Methods and Applications of Analysis.
Vol. 27, No. 4, pp. 359–374, December 2020

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A NOTE ON NODAL SETS ON MANIFOLDS WITH LOWER RICCI BOUND*

JIANCHUN CHU† AND WENSHUAI JIANG‡

Dedicated to the 60th birthday of Professor John Urbas

Abstract. In this note, we study nodal sets of harmonic functions on Riemannian manifolds with lower Ricci curvature bound and the noncollapsing lower volume bound. Both upper and lower bound measure estimates of nodal sets are established.

Key words. Nodal sets, Ricci curvature, Noncollapsing.

Mathematics Subject Classification. Primary 58J05; Secondary 35J15.

1. Introduction. The study of nodal sets for solutions and eigenfunctions is very important for understanding geometric properties of elliptic operators. The upper and lower bound measure estimates for nodal sets have been studied for a long time. Classical results in this area may be found in Donnelly-Fefferman [14, 15], Hardt-Simon [25], Han and Lin [28, 18, 21, 22, 19, 23] and [8, 17, 13, 20, 24]. Some recent developments by Sogge-Zelditch, Colding-Minicozzi, Cheeger-Naber-Valtorta, Naber-Valtorta, Logunov and Lin-Shen could be found in [35, 12, 7, 34, 30, 31, 29] and the references therein. All the presented estimates depend on the bounded geometry of M (e.g. Riemannian curvature two sides bound and injective radius lower bound etc). Under bounded geometry conditions, the Laplacian operator on manifold has Lipschitz coefficients or at least continuous coefficients. Applying standard elliptic estimate in [16], we obtain uniform a priori $C^{1,\alpha}$ estimate for solutions of the corresponding elliptic equations. Such a priori estimates are quite crucial in the measure estimates for nodal sets (see e.g. [25]). In this paper we initiate the study of uniform measure estimates for nodal sets of harmonic function u on Riemannian manifold (M^n, g) with lower Ricci curvature, which leads to lower regularity of solution u. Precisely, under these assumptions, there is no uniform continuous coefficients for Laplacian operator. Hence, there is no uniform a priori $C^{1,\alpha}$ estimate for harmonic functions.

Let (M^n, g, p) be an n-dimensional pointed Riemannian manifold. Consider the harmonic functions u on the geodesic ball $B_2(p)$:

$$\Delta_g u = 0 \quad \text{on } B_2(p).$$

Furthermore, we assume that u is not identically zero. The doubling index for $B_s(x) \subset B_2(p)$ of u is defined by

$$N_u(x,s) := \sup_{B_{2r}(q)\subset B_s(x)} \frac{\sup_{B_{2r}(q)} |u|}{\sup_{B_r(q)} |u|},$$

Based on recent progresses of the study of manifolds with lower Ricci curvature [6, 5], we can prove the following theorem.

*Received March 19, 2020; accepted for publication July 15, 2020.
†Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208, USA (jianchun@math.northwestern.edu).
‡School of Mathematical Sciences, Zhejiang University, Zheda Road 38, Hangzhou, 310027, P. R. China (wsjiang@zju.edu.cn); and School of Mathematics and Statistics, The University of Sydney, NSW, 2006, Australia.

THEOREM 1.1. *Let* (M^n, g, p) *be an* n*-dimensional pointed Riemannian manifold satisfying* $\mathrm{Ric} \geq -\lambda$ *and* $\mathrm{Vol}(B_1(p)) \geq v > 0$. *Suppose that* $u : B_2(p) \to \mathbb{R}$ *is a harmonic function with* $N_u(p, 2) \leq \Lambda$, *then*

(1) *For any* $\varepsilon > 0$ *and* $0 < r < 1$, *there exists a constant* $C_\varepsilon(\varepsilon, \lambda, \Lambda, v, n)$ *such that*

$$\mathrm{Vol}(B_r(u^{-1}(0)) \cap B_1(p)) \leq C_\varepsilon r^{1-\varepsilon}.$$

(2) *If* $u(p) = 0$, *then there exists a positive constant* $c(\lambda, \Lambda, v, n)$ *such that*

$$\mathcal{H}^{n-1}(u^{-1}(0) \cap B_1(p)) \geq c,$$

where $\mathcal{H}^{n-1}$ *denotes the* $(n-1)$*-dimensional Hausdorff measure.*

REMARK 1.2. *Comparing with the estimates for elliptic equations with continuous coefficient in Euclidean space, here we do not have uniform* $C^{1,\alpha}$*-estimate or even* C^1 *estimate for* u. *Actually, the optimal estimate for* u *is the a priori Lipschitz estimate.*

REMARK 1.3. *If we further assume that Ricci curvature has two sided bound, based on the* ϵ*-regularity for balls closing to Euclidean space, our argument actually gives the following upper bound estimate for nodal set:*

$$\mathrm{Vol}(B_r(u^{-1}(0)) \cap B_1(p)) \leq C(\lambda, \Lambda, v, n) r. \tag{1.1}$$

REMARK 1.4. *Theorem 1.1 is the first step to study nodal sets on manifolds with lower Ricci curvature bounds. By developing new techniques, Theorem 1.1 can be improved in a follow up paper [10], where the similar estimate of* (1.1) *is obtained.*

The upper bound estimate is based on the quantitative estimate argument developed by Cheeger-Naber [6] and an approximation argument. The lower bound estimate depends on the metric cone structure of noncollapsing Ricci limit space and the argument of Colding-Minicozzi [12].

One can easily see that the lower volume bound on $B_1(p)$ is necessary for the lower bound measure estimate of nodal sets. Indeed,

EXAMPLE 1.5. *Let* $M = S^1(r) \times \mathbb{R}$ *be a cylinder of radius* r *and* g_r *be the standard Riemannian metric on it. We consider harmonic function*

$$u(x, y) = y \quad \text{on } B_2(p),$$

where $p = (re^{i\theta}, 0)$. *It is clear that* $\mathrm{Ric} = 0$ *and* $N_u(p, 2) \leq 2$. *When* $r \to 0$, *we see that* $\mathrm{Vol}(B_1(p)) \to 0$ *and*

$$\mathcal{H}^1(u^{-1}(0) \cap B_1(p)) = \mathcal{H}^1(S^1(r) \times \{0\}) = 2\pi r \to 0.$$

Similarly, Consider $M = S^1(r) \times \mathbb{R}^2$. For any $N > 0$, let $u(x, y_1, y_2) = f(y_1, y_2)$ where f is a homogenous harmonic polynomial with degree N. Choosing $r \leq r(N)$ sufficiently small, $\mathcal{H}^2(u^{-1}(0) \cap B_1(p))$ could be arbitrarily small. Hence, without the assumption of volume lower bound, we cannot expect $\mathcal{H}^2(u^{-1}(0) \cap B_1(p))$ has a linearly lower bound by frequency or doubling index.

