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002

LOCAL CLASSICAL SOLUTIONS TO THE FULL COMPRESSIBLE NAVIER-STOKES SYSTEM WITH TEMPERATURE-DEPENDENT HEAT CONDUCTIVITY*

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Dedicated to Professor Ling Hsiao on the occasion of her 80th birthday

Abstract. In this paper, we study the full compressible Navier-Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^3$, where the heat conductivity depends on the temperature θ in a power law (θ^b for some constant $b > 0$) of Chapman-Enskog. We first prove the existence of the unique strong solution with non-negative mass density and arbitrarily large data, and then lift the regularities to get a classical one. The corresponding proof is nontrivial due to the appearance of the vacuum and the strong nonlinearity of the temperature-dependent heat conductivity. We introduce a new variable θ^{b+1} to reformulate and simplify the system, and require that the measure of the initial vacuum domain is sufficiently small, for example, the initial vacuum only appears in some one-dimensional curves or two-dimensional surfaces.

Key words. Full compressible Navier-Stokes system, three dimensions, classical solutions, vacuum, temperature-dependent heat conductivity.

Mathematics Subject Classification. 35A01, 35A09, 35B45, 35B65, 35Q35.

1. Introduction. The motion of viscous compressible and heat conductivity fluids is governed by the full compressible Navier-Stokes system (**FCNS**)

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \operatorname{div}\mathbb{T}, \\ (\rho E)_t + \operatorname{div}((\rho E + P)u) = \operatorname{div}(\kappa \nabla \theta) + \operatorname{div}(u\mathbb{T}), \end{cases} \tag{1.1}$$

where $t \geq 0$ is the time variable, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the space variable. ρ is the density, u is the velocity, P is the pressure, for ideal polytropic fluids, the equations of state are

$$P = R\rho\theta, \quad e = c_v\theta = \frac{R}{\gamma - 1}\theta, \tag{1.2}$$

where θ is the absolute temperature, e is the specific internal energy, $R > 0$ is the gas constant, $\gamma > 1$ is the adiabatic exponent, and $c_v > 0$ is the specific heat at unit volume. $E = \frac{1}{2}|u|^2 + e$ is the specific total energy, $\mathbb{T}$ is the viscous stress tensor which is given by

$$\mathbb{T} = 2\mu D(u) + \lambda \operatorname{div} u \mathbb{I}_3, \tag{1.3}$$

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where $\mathbb{I}_3$ is the 3×3 identity matrix, and $D(u)$ is the deformation tensor given by

$$D(u) = \frac{\nabla u + (\nabla u)^\top}{2}, \tag{1.4}$$

in which $(\nabla u)^\top$ is the transpose of matrix ∇u. μ is the shear viscosity coefficient, $\lambda + \frac{2}{3}\mu$ is the bulk viscosity coefficient, and κ is the heat conductivity coefficient.

When the viscosity and heat conductivity coefficients are constants, there are rich literatures on the well-posedness of the isentropic and non-isentropic Navier-Stokes system in the past few decades.

- On one hand, when the initial data are away from the vacuum, system (1.1) satisfies the well-known symmetric hyperbolic-parabolic coupled structure. Based on this observation, in 1962, Nash [30] proved the local existence of 3-D classical solutions to the Cauchy problem of (1.1) (see also Tani [34] for the initial-boundary value problem). Later, Matsumura-Nishida[28, 29] proved the global existence of 3-D classical solutions to the initial-boundary value and Cauchy problems of (1.1) with small initial data. The 1-D global large classical solution of the initial-boundary value problem has been established in Kazhikhov-Shelukhin [19] (see also Kanel′ [18] for the Cauchy problem, Dafermos-Hsiao [7] for the thermoviscoelasticity and Hsiao-Luo [8] for real viscous gases).
- On the other hand, when the initial vacuum is allowed, we need to pay more attention to the analysis of system (1.1)'s mathematical structure. The second order elliptic structure in momentum and energy equations hold since the viscosity and heat conductivity coefficients are all constants. While there is degeneracy of time evolution for velocity and temperature, which leads to an essential difficulty: it is hard to find a reasonable way to extend the definitions of velocity and temperature into vacuum region. For isentropic flow, in 2003, Choe-Kim [6] introduced an initial layer compatibility conditions to overcome the difficulty and proved the local existence of 3-D strong solutions (see also Salvi-Străskraba [32]). In 2006, Cho-Kim [4] extended the local strong solution to a classical one, which, recently, has been shown to be a global one with small energy but large oscillations by Huang-Li-Xin [13] for 3-D space and Li-Xin [20], Luo [27] for 2-D space. For non-isentropic flow, in 2006, Cho-Kim [5] obtained the local existence of strong solutions in 3-D space, which has been extended to a global classical solution by Huang-Li [11] and Wen-Zhu [37].

