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FROM EULER TO THE SEMI-GEOSTROPHIC SYSTEM: CONVERGENCE UNDER UNIFORM CONVEXITY*

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Abstract. We prove that if the initial data is well prepared, then certain solutions to the Euler system converge to a solution of the Semi-Geostrophic system with constant Coriolis force. The main assumptions on the strong solution are the boundedness of the velocity field as well as the uniform convexity of the Legendre-Fenchel transform of the modified pressure.

Key words. SG system, Euler system, flows of maps, optimal mass transport.

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1. Introduction.

The Semi-Geostrophic (abbreviated SG in this paper) system is a model of large scale atmosphere/ocean flows, where "large-scale" means that the flow is rotation-dominated. We are interested in solutions to the SG system which satisfy the convexity principle introduced by Cullen and Purser as a physical stability condition (semi-geostrophic approximation remains accurate for as long as this condition holds):

$$\begin{array}{ll} D_t \mathbf{u}_g + \bar{\nabla} p + J\mathbf{u} = 0 & \\ \mathbf{u}_g = J\nabla p, \ D_t \partial_3 p = 0 & \text{in } [0,T) \times \Omega, \\ \nabla \cdot \mathbf{u} = 0, & \\ \mathbf{u} \cdot \nu = 0 & \text{on } (0,T) \times \partial\Omega, \\ p(0,\cdot) = p_0 & \text{in } \Omega, \end{array} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is open and bounded, $0 < T < \infty$, ∇ denotes the spatial gradient, $\bar{\nabla} = (\partial_1, \partial_2, 0)$, $\nabla\cdot$ the spatial divergence, $D_t := \partial_t + \mathbf{u}\cdot\nabla$ and ν is the (outward) unit vector normal to Ω. The total wind velocity is $\mathbf{u}$, while $\mathbf{u}_g$ is the geostrophic wind. One looks for solutions $(p, \mathbf{u})$ which satisfy the Cullen-Purser stability condition (see, e.g., [8]). This amounts to imposing that $P(t,x) := p(t,x) + (x_1^2 + x_2^2)/2$ be convex as a function of $x = (x_1, x_2, x_3) \in \Omega$ for all $t \in [0,T)$. Here,

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In terms of $(P, \mathbf{u})$, (1.1) can be written

$$\begin{array}{ll} D_t X = J(X - x) & \\ X = \nabla P & \text{in } [0,T) \times \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \\ \mathbf{u} \cdot \nu = 0 & \text{on } (0,T) \times \partial\Omega, \\ P(0,\cdot) = P_0 & \text{in } \Omega, \end{array} \tag{1.2}$$

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If $\nabla P(t,\cdot)_{\#}\chi =: \rho(t,\cdot)$, then one can use the Legendre-Fenchel transform

$$P^*(t,\cdot) := \sup_{x\in\Omega}\{x\cdot X - P(t,x)\}$$

of $P(t,\cdot)$ to formally rewrite (1.1) as the so-called SG in dual variables

$$\begin{aligned}
&\partial_t\rho + \nabla\cdot(U\rho) = 0 \quad \text{in } [0,T)\times\mathbb{R}^3,\\
&\nabla P(t,\cdot)_{\#}\chi = \rho(t,\cdot) \quad \text{for any } t\in[0,T);\\
&U(t,X) = J[X - \nabla P^*(t,X)],\\
&\rho(0,X) = \rho_0(X) \quad \text{in } \mathbb{R}^3
\end{aligned}$$

In [11] one can find a comprehensive bibliography on solutions of SG in dual space. Generally, these solutions are not known to be regular enough to be translated into Eulerian solutions of the problem in physical space. Cullen and Feldman [7] pulled the dual space solutions back to physical space in the form of weak Lagrangian solutions in the case $\rho_0 \in L^p(\mathbb{R}^3)$ for some $p > 1$; the question left open in its generalization (to $p \geq 1$) by Faria et al. [10] regards the even more general case, i.e. the case in which ρ_0 may be a singular probability measure. We settled that in [12], where we introduced an appropriate generalization of weak Lagrangian solutions, namely renormalized relaxed Lagrangian solutions. More recently, existence of Eulerian solutions for a class of initial data, where the conditions include the requirement that the support of $\rho_0 = \nabla P_{0\#}\chi$ in the dual space is the whole space, was obtained by L. Ambrosio, M. Colombo, G. De Philippis, A. Figalli [2, 3] based on the results of G. De Philippis and A. Figalli [9] on regularity of solutions for the Monge-Ampere equation.