The paper is organized as follows. In Section 2, we prove the upper bound measure estimate for nodal sets. The lower bound measure estimate is established in Section 3. In Appendix, we present some preliminary estimates of harmonic functions on Euclidean space.

Acknowlegments. W. Jiang was supported by NSFC no. 11701507 and the Fundamental Research Funds for the Central Universities 2019QNA3001 and DECRA 190101471. Both authors would like to thank the referee for useful suggestions.

2. Upper bound estimate of nodal set. In this section, we prove the upper bound estimate in Theorem 1.1. The main tools are the decomposition argument introduced by Cheeger-Naber [6], and the approximation by Euclidean harmonic functions. Let us begin with the following approximation lemma which is based on the harmonic function convergence along Gromov-Hausdorff convergence.

LEMMA 2.1. *For any $\gamma \in (0,1)$, there exists a constant $\delta(\gamma, \lambda, \Lambda, n)$ such that if*

(1) *u is a nonzero harmonic function on $B_2(x)$ with $N_u(x,2) \leq \Lambda$;*

(2) *$d_{GH}(B_1(x), B_1(0^n)) \leq \delta^3$ and $\mathrm{Ric} \geq -\lambda$ on $B_1(x)$,*

then there exists a nonzero harmonic polynomial P on Euclidean space $\mathbb{R}^n$ with $\deg P \leq \log_2 \Lambda$ and δ^2-GH approximation $\Phi : B_{2\delta}(x) \to B_{2\delta}(0^n)$ such that

$$\Phi\left(B_{\gamma\delta}(u^{-1}(0)) \cap B_\delta(x)\right) \subset B_{2\gamma\delta}(P^{-1}(0)) \cap B_\delta(0^n),$$

where $0^n \in \mathbb{R}^n$ is the origin and $B_1(0^n)$ is the unit ball in $\mathbb{R}^n$.

Proof. We argue by contradiction, assuming that there exists a sequence of harmonic functions u_i on $B_1(x_i)$ with respect to g_i such that

(1) $N_{u_i}(x_i, 2) \leq \Lambda$;

(2) $d_{GH}(B_1(x_i), B_1(0^n)) \leq i^{-3}$ and $\mathrm{Ric}(g_i) \geq -\lambda g_i$ on $B_1(x_i)$;

(3) For each i, there is no i^{-2}-GH approximation Φ and harmonic polynomial P on $\mathbb{R}^n$ such that $\deg P \leq \log_2 \Lambda$ and

$$\Phi\left(B_{\gamma i^{-1}}(u_i^{-1}(0)) \cap B_{i^{-1}}(x_i)\right) \subset B_{2\gamma i^{-1}}(P^{-1}(0)) \cap B_{i^{-1}}(0^n).$$

For convenience, we use $B_r(x_i, g_i)$ to denote the r-geodesic ball centered at x_i with respect to g_i. By (2) and scaling, we obtain

$$d_{GH}\left(B_i(x_i, i^2 g_i), B_i(0^n)\right) \leq i^{-2}$$

and

$$\mathrm{Ric}(i^2 g_i) \geq -\lambda i^{-2}(i^2 g_i) \text{ on } B_i(x_i, i^2 g_i).$$

Then there exists i^{-2}-GH approximation $\Phi_i : B_i(x_i, i^2 g_i) \to B_i(0^n)$. Clearly, u_i is also a harmonic function on $B_i(x_i, i^2 g_i)$ with $N_{u_i}(x_i, 2i) \leq \Lambda$ with respect to $i^2 g_i$. Without loss of generality, we assume that

$$\sup_{B_1(x_i, i^2 g_i)} |u_i| = 1.$$

Since $N_{u_i}(x_i, i) \leq \Lambda$, for any $1 < r \leq i$, we see that

$$\sup_{B_r(x_i, i^2 g_i)} |u_i| \leq C(r, \Lambda).$$

Moreover, by using standard elliptic estimate to the Bochner formula $\Delta|\nabla u_i|^2 = 2|\nabla^2 u_i|^2 + 2\mathrm{Ric}(\nabla u_i, \nabla u_i) \geq -\lambda i^{-2}|\nabla u_i|^2$ we can get uniformly pointwise bound on $|\nabla u_i|$ on any compact set (see Cheng-Yau gradient estimate in [9], also cf. [1, Theorem

7.1]), which means u_i is uniformly Lipschitz on any compact subset. Letting $i \to \infty$, we have pointwisely and uniformly on compact subset that

$$u_i \to h \tag{2.1}$$

for a harmonic function h on $\mathbb{R}^n$ (see [5, Section 4.9]). In particular, the nodal set of u_i would converge to a subset of nodal set of h in the Gromov-Hausdorff sense. Recalling $\sup_{B_1(x_i,i^2g_i)}|u_i| = 1$, we obtain $\sup_{B_1(0^n)}|h| = 1$, which implies h is not identically zero. Thanks to $N_{u_i}(x_i, i) \le \Lambda$, we obtain

$$N_h(0^n, R) \le \Lambda \quad \text{for any } R \in (0, \infty).$$

Using Theorem A.2, the harmonic function h is actually a harmonic polynomial P with $\deg P \le \log_2 \Lambda$. When i is sufficiently large, (2.1) implies that

$$\Phi_i\left(B_\gamma(u_i^{-1}(0), i^2g_i) \cap B_1(x_i, i^2g_i)\right) \subset B_{2\gamma}(P^{-1}(0)) \cap B_1(0^n).$$

By scaling again, we see that

$$\Phi_i\left(B_{\gamma i^{-1}}(u_i^{-1}(0), g_i) \cap B_{i^{-1}}(x_i, g_i)\right) \subset B_{2\gamma i^{-1}}(P^{-1}(0)) \cap B_{i^{-1}}(0^n),$$

which contradicts with (3). Hence we complete the proof. □

As a direct consequence of the above approximation, we can get

LEMMA 2.2. *For any $\gamma, s \in (0,1)$, there exists a constant $\delta(\gamma, \lambda, \Lambda, n)$ such that if*

(1) *u is a harmonic function on $B_s(x)$ with $N_u(x, s) \le \Lambda$;*
(2) *$d_{GH}(B_s(x), B_s(0^n)) \le \delta^3 s$ and $\mathrm{Ric} \ge -\lambda s^{-1}$ on $B_s(x)$,*

then $B_{\gamma\delta s}(u^{-1}(0)) \cap B_{\delta s}(x)$ can be covered by $C(\Lambda, n)\gamma^{1-n}$ balls of radius $4\gamma\delta s$.