However, from a physical point of view, the viscosity and heat conductivity coefficients usually depend on the temperature or the density, or both. Moreover, when we deduce (1.1) from Boltzmann equation through the Chapman-Enskog expansion to the second order (cf. Chapman-Cowling [3] and Li-Qin [21]), we find that, under some proper physical assumptions, the viscosity and heat conductivity coefficients are not constants but functions of temperature, e.g., power functions of temperature like

$$\mu(\theta) = \alpha\theta^b, \qquad \lambda(\theta) = \beta\theta^b, \qquad \kappa(\theta) = \nu\theta^b, \tag{1.5}$$

where α, β, ν and b are constants satisfying

$$\alpha > 0, \quad 2\alpha + 3\beta \geq 0, \quad \nu > 0, \quad \text{and} \quad b > 0. \tag{1.6}$$

- For isentropic flow, such dependence on temperature is reduced to the dependence on density

$$\mu(\rho) = \alpha\rho^c, \qquad \lambda(\rho) = \beta\rho^c, \quad c > 0 \quad \text{is} \quad \text{a} \quad \text{constant}, \tag{1.7}$$

by Boyle and Gay-Lussac law. Assuming that μ is a constant and $\lambda(\rho) = \beta\rho^c$ with $c > 3$, Vaigant-Kazhikhov [35] first proved that there exists a unique global classical solution to the initial-boundary value problem in 2-D space with large initial data away from vacuum. Later, for the Cauchy problem and the periodic boundary conditions with initial vacuum, Huang-Li [10] and Jiu-Wang-Xin [17] obtained the global classical solutions for $c > 4/3$ (see also [40, 15, 12, 16] and the reference therein for other related works). In 1-D space, when $\mu(\rho) = \alpha\rho^c$ with $\alpha > 0$, Lian-Liu-Li-Xiao [25] obtained a unique global piecewise regular solution with strictly positive density and jump discontinuity. Under the assumption of (1.7), the appearance of vacuum leads to the degeneracy of Lamé operator, which provides less regularizing effect of solutions, and the method for constant viscosity case [6] does not work. In 2017, Li-Pan-Zhu [22] considered the case of $c = 1$ without using initial layer compatibility conditions, and obtained the local existence of classical solutions by making use of the "quasi-symmetric hyperbolic"-"elliptic" structure in 2-D space (see also Zhu [42] for 3-D space), and they [23] also obtained local existence for the case of $c > 1$. Recently, Xin-Zhu [38] established the global existence of classical solutions in some homogeneous Sobolev spaces for $c > 1$, and they [39] also obtained the local existence of classical solutions for the case of $0 < c < 1$ with the initial layer compatibility conditions.
- For non-isentropic flow, under the assumption of (1.5), there are possible degeneracy of the second order elliptic structure caused by losing positive lower bound of temperature. In addition, the strong nonlinearity in momentum and energy equations caused by variable coefficients make the problem much more complicated. Thus there are few results on either weak or strong solutions to system (1.1), even in 1-D space. In 2010, when the initial density is strictly positive, Jenssen-Karper [14] proved the global existence of a weak solution to the initial-boundary value problem of (1.1) in 1-D space under conditions

$$\mu = \alpha, \quad \kappa(\theta) = \nu\theta^b \quad \text{for} \quad b \in \Big[0, \frac{3}{2}\Big).$$

