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# DISPERSIVE ESTIMATES FOR THE QHD SYSTEM AND EULER-KORTEWEG EQUATIONS*

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*To Professor Hsiao Ling, with great esteem and appreciation*

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**1. Introduction.** Quantum fluids are systems of highly interacting particles, where the effects of quantum statistics are also relevant at the macroscopic level. This is the case, for instance, in the description of Bose-Einstein condensation, superfluidity, dense astrophysical plasmas or in the modeling of semiconductor devices, see [5] for more details. The evolution of these fluids is governed by the quantum hydrodynamics (QHD) system namely a compressible dispersive Euler fluid

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho v) = 0 \\ \partial_t(\rho v) + \operatorname{div}(\rho v\otimes v) + \nabla p(\rho) = \dfrac{1}{2}\rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), \end{cases} \tag{1.1}$$

where the unknowns are $\rho$ and $J = \rho v$, describing the particle and momentum densities, respectively, whereas $v$ is the associated velocity field. In this paper we always consider the problem (1.1) posed in the whole space $\mathbf{R}^d$. The term $p(\rho)$ denotes the pressure, which for convenience will be assumed to satisfy a power law, namely $p(\rho) = \frac{\gamma-1}{\gamma}\rho^\gamma$ for $\gamma > 1$. The third order term on the right hand side of the equation for the momentum density is a term due to the presence of quantum effects and it provides an extra contribution to the Cauchy stress tensor, as it may be also written in the following way

$$\frac{1}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = \frac{1}{4}\operatorname{div}(\rho\nabla^2\log\rho) = \operatorname{div}(\frac{1}{4}\nabla^2\rho - \nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}). \tag{1.2}$$

This term can also be associated with a quantum pressure correction or a non-linear quantum potential. The associated energy functional to (1.1) is then given by

$$E(t) = \int_{\mathbf{R}^d} \frac{1}{2}\rho|v|^2 + \frac{1}{2}|\nabla\sqrt{\rho}|^2 + f(\rho)\,dx$$

and it is formally conserved along the flow of solutions. The internal energy is given by $f(\rho) = \frac{1}{\gamma}\rho^\gamma$. The only natural bound we have on the velocity field is $v \in L^2(\rho\,dx)$, in particular no control on $v$ is available in the vacuum region $\{\rho = 0\}$. For this

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reason it turns out that, in order to set up a rigorous theory for system (1.1), it is more convenient to deal with the hydrodynamic quantities $(\sqrt{\rho}, \Lambda)$, where $\Lambda = J/\sqrt{\rho}$, see Section 4 in [5]. In this way the total energy reads

$$E(t) = \int_{\mathbf{R}^d} \frac{1}{2}|\Lambda|^2 + \frac{1}{2}|\nabla\sqrt{\rho}|^2 + f(\rho)\, dx \tag{1.3}$$

and moreover, by using identity (1.2), system (1.1) may be written as

$$\begin{cases} \partial_t\rho + \operatorname{div} J = 0 \\ \partial_t J + \operatorname{div}(\Lambda \otimes \Lambda) + \nabla p(\rho) = \frac{1}{4}\nabla\Delta\rho - \operatorname{div}(\nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho}), \end{cases}$$

where all terms now are completely meaningful as distributions by assuming the energy bounds. In what follows we often refer to the pair $(\sqrt{\rho}, \Lambda)$ as the hydrodynamic state, which in turn determine the macroscopic observables $(\rho, J)$ as $\rho = (\sqrt{\rho})^2$ and $J = \sqrt{\rho}\Lambda$.

From a mathematical point of view, the QHD system shares many similarities with the more general class of Euler-Korteweg systems

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho v) = 0 \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla p(\rho) = \rho\nabla\left(\operatorname{div}(\kappa(\rho)\nabla\rho) - \frac{1}{2}\kappa'(\rho)|\nabla\rho|^2\right), \end{cases} \tag{1.4}$$

arising in a different context. Indeed in this case the term on the right hand side of (1.6) describes capillarity effects in diffuse interfaces [23], see also [18] for its derivation. Also in this case the dispersive tensor on the right hand side of the equation for the momentum density can be written as the divergence of a stress tensor,

$$\rho\nabla\left(\operatorname{div}(\kappa(\rho)\nabla\rho) - \frac{1}{2}\kappa'(\rho)|\nabla\rho|^2\right) = \operatorname{div}\mathbb{K},$$

where the Korteweg tensor is given by

$$\mathbb{K} = \left(\operatorname{div}(\rho\kappa(\rho)\nabla\rho) - \frac{1}{2}(\rho\kappa'(\rho) + \kappa(\rho))|\nabla\rho|^2\right)\mathbb{I} - \kappa(\rho)\nabla\rho \otimes \nabla\rho. \tag{1.5}$$

The total energy of the system reads

$$E_\kappa(t) = \int_{\mathbf{R}^d} \frac{1}{2}|\Lambda|^2 + \frac{1}{2}\kappa(\rho)|\nabla\rho|^2 + f(\rho)\, dx, \tag{1.6}$$

where the capillarity coefficient satisfies $\kappa(\rho) \geq 0$. Let us notice that the QHD system (1.1) can be recast from (1.4) by choosing $\kappa(\rho) = \frac{1}{4\rho}$.

One of the most striking features of the QHD system (1.1) consists in its analogy with the nonlinear Schrödinger equation, given by means of the Madelung transformations [28]. More precisely, it can be showed that if $\psi$ is a solution to the following nonlinear Schrödinger (NLS) equation

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi + f'(|\psi|^2)\psi, \tag{1.7}$$

then $(\rho, J)$ defined by $\rho = |\psi|^2$, $J = \operatorname{Im}(\bar{\psi}\nabla\psi)$ solves the QHD system (1.1). This analogy was fully exploited in [1, 2] in order to show the existence of finite energy weak solutions to (1.1). More precisely, by applying a polar factorization method to the wave function, it is possible to rigorously define the hydrodynamic state $(\sqrt{\rho}, \Lambda)$, which in turn determine a finite energy weak solution $(\rho, J)$ to (1.1). Thus, the polar factorization overcomes the difficulty of defining the velocity field in the vacuum region, by rather dealing with the hydrodynamic state $(\sqrt{\rho}, \Lambda)$, which is well-defined a.e. in the whole space.

This approach is quite robust and also allows to deal with dissipative QHD systems [1, 3], however it has the drawback that the initial data must be prescribed by an initial wave function. While, through polar factorization, it is possible to determine a hydrodynamic state associated with a given wave function, the opposite is generally not true. More precisely, it is not clear whether the general hydrodynamic states can determine an associated wave function.

This issue has been solved in [6] for the one-dimensional problem, see also [7] for some partial extensions to the multidimensional case. In particular, in [6] the authors establish a lifting of the wave function which, given a hydrodynamic state, determines an associated wave function. As a byproduct, in the one-dimensional case the existence of finite energy weak solutions can be proved without the initial data being prescribed by a given wave function. Moreover, in [6, 7] we also provide a stability result for weak solutions to (1.1). More precisely, by introducing a new functional relating to the chemical potential, a class is determined in which arbitrary sequences of weak solutions have a strongly convergent subsequence to another weak solution.

The results in [6, 7] provide preliminary steps towards the development of an intrinsically hydrodynamic theory for systems like (1.1) or (1.4). This goal is not only motivated by the above discussion about the assumptions on the initial data. More importantly, the NLS-based approach for the construction of solutions to (1.4) in general fails. Indeed the wave function dynamics associated to (1.4) is given by the following quasilinear Schrödinger equation

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi - \operatorname{div}\left(\tilde{\kappa}(|\psi|^2)\nabla|\psi|^2\right)\psi + \frac{1}{2}\tilde{\kappa}'(|\psi|^2)|\nabla|\psi|^2|^2\psi + f'(|\psi|^2)\psi,$$

where $\tilde{\kappa}(\rho) = \kappa(\rho) - \frac{1}{4\rho}$, for which no satisfactory (global) theory for arbitrarily large initial data is available. This is indeed the reason why the only available results on the existence of solutions to (1.4) consider small, regular perturbations of constant states.