Proof. By scaling, we assume that $s = 1$ without loss of generality. Using Lemma 2.1, there exist constant δ, harmonic polynomial P and δ^2-GH approximation $\Phi : B_{2\delta}(0^n) \to B_{2\delta}(x)$ such that

(1) $\deg P \le \log_2 \Lambda$;
(2) $\Phi\left(B_{\gamma\delta}(u^{-1}(0)) \cap B_\delta(x)\right) \subset B_{2\gamma\delta}(P^{-1}(0)) \cap B_\delta(0^n)$.

By Theorem B.1 (see also [34]), we obtain

$$\mathrm{Vol}(B_{2\gamma\delta}(P^{-1}(0)) \cap B_\delta(0^n)) \le C(\Lambda, n)\gamma\delta^n.$$

This implies that $B_{2\gamma\delta}(P^{-1}(0)) \cap B_\delta(0^n)$ can be covered by $C(\Lambda, n)\gamma^{1-n}$ balls of radius $3\gamma\delta$. To see this, let $\{x_1, \cdots, x_N\}$ be an $\gamma\delta/10$-net of $P^{-1}(0) \cap B_\delta(0^n)$. Then $B_{\gamma\delta/20}(x_i) \cap B_{\gamma\delta/20}(x_j) = \emptyset$ for $i \neq j$ and $B_{\gamma\delta/20}(x_i) \subset B_{2\gamma\delta}(P^{-1}(0)) \cap B_\delta(0^n)$, and $P^{-1}(0) \cap B_\delta(0^n) \subset \cup_{i=1}^N B_{\gamma\delta}(x_i)$. Therefore, $B_{2\gamma\delta}(P^{-1}(0)) \cap B_\delta(0^n)$ can be covered by N balls of radius $3\gamma\delta$. By volume lower bound for each ball, we get

$$N \le \frac{\mathrm{Vol}(B_{2\gamma\delta}(P^{-1}(0)) \cap B_\delta(0^n))}{\min_{i=1,\cdot,N} \mathrm{Vol}(B_{\gamma\delta/20}(x_i))} \le C(\Lambda, n)\gamma^{1-n}. \tag{2.2}$$

Hence $B_{2\gamma\delta}(P^{-1}(0)) \cap B_\delta(0^n)$ can be covered by $C(\Lambda, n)\gamma^{1-n}$ balls of radius $3\gamma\delta$. Combining this with the fact that Φ is a δ^2-GH approximation, we obtain the desired covering of $B_{\gamma\delta}(u^{-1}(0)) \cap B_\delta(x)$. □

The following lemma can be regarded as a special case of the upper bound estimate in Theorem 1.1.

LEMMA 2.3. *Let u be a harmonic function on $B_2(x)$ with $N_u(x,2) \leq \Lambda$. There exists a constant $\delta(\lambda, \Lambda, n)$ such that if $d_{GH}(B_{2s}(y), B_{2s}(0^n)) \leq \delta^3 s$ for any $B_{2s}(y) \subset B_2(x)$ and* Ric $\geq -\lambda$ *on $B_2(x)$, then*

$$\mathrm{Vol}(B_r(u^{-1}(0)) \cap B_1(x)) \leq C(\varepsilon, \delta, n) r^{1-\varepsilon} \quad \textit{for any } 0 < r < 1.$$

Proof. We first assume that

$$\gamma^{i_0}\delta \leq r \leq \gamma^{i_0+1}\delta,$$

where γ is a small constant to be determined later and δ is the constant in Lemma 2.2. For $0 \leq i \leq i_0$, we define the families $\{\mathcal{T}_i\}$ by induction on i, where $\mathcal{T}_i$ is the set of balls of radius $\gamma^i\delta$. When $i = 0$, we take a covering of $B_1(x)$ with δ-balls to be $\mathcal{T}_0$. It then follows that

$$|\mathcal{T}_0| = C(n)\delta^{-n}.$$

When $i > 0$, for each ball $B_{\gamma^i\delta}(y) \in \mathcal{T}_i$, we take a minimal covering of $B_r(u^{-1}(0)) \cap B_{\gamma^i\delta}(y)$ by balls of radius $\gamma^{i+1}\delta$. We define all these balls so obtained to be $\mathcal{T}_{i+1}$.

Applying Lemma 2.2, for each ball $B_{\gamma^i\delta}(y) \in \mathcal{T}_i$, $B_r(u^{-1}(0)) \cap B_{\gamma^i\delta}(y)$ can be covered by $C(\Lambda, n)\gamma^{1-n}$ balls of radius $\gamma^{i+1}\delta$. This implies that

$$\frac{|\mathcal{T}_{i+1}|}{|\mathcal{T}_i|} \leq C(\Lambda, n)\gamma^{1-n}.$$

Now we choose γ sufficiently small such that $C(\Lambda, n) = \gamma^{-\varepsilon}$. Thus,

$$\frac{|\mathcal{T}_{i+1}|}{|\mathcal{T}_i|} \leq \gamma^{1-n-\varepsilon}.$$

It then follows that

$$|\mathcal{T}_{i_0}| \leq \gamma^{(1-n-\varepsilon)i_0}|\mathcal{T}_0| \leq C(n)\delta^{-n}\gamma^{(1-n-\varepsilon)i_0}.$$

Hence, we obtain

$$\mathrm{Vol}(B_r(u^{-1}(0)) \cap B_1(x)) \leq C(n)|\mathcal{T}_{i_0}|\gamma^{i_0 n} \leq C_\varepsilon r^{1-\varepsilon},$$

as required. □

Now we give the proof of upper bound estimate in Theorem 1.1.