While, under this simplified relation, the temperature-dependent heat conductivity still introduces strong degeneracy and nonlinearity causing troubles to the a priori estimates. They define some energy functionals that monitor certain weighted H^1-norms to overcome these difficulties and obtain a global existence of weak solution. In 2015, Pan-Zhang [31] improved Jenssen-Karper's result and obtained a global strong solution for $b \in [0, +\infty)$ in 1-D space. Later, Zhao-Yao [41] extended Pan-Zhang's work to d-D space ($d \geq 2$) with cylindrical or spherical symmetry data. Recently, Wang-Zhao [36] proved the global existence of classical solutions to the Cauchy problem of (1.1) in 1-D space with positive initial density and

$$\mu = \tilde{\mu}h(v)\theta^{\bar{\alpha}}, \quad \kappa = \tilde{\kappa}h(v)\theta^{\bar{\alpha}}, \tag{1.8}$$

where $\tilde{\mu}$ and $\tilde{\kappa}$ are positive constants, $|\bar{\alpha}|$ is a sufficiently small constant and the non-degenerate smooth function $h(v)$ satisfies

$$v^{l_1} + v^{-l_2} \leq Ch(v), \quad h'(v)^2 v \leq Ch(v)^3 \quad \text{for all } v \in (0, +\infty),$$

v is the specific volume, $l_1 \geq 1$ and $l_2 \geq 1$ are positive constants (see also [24] for the corresponding full MHD system). We mention that these results mainly focus on the weak solutions or the non-vacuum classical solutions in 1-D space.

Our goal in this paper is to establish the local existence of strong and classical solutions to (1.1) with initial vacuum and under the assumption

$$\mu = \alpha, \quad \lambda = \beta, \quad \kappa(\theta) = \nu\theta^b, \quad b > 0. \tag{1.9}$$

We will first show the local existence of the unique strong solution to the initial-boundary value problem (IBVP) of (1.1) in a bounded smooth domain $\Omega \subset \mathbb{R}^3$ with the initial data

$$(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0)(x), \quad x \in \Omega, \tag{1.10}$$

and the Dirichlet and thermo-insulated boundary conditions for (u, θ)

$$u = 0, \quad \nabla\theta \cdot n = 0, \quad \text{on } \partial\Omega, \tag{1.11}$$

where $n = (n_1, n_2, n_3)$ is the unit outward normal to $\partial\Omega$. Then we will lift the regularities of the above solution to get a classical solution.

Throughout this paper, we adopt the following notations for homogeneous and inhomogeneous Sobolev spaces (see [9]).

$$\begin{aligned}
&|f|_p = \|f\|_{L^p(\Omega)}, \quad \|f\|_s = \|f\|_{H^s(\Omega)}, \qquad \|f\|_{L^pL^q} := \|f\|_{L^p([0,T];L^q(\Omega))},\\
&D^{k,r}(\Omega) = \left\{f \in L^1_{loc}(\Omega),\ |f|_{D^{k,r}} := \|f\|_{D^{k,r}(\Omega)} = |\nabla^k f|_r < +\infty\right\}, \quad D^k = D^{k,2},\\
&D^1_0(\Omega) = \left\{f \in D^1(\Omega),\ f|_{\partial\Omega} = 0,\ |f|_{D^1_0} := \|f\|_{D^1_0(\Omega)} = |\nabla f|_2 < +\infty\right\}.
\end{aligned}$$

For simplicity, we also use the following notations:

$$\int f := \int_\Omega f\mathrm{d}x \qquad \text{and} \qquad \int_0^t f := \int_0^t f\mathrm{d}s.$$

Before stating the main result, let us first analyze the difficulties in this paper. According to system (1.1), the relations $E = \frac{1}{2}|u|^2 + e$, and (1.2), the time evolution of the temperature can be governed by

$$\rho\theta_t + \rho u \cdot \nabla\theta + \frac{1}{c_v}\big(R\rho\theta\mathrm{div}u - \nu\mathrm{div}(\theta^b\nabla\theta)\big) = \frac{1}{c_v}Q(u), \tag{1.12}$$

where $Q(u) = \dfrac{\alpha}{2}|\nabla u + \nabla u^\top|^2 + \beta(\mathrm{div}u)^2$. Then via introducing the new variable $\psi = \theta^{b+1}$, system (1.1) can be reduced to