Thus, any possible attempt towards establishing a satisfactory existence theory for finite energy weak solutions to system (1.4) would require to derive suitable a priori bounds. The lack of those bounds is a major problem for conservative systems like (1.1) or (1.4). This is in sharp contrast with viscous systems (like the compressible Navier-Stokes system [26, 19], for instance), where the energy dissipation gives a Sobolev control on the velocity field. Also in the case of degenerate viscosity [24, 25], where the energy dissipation does not yield sufficient bounds due to the degeneracy in the vacuum region, further a priori estimates are available [12, 13], which allow to infer a compactness property for weak solutions, see for instance [8] for the QNS equations, namely system (1.1) augmented by a degenerate viscosity, or [9] for the analysis of a particular Navier-Stokes-Korteweg system.

The main goal of this paper is to collect some dispersive estimates available for (1.1) and (1.4). Some of these estimates have their analogue also in classical compressible fluid dynamics on in the context of NLS-type equations. This is the case for instance of the estimates presented in Proposition 4 or the interaction Morawetz estimates of Proposition 6. However some other estimates that we present here appear to be completely new, see for example the Morawetz-type estimates in Proposition 13.

This paper is structured as follows. In Section 2 we review the dispersive estimates based on a modulated energy functional, see (2.4), and the interaction Morawetz estimates. Section 3 is devoted to the functional $I(t)$ defined in (3.4). Finally in Section 4 we prove a second Morawetz-type estimate, first introduced in [6] to study the stability of weak solutions to the one dimensional QHD system.

**1.1. Preliminaries.** First of all, we recall the definition of finite energy weak solutions for the QHD system.

Definition 1. Let $0 < T \leq \infty$, $\rho_0 \in L^1_{loc}(\mathbf{R}^d)$, $J_0 \in L^1_{loc}(\mathbf{R}^d)$, we say that $(\rho, J)$ is a weak solution to (1.1) in $[0,T) \times \mathbf{R}^d$ if there exists a pair of functions $(\sqrt{\rho}, \Lambda)$, with $\sqrt{\rho} \in L^2_{loc}(0,T; H^1_{loc}(\mathbf{R}^d))$, $\Lambda \in L^2_{loc}(0,T; L^2_{loc}(\mathbf{R}^d))$, such that

- $\rho = (\sqrt{\rho})^2$, $J = \sqrt{\rho}\Lambda$;
- (continuity equation) for any $\eta \in \mathcal{C}^\infty_c([0,T) \times \mathbf{R}^d)$, we have
$$\int_0^T \int_{\mathbf{R}^d} \rho \partial_t \eta + J \cdot \nabla \eta \, dx dt + \int_{\mathbf{R}^d} \rho_0 \eta(0, \cdot) \, dx = 0;$$
- (momentum equation) for any test function $\zeta \in \mathcal{C}^\infty_c([0,T) \times \mathbf{R}^d; \mathbf{R}^d)$, we have
$$\int_0^T \int_{\mathbf{R}^d} J \cdot \partial_t \zeta + \Lambda \otimes \Lambda : \nabla \zeta + p(\rho) \operatorname{div} \zeta + \nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho} : \nabla \zeta$$
$$- \rho \frac{1}{4} \Delta \operatorname{div} \zeta \, dx dt + \int_{\mathbf{R}^d} J_0 \cdot \zeta(0, \cdot) \, dx = 0;$$
- (generalized irrotationality condition) for a.e. $t \in [0,T)$ the identity
$$\nabla \wedge J = 2 \nabla\sqrt{\rho} \wedge \Lambda,$$
is satisfied in the sense of distributions.

We say that $(\rho, J)$ is a *finite energy weak solution* if moreover for a.e. $t \in [0,T)$ the total mass $M(t) \int \rho(t,x) \, dx$ is finite and the total energy satisfies $E(t) \leq E(0)$, where $E(t)$ is defined in (1.3).

In order to give the analogue definition also for the Euler-Korteweg system (1.4) we need some additional integrability assumptions, depending on the capillarity coefficient. For this purpose, it is convenient to introduce the following state functions. We define
$$K(\rho) = \int_0^\rho s\kappa(s) \, ds, \quad \beta(\rho) = \int_0^\rho \sqrt{\kappa(s)} \, ds. \tag{1.8}$$
By using these additional state functions and by also recalling identity (1.5) we can rewrite system (1.4) as
$$\begin{cases} \partial_t \rho + \operatorname{div} J = 0 \\ \partial_t J + \operatorname{div}(\Lambda \otimes \Lambda) + \nabla p(\rho) = \nabla \Delta K(\rho) - \frac{1}{2} \nabla (K''(\rho) |\nabla \rho|^2) - \operatorname{div}(\nabla \beta(\rho) \otimes \nabla \beta(\rho)). \end{cases} \tag{1.9}$$

Definition 2. Let $\kappa(\cdot) \in C^1(\mathbf{R}_+)$. Let $0 < T \leq \infty$, $\rho_0 \in L^1_{loc}(\mathbf{R}^d)$, $J_0 \in L^1_{loc}(\mathbf{R}^d)$, we say that $(\rho, J)$ is a weak solution to (1.1) in $[0,T) \times \mathbf{R}^d$ if there exists a pair of functions $(\sqrt{\rho}, \Lambda)$ such that

- $\rho = (\sqrt{\rho})^2$, $J = \sqrt{\rho}\Lambda$;
- $\rho \in L^1_{loc}([0,T] \times \mathbf{R}^d)$, $\Lambda \in L^2_{loc}([0,T] \times \mathbf{R}^d)$;
- $\beta(\rho) \in L^2_{loc}(0,T; H^1_{loc}(\mathbf{R}^d))$, where $\beta(\rho)$ is defined in (1.8);
- $K(\rho) \in L^1_{loc}([0,T] \times \mathbf{R}^d)$, where $K(\rho)$ is defined in (1.8);
- $K''(\rho)|\nabla\rho|^2 \in L^1_{loc}([0,T] \times \mathbf{R}^d)$;
- (continuity equation) for any $\eta \in \mathcal{C}^\infty_c([0,T) \times \mathbf{R}^d)$, we have

$$\int_0^T \int_{\mathbf{R}^d} \rho\partial_t\eta + J \cdot \nabla\eta \, dxdt + \int_{\mathbf{R}^d} \rho_0\eta(0,\cdot)\, dx = 0; \tag{1.10}$$

- (momentum equation) for any test function $\zeta \in \mathcal{C}^\infty_c([0,T) \times \mathbf{R}^d; \mathbf{R}^d)$, we have

$$\begin{aligned} \int_0^T \int_{\mathbf{R}^d} & J \cdot \partial_t\zeta + \Lambda \otimes \Lambda : \nabla\zeta + p(\rho)\operatorname{div}\zeta + \nabla\beta(\rho) \otimes \nabla\beta(\rho) : \nabla\zeta \\ & + \frac{1}{2}K''(\rho)|\nabla\rho|^2 \operatorname{div}\zeta - K(\rho)\Delta\operatorname{div}\zeta \, dxdt \\ & + \int_{\mathbf{R}^d} J_0 \cdot \zeta(0,\cdot)\, dx = 0; \end{aligned} \tag{1.11}$$

We say that $(\rho, J)$ is a *finite energy weak solution* if moreover for a.e. $t \in [0,T)$ the total mass $M(t) = \int \rho(t,x)\, dx$ is finite and the total energy satisfies $E_\kappa(t) \le E_\kappa(0)$, where $E_\kappa(t)$ is defined in (1.6).

