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SOME IMPLICATIONS OF THE 2-FOLD BAILEY LEMMA*

ALEXANDER E. PATKOWSKI†

Abstract. The 2-fold Bailey lemma is a special case of the s-fold Bailey lemma introduced by Andrews in 2000. We examine this special case and its applications to partitions and recently discovered q-series identities. Our work provides a general comparison of the utility of the 2-fold Bailey lemma and the more widely applied 1-fold Bailey lemma. We also offer a discussion of the $\mathrm{spt}_M(n)$ function and related identities.

Key words. partitions, q-series, Bailey's lemma.

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1. Introduction.

The Symmetric Bilateral Bailey transform [4], states that if

$$B_n = \sum_{j=-n}^{n} A_j u_{n-j} v_{n+j}, \tag{1.1}$$

and

$$\gamma_n = \sum_{j=|n|}^{\infty} \delta_j u_{j-n} v_{n+j}, \tag{1.2}$$

then

$$\sum_{n=-\infty}^{\infty} A_n \gamma_n = \sum_{n=0}^{\infty} B_n \delta_n. \tag{1.3}$$

Here we say that (A_n, B_n) is a Bailey pair (in the symmetric sense), and (γ_n, δ_n) is a conjugate Bailey pair. If we break symmetry, we say a pair (α_n, β_n) is a Bailey pair if

$$\beta_n(a) = \sum_{j=0}^{n} \alpha_j u_{n-j} v_{n+j}. \tag{1.4}$$

Bailey [6] chose $v_n = (aq)_n^{-1}$, $u_n = (q)_n^{-1}$, (where we use standard notation [8] for q-shifted factorials) giving the conjugate pair (γ_n, δ_n)

$$\delta_n = (\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n, \tag{1.5}$$

$$\gamma_n = \frac{(aq/\rho_1)_\infty(aq/\rho_2)_\infty}{(aq)_\infty(aq/\rho_1\rho_2)_\infty} \frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n}{(aq/\rho_1)_n(aq/\rho_2)_n}. \tag{1.6}$$

In [3] Andrews considered an s-fold extension of the Bailey lemma, and gave the definition of the s-fold Bailey pair $(A_{n_1,n_2,\dots,n_s}, B_{n_1,n_2,\dots,n_s})$ relative to $(a_1, a_2, \dots, a_s)$,

$$B_{n_1,\dots,n_s} = \sum_{r_1=-n_1}^{n_1} \cdots \sum_{r_s=-n_s}^{n_s} \frac{A_{r_1,\dots,r_s}}{(a_1q)_{n_1+r_1}(q)_{n_1-r_1}\dots(a_sq)_{n_s+r_s}(q)_{n_s-r_s}}.$$

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†1390 Bumps River Rd., Centerville, MA 02632, USA (alexpatk@hotmail.com, alexepatkowski@gmail.com).

Here we focus on the $s = 2$ case, in which case Andrews' generalisation of (1.4) is given by

$$B_{n_1,n_2} = \sum_{r_1=-n_1}^{n_1} \sum_{r_2=-n_2}^{n_2} \frac{A_{r_1,r_2}}{(a_1q)_{n_1+r_1}(q)_{n_1-r_1}(a_2q)_{n_2+r_2}(q)_{n_2-r_2}}.$$

For this symmetric 2-fold Bailey pair relative to (a_1, a_2) we have the identity

$$\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} A_{n_1,n_2}\gamma_{n_1}(a_1)\gamma_{n_2}(a_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} B_{n_1,n_2}\delta_{n_1}(a_1)\delta_{n_2}(a_2), \tag{1.7}$$

where both $(\gamma_{n_1}(a_1), \delta_{n_1}(a_1))$ and $(\gamma_{n_2}(a_2), \delta_{n_2}(a_2))$ are given by (1.5)–(1.6). In [3] Andrews' main focus is the application of the conjugate pair (1.5)–(1.6) for both $(\gamma_{n_1}(1), \delta_{n_1}(1))$ and $(\gamma_{n_2}(1), \delta_{n_2}(1))$ in (1.7) with $\rho_1,\ \rho_2 \to \infty$. In [10] q-series were discovered related to both positive definite quadratic forms and indefinite ternary quadratic forms using only the 1-fold Bailey lemma. A nice consequence of [10] is that it is clear that identities that arise naturally from inserting 2-fold Bailey pairs into the 2-fold Bailey lemma may also be obtained using Bailey pairs with the 1-fold Bailey lemma. One particularly nice example from that study is the new expansion