$$\begin{cases}
\rho_t + \mathrm{div}(\rho u) = 0,\\
\rho u_t + \rho u \cdot \nabla u + \nabla P + Lu = 0,\\
\rho\psi_t + \rho u \cdot \nabla\psi + a_1\rho\psi\mathrm{div}u - a_2\psi^{\frac{b}{b+1}}\Delta\psi = a_3\psi^{\frac{b}{b+1}}Q(u),
\end{cases} \tag{1.13}$$

where $P = R\rho\psi^{\frac{1}{b+1}}$, $Lu = -\alpha\Delta u - (\alpha+\beta)\nabla\mathrm{div}u$, and

$$a_1 = \frac{1}{c_v}R(b+1), \quad a_2 = \frac{1}{c_v}\nu, \quad a_3 = \frac{1}{c_v}(b+1).$$

The difficulties and strategies in our proof can be summarized as follows.

For the local existence of **strong** solutions:

1. **Degeneracy caused by vacuum**. Due to the appearance of vacuum, there are degeneracy of time evolution in $(1.13)_2$ and $(1.13)_3$, as the constant heat conductivity case [5], which will be overcome by the initial compatibility conditions.
2. **Degeneracy caused by variable heat conductivity**. In [5] (i.e., $b = 0$), the uniform elliptic structure in energy equation holds, which plays an essential role in the estimates of $|\psi|_{D^2}$. While for our case $b > 0$, according to $(1.13)_3$ we have

$$|\psi|_{D^2} \leq C|Q(u)|_2 + C\Big|\psi^{-\frac{b}{b+1}}\left(\rho\psi_t + \rho u\cdot\nabla\psi + a_1\rho\psi\mathrm{div}u\right)\Big|_2.$$

Then it is obvious that we need to study the positivity of ψ. Notice that $(1.13)_3$ is a degenerate parabolic equation, thus by dividing $(1.13)_3$ by ρ and establishing the uniform minimum principle that is independent of the lower bound of ρ to the resulted equation, we deduced the positive lower bound of ψ in a short time when θ_0 is strictly positive.
3. **Nonlinearity caused by variable heat conductivity**. During the a priori estimates, there are additional nonlinear terms caused by temperature-dependent heat conductivity, some of these terms were not friendly with our estimates. For these terms, we need to have restrictions of the size of the initial vacuum domain.

<u>For the local existence of **classical** solutions</u>:

1. **Strong coupling between u and ψ in higher-order estimates**. First, comparing with the isentropic case, the additional energy equation in our non-isentropic system makes it much harder to obtain the higher-order regularity. We take the estimate of P as an example. For isentropic case, according to the mass equation one has

$$P_t + u\cdot\nabla P + \gamma P\mathrm{div}u = 0.$$

Then based on the regularity of u, one can obtain the regularity of P by a standard regularity theory of hyperbolic equations, the work in [4] depends heavily on this fact. While for non-isentropic case, according to $(1.13)_3$ one has

$$P_t + u\cdot\nabla P + (R+1)P\mathrm{div}u = RQ(u) + R\nu\Delta\psi,$$

which implies that the regularity of P depends heavily on the regularities of u and ψ. Moreover, not like the first and second order regularity estimates, due to the presence of quadratic nonlinear term $Q(u)$ in $(1.13)_3$, it is hard to obtain the higher-order estimates for u and ψ separately. Second, we find that there are many extra terms generated from the nonlinear terms P, $\psi^{\frac{b}{b+1}}\Delta\psi$ and $\psi^{\frac{b}{b+1}}Q(u)$ during the higher-order regularity estimates, which have caused the strong coupling between the higher-order regularity estimates of u and ψ. For example, during the estimates of $\|\nabla u_{tt}\|_{L^2L^2}$, we find that

$$\|\nabla u_{tt}\|_{L^2L^2} \qquad \text{depends on} \qquad \|\nabla\psi_{tt}\|_{L^2L^2},$$

and vice versa. Some new estimates are developed to deal with the cross estimates between u and ψ.