**2. Dispersive estimates for finite energy weak solutions and their asymptotic behavior.** In this Section we present some dispersive estimates which provide information about the asymptotic behavior of finite energy weak solutions for large times. The results of this Section will be given for a general Euler–Korteweg system (1.4), the analogous statements for the QHD system can be recast by setting $\kappa(\rho) = \frac{1}{4\rho}$ in the subsequent analysis.

The first dispersive estimate we discuss here is based on the total momentum of inertia functional, given by

$$V(t) = \int \frac{|x|^2}{2}\rho(t,x)\, dx. \tag{2.1}$$

It is straightforward to check that formally we have

$$\frac{d}{dt}V(t) = \int x \cdot J(t,x)\, dx \tag{2.2}$$

and moreover

$$\frac{d}{dt}\int x \cdot J(t,x)\, dx = 2E_\kappa(t) + (d(\gamma-1)+2)\int \rho^\gamma\, dx + \frac{d}{2}\int K''(\rho)|\nabla\rho|^2\, dx, \tag{2.3}$$

where $E_\kappa$ is the total energy defined in (1.6) with the internal energy density $f(\rho) = \frac{1}{\gamma}\rho^\gamma$. Given the above identities it is therefore natural to consider the following functional

$$H_\kappa(t) = \int \frac{|x|^2}{2}\rho(t,x)\, dx - t\int x \cdot J(t,x)\, dx + t^2 E_\kappa(t).$$

Let us notice that at initial time we have $V(0) = \int \frac{|x|^2}{2}\rho_0(x)\,dx$ and that, for any $t \in \mathbf{R}$, $V(t)$ is actually non-negative. Indeed we may write

$$H_\kappa(t) = t^2 \int \frac{1}{2}\left|\Lambda - \frac{x}{t}\sqrt{\rho}\right|^2 + \frac{1}{2}\kappa(\rho)|\nabla\rho|^2 + \frac{1}{\gamma}\rho^\gamma\,dx. \tag{2.4}$$

In the next Proposition we show that, by assuming the initial momentum of inertia (2.1) to be finite at initial time and that $E(t)$ is non-increasing, the functional $H_\kappa(t)$ will be bounded for weak solutions to Euler-Korteweg equations. The functional $H_\kappa(t)$ can be interpreted as a modulated energy functional (or relative entropy) centered on the rarefaction wave for the velocity $v = \frac{x}{t}$.

PROPOSITION 3. *Let $(\rho, J)$ be a finite energy weak solution to* (1.4) *as in Definition 2 and let us assume that the total energy $E(t)$ is non-increasing and that*

$$\int_{\mathbf{R}^d} |x|^2\rho_0(x)\,dx < \infty.$$

*Then we have*

$$\begin{aligned} H_\kappa(t) &+ (d(\gamma-1)-2)\int_0^t s\int_{\mathbf{R}^d}\rho^\gamma(s,x)\,dxds \\ &+\frac{d}{2}\int_0^t s\int_{\mathbf{R}^d} K''(\rho)(s,x)|\nabla\rho(s,x)|^2dxds \le \int_{\mathbf{R}^d}\frac{|x|^2}{2}\rho_0(x)\,dx. \end{aligned} \tag{2.5}$$

*Proof.* The proof is rather standard, so we just sketch it out here. We address to Proposition 28 in [6] or Proposition 12 in [5] for more details.

It is immediate to show this result for smooth solutions when the energy is conserved, since in this situation, by using the identities (2.1), (2.2) and (2.3) it follows

$$\frac{d}{dt}H_\kappa(t) = -(d(\gamma-1)-2)t\int\rho^\gamma\,dx - \frac{d}{2}t\int K''(\rho)|\nabla\rho|^2\,dx.$$

Then by integrating the previous identity in time we get formula (2.5), wwith an equality sign. Now let us consider the case of finite energy weak solutions which satisfy the assumptions of Proposition 3. We begin by proving that the total momentum of inertia is finite for any finite time, provided that it is finite at time $t = 0$ Then, combining the identities (1.10) and (1.11) with a suitable choice of test functions, it is possible to show a truncated version of (2.5), therefore the complete inequality (2.5) is obtained by passing into the limit the truncated ones. □ As already pointed out the Introduction, the dispersive estimates presented here have their analogue also in other contexts. In particular similar calculations as the ones presented in Proposition 3 are used in compressible fluid dynamics [32, 31, 33], for the Vlasov-Poisson system [11, 30, 22] and for NLS equations [21, 10, 20]. There is a connected result for a particular class of Euler–Korteweg systems, somehow related to the dispersive logarithmic Schrödinger equation ($\gamma = 1$), given in [15]. A similar result appeared also in the Appendix of the review [16]. Moreover, the Morawetz-type estimates of Proposition 6 below are the analogue of the interaction Morawetz estimates for NLS equations [17], see also [29].

PROPOSITION 4. *Assume the same hypotheses of Proposition 3 and moreover $K''(\rho) \ge 0$, then one has*

$$H_\kappa(t) \lesssim t^{2(1-\sigma)} + \int_{\mathbf{R}^d}\frac{|x|^2}{2}\rho_0(x)\,dx, \tag{2.6}$$

*where* $\sigma = \min\{1, \frac{d}{2}(\gamma - 1)\}$. *In particular it follows*

$$\int_{\mathbf{R}^d} \kappa(\rho(t,x))|\nabla\rho(t,x)|^2\,dx \lesssim t^{-2\sigma},$$

$$\int_{\mathbf{R}^d} \rho^\gamma(t,x)\,dx \lesssim t^{-2\sigma}.$$

$$\int_{\mathbf{R}^d} |\Lambda(t,x) - \frac{x}{t}\sqrt{\rho}(t,x)|^2\,dx \lesssim t^{-2\sigma}.$$

*Proof.* Let us first consider the case $\gamma \geq 1 + \frac{2}{d}$, then from (2.5) we have

$$H_\kappa(t) + \int_0^t s\int_{\mathbf{R}^d} \frac{K''(\rho)}{2}|\nabla\rho|^2 dxds \leq \int \frac{|x|^2}{2}\rho_0(x)\,dx.$$

Thus (2.6) holds for $\sigma = 1$. Let us now consider the case $1 < \gamma < 1 + \frac{2}{d}$, if we define

$$F(t) = \frac{t^2}{\gamma}\int \rho^\gamma(t,x)\,dx$$

then by (2.5) we have

$$F(t) \leq H_\kappa(t) \leq [2 + d(1-\gamma)]\int_0^t \frac{1}{s}F(s)\,ds + \int \frac{|x|^2}{2}\rho_0(x)\,dx.$$