Loeper [15] first proved uniqueness for classical solutions in the class of functions which are Hölder continuous in space uniformly with respect to time. Cheng, Cullen and Feldman [6], besides the short time existence of classical solutions for SG with variable Coriolis force, prove uniqueness of such solutions in the class of sufficiently regular classical solutions. There is also work by Brenier and Cullen [5] where a formal proof of convergence of solutions for a Navier-Stokes with Boussinesq approximation to a solution of $x-z$ SG is given. Both systems are assumed to have smooth solutions, which is unknown in general even for "nice" initial data. Also, it is assumed that $P(t,\cdot)$ has a smooth convex extension to $\mathbb{R}^3$ such that $\nabla^2 P^*(t,\cdot)$ is bounded away from zero and infinity uniformly for $t \in [0,T]$ (for some $T > 0$). A weak-strong uniqueness result in the case of uniformly convex P^* was recently obtained in [13] based on the relative entropy approach from [5] .

In this paper we show that the methods of [5], [13] can be extended to prove a convergence result of classical solutions for 3D Euler satisfying an a priori estimate on the second convective derivative of the velocity to some special solutions (satisfying some uniform convexity condition for the Legendre transform of the modified potential) for 3D SG.

Assume we have a C^2 classical solution for the initial-value problem for the three-dimensional Euler system with Coriolis force

$$\begin{aligned}
&D_t\mathbf{v} + \nabla\tilde{p} + J\mathbf{v} = 0 && \\
&\nabla\cdot\mathbf{v} = 0 && \text{in } [0,T)\times\Omega,\\
&\mathbf{v}\cdot\nu = 0 && \text{on } (0,T)\times\partial\Omega\\
&\mathbf{v}(0,\cdot) = \mathbf{v}_0 && \text{in } \Omega,
\end{aligned} \tag{1.3}$$

where $D_t := \partial_t + \mathbf{v} \cdot \nabla$, and $\mathbf{v}_0$ is some smooth, divergence-free function on Ω. Let

$$\tilde{P}(t,x) := \tilde{p}(t,x) + (x_1^2 + x_2^2)/2.$$

Then (1.3) can also be recast in terms of $(\tilde{P}, \mathbf{v})$ as

$$\begin{array}{ll} D_t\mathbf{v} + \tilde{X} + J\mathbf{v} = -J^2 x & \\ \tilde{X} = \nabla \tilde{P} & \text{in } [0,T) \times \Omega, \\ \nabla \cdot \mathbf{v} = 0 & \\ \mathbf{v} \cdot \nu = 0 & \text{on } (0,T) \times \partial\Omega \\ \mathbf{v}(0,\cdot) = \mathbf{v}_0 & \text{in } \Omega. \end{array} \tag{1.4}$$

Upon differentiating the first equation of (1.4) in time along particle paths we obtain

$$D_t^2 \mathbf{v} + D_t \tilde{X} + J D_t \mathbf{v} = -J^2 \mathbf{v}.$$

Using the same equation to compute $D_t\mathbf{v}$ and plugging in the latter, and noting that $-J^3 = J$, yields

$$D_t^2 \mathbf{v} + D_t \tilde{X} = -J D_t \mathbf{v} - J^2 \mathbf{v} = J(\tilde{X} - x). \tag{1.5}$$

Hoskins [14] noted that an alternative derivation of SG from Euler can be obtained by assuming that the term $D_t^2\mathbf{v}$ is componentwise very small compared to $\mathbf{v}$ in (1.5) above. Note that if one throws that term away, then (1.5) becomes the first equation in (1.1). In this paper we prove that a slightly stronger result can be obtained rigorously in the following sense: if some solution of (1.1) is uniformly convex and regular in some precise sense, then the L^2-norm of the difference $\nabla p(t,\cdot) - \nabla \tilde{p}(t,\cdot)$ can be estimated uniformly for $t \in [0,T)$ in terms of $\|\nabla p_0 - \nabla \tilde{p}_0\|_{L^2(\Omega;\mathbb{R}^3)}$ and the $L^1((0,T) \times \Omega; \mathbb{R}^3)$-norm of $D_t^2\mathbf{v}$.

The most important assumption on the SG solution is the uniform (in time) boundedness away from zero and infinity of $\nabla^2 P^*(t,\cdot)$. This yields the metric equivalence between the L^2-norm and the relative entropy functional which is instrumental to our proof.