2. **Strong nonlinearity of the operator** $\psi^{\frac{b}{b+1}}\Delta\psi$. We emphasize that, unlike the linear elliptic operator, the nonlinear elliptic operator $\psi^{\frac{b}{b+1}}\Delta\psi$ brings many extra terms during the high-order regularity estimates, which are possibly two-order higher than the terms on the left-hand side. For example, the extra term $\psi^{-\frac{1}{b+1}}\psi_t\Delta\psi\psi_{tt}$ that comes from $\psi^{\frac{b}{b+1}}\Delta\psi$ cause the appearance of $\|\nabla\psi_{tt}\|_{L^2L^2}$, which is two-order higher than the term $\|\nabla\psi_t\|_{L^\infty L^2}$ on the left-hand side of the estimates. This is one of the main difficulties, we need to find some "smallness" mechanism to absorb this higher order term, see Section 4 for details.

3. **Time continuity of** u **and** ψ. The time continuity of the first and second order derivatives of u and ψ are not trivial, we need to establish the time weighted energy estimates of $t^{\frac{1}{2}}u_t, t\psi_t \in L^\infty([0,T];D^2)$, $tu_{tt}, t^{\frac{3}{2}}\psi_{tt} \in L^\infty([0,T];D^1)$, etc. by using the structure of the reformulated system (1.13) and carefully matching the weights of time. This is different from the existence of strong solutions, where we only need to get the time continuity of u and ψ themselves.

Now, we first introduce the following definition of strong solutions to IBVP (1.1) with (1.10)-(1.11).

Definition 1.1. *Let* $q\in(3,6]$, (ρ,u,θ) *is called a strong solution on* $[0,T]\times\Omega$ *to IBVP* (1.1) *with* (1.10)-(1.11), *if*

(1) (ρ,u,θ) *satisfies the system* (1.1) *a.e. in* $(0,T)\times\Omega$*;*

(2) (ρ,u,θ) *belongs to the following class* Φ:

$$
\begin{aligned}
\Phi=\Big\{(\rho,u,\theta)|\ &\rho\geq 0,\ \rho\in C([0,T];H^1\cap W^{1,q}),\ \rho_t\in C([0,T];L^2\cap L^q),\\
&(u,\theta)\in C([0,T];D^1\cap D^2)\cap L^2([0,T];D^{2,q}),\\
&(u_t,\theta_t)\in L^2([0,T];D^1),\ (\sqrt{\rho}u_t,\sqrt{\rho}\theta_t)\in L^\infty([0,T];L^2)\Big\};
\end{aligned}
\tag{1.14}
$$

(3) (ρ,u,θ) *satisfies the corresponding initial conditions a.e. on* $\{t=0\}\times\Omega$, *and also satisfies the corresponding boundary conditions in the sense of traces.*

Then state our main results as follows.

Theorem 1.1 (Local existence of **strong** solutions). *Let* $\underline{\theta}>0$ *be some real constant. Assume that the initial data* (ρ_0,u_0,θ_0) *satisfy the following regularity conditions*

$$
\rho_0\geq 0,\quad \rho_0\in H^1\cap W^{1,q},\quad u_0\in D_0^1\cap D^2,\quad \theta_0\in D^1\cap D^2,\quad \theta_0\geq\underline{\theta},
\tag{1.15}
$$

for $q\in(3,6]$ *and the initial layer compatibility conditions*

$$
\begin{cases}
Lu_0+\nabla(R\rho_0\theta_0)=\rho_0^{\frac{1}{2}}g_1,\\
-\dfrac{1}{c_v}\Big(\dfrac{\nu}{b+1}\Delta\theta_0^{b+1}+Q(u_0)\Big)=\rho_0^{\frac{1}{2}}g_2,
\end{cases}
\tag{1.16}
$$

for some $(g_1,g_2)\in L^2$. *Then there exists a positive time* $T_*>0$, *such that the IBVP* (1.1) *with* (1.10)-(1.11) *admits a unique strong solution* (ρ,u,θ) *on* $[0,T_*]\times\Omega$ *under the condition*

$$
|V|\leq\epsilon_0,
\tag{1.17}
$$

where V denotes the initial vacuum domain:

$$V = \{x \in \Omega \mid \rho_0(x) = 0\},$$

$|V|$ is the measure of V, and $\epsilon_0 = \epsilon_0(\rho_0, u_0, \theta_0, \alpha, \beta, \nu, R, c_v, b)$ is a small constant.