By Gronwall we then have

$$F(t) \lesssim t^{2+d(1-\gamma)}F(1) + \int \frac{|x|^2}{2}\rho_0(x)\,dx,$$

which also implies

$$\int \rho^\gamma(t,x)\,dx \lesssim t^{d(1-\gamma)}.$$

We can now plug the above estimate in (2.5) in order to obtain

$$H_\kappa(t) + \int_0^t s\int_{\mathbf{R}^d} \frac{K''(\rho)}{2}|\nabla\rho|^2 dxds \lesssim \int_0^t s^{1+d(1-\gamma)}\,ds + \int \frac{x^2}{2}\rho_0(x)\,dx,$$

which then implies (2.6). □

Remark 5. Proposition 4 shows that $\kappa(\rho)|\nabla\rho|^2$ converges to zero and formally the velocity field asymptotically approaches a rarefaction wave, namely $v(t,x) \sim x/t$ as $t \to \infty$. If the total energy is conserved, we also infer that

$$\lim_{t\to\infty} \frac{1}{2}\|\Lambda(t)\|_{L^2}^2 = E(0),$$

i.e. for large times all the energy is transferred to the kinetic part. Moreover, in the case of QHD system, namely $\kappa(\rho) = \frac{1}{4\rho}$, by Gagliardo-Nirenberg the decay of $\|\kappa^{\frac{1}{2}}(\rho)\nabla\rho\|_{L^2} = \|\nabla\sqrt{\rho}(t)\|_{L^2}$ implies the following dispersive estimate

$$\|\sqrt{\rho}(t)\|_{L^p} \lesssim \|\sqrt{\rho}(t)\|_{L^2}^{1-\alpha}\|\nabla\sqrt{\rho}(t)\|_{L^2}^{\alpha} \lesssim t^{-\alpha\sigma},$$

where $0 \le \alpha \le 1$ such that

$$\frac{1}{p} = \frac{1}{2} - \frac{\alpha}{d},$$

and $2 \le p < \infty$ for $d = 2$, $2 \le p \le 6$ for $d = 3$. Notice that in the case $\gamma \ge 1 + 2/d$ this is still consistent with the dispersive estimate for the free Schrödinger equation.

The next Proposition is the analogue of the interaction Morawetz estimates for NLS equations [17], see also [29, 27].

PROPOSITION 6. *Let* $(\rho, J)$ *be a weak solution to the system* (1.4) *such that*

$$M(t) + E(t) \le M_1, \quad a.e.\ t \in [0, T)$$

*for a constant* $0 < M_1 < \infty$*, and let us further assume that* $K''(\rho) \ge 0$*, where* $K(\rho)$ *is defined in* (1.8)*. Then we have*

$$\| |\nabla|^{\frac{3-d}{2}} \sqrt{\rho K(\rho)} \|^2_{L^2(\mathbf{R}_t \times \mathbf{R}^d_x)} + \| |\nabla|^{\frac{1-d}{2}} \sqrt{\rho p(\rho)} \|^2_{L^2(\mathbf{R}_t \times \mathbf{R}^d_x)} \lesssim M_1^2.$$

*Proof.* Analogously to the proof of Proposition 3, also here we provide a formal proof. The rigorous argument, as before, requires to combine the identities (1.10), (1.11) by suitably choosing the test functions. We omit these technical details here.

Let us consider the interaction functional

$$M^{int}(t) = \int_{\mathbf{R}^{2d}} \rho(t, y) \frac{x - y}{|x - y|} \cdot J(t, x) dx dy.$$

For $(\rho, J)$ with finite total mass and energy, the functional $M^{int}(t)$ is well-defined on $[0, T]$ with bound

$$|M^{int}(t)| \le \|\rho\|_{L^\infty_t L^1_x} \|J\|_{L^\infty_t L^1_x} \le M_1^2.$$

By differentiating $M^{int}(t)$ in time using the system (1.9) and integrating by parts we obtain

$$\begin{aligned} \frac{d}{dt} M^{int}(t) = &- \int_{\mathbf{R}^{2d}} J(y) \cdot B(x, y) \cdot J(x) dx dy + \int_{\mathbf{R}^{2d}} \rho(y) \Lambda(t, x) \cdot B(x, y) \cdot \Lambda(x) dx dy \\ &+ \int_{\mathbf{R}^{2d}} \rho(y) \nabla \beta(\rho)(x) \cdot B(x, y) \cdot \nabla \beta(\rho)(x) dx dy \\ &+ \int_{\mathbf{R}^{2d}} \rho(y) \frac{d-1}{|x - y|} \left[ p(\rho) + \frac{1}{2} K''(\rho) |\nabla \rho|^2 \right] (x) dx dy \\ &- \int_{\mathbf{R}^{2d}} \rho(y) \frac{d-1}{|x - y|} \triangle K(\rho)(x) dx dy, \end{aligned} \tag{2.7}$$

where in the right hand side we omitted the time dependence for simplicity and the matrix $B$ is given by

$$B(x, y) = \nabla_x \left( \frac{x - y}{|x - y|} \right) = \frac{1}{|x - y|} \left[ Id - \frac{(x - y) \otimes (x - y)}{|x - y|^2} \right].$$

Since $B(x,y)$ is a semi-positive definite matrix and symmetric in $x$ and $y$, we have

$$\begin{aligned}\int_{\mathbf{R}^{2d}} J(t,y)\cdot B(x,y)\cdot J(t,x)dxdy =&\int_{\mathbf{R}^{2d}}(\sqrt{\rho}\Lambda)(t,y)\cdot B(x,y)\cdot(\sqrt{\rho}\Lambda)(t,x)dxdy\\ \le&\frac{1}{2}\int_{\mathbf{R}^{2d}}\rho(t,y)\Lambda(t,x)\cdot B(x,y)\cdot\Lambda(t,x)dxdy\\ &+\frac{1}{2}\int_{\mathbf{R}^{2d}}\rho(t,x)\Lambda(t,y)\cdot B(x,y)\cdot\Lambda(t,y)dxdy\\ =&\int_{\mathbf{R}^{2d}}\rho(t,y)\Lambda(t,x)\cdot B(x,y)\cdot\Lambda(t,x)dxdy.\end{aligned}$$

Therefore the first and second lines in the right hand side of (2.7) are non-negative. Also by our assumption

$$\int_{\mathbf{R}^{2d}}\rho(t,y)\frac{d-1}{|x-y|}\left[K''(\rho)|\nabla\rho|^2\right](t,x)dxdy\ge 0.$$

To estimate the remaining integrals in the right hand side of (2.7), we first notice that by the symmetry in $x$ and $y$, we have

$$\begin{aligned}\int_{\mathbf{R}^{2d}}\rho(t,y)\frac{d-1}{|x-y|}p(\rho)(t,x)dxdy =&\frac{1}{2}\int_{\mathbf{R}^{2d}}\rho(t,y)\frac{d-1}{|x-y|}p(\rho)(t,x)dxdy\\ &+\frac{1}{2}\int_{\mathbf{R}^{2d}}\rho(t,x)\frac{d-1}{|x-y|}p(\rho)(t,y)dxdy\\ \ge&\int_{\mathbf{R}^{2d}}\sqrt{\rho\, p(\rho)}(t,y)\frac{d-1}{|x-y|}\sqrt{\rho\, p(\rho)}(t,x)dxdy\end{aligned}$$

The convolution of the kernel $|x-y|^{-1}$ gives

$$\int_{\mathbf{R}^{d}}\frac{1}{|x-y|}\sqrt{\rho\, p(\rho)}(t,x)dx = C|\nabla|^{1-d}\sqrt{\rho\, p(\rho)}(t,y),$$

then we obtain

$$\begin{aligned}\int_{\mathbf{R}^{2d}}\rho(t,y)\frac{d-1}{|x-y|}p(\rho)(t,x)dxdy \ge& C\int_{\mathbf{R}^{d}}\sqrt{\rho\, p(\rho)}(t,y)|\nabla|^{1-d}\sqrt{\rho\, p(\rho)}(t,y)dy\\ =&C\||\nabla|^{\frac{1-d}{2}}\sqrt{\rho\, p(\rho)}\|^2_{L^2_x}(t).\end{aligned}$$

Similarly the last line of (2.7) can be written as

$$-\int_{\mathbf{R}^{2d}}\rho(t,y)\frac{d-1}{|x-y|}\triangle K(\rho)(t,x)dxdy = C\int_{\mathbf{R}^{d}}\rho(t,y)|\nabla|^{3-d}K(\rho)(t,y)dy,$$

and again by the symmetry we obtain

$$-\int_{\mathbf{R}^{2d}}\rho(t,y)\frac{d-1}{|x-y|}\triangle K(\rho)(t,x)dxdy \ge C\||\nabla|^{\frac{3-d}{2}}\sqrt{\rho K(\rho)}\|^2_{L^2_x}(t).$$