Proof of (1) in Theorem 1.1. For convenience, we use δ to denote the constant δ^3 in Lemma 2.3 and introduce some notations:

$$r_{\mathrm{Reg},\delta}(x) = \max\{r \mid d_{GH}(B_s(x), B_s(0^n)) \leq \delta s \text{ for all } 0 < s \leq r\},$$

$$D_s = \{x \in B_1(p) \mid r_{\mathrm{Reg},\delta}(x) < s\}.$$

Combining this with almost Reifenberg theorem [11, Theorem 0.8], it was proved in Cheeger-Jiang-Naber [5, Theorem 1.3] (see also [6]) that for any $0 < s \leq 1$,

$$\mathrm{Vol}(D_s) \leq C(\lambda, v, n)s^2.$$

Then D_s can be covered by $C(\lambda, v, n)s^{2-n}$ balls of radius s. Let $A_0 = B_1(p) \setminus D_{2^{-1}}$ and $A_i = D_{2^{-i}} \setminus D_{2^{-i-1}}$ for $i \geq 1$. We assume that the covering of $D_{2^{-i}}$ is $\{B_{2^{-i}}(x_{ij})\}$ where $x_{ij} \in D_{2^{-i}}$, i.e.,

$$D_{2^{-i}} \subset \bigcup_j B_{2^{-i}}(x_{ij}).$$

For any $r > 0$, we compute

$$\begin{aligned}\mathrm{Vol}(B_r(u^{-1}(0)) \cap B_1(p)) &= \sum_i \mathrm{Vol}(B_r(u^{-1}(0)) \cap A_i) \\ &\leq \sum_i \sum_j \mathrm{Vol}(B_r(u^{-1}(0)) \cap B_{2^{-i}}(x_{ij})).\end{aligned}$$

Suppose that $2^{-i_0} \leq r < 2^{-i_0+1}$ for some i_0. When $i \geq i_0$, it is clear that

$$\mathrm{Vol}(B_r(u^{-1}(0)) \cap B_{2^{-i}}(x_{ij})) \leq \mathrm{Vol}(B_{2^{-i}}(x_{ij})) \leq C2^{-in},$$

which implies

$$\begin{aligned}\sum_{i \geq i_0} \sum_j \mathrm{Vol}(B_r(u^{-1}(0)) \cap B_{2^{-i}}(x_{ij})) &\leq C \sum_{i \geq i_0} 2^{i(n-2)-in} \\ &\leq C2^{-2i_0} \leq Cr^2.\end{aligned}$$

When $i < i_0$, by scaling and Lemma 2.3, we see that

$$\mathrm{Vol}(B_r(u^{-1}(0)) \cap B_{2^{-i}}(x_{ij})) \leq C_\varepsilon (2^i r)^{1-\varepsilon} (2^{-i})^n.$$

Here we used $x_{ij} \in D_{2^{-i}}$.

It then follows that

$$\begin{aligned}\sum_{i < i_0} \sum_j \mathrm{Vol}(B_r(u^{-1}(0)) \cap B_{2^{-i}}(x_{ij})) &\leq C_\varepsilon r^{1-\varepsilon} \sum_{i < i_0} 2^{i(n-2)+i(1-\varepsilon-n)} \\ &\leq C_\varepsilon r^{1-\varepsilon} \sum_i 2^{-i(1+\varepsilon)} \leq C_\varepsilon r^{1-\varepsilon}.\end{aligned}$$

Therefore, combining the above inequalities, we have

$$\mathrm{Vol}(B_r(u^{-1}(0)) \cap B_1(p)) \leq C_\varepsilon r^{1-\varepsilon},$$

as required. □

3. Lower bound estimate of nodal set. In this section, we give the proof of lower bound estimate in Theorem 1.1. The argument is motivated by Colding-Minicozzi's lower bound estimate on nodal set for eigenfunctions (see [12]). Although lacking the monotone frequency, the following lemma can be proved by using metric cone structure.

PROPOSITION 3.1. *Let u be a harmonic function on $B_2(p)$ with $N_u(p, 2) \leq \Lambda$ and $u(p) = 0$. Suppose that* $\mathrm{Ric} \geq -\lambda$ *on $B_2(p)$ and* $\mathrm{Vol}(B_1(p)) \geq v$. *Then there exists a constant $R = R(\lambda, \Lambda, v, n)$ such that*

$$\left| \int_{B_R(p)} u \right| \leq \frac{1}{3} \int_{B_R(p)} |u|.$$

To prove this proposition, the following approximation result is important. One observation is that if a harmonic function on a metric cone with polynomial growth vanishes at cone vertex then the integral of the harmonic function over any metric ball with center at cone vertex must vanish.

LEMMA 3.2. *Let u be a harmonic function on $B_2(p)$ with $N_u(p,2) \le \Lambda$ and $u(p) = 0$. Suppose that* $\mathrm{Ric} \ge -\lambda$ *on $B_2(p)$ and* $\mathrm{Vol}(B_1(p)) \ge v$. *There exists a constant $\delta = \delta(\lambda, \Lambda, n, v)$ such that if*

$$d_{GH}(B_2(p), B_2(x^*)) \le \delta^3$$

for some metric cone $(C(X), x^*)$, then

$$\left| \int_{B_\delta(p)} u \right| \le \frac{1}{3} \int_{B_\delta(p)} |u|.$$

Proof. We argue by contradiction, assuming that there exists a sequence of harmonic functions u_i on $B_2(p_i) \subset M_i$ such that

(a) $u_i(p_i) = 0$ and $N_{u_i}(p_i, 2) \le \Lambda$;
(b) $\mathrm{Ric} \ge -\lambda$ on $B_2(p_i)$ and $\mathrm{Vol}(B_2(p_i)) \ge v$;
(c) There exists a metric cone $(C(X_i), x_i^*)$ such that

$$d_{GH}(B_2(p_i), B_2(x_i^*)) \le i^{-3};$$

(d) $\left|\int_{B_{i^{-1}}(p_i)} u_i\right| > \frac{1}{3}\int_{B_{i^{-1}}(p_i)} |u_i|$.