2. Main result.

2.1. 3D case. The starting point will be the weak Lagrangian solutions of SG in the physical space. Such solutions were introduced by Cullen & Feldman [7], and existence of Lagrangian solutions of (1.1) was shown in [7] for any $\rho_0 = \nabla P_{0\#}\chi \in L^q(\mathbb{R}^3)$ for $q > 1$; this result was then extended to the case $q = 1$ in [10]. The restriction $\rho_0 \ll \mathcal{L}^3$ amounts to requiring some *strict convexity* property of the potentials $P(t,\cdot)$, which is certainly more than what Cullen and Purser's stability condition imposes (which is simply the convexity of $P(t,\cdot)$).

The following definition (of weak Lagrangian solution) can be found in [13]. It uses the form (1.2) of SG system, and extends the original definition of weak Lagrangian solution in physical space from [7] (where $P_0 \in W^{1,\infty}(\Omega)$) to the case $P_0 \in H^1(\Omega)$.

DEFINITION 2.1. *Let $P_0 \in H^1(\Omega)$ be convex. Let $P : [0,T) \times \Omega \to \mathbb{R}$ satisfy*

$$P \in C([0,T); H^1(\Omega)), \tag{2.1}$$

$$P(t,\cdot) \text{ is convex in } \Omega \text{ for each } t \in [0,T). \tag{2.2}$$

Let $F : [0,T) \times \Omega \to \Omega$ be a Borel map such that

$$F \in C([0,T); L^2(\Omega;\mathbb{R}^3)). \tag{2.3}$$

Then the pair (P,F) is called a weak Lagrangian solution of (1.1) in $[0,T) \times \Omega$ if

(a) *$F(0,x) = x$, $P(0,x) = P_0(x)$ for a.e. $x \in \Omega$,*

(b) *for any $t > 0$ the mapping $F(t,\cdot) : \Omega \to \Omega$ is Lebesgue measure preserving, in the sense that $F(t,\cdot)_{\#}\chi = \chi$;*

(c) *There exists a Borel map $F^* : [0,T) \times \Omega \to \Omega$ such that for every $t \in (0,T)$ the map $F^*(t,\cdot) : \Omega \to \Omega$ is Lebesgue measure preserving: $F^*(t,\cdot)_{\#}\chi = \chi$, and satisfies $F^*(t,F(t,x)) = x$ and $F(t,F^*(t,x)) = x$ for a.e. $x \in \Omega$;*

(d) *The function*

$$Z(t,x) = \nabla P(t,F(t,x)) \tag{2.4}$$

is a distributional solution of

$$\begin{aligned} &\partial_t Z(t,x) = J\big[Z(t,x) - F(t,x)\big] && \text{in } [0,T) \times \Omega, \\ &Z(0,x) = \nabla P_0(x) && \text{in } \Omega. \end{aligned} \tag{2.5}$$

Our goal is to prove:

THEOREM 2.2. *Let (P,F) be a weak Lagrangian solution for* (1.1) *in* $[0,T) \times \Omega$ *such that the following properties are satisfied:*

(i) *There exists $\mathcal{R} \in (0,\infty)$ and $0 < \delta < \mathcal{R}$ such that $\overline{\nabla P(t,\Omega)} \subset B(0,\mathcal{R}-\delta)$ (open ball centered at the origin) for all $t \in [0,T)$.*

(ii) *For each $t \in [0,T)$ there exists an extension (still denoted by $P^*(t,\cdot)$) of $P^*(t,\cdot)$ from $\nabla P(t,\Omega)$ to $B(0,\mathcal{R})$ and there exists $\lambda \in (0,\infty)$ such that*

$$\lambda I_3 \le \nabla^2 P^*(t,X) \text{ for each } t \in [0,T) \text{ and all } X \in B(0,\mathcal{R}). \tag{2.6}$$

(iii) *$P^* \in W^{1,\infty}(0,T; W^{2,\infty}(B(0,\mathcal{R})) \cap L^\infty(0,T; W^{3,\infty}(B(0,\mathcal{R}))$.*

Then there exists a real constant C depending only on P such that for any classical solution $(\tilde{P},\mathbf{v})$ of (1.4) *which satisfies*

$$(\tilde{P},\mathbf{v}) \in C^2((0,T) \times \overline{\Omega}; \mathbb{R}^4),$$
$$\overline{\nabla \tilde{P}(t,\Omega)} \subset B(0,\mathcal{R}-\delta) \text{ for all } t \in [0,T),$$

we have

$$\|\nabla P(t,\cdot) - \nabla \tilde{P}(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^3)} \le C\big[\|\nabla P_0 - \nabla \tilde{P}_0\|_{L^2(\Omega;\mathbb{R}^3)} + \|D_t^2 \mathbf{v}\|_{L^1((0,T)\times\Omega;\mathbb{R}^3)}\big] \tag{2.7}$$

for all $t \in [0,T)$.