By summarising the argument above, we get

$$\frac{d}{dt}M^{int}(t)\ge C\||\nabla|^{\frac{1-d}{2}}\sqrt{\rho\, p(\rho)}\|^2_{L^2_x}(t)+C\||\nabla|^{\frac{3-d}{2}}\sqrt{\rho K(\rho)}\|^2_{L^2_x}(t), \tag{2.8}$$

then integrating (2.8) in time shows

$$\||\nabla|^{\frac{1-d}{2}}\sqrt{\rho\, p(\rho)}\|^2_{L^2_tL^2_x}+C\||\nabla|^{\frac{3-d}{2}}\sqrt{\rho K(\rho)}\|^2_{L^2_tL^2_x}\lesssim M^{int}(T)-M^{int}(0)\lesssim M_1^2.$$

□

**3. The weak entropy inequality and a higher order energy estimate.** In this Section we discuss the functional first introduced in [6] related to the chemical potential associated to system (1.1). Formally, the chemical potential is determined by the first variation of the total energy functional with respect to the mass density,

$$\mu = \frac{\delta E}{\delta \rho} = -\frac{1}{2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} + \frac{1}{2}|v|^2 + f'(\rho). \tag{3.1}$$

By multiplying the above definition by $\rho$, we then have

$$\xi = \rho\mu = -\frac{1}{4}\Delta\rho + e + p(\rho), \tag{3.2}$$

where $e$ is the total energy density given by

$$e = \frac{1}{2}|\nabla\sqrt{\rho}|^2 + \frac{1}{2}|\Lambda|^2 + f(\rho). \tag{3.3}$$

Formally, the functional we consider is given by

$$I(t) = \int_{\mathbf{R}^d} \frac{1}{2}\rho\mu^2 + \frac{1}{2}(\partial_t\sqrt{\rho})^2\,dx.$$

As thoroughly discussed in [6], the chemical potential $\mu$ is too singular in the framework of weak solutions to (1.1). For this reason in [6] we rather consider the function $\lambda$, which is implicitly given by $\sqrt{\rho}\lambda = \xi$, by further setting $\lambda = 0$ a.e. in the vacuum region. In this way the functional $I(t)$ can be written as

$$I(t) = \int_{\mathbf{R}^d} \frac{1}{2}\lambda^2 + \frac{1}{2}(\partial_t\sqrt{\rho})^2\,dx, \tag{3.4}$$

so that $I(t) < \infty$ implies $\lambda \in L^2(\mathbf{R}^d)$. Let us remark that this is somehow analogous to what we already discuss about the velocity field, which is not well defined in the vacuum region, whereas we have $\Lambda \in L^2(\mathbf{R}^d)$.

By exploiting the analogy with NLS dynamics for the QHD system given by means of the Madelung transformations (or more rigorously, by the polar factorization), then we can see that, for Schrödinger-generated solutions we actually have

$$I(t) = \int_{\mathbf{R}^d} \frac{1}{2}|\partial_t\psi|^2\,dx.$$

Indeed by using the polar factorization (see Lemma 2.1 in [4] for instance) we have

$$|\partial_t\psi|^2 = (\partial_t\sqrt{\rho})^2 + \lambda^2,$$

where clearly $\partial_t\sqrt{\rho} = \mathrm{Re}(\bar{\phi}\partial_t\psi)$ and we defined $\lambda := -\mathrm{Im}(\bar{\phi}\partial_t\psi)$. Moreover, let us notice that, by using equation (1.7), we also have

$$\sqrt{\rho}\lambda = -\mathrm{Im}(\bar{\psi}\partial_t\psi) = \frac{1}{4}\Delta\rho + \frac{1}{2}|\nabla\psi|^2 + \rho f'(\rho),$$

so that we obtain again identity (3.2) from the polar factorization. Consequently, the functional $I(t)$ gives an $H^2$ control for solutions to (1.7). On the contrary, let us notice that $I(t)$ does not provide additional regularity estimates for the hydrodynamic state $(\sqrt{\rho}, \Lambda)$ even though, as we see in Proposition 13 later, the boundedness of $I(t)$ will yield better integrability properties. For more details about the generalized chemical potential $\lambda$ and the functional $I(t)$ we address the reader to [6, 7].

First of all, we derive the conservation law for the total energy density, defined in (3.3).

LEMMA 7. *Let* $(\rho, J)$ *be a smooth solution to* (1.1) *such that* $\rho > 0$. *Then the energy density* $e$ *satisfies the following conservation law*

$$\partial_t e + \operatorname{div}(\Lambda\lambda - \partial_t\sqrt{\rho}\nabla\sqrt{\rho}) = 0. \tag{3.5}$$

*Proof.* Since we are dealing with smooth solutions and for which $\rho > 0$, then we can write system (1.1) as

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho v) = 0 \\ \rho\partial_t v + \rho v\cdot\nabla v + \rho\nabla f'(\rho) = \frac{1}{2}\rho\nabla\left(\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}}\right) \end{cases}$$

and we notice that the equation for the momentum density can be equivalently written as

$$\rho\partial_t v + \rho\nabla\mu = 0,$$

where $\mu$ is the chemical potential defined in (3.1). By differentiating the energy density with respect to time we obtain

$$\begin{aligned} \partial_t e =& \nabla\sqrt{\rho}\cdot\partial_t\nabla\sqrt{\rho} + \left(\frac{1}{2}|v|^2 + f'(\rho)\right)\partial_t\rho + \rho v\partial_t v \\ =& \operatorname{div}\left(\nabla\sqrt{\rho}\partial_t\sqrt{\rho}\right) + \left(-\frac{1}{2}\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}} + \frac{1}{2}|v|^2 + f'(\rho)\right)\partial_t\rho + \rho v\partial_t v. \end{aligned}$$

By using the evolution equations above and definition (3.1) we then have

$$\begin{aligned} \partial_t e =& \operatorname{div}\left(\nabla\sqrt{\rho}\partial_t\sqrt{\rho}\right) - \mu\operatorname{div}(\rho v) - \rho v\cdot\nabla\mu \\ =& \operatorname{div}\left(\nabla\sqrt{\rho}\partial_t\sqrt{\rho} - \rho v\mu\right). \end{aligned}$$

□ By using the previous Lemma we can now compute the time derivative of the functional $I(t)$.

PROPOSITION 8. *Let* $(\rho, J)$ *be a smooth solution to* (1.1) *such that* $\rho > 0$. *Then we have*

$$\frac{d}{dt}I(t) = 2\int_{\mathbf{R}^d}\lambda\partial_t\sqrt{\rho}\,p'(\rho)\,dx. \tag{3.6}$$

*Proof.* Since we consider smooth, positive solutions to (1.1), then we write the functional $I(t)$ as

$$I(t) = \frac{1}{2}\int_{\mathbf{R}^d}\rho(\mu^2 + \sigma^2)\,dx,$$

where the chemical potential is given in (3.1) and we defined $\sigma = \partial_t \log\sqrt{\rho}$. In this framework we can formally derive the evolution equations for $\mu, \sigma$. By using the identity (3.2) we have

$$\begin{aligned}\partial_t(\rho\mu) =& \partial_t\left(e - \frac{1}{4}\triangle\rho + p(\rho)\right)\\ =& \operatorname{div}(\nabla\sqrt{\rho}\partial_t\sqrt{\rho} - \rho v\mu) - \frac{1}{4}\partial_t\triangle\rho + \partial_t p(\rho).\end{aligned}$$