$$(q)_\infty^3 = \sum_{N\in\mathbb{Z}} (-1)^N q^{N(N-1)/2} \sum_{\substack{n=0 \\ 2|j|\le n}}^{\infty} (-1)^{n+j} q^{n(n+1)/2-j(3j-1)/2+jN}. \tag{1.8}$$

In fact one may summarize the simple method from [10] in the following proposition.

Proposition 1. *If (γ_n, δ_n) is a conjugate Bailey pair relative to $a = 1$, then (α_N, β_N) is a Bailey pair relative to $a = q$ where*

$$\alpha_N = \frac{1-q^{2N+1}}{1-q} \sum_{j\in\mathbb{Z}} \gamma_j (-1)^{N+j} q^{\binom{j+N}{2}+j}, \tag{1.9}$$

$$\beta_N = \frac{\delta_N q^{-N}}{(q)_{2N}}. \tag{1.10}$$

Given [10], one might wonder if there is a direct relation between a 1-fold Bailey pair given a 2-fold Bailey pair. As it turns out, there are many such relations and, in general, we have the following relationship between the 1-fold and the $(s+1)$–fold Bailey pair.

Lemma 2. *Let $(A_{n_1,\dots,n_s,n}, B_{n_1,\dots,n_s,n})$ form an $(s+1)$-fold Bailey pair relative to $(a_{n_1}, a_{n_2}, \dots, a_{n_s}, a)$, and let $(\gamma^{(k)}, \delta^{(k)})$ for $k = 1, \dots, s$ be s conjugate Bailey pairs relative to a_{n_k}. Then (A_n, B_n) defined by*

$$\begin{aligned} A_n &:= \sum_{n_1,\dots,n_s\in\mathbb{Z}} \gamma_{n_1}^{(1)} \cdots \gamma_{n_s}^{(s)} A_{n_1,\dots,n_s,n} \\ B_n &:= \sum_{n_1,\dots,n_s=0}^{\infty} \delta_{n_1}^{(1)} \cdots \delta_{n_s}^{(s)} B_{n_1,\dots,n_s,n} \end{aligned} \tag{1.11}$$

forms a 1-fold Bailey pair relative to a.

Proof. We write out the $s = 1$ case. We have,

$$\begin{aligned}
&\sum_{n_1=0}^{\infty} \delta_{n_1}^{(1)}(a_1) B_{n_1,n}(a_1, a) \\
&= \sum_{n_1=0}^{\infty} \delta_{n_1}^{(1)}(a_1) \sum_{r_1=-n_1}^{n_1} \sum_{r_2=-n}^{n} \frac{A_{r_1,r_2}}{(a_1q)_{n_1+r_1}(q)_{n_1-r_1}(aq)_{n+r_2}(q)_{n-r_2}} \\
&= \sum_{r_2=-n}^{n} \frac{1}{(aq)_{n+r_2}(q)_{n-r_2}} \sum_{n_1=0}^{\infty} \delta_{n_1}^{(1)}(a_1) \sum_{r_1=-n_1}^{n_1} \frac{A_{r_1,r_2}}{(a_1q)_{n_1+r_1}(q)_{n_1-r_1}} \\
&= \sum_{r_2=-n}^{n} \frac{1}{(aq)_{n+r_2}(q)_{n-r_2}} \sum_{r_1=-\infty}^{\infty} A_{r_1,r_2} \sum_{n_1=|r_1|}^{\infty} \frac{\delta_{n_1}^{(1)}(a_1)}{(a_1q)_{n_1+r_1}(q)_{n_1-r_1}} \\
&= \sum_{r_2=-n}^{n} \frac{1}{(aq)_{n+r_2}(q)_{n-r_2}} \sum_{r_1=-\infty}^{\infty} A_{r_1,r_2} \gamma_{r_1}^{(1)}(a_1) \\
&= \sum_{r_2=-n}^{n} \frac{A_{r_2}(a)}{(aq)_{n+r_2}(q)_{n-r_2}} = B_n(a).
\end{aligned} \tag{1.12}$$