By rescaling $\tilde g_i = i^2 g_i$ we have

(1) $u_i(p_i) = 0$ and $N_{u_i}(p_i, 2i) \le \Lambda$;
(2) $\mathrm{Ric} \ge -\lambda i^{-2}$ on $B_{2i}(p_i, \tilde g_i)$ and $\mathrm{Vol}(B_{2i}(p_i, \tilde g_i)) \ge v i^n$;
(3) There exists a metric cone $(C(X_i), x_i^*)$ such that

$$d_{GH}(B_{2i}(p_i, \tilde g_i), B_{2i}(x_i^*)) \le i^{-2};$$

(4) $\left|\int_{B_1(p_i,\tilde g_i)} u_i\right| > \frac{1}{3}\int_{B_1(p_i,\tilde g_i)} |u_i|$.

Without loss of generality, we assume that $\sup_{B_1(p_i,\tilde g_i)} |u_i| = 1$. By (1), we have $\sup_{B_r(p_i,\tilde g_i)} |u_i| \le C(r, \Lambda)$ for any $r \le i$. By compactness, passing to a subsequence, we assume that u_i converges to h, where h is a nonzero harmonic function on $C(X)$ with $h(x^*) = 0$. Combining $N_{u_i}(p_i, 2) \le \Lambda$ and the mean value inequality for subharmonic functions (see [27, Theorem 2.1]), we obtain

$$\sup_{B_1(p_i,\tilde g_i)} u_i^2 \le \Lambda^2 \sup_{B_{1/2}(p_i,\tilde g_i)} u_i^2 \le C\Lambda^2 \int_{B_1(p_i,\tilde g_i)} u_i^2 \le C\Lambda^2 \left(\sup_{B_1(p_i,\tilde g_i)} |u_i|\right)\left(\int_{B_1(p_i,\tilde g_i)} |u_i|\right).$$

This implies

$$\sup_{B_{\frac{1}{2}}(p_i,\tilde g_i)} |u_i| \le \sup_{B_1(p_i,\tilde g_i)} |u_i| \le C\Lambda^2 \int_{B_1(p_i,\tilde g_i)} |u_i|. \tag{3.1}$$

Recalling $\sup_{B_1(p_i,\tilde g_i)} |u_i| = 1$ and using $N_{u_i}(p_i, 2) \le \Lambda$ again, we see that

$$\sup_{B_{1/2}(p_i,\tilde g_i)} |u_i| \ge \frac{1}{\Lambda}.$$

It then follows that

$$\int_{B_1(p_i,\tilde{g}_i)} |u_i| \geq \frac{1}{C\Lambda^3}. \tag{3.2}$$

On the other hand, by the expansion of harmonic function with order at most $C(\Lambda)$ on metric cone (see [5]) and $h(x^*) = 0$, we have

$$\lim_{i\to\infty} \int_{B_1(p_i,\tilde{g}_i)} u_i = \int_{B_1(x^*)} h = 0.$$

Thanks to (4), we obtain

$$0 = \lim_{i\to\infty} \left| \int_{B_1(p_i,\tilde{g}_i)} u_i \right| \geq \lim_{i\to\infty} \frac{1}{3} \int_{B_1(p_i,\tilde{g}_i)} |u_i| \geq 0,$$

which contradicts with (3.2). □

Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. As in [6], we define the volume ratio and the (t, s)-volume energy:

$$\mathcal{V}_r = \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}_{-\lambda}(r)} \quad \text{and} \quad \mathcal{W}_{t,s} = \log \frac{\mathcal{V}_s}{\mathcal{V}_t},$$

where $\mathrm{Vol}_{-\lambda}(r)$ denotes the volume of a ball of radius r in n-dimensional hyperbolic space of curvature $-\lambda/(n-1)$.

By the "almost volume cone implies almost metric cone" theorem of [2] (see also [1, 3, 4]), for constant δ in Lemma 3.2, there exists a constant $\varepsilon = \varepsilon(\delta, n, v, \lambda)$ such that if

$$\mathcal{W}_{r,2r} \leq \varepsilon,$$

then

$$d_{GH}(B_r(p), B_r(x^*)) \leq \delta r, \tag{3.3}$$

for some metric cone $(C(X), x^*)$.

Let

$$I_0 = \min\{i \in \mathbb{Z}_{\geq 0} \mid \mathcal{W}_{2^{-i-1},2^{-i}} \leq \varepsilon\}.$$

Thus,

$$\log \frac{\mathrm{Vol}_{-\lambda}(1)}{\mathrm{Vol}(B_1(p))} = \log \frac{1}{\mathcal{V}_1} \geq \sum_{i=0}^{\infty} \mathcal{W}_{2^{-i-1},2^{-i}} \geq \sum_{i=0}^{I_0} \mathcal{W}_{2^{-i-1},2^{-i}} \geq (I_0+1)\varepsilon.$$

It then follows that

$$I_0 \leq C(v, \varepsilon) = C(\lambda, \Lambda, v, n).$$

We define $R = 2^{-I_0-1}$. By (3.3) and the definition of I_0, we have

$$d_{GH}(B_R(p), B_R(x^*)) \leq \delta R,$$

for some metric cone $(C(X), x^*)$. Thus Proposition 3.1 follows directly from Lemma 3.2. □

Using Proposition 3.1, we can apply the argument of [12] to prove the lower bound estimate in Theorem 1.1.

Proof of (2) in Theorem 1.1. Let u^+ and u^- be the positive and negative parts of u, i.e.,

$$u = u^+ - u^- \quad \text{and} \quad |u| = u^+ + u^-.$$

We define

$$B^+ = \{u > 0\} \cap B_R(p) \quad \text{and} \quad B^- = \{u < 0\} \cap B_R(p),$$

where R is chosen from Proposition 3.1. To prove Theorem 1.1, by the isoperimetric inequality (see e.g. [26]), it suffices to prove

$$\mathrm{Vol}(B^+) \geq c \quad \text{and} \quad \mathrm{Vol}(B^-) \geq c$$

for some constant c. Here we just prove the lower bound of $\mathrm{Vol}(B^+)$, as the argument for B^- is almost identical.