By using the continuity equation we may write

$$\rho\partial_t\mu + \rho v\cdot\nabla\mu = \operatorname{div}(\nabla\sqrt{\rho}\partial_t\sqrt{\rho}) - \frac{1}{4}\partial_t\triangle\rho + \partial_t p(\rho). \tag{3.7}$$

We now formally derive the equation for $\sigma$. Let us write the continuity equation as below

$$\partial_t\rho + v\cdot\nabla\rho + \rho\operatorname{div} v = 0,$$

we find the equation for $\log\sqrt{\rho}$, namely

$$\partial_t\log\sqrt{\rho} + v\cdot\nabla\log\sqrt{\rho} + \frac{1}{2}\operatorname{div} v = 0.$$

By differentiating the previous equation with respect to time and by using $\partial_t v = -\nabla\mu$ we then obtain

$$\partial_t\sigma + v\cdot\nabla\sigma - \nabla\mu\cdot\nabla\log\sqrt{\rho} - \frac{1}{2}\triangle\mu = 0.$$

By multiplying this by $\rho$ we get

$$\rho\partial_t\sigma + \rho v\cdot\nabla\sigma = \frac{1}{2}\operatorname{div}(\rho\nabla\mu). \tag{3.8}$$

Now we can use the equations (3.7) and (3.8) to compute the time derivative of the functional $I(t)$. By differentiating the functional $I(t)$ in time using the continuity equation of $\rho$, the evolution (3.7) and (3.8) of $\mu$ and $\sigma$, we obtain

$$\begin{aligned}\frac{d}{dt}I(t) =& \int_{\mathbf{R}^d}\frac{1}{2}(\mu^2+\sigma^2)\partial_t\rho + \mu\rho\partial_t\mu + \sigma\rho\partial_t\sigma\,dx\\ =& \int_{\mathbf{R}^d}\mu\left(\operatorname{div}(\nabla\sqrt{\rho}\partial_t\sqrt{\rho}) - \frac{1}{4}\triangle\partial_t\rho + \partial_t p(\rho))\right)\\ &+ \frac{\sigma}{2}\operatorname{div}(\rho\nabla\mu)\,dx.\end{aligned}$$

By using that $\sqrt{\rho}\sigma = \partial_t\sqrt{\rho}$, $\rho\sigma = \frac{1}{2}\partial_t\rho$ and integration by parts, we get

$$\frac{d}{dt}I(t) = \int_{\mathbf{R}^d}\mu\partial_t p(\rho)dx = 2\int_{\mathbf{R}^d}\lambda\partial_t\sqrt{\rho}\,p'(\rho)dx.$$

□ It is clear that in order to infer a bound on $I(t)$, we would need to use Gronwall-type estimate in (3.6). We first deal with the one dimensional case, which is more immediate, then the two dimensional case will be treated in Proposition 12.

Proposition 9. *Let $d = 1$. Let $(\rho, J)$ be a positive, smooth solution to* (1.1) *with finite total mass and total energy, namely there exists a $0 < M_1 < \infty$ such that*

$$M(0) + E(0) \leq M_1.$$

*Then it follows*

$$\sup_{t\in[0,T]} I(t) \leq C(M_1, T)I(0).$$

*Proof.* By using (3.6), it follows

$$\begin{aligned}\frac{d}{dt}I(t) =& 2(\gamma-1)\int_{\mathbf{R}} \rho^{\gamma-1}\lambda\partial_t\sqrt{\rho}\,dx\\ \lesssim& \|\sqrt{\rho}\|^{\gamma-1}_{L^\infty(\mathbf{R})} I(t)\\ \lesssim& \|\sqrt{\rho}\|^{\gamma-1}_{H^1(\mathbf{R})} I(t) \lesssim M_1^{\frac{\gamma-1}{2}} I(t).\end{aligned}$$

Then the conservation of $M(t)$, $E(t)$ and Gronwall's inequality imply the bound of $I(t)$. □

In the two dimensional case, we will show that an analogue of Proposition 9 holds for $\gamma \leq \frac{5}{3}$, but we need more information on the solutions since the $L^\infty_{t,x}$ bound of $\rho$ doesn't come directly from the mass and energy bounds any more. The next lemma shows that in the framework of the functional $I(t)$, the hydrodynamic functions $(\rho, J)(t)$ can be associated to a complex function $\psi(t,\cdot) \in H^2(\mathbf{R}^d)$ via a method polar factorization for any $t$, and for the details we refer to Proposition 22 in [7].

Lemma 10. *Let $(\rho, J)$ be a smooth solution to* (1.1) *such that $\rho > 0$ and*

$$M(t) + E(t) + I(t) < \infty, \quad t \in [0, T).$$

*Then for any $t \in [0, T)$, there exists a complex function $\psi(t,\cdot) \in H^2(\mathbf{R}^d)$ such that*

$$\sqrt{\rho} = |\psi|, \quad \Lambda = \mathrm{Im}(\bar{\phi}\nabla\psi), \quad \nabla\psi = (\nabla\sqrt{\rho} + i\Lambda)\phi, \tag{3.9}$$

*where $\phi$ is a polar factor of $\psi$. Moreover, for $t \in [0, T)$ we have the bound*

$$\|\nabla^2\psi(\cdot;t)\|^2_{L^2(\mathbf{R}^d)} \leq C\,I(t) + C(M(t), E(t)).$$

Another technical tool we need for our computation is the Brezis-Gallouet inequality [14].

Lemma 11. *For any $f \in H^2(\mathbf{R}^2)$ we have*

$$\|f\|_{L^\infty(\mathbf{R}^2)} \leq C(1 + \sqrt{\log(1 + \|f\|_{H^2(\mathbf{R}^2)})}). \tag{3.10}$$

Now we can show the boundedness of $I(t)$ in the 2-dimensional case.

Proposition 12. *Let $(\rho, J)$ be a positive ($\rho > 0$), smooth solution to* (1.1) *in 2-dimensional space such that*

$$M(0) + E(0) \leq M_1 < \infty,$$

*and we further assume* $1 < \gamma \le \frac{5}{2}$, *then we have*

$$\sup_{t\in[0,T]} I(t) \le C(M_1, T)I(0).$$

*Proof.* By using formula (3.6) we have

$$\begin{aligned}\frac{d}{dt}I(t) =& 2\int_{\mathbf{R}^2} \lambda\partial_t\sqrt{\rho}\,p'(\rho)dx = \int_{\mathbf{R}^2} \sqrt{\rho}\lambda\partial_t\rho^{\gamma-1}\,dx\\ =& \int_{\mathbf{R}^2}\left(-\frac{1}{4}\Delta\rho+\frac{1}{2}|\nabla\sqrt{\rho}|^2+\frac{1}{2}|\Lambda|^2+\rho^\gamma\right)\partial_t\rho^{\gamma-1}\,dx,\end{aligned}$$

where in the last equality we used formula (3.2). By using the identity

$$\frac{\gamma-1}{4\gamma}\frac{d}{dt}\int|\nabla\rho^{\frac{\gamma}{2}}|^2\,dx = -\frac{1}{4}\int\Delta\rho\partial_t\rho^{\gamma-1}\,dx-\frac{\gamma-2}{2}\int|\nabla\sqrt{\rho}|^2\partial_t\rho^{\gamma-1}\,dx,$$

we have that

$$\begin{aligned}\frac{d}{dt}I(t) = \frac{\gamma-1}{4\gamma}\frac{d}{dt}\int|\nabla\rho^{\frac{\gamma}{2}}|^2\,dx + \int\left(\frac{\gamma-1}{2}|\nabla\sqrt{\rho}|^2+\frac{1}{2}|\Lambda|^2\right)\partial_t\rho^{\gamma-1}\,dx\\ + \frac{\gamma-1}{2\gamma-1}\frac{d}{dt}\int\rho^{2\gamma-1}\,dx.\end{aligned}$$