REMARK 2.3. *Note that under the assumptions on the SG solution from Theorem 2.2, its uniqueness in the class of weak solutions of* (1.1) *was proved recently in [13]. The same reference contains nontrivial examples of solutions of* (1.1) *satisfying the assumptions listed.*

Before proving the theorem, we need the following fact, which can be found (along with its proof) in [13]:

PROPOSITION 2.4. *Let (P,F) as in Theorem 2.2 and denote*

$$\mathbf{u}(t,x) := (\partial_t \nabla P^*)(t,\nabla P(t,x)) + \nabla^2 P^*(t,\nabla P(t,x)) J[\nabla P(t,x) - x]. \tag{2.8}$$

Then the following hold:

(a) $\mathbf{u} \in L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^3))$ *and*

$$\nabla \cdot \mathbf{u}(t,\cdot) \equiv 0 \text{ and } \mathbf{u}(t,\cdot)\cdot\nu = 0 \text{ on } \partial\Omega \text{ for a.e. } t\in[0,T). \tag{2.9}$$

(b) *The map F satisfies $F \in W^{1,\infty}(0,T;L^\infty(\Omega;\mathbb{R}^3))$ and*

$$\partial_t F(t,x) = \mathbf{u}(t,F(t,x)), \quad F(0,x) = x \text{ for a.e. } (t,x)\in(0,T)\times\Omega. \tag{2.10}$$

We now have all the necessary ingredients to prove Theorem 2.2:

Proof of Theorem 2.2. Define, as in [5], the so-called Bregman (or relative entropy) functional

$$\eta[P^*](t,x) = P^*(t,\tilde{X}(t,x)) - P^*(t,X(t,x)) - \nabla P^*(t,X(t,x))\cdot\big[\tilde{X}(t,x)-X(t,x)\big]$$

for

$$X(t,x) := \nabla P(t,x), \quad \tilde{X} := \nabla\tilde{P}(t,x).$$

Let us, for notational relief, denote by $\|\cdot\|$ the $\|\cdot\|_{L^2(\Omega;\mathbb{R}^3)}$-norm. From (ii) and (iii) we deduce

$$\lambda_0\|\nabla P(t,\cdot)-\nabla\tilde{P}(t,\cdot)\|^2 \le \int_\Omega \eta[P^*](t,x)dx \le \lambda_0^{-1}\|\nabla P(t,\cdot)-\nabla\tilde{P}(t,\cdot)\|^2 \tag{2.11}$$

for some $\lambda_0\in(0,1)$ and each $t\in[0,T)$. Therefore, if we denote

$$E(t) := \int_\Omega \eta[P^*](t,x)dx,$$

it suffices to prove that

$$\dot{E}(t) \le C\|\nabla P(t,\cdot)-\nabla\tilde{P}(t,\cdot)\|^2 + C\|D_t^2\mathbf{v}(t,\cdot)\|_{L^1(\Omega;\mathbb{R}^3)} \text{ for all } t\in[0,T]$$

for a constant C (depending only on P) to get the desired inequality.

Let $\tilde{F}$ be the flow map of $\mathbf{v}$. Using regularity assumptions for $\mathbf{v}$ and the properties (1.4), we obtain that $\tilde{F}(t,\cdot)$ is a Lebesgue measure preserving C^2-diffeomorphism of Ω for each $t\in[0,t)$, and $\tilde{F}(\cdot,x)\in C^3([0,t))$ for each $x\in\Omega$. Introduce Lagrangian coordinates x_0 corresponding to the flow map $\tilde{F}$, they are defined by $x=\tilde{F}(t,x_0)$. Note that (1.5) in Lagrangian coordinates (which we now denote by x) reads

$$\partial_t^3\tilde{F}(t,x) + \partial_t\tilde{Z}(t,x) = J[\tilde{Z}(t,x)-\tilde{F}(t,x)], \tag{2.12}$$

