\partial N_2^q}{\partial q} + N_1^t \frac{\partial N_2^q}{\partial t} - (N_2^q \frac{\partial N_1^q}{\partial q} + N_2^t \frac{\partial N_1^q}{\partial t})\\
&= \frac{1}{2}\dot{T}_{N_2}(\frac{1}{2}\dot{T}_{N_1}q + \ell_{N_1}(t)) + T_{N_1}(\frac{1}{2}\ddot{T}_{N_2}q + \dot{\ell}_{N_2}) - \frac{1}{2}\dot{T}_{N_1}(\frac{1}{2}\dot{T}_{N_2}q + \ell_{N_2}(t))\\
&- T_{N_2}(\frac{1}{2}\ddot{T}_{N_1}q + \dot{\ell}_{N_1})\\
&= \left(T_{N_1}(t)\dot{\ell}_{N_2} - T_{N_2}(t)\dot{\ell}_{N_1} + \frac{1}{2}(\ell_{N_1}(t)\dot{T}_{N_2} - \ell_{N_2}(t)\dot{T}_{N_1})\right)\\
&+ \frac{1}{2}(T_{N_1}(t)\ddot{T}_{N_2}(t) - T_{N_2}(t)\ddot{T}_{N_1}(t))q
\end{aligned}$$

and

$$\ell_{[N_1,N_2]} = T_{N_1}(t)\dot{\ell}_{N_2} - T_{N_2}(t)\dot{\ell}_{N_1} + \frac{1}{2}\left(\ell_{N_1}(t)\dot{T}_{N_2} - \ell_{N_2}(t)\dot{T}_{N_1}\right) = 0.$$

Then $\mathcal{K}_V$ is stable by the Lie bracket, therefore $\mathcal{K}_V$ is a Lie subalgebra of $\mathcal{H}_V$. □

LEMMA 2.2. *If $V_1(t,q) = V_2(t,q) + \frac{C}{q^2}$ (for C a constant), then $\mathcal{K}_{V_1} = \mathcal{K}_{V_2}$.*

Proof. Replacing $V_1(t,q)$ by its value in equation (2.1), we get:

$$-\frac{1}{4}\dddot{T}_N\, q^2 - \ddot{\ell}q + \dot{\sigma} + T_N\frac{\partial V_2}{\partial t} + \frac{1}{2}\dot{T}_N q\left(\frac{\partial V_2}{\partial q} - 2\frac{C}{q^3}\right) + \ell\left(\frac{\partial V_2}{\partial q} - 2\frac{C}{q^3}\right)$$
$$+ \dot{T}_N\left(V_2 + \frac{C}{q^2}\right) - \frac{\theta^2}{4}\ddot{T}_N = 0$$

$\Leftrightarrow$

$$-\frac{1}{4}\dddot{T}_N\, q^2 - \ddot{\ell}q + \dot{\sigma} + T_N\frac{\partial V_2}{\partial t} + \frac{1}{2}\dot{T}_N q\frac{\partial V_2}{\partial q} + \ell\frac{\partial V_2}{\partial q} - \frac{2lC}{q^3} + \dot{T}_N V_2 - \frac{\theta^2}{4}\ddot{T}_N = 0.$$

Therefore, if $N \in \mathcal{K}_{V_2}$, as $l_N = 0$, we see that N satisfies (2.1) for $V = V_1$. Then V_1 et V_2 give the same isovectors such as $\ell_N = 0$.

Therefore:

$$\mathcal{K}_{V_1} = \mathcal{K}_{V_2}.$$

□

COROLLARY 2.3. *If $V_1 = \frac{C}{q^2}(C \neq 0)$ and $V_2 = 0$, then $\mathcal{H}_{V_1} = \mathcal{K}_{V_1} = \mathcal{K}_{V_2}$ and therefore: $\mathcal{H}_{V_1} \subset \mathcal{H}_{V_2}$.*

Proof. Let $N \in \mathcal{H}_{V_1}$, the equation (2.1) becomes:

$$-\frac{1}{4}\dddot{T}_N\, q^2 - \ddot{\ell}q + \dot{\sigma} + \frac{1}{2}\dot{T}_N q(-2\frac{C}{q^3}) + \ell(-2\frac{C}{q^3}) + \dot{T}_N(\frac{C}{q^2}) - \frac{\theta^2}{4}\ddot{T}_N = 0,$$

that is

$$-\frac{1}{4}\dddot{T}_N\, q^2 - \ddot{\ell}q + \dot{\sigma} - \frac{\theta^2}{4}\ddot{T}_N - 2\frac{\ell C}{q^3} = 0\,.$$

As T_N, ℓ and σ depend only on t, the system is equivalent to:

$$\begin{cases} 2C\ell = 0 \\ \ddot{\ell} = 0 \\ \dot{\sigma} = \frac{\theta^2}{4}\ddot{T}_N \\ \dddot{T}_N = 0\,. \end{cases} \tag{2.2}$$

But $C \neq 0$, therefore $\ell_N = 0$ and

$$\mathcal{H}_{V_1} = \mathcal{K}_{V_1} \underset{\text{(Lemma 2.2)}}{=} \mathcal{K}_{V_2} \subset \mathcal{H}_{V_2}\,.$$

□

Remark 2.4. Thus, we recover another result of Lescot and Zambrini(cf.[4], p.220, (3)).

REFERENCES

[1] M. Flato, J. Simon, H. Snellmann and D. Sternheimer, *Simple facts about analytic vectors and integrability*, Annales Scientifiques de l'E.N.S. 4^e série, 5:3 (1972), pp. 423–434.
[2] P.Lescot, H. Quintard and J.-C. Zambrini, *Solving stochastic differential equations with Cartan's exterior differential system* (preprint, 2015).
[3] P. Lescot and J.-C.Zambrini, *Isovectors for the Hamilton–Jacobi–Bellman Equation, Formal Stochastic Differentials and First Integrals in Euclidean Quantum Mechanics*, Proceedings of the Ascona conference (2002), pp. 187–202. Birkhaüser (Progress in Probability, vol 58), 2004.
[4] P. Lescot and J.-C. Zambrini, *Probabilistic deformation of contact geometry, diffusion processes and their quadratures*, Seminar on Stochastic Analysis, Random Fields and applications V, pp. 203–226, Birkhaüser (Progress in Probability, vol. 59), 2008.
[5] W. Miller, Jr., *Symmetry and separation of variables*, Encyclopedia of Mathematics and its Applications 4, Cambridge University Press (2012).
[6] S. I. Rosencrans, *Perturbation Algebra of an Elliptic Operator*, Journal of Mathematical Analysis and Applications, 56 (1976), pp. 317–329.
[7] J.-C. Zambrini, *From the geometry of parabolic PDE to the geometry of SDE*, Seminar on Mathematical analysis of random phenomena, pp. 213-230, World Sci. Publ. Hackensack, NJ, 2007.