Consequently we obtain

$$\begin{aligned}\frac{d}{dt}\left[I(t)-\frac{\gamma-1}{4\gamma}\int|\nabla\rho^{\frac{\gamma}{2}}|^2\,dx-\frac{\gamma-1}{2\gamma-1}\int\rho^{2\gamma-1}\,dx\right]\\ = \int\left(\frac{\gamma-1}{2}|\nabla\sqrt{\rho}|^2+\frac{1}{2}|\Lambda|^2\right)\partial_t\rho^{\gamma-1}\,dx. \quad (3.11)\end{aligned}$$

To control the term on the right hand side, we use Lemma 10. By the polar factorization (3.9), we have

$$\int\left(\frac{\gamma-1}{2}|\nabla\sqrt{\rho}|^2+\frac{1}{2}|\Lambda|^2\right)\partial_t\rho^{\gamma-1}\,dx \le C\int_{\mathbf{R}^2}|\nabla\psi(x;t)|^2|\psi(x;t)|^{2\gamma-3}\partial_t\sqrt{\rho}\,dx.$$

Moreover by using Gagliardo-Nirenberg inequality and log −type Sobolev inequality (3.10), it follows that

$$\begin{aligned}\mathrm{II} \le& C\|\nabla\psi(\cdot;t)\|^2_{L^4(\mathbf{R}^2)}\|\psi(\cdot;t)\|^{2\gamma-3}_{L^\infty}\|\partial_t\sqrt{\rho}(t)\|_{L^2(\mathbf{R}^2)}\\ \le& C\|\nabla\psi(\cdot;t)\|_{L^2(\mathbf{R}^2)}\|\nabla^2\psi(\cdot;t)\|_{L^2(\mathbf{R}^2)}\\ &(1+\sqrt{\log(1+\|\psi(\cdot;t)\|_{H^2(\mathbf{R}^2)})})^{2\gamma-3}\|\partial_t\sqrt{\rho}(t)\|_{L^2(\mathbf{R}^2)}\\ \le& C\,I(t)\,(\log I(t))^{\gamma-\frac{3}{2}}+C(M(t),E(t)).\end{aligned}$$

On the other hand, it is straightforward to check that the functional appearing on the left hand side of (3.11) is equivalent to $I(t)$. Indeed we have

$$\int_{\mathbf{R}^2}\rho^{2\gamma-1}dx \le C(\|\sqrt{\rho}(t)\|_{H^1(\mathbf{R}^2)}) \le C(M(t),E(t)),$$

and

$$\int_{\mathbf{R}^2} \rho^{\gamma-1}|\nabla\sqrt{\rho}|^2 dx \le \|\psi(\cdot;t)^{\gamma-1}(t)\|_{L^2(\mathbf{R}^2)}\|\nabla\psi(\cdot;t)\|^2_{L^4(\mathbf{R}^2)}$$
$$\le C(M(t),E(t))I(t)^{\frac{1}{2}} \le \frac{1}{2}I(t) + C(M(t),E(t)).$$

Therefore we can conclude

$$\frac{d}{dt}I(t) \le C\, I(t)\,(\log I(t))^{\gamma-\frac{3}{2}} + C(M(t),E(t)),$$

and the boundedness of $I(t)$ follows the conservation of $M(t)$, $E(t)$ and Gronwall's inequality for $\gamma - \frac{3}{2} \le 1$, namely $\gamma \le \frac{5}{2}$. □

**4. A second Morawetz-type estimate.** This last Section is devoted to present a Morawetz-type estimate for the energy density for solutions to the one dimensional QHD system. In particular we show that from the functional $I(t)$ we obtain an improved integrability for the energy density. This fact has been exploited in [6] to show a stability property regarding sequences of weak solutions to the one dimensional QHD system, see Theorem 4 in [6].

In the previous Section we proved that (3.3) holds, for smooth, positive solutions satisfying the conservation law (3.5), for the energy density and moreover it provides uniform bounds for $I(t)$, over compact time intervals.

On the contrary, in this Section we deal with arbitrary weak solutions. By assuming that those solutions satisfy the bounds given by $I(t)$ and they satisfy a weaker version of (3.5), then we gain some integrability properties.

Proposition 13. *Let $d = 1$. Consider a weak solution $(\rho, J)$ to* (1.1), *which satisfies*

$$\begin{aligned} &\|\sqrt{\rho}\|_{L^\infty(0,T;H^1(\mathbf{R}))} + \|\Lambda\|_{L^\infty(0,T;L^2(\mathbf{R}))} \le M_1 \\ &\|\partial_t\sqrt{\rho}\|_{L^\infty(0,T;L^2(\mathbf{R}))} + \|\lambda\|_{L^\infty(0,T;L^2(\mathbf{R}))} \le M_2. \end{aligned} \tag{4.1}$$

*Moreover we assume the following weak entropy inequality holds, in the sense of distributions*

$$\partial_t e + \partial_x(\Lambda\lambda - \partial_t\sqrt{\rho}\,\partial_x\sqrt{\rho}) \le 0. \tag{4.2}$$

*Then it follows*

$$\|e\|_{L^2_{t,x}} + \|\partial_x^2\rho\|_{L^2_{t,x}} + \|\partial_x J\|_{L^\infty_t L^2_x} \le C(M_1,M_2)(1+T)^{\frac{1}{2}}.$$

*Proof.* From (4.1) it follows $\rho$ is continuous in $x$ and $\|\sqrt{\rho}\|_{L^\infty_{t,x}} \le C(M_1)$. Then because of (1.1) we get

$$\|\partial_x J\|_{L^\infty_t L^2_x} = \|\partial_t\rho\|_{L^\infty_t L^2_x} \le 2\|\sqrt{\rho}\|_{L^\infty_{t,x}}\|\partial_t\sqrt{\rho}\|_{L^\infty_t L^2_x} \le C(M_1,M_2).$$

To prove the space-time estimate for the energy density, we define

$$U(t,x) = \int_{-\infty}^{x} J(t,s)ds + C_1,$$

where $C_1$ is a suitable constant such that $U \geq 0$, for instance it is sufficient to choose $C_1 > \|J\|_{L_t^\infty L_x^1}$. By integrating in space the momentum equation in (1.1), it follows

$$\partial_t U + \Lambda^2 + (\partial_x \sqrt{\rho})^2 + p(\rho) = \frac{1}{4}\partial_x^2 \rho,$$

namely

$$\partial_t U + (\sqrt{\rho}\lambda + e) - 2f(\rho) = 0. \tag{4.3}$$

The bounds in (4.1) imply the equation (4.3) holds in the space $L_t^\infty L_x^1$. We want to deal with the interaction functional of $T$ and the energy density $e$. The entropy inequality (4.2) only holds in the sense of distribution, hence the rigorous justification of our computations requires the use of a standard mollification procedure. Let us denote by $\{\chi_\varepsilon\}_{\varepsilon>0}$ a sequence of smooth, positive space-time mollifiers supported on $(-\varepsilon, \varepsilon)^2$, then for any function $h \in L^1([0,T); L^1_{loc}(\mathbf{R}))$ denote by

$$h_\varepsilon(t,x) = \int_0^T \int_{\mathbf{R}} h(t-s, x-y)\chi_\varepsilon(s,y)dyds, \quad (t,x) \in (\varepsilon, T-\varepsilon) \times \mathbf{R}.$$