where $\tilde{Z}(t,x) := \tilde{X}(t,\tilde{F}(t,x))$. Next we rewrite expression of $E(t)$ by changing variables in the first two terms in the expression of $\eta[P^*]$ to Lagrangian coordinates determined by the flow maps $\tilde{F}(t,x)$ and $F(t,x)$, respectively. Then we obtain, using the measure-preserving properties of $\tilde{F}$ and F:

$$E(t) = \int_\Omega \big\{P^*(t,\tilde{Z}(t,x)) - P^*(t,Z(t,x)) - x\cdot\big[\tilde{X}(t,x)-X(t,x)\big]\big\}dx$$

We differentiate $E(t)$ with respect to time and use (2.5) and (2.12) to get, switching back to Eulerian coordinates:

$$\begin{aligned}\dot{E}(t) = &\int_\Omega \big[(\partial_t P^*)(t,\tilde{X}) - (\partial_t P^*)(t,X)\big]dx \\ &+ \int_\Omega \nabla P^*(t,\tilde{X})\cdot G(x,\tilde{X})dx + \int_\Omega Jx\cdot \tilde{X}dx \\ &- \int_\Omega \nabla P^*(t,X)\cdot G(x,X)dx - \int_\Omega Jx\cdot X dx \\ &- \int_\Omega \nabla P^*(t,\tilde{X})\cdot D_t^2\mathbf{v}dx,\end{aligned}$$

where we have denoted $G(x,y) := J(y-x)$. After a cancellation due to

$$\nabla P^*(t,\nabla P(t,x)) = x \text{ for all } t\in[0,T] \text{ and all } x\in\Omega \tag{2.13}$$

we get

$$\begin{aligned}\dot{E}(t) = &\int_\Omega \big[(\partial_t P^*)(t,\tilde{X}) - (\partial_t P^*)(t,X)\big]dx + \int_\Omega Jx\cdot\tilde{X}dx \\ &+ \int_\Omega \nabla P^*(t,\tilde{X})\cdot G(x,\tilde{X})dx - \int_\Omega \nabla P^*(t,\tilde{X})\cdot D_t^2\mathbf{v}dx.\end{aligned} \tag{2.14}$$

Let us now estimate the second and third terms in the right hand side. We begin by noticing that the boundedness of $\nabla^2 P^*$ in space-time (by (iii)) and (2.11) imply that

$$\int_\Omega \big[\nabla P^*(t,\tilde{X}) - \nabla P^*(t,X)\big]\cdot\big[G(x,\tilde{X}) - G(x,X)\big]dx$$

is bounded from above by $CE(t)$. By (2.13) and the fact that $J^T = -J$, we have

$$\int_\Omega \nabla P^*(t,X)\cdot\big[G(x,\tilde{X}) - G(x,X)\big]dx = -\int_\Omega Jx\cdot(\tilde{X}-X)dx.$$

We consequently infer

$$\begin{aligned}&\int_\Omega \nabla P^*(t,\tilde{X})\cdot G(x,\tilde{X})dx + \int_\Omega Jx\cdot\tilde{X}dx \\ &\qquad\le CE(t) + \int_\Omega \nabla P^*(t,\tilde{X})\cdot G(x,X)dx + \int_\Omega Jx\cdot X dx.\end{aligned} \tag{2.15}$$

Now let us deal with the first term in the right hand side of (2.14). By (iii) we have that $\partial_t\nabla^2 P^*$ is bounded in space-time, so

$$\int_\Omega \eta[\partial_t P^*](t,x)dx \le \tilde{C}\|\tilde{X} - X\|^2 \le CE(t),$$

where (2.11) was used in the last inequality. This yields

$$\begin{aligned}&\int_\Omega \big[(\partial_t P^*)(t,\tilde{X}) - (\partial_t P^*)(t,X)\big]dx \\ &\qquad\le CE(t) + \int_\Omega (\partial_t \nabla P^*)(t,X)\cdot(\tilde{X}-X)dx.\end{aligned} \tag{2.16}$$

In view of (2.14), (2.15) and (2.16) we see that

$$\dot{E}(t) = CE(t) + S(t) - \int_\Omega \nabla P^*(t, \tilde{X}) \cdot D_t^2 \mathbf{v} dx, \tag{2.17}$$

where

$$S(t) := \int_\Omega \nabla P^*(t, \tilde{X}) \cdot G(x, X) dx + \int_\Omega (\partial_t \nabla P^*)(t, X) \cdot (\tilde{X} - X) dx + \int_\Omega Jx \cdot X dx.$$