By the mean value inequality for subharmonic functions (see [27, Theorem 2.1]), we obtain the interior estimate

$$\sup_{B_{\frac{R}{2}}(p)} u^2 \leq CR^{-n} \int_{B_R(p)} u^2.$$

By the similar argument of (3.1), we obtain

$$\sup_{B_R(p)} |u| \leq C\Lambda^2 R^{-n} \int_{B_R(p)} |u|,$$

which implies

$$\int_{B_R(p)} u^2 \leq C\Lambda^2 R^{-n} \left(\int_{B_R(p)} |u| \right)^2. \tag{3.4}$$

On the other hand, from Proposition 3.1, we have

$$\left| \int_{B_R(p)} u^+ - \int_{B_R(p)} u^- \right| \leq \frac{1}{3} \int_{B_R(p)} u^+ + \frac{1}{3} \int_{B_R(p)} u^-.$$

It then follows that

$$\int_{B_R(p)} u^- \leq 2 \int_{B_R(p)} u^+.$$

Thus,

$$\int_{B_R(p)} |u| = \int_{B_R(p)} u^+ + \int_{B_R(p)} u^- \leq 3 \int_{B_R(p)} u^+. \tag{3.5}$$

Combining (3.4) and (3.5), we have

$$\left(\int_{B_R(p)} |u|\right)^2 \leq 9\left(\int_{B^+} u^+\right)^2 \leq 9\mathrm{Vol}(B^+)\int_{B^+} u^2 \leq C\Lambda^2 R^{-n}\mathrm{Vol}(B^+)\left(\int_{B_R(p)} |u|\right)^2.$$

Then we get

$$\mathrm{Vol}(B^+) \geq \frac{R^n}{C\Lambda^2},$$

as required. □

Appendix A. Some results of harmonic functions on $\mathbb{R}^n$. We collect some results of harmonic functions on $\mathbb{R}^n$. The following theorem is the interior estimates of derivatives.

THEOREM A.1 (Theorem 2.10 of [16]). *Let h be a harmonic function on $B_{2R}(0^n)$. Then for any k, we have*

$$\sup_{B_R(0^n)} |\nabla^k u| \leq \left(\frac{nk}{R}\right)^k \sup_{B_{2R}(0^n)} |u|.$$

THEOREM A.2. *Let h be a harmonic function on $\mathbb{R}^n$. If there exists a constant Λ such that*

$$N_h(0^n, R) \leq \Lambda \quad \textit{for any } R \in (0, \infty),$$

then h is a harmonic polynomial with $\deg h \leq \log_2 \Lambda$.

Proof. Without loss of generality, we assume that $\sup_{B_1(0^n)} |u| = 1$. Since the doubling index is bounded by Λ, for any positive integer i, we have

$$\sup_{B_{2^i}(0^n)} |u| \leq \Lambda^i.$$

It then follows that

$$\sup_{B_{2R}(0^n)} |u| \leq \Lambda^2 R^{\log_2 \Lambda}.$$

Choosing $k = [\log_2 \Lambda] + 1$ ($[\cdot]$ denotes the greatest integer function), by Theorem A.1, we obtain

$$\sup_{B_R(0^n)} |\nabla^k u| \leq \left(\frac{nk}{R}\right)^k \Lambda^2 R^{\log_2 \Lambda} = \frac{(nk)^k \Lambda^2}{R^{k-\log_2 \Lambda}}.$$

Letting $R \to \infty$, it is clear that $\nabla^k u = 0$, which implies h is a polynomial with $\deg h \leq [\log_2 \Lambda] \leq \log_2 \Lambda$. □

Appendix B. Volume estimates of real varieties. Let P be a nonzero polynomial of degree d on $\mathbb{R}^n$. In [36], Wongkew established the following volume estimate of tubular neighbourhood of real algebraic variety $P^{-1}(0)$ (see also [32]):

THEOREM B.1 (Main Theorem of [36]). *For any $r \in (0, 1)$, there exists a constant $C(d, n)$ such that*

$$\mathrm{Vol}(B_r(P^{-1}(0)) \cap B_1(0^n)) \leq Cr.$$

We use $Q_1(0^n)$ to denote the n-cube centered at the origin 0^n with radius 1, i.e.,

$$Q_1(0^n) = [-1,1]^n.$$

Then the following theorem is equivalent to Theorem B.1.

Theorem B.2. *For any $r \in (0,1)$, there exists a constant $C(d,n)$ such that*

$$\mathrm{Vol}(B_r(P^{-1}(0)) \cap Q_1(0^n)) \leq Cr. \tag{B.1}$$

Remark B.3. *Actually, in [36], the constant $C(d,n)$ has explicit expression. But (B.1) is enough for our use. Because of this, compared to [36], the proof of Theorem B.2 is more simple and clear.*

Following the argument of [36], we give a proof of Theorem B.2 for the reader's convenience.

Proof. We use V to denote the real algebraic variety $P^{-1}(0)$. For $r \in (0,1)$, we divide $Q_1(0^n)$ into r^{-n} subcubes with radius r and denote them by $Q_{i_1\cdots i_n}$ for $i_k = 1,\cdots,r^{-1}$ and $k = 1,\cdots,n$. Here we assume that r^{-1} is an integer without loss of generality.

Lemma B.4. *There exists a constant $C(d,n)$ such that $V \cap Q_1(0^n)$ can be covered by Cr^{1-n} subcubes.*

Proof of Lemma B.4. We argue by induction on dimension n. When $n = 1$, Lemma B.4 is a direct consequence of the fundamental theorem of algebra. Next, we assume that this lemma is true for $n-1$ dimensional case.

Each subcubes $Q_{i_1\cdots i_n}$ has $2n$ faces. We denote these faces by $F_{i_1\cdots i_n,\gamma}$ for $\gamma = 1,\cdots,2n$. It is clear that each face can be regarded as a $(n-1)$-cube with radius r in $\mathbb{R}^{n-1}$.

Let $\{e_\alpha\}_{\alpha=1}^n$ be the standard basis of $\mathbb{R}^n$, i.e., the α-th component of e_α is one and the others are zero. We define

$$\begin{aligned}
W_\alpha &= \bigcup \{F_{i_1\cdots i_n,\gamma} \mid \text{The normal vector of } F_{i_1\cdots i_n,\gamma} \text{ is } e_\alpha\},\\
J_\alpha &= \{F_{i_1\cdots i_n,\gamma} \mid F_{i_1\cdots i_n,\gamma} \subset W_\alpha \text{ and } V \cap F_{i_1\cdots i_n,\gamma} \neq \emptyset\},\\
J &= \bigcup_\alpha J_\alpha = \{F_{i_1\cdots i_n,\gamma} \mid V \cap F_{i_1\cdots i_n,\gamma} \neq \emptyset\}.
\end{aligned}$$

In fact, J_α is the set of faces $F_{i_1\cdots i_n,\gamma}$ that intersects V and has normal vector e_α, and $J = \cup_\alpha J_\alpha$ is just the set of faces $F_{i_1\cdots i_n,\gamma}$ that intersects V. The set W_α consists of $(r^{-1}+1)$ slices $H_{\alpha,\beta}$. Each $H_{\alpha,\beta}$ can be regarded as a $(n-1)$-cube with radius 1 in $\mathbb{R}^{n-1}$.