By taking the convolution of $\chi_\varepsilon$ with (4.2) and (4.3), one has

$$\partial_t e_\varepsilon + \partial_x(\Lambda\lambda - \partial_t\sqrt{\rho}\partial_x\sqrt{\rho})_\varepsilon \leq 0, \tag{4.4}$$

$$\partial_t U_\varepsilon + (\sqrt{\rho}\lambda + e)_\varepsilon - 2f(\rho)_\varepsilon = 0. \tag{4.5}$$

If we multiply (4.5) by $(\sqrt{\rho}\lambda + e)_\varepsilon$, then

$$\begin{aligned}\int_\varepsilon^{T-\varepsilon}\int_{\mathbf{R}} (\sqrt{\rho}\lambda + e)_\varepsilon^2 dxdt = &-\int_\varepsilon^{T-\varepsilon}\int_{\mathbf{R}} (\sqrt{\rho}\lambda + e)_\varepsilon \partial_t U_\varepsilon\, dxdt \\ &+ 2\int_\varepsilon^{T-\varepsilon}\int_{\mathbf{R}} (\sqrt{\rho}\lambda + e)_\varepsilon f(\rho)_\varepsilon dxdt = I + II.\end{aligned} \tag{4.6}$$

Therefore

$$\begin{aligned} I = -\int_\varepsilon^{T-\varepsilon}\int_{\mathbf{R}} (\sqrt{\rho}\lambda + e)_\varepsilon \partial_t U_\varepsilon dxdt = &-\int_{\mathbf{R}} (\sqrt{\rho}\lambda + e)_\varepsilon(t) U_\varepsilon(t) dx \Big|_{t=\varepsilon}^{T-\varepsilon} \\ &+ \int_\varepsilon^{T-\varepsilon}\int_{\mathbf{R}} U_\varepsilon \partial_t(\sqrt{\rho}\lambda + e)_\varepsilon dxdt \\ = &-\int_{\mathbf{R}} (\sqrt{\rho}\lambda + e)_\varepsilon(t) U_\varepsilon(t) dx \Big|_{t=\varepsilon}^{T-\varepsilon} \\ &+ \int_\varepsilon^{T-\varepsilon}\int_{\mathbf{R}} U_\varepsilon \partial_t(-\frac{1}{4}\partial_x^2\rho_\varepsilon + 2e_\varepsilon + p(\rho)_\varepsilon) dxdt \\ = &I_1 + I_2. \end{aligned}$$

We have

$$I_1 = \int_{\mathbf{R}} (\sqrt{\rho}\lambda + e)_\varepsilon(t) U_\varepsilon(t) dx \leq \|U\|_{L^\infty_{t,x}} (\|e\|_{L_t^\infty L_x^1} + \|\sqrt{\rho}\|_{L_t^\infty L_x^2}\|\lambda\|_{L_t^\infty L_x^2}) \leq C(M_1, M_2).$$

While

$$I_2 = \int_\varepsilon^{T-\varepsilon} \int_{\mathbf{R}} U_\varepsilon \partial_t (-\frac{1}{4}\partial_x^2 \rho_\varepsilon + 2e_\varepsilon + p(\rho)_\varepsilon) dxdt = I_{21} + I_{22} + I_{23}. \quad (4.7)$$

By using the continuity equation one has

$$I_{21} = \int_\varepsilon^{T-\varepsilon} \int_{\mathbf{R}} U_\varepsilon \partial_t (-\frac{1}{4}\partial_x^2 \rho_\varepsilon) dxdt = -\frac{1}{4}\int_\varepsilon^{T-\varepsilon} \int_{\mathbf{R}} \partial_x^2 U_\varepsilon \partial_t \rho_\varepsilon \, dxdt$$
$$= \frac{1}{4}\int_\varepsilon^{T-\varepsilon} \int_{\mathbf{R}} (\partial_x J_\varepsilon)^2 dxdt \le C(M_1, M_2)T.$$

Since $T_\varepsilon \ge 0$ and by using the mollified entropy inequality (4.4),w it follows

$$I_{22} = \int_\varepsilon^{T-\varepsilon} \int_{\mathbf{R}} U_\varepsilon \, \partial_t e_\varepsilon \, dxdt \le -\int_\varepsilon^{T-\varepsilon} \int_{\mathbf{R}} U_\varepsilon \, \partial_x (\lambda\Lambda - \partial_x \sqrt{\rho}\partial_t \sqrt{\rho})_\varepsilon \, dxdt$$
$$= \int_\varepsilon^{T-\varepsilon} \int_{\mathbf{R}} J_\varepsilon \, (\lambda\Lambda - \partial_x \sqrt{\rho}\partial_t \sqrt{\rho})_\varepsilon \, dxdt$$
$$\le T \, \|J\|_{L^\infty_{t,x}} \big( \|\lambda\|_{L^\infty_t L^2_x} \|\Lambda\|_{L^\infty_t L^2_x}$$
$$+ \|\partial_x \sqrt{\rho}\|_{L^\infty_t L^2_x} \|\partial_t \sqrt{\rho}\|_{L^\infty_t L^2_x} \big) \le C(M_1, M_2)T.$$

Moreover to estimate $I_{23}$ we have

$$I_{23} = \int_\varepsilon^{T-\varepsilon} \int_{\mathbf{R}} T_\varepsilon \partial_t p(\rho)_\varepsilon dxdt$$
$$= 2\int_\varepsilon^{T-\varepsilon} \int_{\mathbf{R}} T_\varepsilon [p'(\rho)\sqrt{\rho}\partial_t \sqrt{\rho}]_\varepsilon dxdt$$
$$\le 2T \|T\|_{L^\infty_{t,x}} \|p'(\rho)\sqrt{\rho}\|_{L^\infty_t L^2_x} \|\partial_t \sqrt{\rho}\|_{L^\infty_t L^2_x} \le C(M_1, M_2)T.$$

Hence (4.7) is bounded by $I_2 = I_{21} + I_{22} + I_{23} \le C(M_1, M_2)T$, therefore the first term $I$ on the right hand side of (4.6) satisfyies $I \le I_1 + I_2 \le C(M_1, M_2)T$

It remains to estimate $II$ the last term on the right hand side of (4.6). By using (4.1) and the $L^\infty_{t,x}$ bound of $\sqrt{\rho}$, one has

$$II = \int_\varepsilon^{T-\varepsilon} \int_{\mathbf{R}} (\sqrt{\rho}\lambda + e)_\varepsilon f(\rho)_\varepsilon dxdt \le C(M_1, M_2)T.$$

Therefore by using the previous estimates, the convergence properties of the convolution operator in $L^1$ and the semicontinuity, it follows

$$\|\sqrt{\rho}\lambda + e\|^2_{L^2_{t,x}} \le \liminf_{\varepsilon \to 0} \int_\varepsilon^{T-\varepsilon} \int_{\mathbf{R}} (\sqrt{\rho}\lambda + e)^2_\varepsilon dxdt \le C(M_1, M_2)(1+T),$$

hence

$$\|e\|^2_{L^2_{t,x}} \le \|\sqrt{\rho}\lambda + e\|^2_{L^2_{t,x}} + \|\sqrt{\rho}\lambda\|^2_{L^2_{t,x}}$$
$$\le \|\sqrt{\rho}\lambda + e\|^2_{L^2_{t,x}} + T\|\sqrt{\rho}\|^2_{L^\infty_{t,x}} \|\lambda\|^2_{L^\infty_t L^2_x} \le C(M_1, M_2)(1+T).$$

The bound of $\partial_x^2 \rho$ concludes our argument. By using the definition of $\lambda$ and the bounds above, it follows that

$$\|\partial_x^2 \rho\|_{L^2_{t,x}} \le 4\|\sqrt{\rho}\lambda + e\|_{L^2_{t,x}} + 4\|p(\rho)\|_{L^2_{t,x}} \le C(M_1, M_2)(1+T)^{\frac{1}{2}}.$$

□

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