It remains to estimate $S(t)$ in terms of $E(t)$ in order to finish the proof. Due to (i), (iii) and (2.11) again, we have that

$$\int_\Omega \left[\nabla P^*(t, \tilde{X}) - \nabla P^*(t, X) - \nabla^2 P^*(t, X)(\tilde{X} - X)\right] \cdot G(x, X) \leq CE(t).$$

Since we have

$$\int_\Omega \nabla P^*(t, X) \cdot G(x, X) dx = \int_\Omega x \cdot J(X - x) dx = -\int_\Omega Jx \cdot X dx,$$

we get that the previously displayed inequality is equivalent to

$$\begin{aligned} \int_\Omega \nabla P^*(t, \tilde{X}) \ &\cdot \ G(x, X) dx \\ &\leq CE(t) + \int_\Omega \nabla^2 P^*(t, X)(\tilde{X} - X) \cdot G(x, X) dx - \int_\Omega Jx \cdot X dx. \end{aligned}$$

Thus,

$$S(t) \leq CE(t) + \int_\Omega \left[(\partial_t \nabla P^*)(t, X) + \nabla^2 P^*(t, X) G(x, X)\right] \cdot (\tilde{X} - X) dx. \tag{2.18}$$

But, by (2.8), we have

$$(\partial_t \nabla P^*)(t, X(t, x)) + \nabla^2 P^*(t, X(t, x)) G(x, X(t, x)) = \mathbf{u}(t, x).$$

Since for all $t \in [0, T)$ we have that $\tilde{X}(t, \cdot) - X(t, \cdot)$ is the gradient of a $W^{1,\infty}(\Omega)$ function, we use Proposition 2.4 (a) to infer that the integral in the right-hand-side of (2.18) vanishes. Finally, we use (2.17) to conclude

$$\dot{E}(t) \leq CE(t) + \|\nabla P^*\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)} \|D_t^2 \mathbf{v}(t, \cdot)\|_{L^1(\Omega;\mathbb{R}^3)},$$

which yields the thesis by a standard application of Gronwall's Lemma. □

2.2. The 2D periodic case. The assumptions on P take a lighter form in the case of the 2D torus. It turns out that if we consider the problem on the two-dimensional torus $\mathbb{T}^2$ (instead of a bounded and open $\Omega \subset \mathbb{R}^3$), then the uniform convexity assumptions in Theorem 2.2 become the following:

$$0 < m \leq \nabla P(t, \cdot)_\# \mathcal{L}^2|_{\mathbb{T}^2} \leq M < \infty$$

for some fixed $m,\ M$ and all $t \in [0, T]$. Rewritten in terms of the dual space solutions this reads $m \leq \rho(t, \cdot) \leq M$ for all $t \in [0, T]$; fortunately, it can be showed that this is satisfied by dual space solutions under the mere assumption that the initial data is like that (see, e.g. [2]). The physical case of a bounded $\Omega \subset \mathbb{R}^3$ is, as seen above, somewhat more involved as it requires some extra-assumptions on P. The periodic 2D SG reads

$$\begin{aligned} &D_t \mathbf{u}_g + \nabla p + J\mathbf{u} = 0 \text{ in } [0,T) \times \mathbb{T}^2, \\ &\mathbf{u}_g := J\nabla p, \\ &\nabla \cdot \mathbf{u} = 0 \text{ in } [0,T) \times \mathbb{T}^2, \\ &p(0,\cdot) = p_0 \text{ in } \mathbb{T}^2, \end{aligned} \tag{2.19}$$

while the periodic 2D Euler with rotation is

$$\begin{aligned} &D_t \mathbf{v} + \nabla \tilde{p} + J\mathbf{v} = 0 \text{ in } [0,T) \times \mathbb{T}^2, \\ &\nabla \cdot \mathbf{v}(t,\cdot) = 0 \text{ in } \mathbb{T}^2 \text{ for } t \in [0,T), \\ &\mathbf{v}(0,\cdot) = \mathbf{v}_0 \quad \text{in } \mathbb{T}^2, \end{aligned} \tag{2.20}$$

where $\mathbf{v}_0$ is some smooth, divergence-free function on $\mathbb{T}^2$. The rotation is now given by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Introduce the functions

$$P(t,x) := p(t,x) + (x_1^2 + x_2^2)/2, \qquad \tilde{P}(t,x) := \tilde{p}(t,x) + (x_1^2 + x_2^2)/2.$$