Since these steps can be repeated separately for each k, the more general case follows. □

Lemma 2 appears to be new, and allows one to prove all of Andrews' pentagonal number theorem identities in [3] using only the 1-fold Bailey lemma, by choosing $\gamma_{n_k}^{(k)} = q^{n_k^2}/(q)_\infty$, and $\delta_{n_k}^{(k)} = q^{n_k^2}$, for every k between 1 and s.

As an application of Lemma 2 we will show that Slater's well known Bailey pair (A5), see [13, p. 463], follows from the "diagonal" 2-fold Bailey pair [3, Eqs. (4.1) and (4.4)]

$$\begin{aligned}
A_{n_1,n_2} &= (-1)^{n_1+n_2} q^{\binom{n_1+n_2}{2}} \\
B_{n_1,n_2} &= \begin{cases} \dfrac{1}{(q)_{2n_1}} & \text{if } n_1 = n_2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{1.13}$$

relative to 1. First we obtain Slater's B_n:

$$\begin{aligned}
B_n &= \sum_{k=0}^{\infty} B_{k,n} q^{k^2} \\
&= \sum_{k=0}^{\infty} \chi(k = n) \frac{q^{k^2}}{(q)_{2n}} \\
&= \frac{q^{n^2}}{(q)_{2n}}.
\end{aligned} \tag{1.14}$$

For A_n we have to work a little bit harder:

$$\begin{aligned} A_n &= \frac{1}{(q;q)_\infty} \sum_{k=-\infty}^{\infty} A_{k,n} q^{k^2} \\ &= \frac{(-1)^n q^{n(n-1)/2}}{(q;q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2+nj}. \end{aligned} \tag{1.15}$$

By the Jacobi triple product identity [8, Eq. (1.6.1)]

$$\sum_{j=-\infty}^{\infty} (-a)^j q^{\binom{j}{2}} = (q)_\infty (a)_\infty (q/a)_\infty$$

this yields

$$A_n = \frac{(-1)^n q^{n(n-1)/2} (q^3,q^3)_\infty (q^{n+1};q^3)_\infty (q^{2-n};q^3)_\infty}{(q;q)_\infty}.$$

Considering the three congruence classes of n modulo 3 this finally simplifies to

$$\alpha_n = \begin{cases} 1 & \text{if } n=0 \\ q^{3k^2+k} + q^{3k^2-k} & \text{if } n=3k \\ -q^{3k^2 \pm k} & \text{if } n = 3k \pm 1. \end{cases} \tag{1.16}$$

Now given our computations, we can exploit the uniqueness of Bailey pairs to obtain a 2-fold Bailey pair from Slater's $A(3)$ Bailey pair. We show this by first observing that the α_n corresponding to Slaters [13, A(3)] is the same as α_n in (1.16) but with q replaced by q^2,

$$\alpha_n = \begin{cases} 1 & \text{if } n=0 \\ q^{6k^2+2k} + q^{6k^2-2k} & \text{if } n=3k \\ -q^{6k^2 \pm 2k} & \text{if } n = 3k \pm 1. \end{cases} \tag{1.17}$$

That is, suppose

$$B_n = \frac{q^n}{(q)_{2n}},$$

and in Lemma 2, set $s=1$, and $\gamma_{n_1}^{(1)} = q^{n_1^2}/(q)_\infty$, and $\delta_{n_1}^{(1)} = q^{n_1^2}$. The uniqueness of Bailey pairs with (1.17) and (1.15) tells us we must have

$$\begin{aligned} A_n &= \frac{(-1)^n q^{n(n-1)}}{(q^2;q^2)_\infty} \sum_{j \in \mathbb{Z}} (-1)^j q^{j(3j-1)+2nj} \\ &= \frac{1}{(q)_\infty} \sum_{r=-\infty}^{\infty} q^{r^2} A_{r,n}, \end{aligned}$$

and

$$\begin{aligned} B_n &= \frac{q^n}{(q)_{2n}} \\ &= \sum_{r=0}^{\infty} q^{r^2} B_{r,n}. \end{aligned}$$