Fixing α, we define

$$\begin{aligned}
U_{\alpha,b} &= \{H_{\alpha,\beta} \mid H_{\alpha,\beta} \subset V\},\\
U_{\alpha,g} &= \{H_{\alpha,\beta} \mid H_{\alpha,\beta} \text{ is not contained in } V\},
\end{aligned}$$

and

$$\begin{aligned}
J_{\alpha,b} &= \{F_{i_1\cdots i_n,\gamma} \mid F_{i_1\cdots i_n,\gamma} \subset H_{\alpha,\beta} \text{ for some } H_{\alpha,\beta} \in U_{\alpha,b}\},\\
J_{\alpha,g} &= \{F_{i_1\cdots i_n,\gamma} \mid F_{i_1\cdots i_n,\gamma} \subset H_{\alpha,\beta} \text{ for some } H_{\alpha,\beta} \in U_{\alpha,g}\}.
\end{aligned}$$

By definition of $U_{\alpha,b}$, for each $H_{\alpha,\beta} \in U_{\alpha,b}$, the polynomial P is identically zero on $H_{\alpha,\beta}$. We claim that

$$|U_{\alpha,b}| \leq d.$$

If not, then $|U_{\alpha,b}| \geq d+1$. Then for any line L paralleled to e_α, there are at least $d+1$ intersection point of L and V. This implies that $P|_L$ (the polynomial P restricted on L) has at least $d+1$ real roots. Since the degree of P is d, then the degree of $P|_L$ is at most d. Using the fundamental theorem of algebra, we see that $P|_L = 0$. Since L is arbitrary, we obtain P is identically zero, which contradicts with $P \not\equiv 0$.

For any $H_{\alpha,\beta} \in U_{\alpha,b}$, $H_{\alpha,\beta}$ consist of Cr^{1-n} faces $F_{i_1\cdots i_n,\gamma}$. Since $H_{\alpha,\beta} \subset V$, then each $F_{i_1\cdots i_n,\gamma} \in J_\alpha$. It then follows that

$$|J_{\alpha,b}| \leq |U_{\alpha,b}| \cdot Cr^{1-n} \leq Cdr^{1-n}. \tag{B.2}$$

For any $H_{\alpha,\beta} \in U_{\alpha,g}$, we regard $H_{\alpha,\beta}$ as a $(n-1)$-cube with radius 1. The face $F_{i_1\cdots i_n,\gamma} \subset H_{\alpha,\beta}$ can be regarded as a $(n-1)$-cube with radius r. The degree of $P|_{H_{\alpha,\beta}}$ is at most d and we have

$$(P|_{H_{\alpha,\beta}})^{-1}(0) \cap H_{\alpha,\beta} = V \cap H_{\alpha,\beta}.$$

Applying the induction hypothesis to $P|_{H_{\alpha,\beta}}$, $V \cap H_{\alpha,\beta}$ can be covered by Cr^{2-n} $(n-1)$-cubes (faces) $F_{i_1\cdots i_n,\gamma}$. Hence,

$$|J_{\alpha,g}| \leq |U_{\alpha,g}| \cdot Cr^{2-n} \leq C(r^{-1}+1)r^{2-n} \leq Cr^{1-n}. \tag{B.3}$$

Combining (B.2) and (B.3), we have

$$|J_\alpha| = |J_{\alpha,b}| + |J_{\alpha,g}| \leq Cr^{1-n},$$

which implies

$$|J| = \sum_\alpha |J_\alpha| \leq Cr^{1-n}. \tag{B.4}$$

We define

$$\begin{aligned}
I &= \{Q_{i_1\cdots i_n} \mid V \cap Q_{i_1\cdots i_n} \neq \emptyset\}, \\
I_b &= \{Q_{i_1\cdots i_n} \in I \mid V \cap F_{i_1\cdots i_n,\gamma} = \emptyset \text{ for any } \gamma\}, \\
I_g &= \{Q_{i_1\cdots i_n} \in I \mid V \cap F_{i_1\cdots i_n,\gamma} \neq \emptyset \text{ for some } \gamma\}.
\end{aligned}$$

To prove Lemma B.4, it suffices to prove

$$|I| \leq Cr^{1-n}.$$

It is clear that

$$I = I_b \cup I_g \quad \text{and} \quad |I| = |I_b| + |I_g|.$$

By the definition of I_b, V has at least $|I_b|$ components. Using [33, Theorem 2], the sum of the Betti number of V is less than $d(2d-1)^{n-1}$. In particular, V has at most $d(2d-1)^{n-1}$ components. This implies that

$$|I_b| \leq d(2d-1)^{n-1}. \tag{B.5}$$

For the term $|I_g|$, since every face can be contained in at most 2^n subcubes. It then follows that

$$|I_g| \leq 2^n |J| \leq C r^{1-n}, \tag{B.6}$$

where we used (B.4) in the second inequality. Combining (B.5) and (B.6), we have

$$|I| = |I_b| + |I_g| \leq C r^{1-n},$$

as desired. □

We now complete the proof of Theorem B.2. Using Lemma B.4, $V \cap Q_1(0^n)$ can be covered by Cr^{1-n} subcubes. This implies that $B_r(V) \cap Q_1(0^n)$ can also be covered by Cr^{1-n} subcubes. Therefore, we obtain

$$\mathrm{Vol}(B_r(V) \cap Q_1(0^n)) \leq C r^{1-n} \cdot C r^n \leq C r,$$

as required. □

Appendix C. Doubling Index. We do not use the following estimate (Lemma C.1) in the proof of Theorem 1.1, but it may have independent interest. The proof follows the argument of Lin-Shen [29] (see also [25]).