Then we can rewrite (2.19) in the form

$$\begin{aligned} &D_t X = J(X - x) \\ &X = \nabla P \qquad\qquad \text{in } [0,T) \times \mathbb{T}^2, \\ &\nabla \cdot \mathbf{u} = 0 \\ &P(0,\cdot) = P_0 \qquad\quad \text{in } \mathbb{T}^2, \end{aligned} \tag{2.21}$$

while 2D Euler system (2.20) can be written in the form (1.5) on $[0,T) \times \mathbb{T}^2$.

The following definition (of weak Lagrangian solution) of SG system (2.19) can be found in [13], it uses the form (2.21) of the system, and extends the original definition of weak Lagrangian solution in physical space from [7] to the case of $\mathbb{T}^2$. In [2] the authors prove existence of weak Eulerian solutions for SG on the 2-d flat torus $\mathbb{T}^2$ under the requirement that the dual-space densities ρ stay bounded away from zero and infinity. In [2] Section 5 they show that their weak Eulerian solutions yield weak Lagrangian solutions in the spirit of [7].

DEFINITION 2.5. *Let $P_0 : \mathbb{R}^2 \to \mathbb{R}$ be convex and such that $P_0 - |\mathrm{Id}|^2/2$ is periodic. Let $r \in [1,\infty)$ and $P : [0,T) \times \Omega \to \mathbb{R}$ satisfy*

$$P \in L^\infty([0,T); W^{1,\infty}_{loc}(\mathbb{R}^2)) \cap C([0,T); W^{1,r}_{loc}(\mathbb{R}^2)), \tag{2.22}$$

$$P(t,\cdot) \textit{ is convex and } P(t,\cdot) - |\mathrm{Id}|^2/2 \textit{ is periodic for each } t \in [0,T). \tag{2.23}$$

Let $F : [0,T) \times \mathbb{T}^2 \to \mathbb{T}^2$ be a Borel map such that

$$F \in C([0,T); L^r_{loc}(\mathbb{R}^2; \mathbb{R}^2)). \tag{2.24}$$

Then the pair (P,F) is called a weak Lagrangian solution of (2.19) in $[0,T) \times \mathbb{T}^2$ if

(a) *$F(0, x) = x$, $P(0, x) = P_0(x)$ for a.e. $x \in \mathbb{T}^2$,*
(b) *for any $t > 0$ the mapping $F(t, \cdot) : \mathbb{T}^2 \to \mathbb{T}^2$ is Lebesgue measure preserving, in the sense that $F_{t\#}\chi = \chi$;*
(c) *There exists a Borel map $F^* : [0, T) \times \mathbb{T}^2 \to \mathbb{T}^2$ such that for every $t \in (0, T)$ the map $F^*(t, \cdot) : \mathbb{T}^2 \to \mathbb{T}^2$ is Lebesgue measure preserving: $F^*_{t\#}\chi = \chi$, and satisfies $F^*_t \circ F_t(x) = x$ and $F_t \circ F^*_t(x) = x$ for a.e. $x \in \mathbb{T}^2$;*
(d) *The function*

$$Z(t, x) = \nabla P(t, F(t, x)) \tag{2.25}$$

is a distributional solution of

$$\begin{aligned} &\partial_t Z(t, x) = J\big[Z(t, x) - F(t, x)\big] && \text{in } [0, T) \times \mathbb{T}^2, \\ &Z(0, x) = \nabla P_0(x) && \text{in } \mathbb{T}^2. \end{aligned} \tag{2.26}$$

The counterpart of Theorem 2.2 in this setting is:

THEOREM 2.6. *Let (P, F) be a weak Lagrangian solution for (2.19) in $[0, T) \times \mathbb{T}^2$. Assume further:*
(i) *There exists $\lambda \in (0, \infty)$ such that*

$$\lambda I_2 \leq \nabla^2 P^*(t, x) \text{ for each } t \in [0, T) \text{ and all } x \in \mathbb{T}^2. \tag{2.27}$$