Therefore, the $A_{r,n}$ and $B_{r,n}$ are forced once $(\gamma_{n_1}^{(1)}, \delta_{n_1}^{(1)})$ is chosen, and we have proven that we must have the symmetric 2-fold Bailey pair

$$A_{n_1,n_2} = \frac{(-1)^{n_1+n_2}}{(-q)_\infty} q^{4\binom{n_1}{2}+n_1+2\binom{n_2}{2}+2n_1n_2}, \tag{1.18}$$

$$B_{n_1,n_2} = \begin{cases} 0, & \text{if } n_2 \neq n_1, \\ \dfrac{q^{n_1-n_1^2}}{(q)_{2n_1}}, & \text{if } n_1 = n_2. \end{cases} \tag{1.19}$$

This pair gives us the identity due to L. J. Rogers [12, pg. 332, Eq. (13)] (from the $\rho_1, \rho_2 \to \infty$ case of (1.5)–(1.6))

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_{2n}} &= \frac{1}{(q)_\infty (q^2;q^2)_\infty} \sum_{n\in\mathbb{Z}} (-1)^n q^{n(2n-1)} \sum_{j\in\mathbb{Z}} (-1)^j q^{j(3j-1)+2nj} \\ &= \frac{1}{(q^3,q^4,q^5,q^6,q^7;q^{10})_\infty (q^2,q^{18};q^{20})_\infty}. \end{aligned} \tag{1.20}$$

It is important to note that the literature has a large volume of 1-fold Bailey pairs, and so our argument in obtaining (1.18)–(1.19) is a more natural and potent strategy in obtaining further 2-dimensional identities.

Andrews used (1.13) to obtain a nice two-dimensional pentagonal number theorem identity [3, Theorem 2]. For more 2-fold and 3-fold Bailey pairs see Berkovich [7].

Tactically speaking, the pair (1.14)–(1.15) presently has more utility, as there are more known conjugate Bailey pairs for the 1-fold Bailey lemma (e.g. [4]). However, (1.13) would appear to encompass a larger pool of identities overall, as (1.14)–(1.15) is obtained from the limiting case $\rho_1, \rho_2 \to \infty$ of (1.5)–(1.6) with (1.13).

2. The $\mathrm{spt}(n)$ **function of Andrews.** In [11], we encountered the double sum

$$\sum_{n_1,n_2=1}^{\infty} \frac{q^{n_1+n_2}}{(1-q^{n_1})^2 (q^{n_1+1})_\infty (1-q^{n_2})^2 \dots (1-q^{n_1+n_2})},$$

and asked if the sum over n_2 had any origin from the 1-fold Bailey lemma. The proof relied on a 2-fold Bailey pair from [9]. It was also suggested there was a new generalized form of Andrews' relation [5] $\mathrm{spt}(n) = np(n) - \frac{1}{2}N_2(n)$. Here $\mathrm{spt}(n)$ is the total number of appearances of the smallest parts of all the partitions of n, $p(n)$ is the classical unrestricted partition function, and $N_2(n)$ is the second Atkin-Garvan moment (see (2.6) and [5] for the generating function).

Lemma 3. *For n and M non-negative integers, (α_n, β_n) forms a Bailey pair relative to $a = 1$, where*

$$\alpha_n = \frac{(q)_M^2 (-1)^n (1+q^n) q^{n(3n-1)/2}}{(q)_{M-n}(q)_{M+n}}, \tag{2.1}$$

for $1 \le n \le M$, $\alpha_n = 0$ if $n > M$,

$$\alpha_0 = 1,$$

and

$$\beta_n = \frac{(q)_M}{(q)_n(q)_{n+M}}. \tag{2.2}$$

Proof. Using the inverse relation of a Bailey pair [2] (or [15, Eq. (2.4)]),

$$\alpha_n = \frac{(1-aq^{2n})(a)_n(-1)^n q^{n(n-1)/2}}{(1-a)(q)_n}\sum_{k=0}^{n}(q^{-n})_k(aq^n)_k q^k\beta_k, \tag{2.3}$$