LEMMA C.1. *Let u be a harmonic function on $B_{3r}(x)$ and m be a positive integer. There exists a constant $\delta(m, \lambda, n)$ such that if* Ric $\geq -\lambda$ *on $B_{2r}(x)$, $d_{GH}(B_{2r}(x), B_{2r}(0^n)) \leq \delta r$ and*

$$\fint_{B_{2r}(x)} u^2 \leq 2^{2m+1} \fint_{B_r(x)} u^2,$$

then

$$\fint_{B_r(x)} u^2 \leq 2^{2m+1} \fint_{B_{r/2}(x)} u^2,$$

where $\fint$ denotes the average of the integral.

Proof. By scaling, it suffices to prove the case $r = 1$. We argue by contradiction, assuming that there exists a sequence of harmonic functions u_i on $B_3(p_i) \subset M_i$ such that

(1) $d_{GH}(B_2(x_i), B_2(0^n)) \leq i^{-1}$ and $\mathrm{Ric}(g_i) \geq -\lambda g_i$ on $B_2(x_i)$;
(2) $\fint_{B_2(x_i)} u_i^2 \leq 2^{2m+1} \fint_{B_1(x_i)} u_i^2$ and $\fint_{B_1(x_i)} u_i^2 > 2^{2m+1} \fint_{B_{1/2}(x_i)} u_i^2$.

Without loss of generality, we assume that $\fint_{B_2(x_i)} u_i^2 = 1$. By compactness, passing to a subsequence, we assume that u_i converges to h over $B_2(x_i) \xrightarrow{d_{GH}} B_2(0^n)$, where h is a harmonic function on $B_2(0^n)$ (see [5, Section 4.9]). Hence, we have

$$\lim_{i\to\infty} \int_{B_t(x_i)} u_i^2 = \int_{B_t(0^n)} h^2 \quad \text{for any } t \in (0, 2). \tag{C.1}$$

Combining this with $\fint_{B_2(x_i)} u_i^2 = 1$ and the volume convergence (see [11, Theorem 0.1]), for any $t \in (0, 2)$, we have

$$\int_{B_t(0^n)} h^2 = \lim_{i\to\infty} \int_{B_t(x_i)} u_i^2 \leq \lim_{i\to\infty} \mathrm{Vol}(B_2(x_i)) = \mathrm{Vol}(B_2(0^n)).$$

Letting $t \to 2$, it then follows that

$$\int_{B_2(0^n)} h^2 \leq \mathrm{Vol}(B_2(0^n)).$$

Combining this with (2), $⨍_{B_2(x_i)} u_i^2 = 1$ and (C.1), we see that

$$⨍_{B_2(0^n)} h^2 \leq 1 \leq 2^{2m+1} ⨍_{B_1(0^n)} h^2 \tag{C.2}$$

and

$$⨍_{B_1(0^n)} h^2 \geq 2^{2m+1} ⨍_{B_{1/2}(0^n)} h^2. \tag{C.3}$$

By the three spheres theorem for harmonic functions, the function

$$f(s) = \log_2 \left(⨍_{B_{2^s}(0^n)} h^2 \right)$$

is convex and analytic on $(-\infty, 1)$. By (C.2) and (C.3), it is clear that

$$f(1) \leq 2m + 1 + f(0) \ \text{ and } \ f(0) \geq 2m + 1 + f(-1).$$

Thus,

$$2f(0) \geq f(1) + f(-1).$$

Combining this with the convexity of f, we obtain

$$2f(0) = f(1) + f(-1),$$

which implies that f is linear on $[-1, 1]$. More precisely, we have

$$f(s) = f(-1) + (2m + 1)(s + 1) \ \text{ on } [-1, 1].$$

Since f is analytic on $(-\infty, 1)$, we see that

$$f(s) = f(-1) + (2m + 1)(s + 1) \ \text{ on } (-\infty, 1].$$

It then follows that

$$\lim_{s \to -\infty} \frac{f(s)}{2s} = m + \frac{1}{2}.$$

On the other hand, $\lim_{s\to-\infty} \frac{f(s)}{2s}$ is equal to the vanishing order of h at 0^n. This implies that $m + \frac{1}{2}$ is an integer, which is a contradiction. □

Lemma C.2. *Let u be a harmonic function on $B_2(x)$. There exists a constant $\delta(\lambda, \Lambda, n)$ such that if $r_{Reg,\delta} \geq 1$,* Ric $\geq -\lambda$ *on $B_2(x)$ and*

$$⨍_{B_1(x)} u^2 \leq \Lambda ⨍_{B_{1/2}(x)} u^2,$$

then

$$\fint_{B_r(x)} u^2 \le C(\lambda, n)\Lambda^4 \fint_{B_{r/2}(x)} u^2 \quad \text{for } r \in (0,1).$$

Proof. Let m be an integer such that $2^{2m-1} \le \Lambda \le 2^{2m+1}$. For any $r \in (0,1)$, let k be the integer such that $2^{-k} \le r \le 2^{-k+1}$. Using Lemma C.1 repeatedly, we obtain

$$\fint_{B_{2^{-k+2}}(x)} u^2 \le 2^{8m+4} \fint_{B_{2^{-k-2}}(x)} u^2.$$

Applying [27, Theorem 2.1], we get

$$\begin{aligned}\fint_{B_r(x)} u^2 &\le \sup_{B_{2^{-k+1}}(x)} u^2 \le C(\lambda, n) \fint_{B_{2^{-k+2}}(x)} u^2 \\ &\le C(\lambda, n) 2^{8m+4} \fint_{B_{2^{-k-2}}(x)} u^2 \le C(\lambda, n) 2^{8m+4} \sup_{B_{2^{-k-2}}(x)} u^2 \\ &\le C(\lambda, n) 2^{8m+4} \fint_{B_{r/2}(x)} u^2 \le C(\lambda, n)\Lambda^4 \fint_{B_{r/2}(x)} u^2,\end{aligned}$$

as required. □

By [27, Theorem 2.1], we have the following equivalent form of Lemma C.2:

PROPOSITION C.3. *Let u be a harmonic function on $B_2(x)$. There exists a constant $\delta(\lambda, \Lambda, n)$ such that if $r_{Reg,\delta} \ge 1$, $\mathrm{Ric} \ge -\lambda$ on $B_2(x)$ and*

$$\sup_{B_1(x)} |u| \le \Lambda \sup_{B_{1/2}(x)} |u|,$$

then

$$\sup_{B_r(x)} |u| \le C(n, \lambda, \Lambda) \sup_{B_{r/2}(x)} |u| \quad \text{for } r \in (0,1).$$

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