(ii) *$P^* \in W^{1,\infty}(0, T; W^{2,\infty}(\mathbb{T}^2)) \cap L^\infty(0, T; W^{3,\infty}(\mathbb{T}^2))$.*
Then there exists a real constant C depending only on P such that for any solution $(\mathbf{v}, \tilde{p}) \in C^2((0, T) \times \mathbb{T}^2)$ of (2.20) we have

$$\|\nabla p(t, \cdot) - \nabla\tilde{p}(t, \cdot)\|_{L^2(\mathbb{T}^2;\mathbb{R}^2)} \leq C\big[\|\nabla p_0 - \nabla\tilde{p}_0\|_{L^2(\mathbb{T}^2;\mathbb{R}^2)} + \|D_t^2\mathbf{v}\|_{L^1((0,T)\times\mathbb{T}^2;\mathbb{R}^2)}\big] \tag{2.28}$$

for all $t \in [0, T)$.

The proof is a minor modification of the proof of the main result from the previous subsection (see [13]).

3. A convergence result. Clearly, Theorem 2.2 immediately implies the following:

COROLLARY 3.1. *Assume (P, F) is a weak Lagrangian solution for (1.1) satisfying the assumptions from Theorem 2.2. Moreover, assume that there exists a sequence $(\tilde{p}_n, \mathbf{v}_n)$ of classical solutions for (1.3) such that*

$$\lim_{n\to\infty} \nabla\tilde{p}_{0,n} = \nabla p_0 \text{ in } L^2(\Omega; \mathbb{R}^3) \tag{3.1}$$

and

$$\lim_{n\to\infty} D_t^2\mathbf{v}_n = 0 \text{ in } L^1((0, T) \times \Omega; \mathbb{R}^3). \tag{3.2}$$

Then

$$\lim_{n\to\infty} \nabla\tilde{p}_n(t, \cdot) = \nabla p(t, \cdot) \text{ in } L^2(\Omega; \mathbb{R}^3) \text{ for each } t \in [0, T). \tag{3.3}$$

Below we show that (3.2) follows easily from a uniform bound on $\mathbf{v}_{0,n}$ and a condition reminiscent of Hoskins' condition $D_t^2\mathbf{v} \ll \mathbf{v}$, namely (3.5) below. Note that this is a weaker condition, as Hoskins [14] assumes it to be valid componentwise.

COROLLARY 3.2. *Assume (P, F) is a weak Lagrangian solution for (1.1) satisfying the assumptions from Theorem 2.2. Moreover, assume that there exists a sequence $(\tilde{p}_n, \mathbf{v}_n)$ of classical solutions for (1.3) such that (3.1) holds, and*

$$\sup_{n\geq 1} \|\mathbf{v}_{0,n}\|_{L^2(\Omega;\mathbb{R}^3)} < \infty. \tag{3.4}$$

Further assume that there exists a sequence $\{\epsilon_n\}_n \to 0^+$ such that

$$\|D_t^2 \mathbf{v}_n\|_{L^1((0,T)\times\Omega;\mathbb{R}^3)} \leq \epsilon_n \|\mathbf{v}_n\|_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \text{ for all } n. \tag{3.5}$$

Then (3.3) holds.

Proof. In light of (3.4) and the standard energy estimate for (1.3) we have

$$\sup_{n\geq 1,\ t\in[0,T)} \|\mathbf{v}_n(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^3)} < \infty,$$

which combines with (3.5) to yield (3.2). We conclude the proof by applying Corollary 3.1 to $(\tilde{p}_n, \mathbf{v}_n)$. □

Finally, we provide a simple example illustrating the situation described by Corollary 3.2. Let $\phi \in C_c^2(0, T; \mathbb{R}^3)$ and a sequence $\mathbf{v}_{0,n} \to \mathbf{v}_0 \in \mathbb{R}^3$. Note that for any positive integer n, the pair

$$(\tilde{p}_n, \mathbf{v}_n) = \big([\mathbf{v}_{0,n} + \phi(t/n) - J\dot{\phi}(t/n)/n] \cdot Jx, \mathbf{v}_{0,n} + \phi(t/n)\big)$$

solves (1.3) with initial data $\mathbf{v}_{0,n}$. Furthermore, observe that

$$(p, \mathbf{u}) = \big(\mathbf{v}_0 \cdot Jx, \mathbf{v}_0\big)$$

is a stationary solution for (1.1), and if we put

$$P(t,x) := \frac{1}{2}(x_1^2 + x_2^2) + \mathbf{v}_0 \cdot Jx$$

we can easily check that the hypotheses of Corollary 3.2 are trivially satisfied, along with the conclusion (3.3).

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