we choose our β_n to be (2.2), insert into (2.3), and write

$$(-1)^N(1+q^N)q^{\binom{N}{2}}\sum_{j=0}^{N}\frac{(q^N)_j(q^{-N})_j q^j}{(q)_j(q)_{j+M}} = \frac{(q)_M(-1)^N(1+q^N)q^{N(3N-1)/2}}{(q)_{M-N}(q)_{M+N}} = \alpha_N,$$

for $N > 0$, and

$$\alpha_0 = \frac{1}{(q)_M}.$$

This follows from the q–Chu–Vandermonde theorem [8, Eq. (II.6)] with $(a, c, n) \mapsto (q^N, q^{M+1}, N)$, because

$$(q)_M\sum_{j=0}^{N}\frac{(q^N)_j(q^{-N})_j q^j}{(q)_j(q)_{j+M}} = \frac{(q^{M-N+1})_N q^{N^2}}{(q^{M+1})_N} = \frac{(q)_M(q)_M q^{N^2}}{(q)_{M-N}(q)_{M+N}}.$$

Finally, to respect the convention that $\alpha_0 = 1$, we multiply through by $(q)_M$. □

Corollary 4. *We have, for each natural number M,*

$$\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2(1-q^{n+1})\cdots(1-q^{n+M})} \tag{2.4}$$

$$= \frac{1}{(q)_M}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + (q)_M\sum_{n=1}^{M}\frac{(-1)^n(1+q^n)q^{n(3n+1)/2}}{(q)_{M-n}(q)_{M+n}(1-q^n)^2}.$$

Proof. Using the conjugate pair (1.5)–(1.6), differentiating with respect to ρ_1, ρ_2 and then putting $\rho_1, \rho_2 = 1$, we obtain

$$\sum_{n=1}^{\infty}(q;q)_{n-1}^2\beta_n q^n = \alpha_0\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + \sum_{n=1}^{\infty}\frac{\alpha_n q^n}{(1-q^n)^2}. \tag{2.5}$$

Applying the Bailey pair contained in Lemma 3 to (2.5) now gives the theorem. □

The q-series on the left side of (2.4) appeared in [11], and is the generating function for $\mathrm{spt}^*_M(n)$, the total number of appearances of the smallest parts of the number of partitions of n where parts greater than the smallest plus M do not occur. The first sum on the right side of (2.4) may be interpreted as $\sum_{k=0}^{n}\sigma_1(k)p_M(n-k)$, where $\sigma_1(n) = \sum_{d|n} d$, and $p_M(n)$ is the number of partitions of n into parts $\leq M$. The limiting case $M \to \infty$ is Euler's well known formula $np(n) = \sum_{k=0}^{n}\sigma_1(k)p(n-k)$, which is also an observation used by Andrews to obtain his spt(n) identity [5]. By Tannery's theorem [14, pg. 292] and [5, Eq. (3.4)], it can be seen that the limit of the second sum on the right side of Corollary 4 is

$$\frac{1}{2}\sum_{n=1}^{\infty} N_2(n)q^n = -\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{(1-q^n)^2}. \tag{2.6}$$

The second sum on the right hand side of Corollary 4 is more complicated, and is worthy of a separate study, as any information on $\mathrm{spt}_M^*(n)$ is important to better understand spt(n). The case $M \to \infty$ of Corollary 4 can now be seen as $\mathrm{spt}(n) = np(n) - \frac{1}{2}N_2(n)$. While Corollary 4 is important in its own right, it also implies the following Bailey pair.

Lemma 5. *For n and M non-negative integers, (α_M, β_M) forms a Bailey pair relative to $a = 1$, where*

$$\alpha_M = \frac{(-1)^M(1+q^M)q^{M(3M+1)/2}}{(1-q^M)^2}, \tag{2.7}$$

for $M > 0$, and

$$\alpha_0 = \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n},$$

$$\beta_M = \frac{1}{(q)_M}\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2(1-q^{n+1})\cdots(1-q^{n+M})}. \tag{2.8}$$

The point of this section is that no discussion of $\mathrm{spt}_M^*(n)$ (or Corollary 4) arose until studying some identities using the 2-fold Bailey lemma.

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