If we improve the regularity of the initial data, the strong solution in Theorem 1.1 can be lift to a classical solution, i.e., we have the local existence of classical solutions.

THEOREM 1.2 (Local existence of **classical** solutions). *Let $\underline{\theta} > 0$ be some real constant. Assume that the initial data (ρ_0, u_0, θ_0) satisfy the following regularity conditions*

$$\rho_0 \geq 0, \quad \rho_0 \in H^3, \quad u_0 \in D_0^1 \cap D^3, \quad \theta_0 \in D^1 \cap D^3, \quad \theta_0 \geq \underline{\theta}, \tag{1.18}$$

and the initial layer compatibility conditions

$$\begin{cases} Lu_0 + \nabla(R\rho_0\theta_0) = \rho_0 g_1, \\ -\dfrac{1}{c_v}\Big(\dfrac{\nu}{b+1}\Delta\theta_0^{b+1} + Q(u_0)\Big) = \rho_0 g_2, \end{cases} \tag{1.19}$$

for some $g_1 \in D^1$ with $\big(\rho_0^{\frac{1}{2}} g_1, \rho_0^{\frac{1}{2}} g_2\big) \in L^2$. Then there exists a positive time $T_* > 0$, such that the IBVP (1.1) with (1.10)-(1.11) admits a unique classical solution (ρ, u, θ) on $(0, T_*] \times \Omega$ under the condition

$$|V| \leq \epsilon_0, \tag{1.20}$$

where $\epsilon_0 = \epsilon_0(\rho_0, u_0, \theta_0, \alpha, \beta, \nu, R, c_v, b)$ is a small constant. Moreover, (ρ, u, θ) satisfy the regularity (4.3) on $[0, T_*] \times \Omega$.

REMARK 1.3. *The local existence results also hold for the Cauchy problem of the* **FCNS** *system with non-vacuum far field state.*

REMARK 1.4. *For the full MHD system with positive constant magnetic diffusivity, since it has similar structure as the* **FCNS** *system, we can also establish the local existence results for the initial boundary value problem and the Cauchy problem with the corresponding conditions on the magnetic field: no-slip boundary condition or zero far field condition, see [2] for details.*

The structure of this paper is organized as follows. In Section 2, we introduce some important lemmas that are frequently used in our proof. In Section 3 and Section 4, we prove the local existence of strong solutions and classical solutions to the IBVP of the **FCNS** system, respectively.

2. Preliminaries. In this section, we present some important lemmas that are frequently used in our proof. The first one follows from the classical Sobolev embedding results.

LEMMA 2.1. *There exists a constant $C > 0$ such that for $f \in D_0^1$, $g \in D_0^1 \cap D^2$ or $g \in W^{1,q}$, one has*

$$|f|_6 \leq C|f|_{D_0^1}, \quad |g|_\infty \leq C|g|_{D_0^1 \cap D^2}, \quad |g|_\infty \leq C|g|_{W^{1,q}} \text{ for } q > 3.$$

The second one is the Poincaré type inequality (see Chapter 8 in [26]).

Lemma 2.2 ([26]). *There exists a constant C depending only on Ω, $|\rho|_r$ with $r \geq 1$, where $\rho \geq 0$ is a real function satisfying $|\rho|_1 > 0$, such that for every $f \geq 0$ satisfying*

$$\rho f \in L^1, \quad \sqrt{\rho} f \in L^2, \quad \nabla f \in L^2,$$

one has

$$|f|_6 \leq C\big(|\rho f|_1 + (1 + |\rho|_2 |\nabla f|_2)\big) \leq C\big(|\rho f|_2 + (1 + |\rho|_2)|\nabla f|_2\big).$$

Proof. We first denote that

$$\overline{F} = \frac{1}{|\Omega|}\int F,$$

then via the classical Poincaré inequality, one has

$$\overline{F}\int \rho = \int \rho(\overline{F} - F) + \int \rho F \leq C\left(|\rho F|_1 + |\rho|_2|\nabla F|_2\right) \leq C\big(|\rho|_1^{\frac{1}{2}}|\sqrt{\rho}F|_2 + |\rho|_2|\nabla F|_2\big),$$

which implies that

$$\overline{F} \leq C\big(|\sqrt{\rho}F|_2 + |\rho|_2|\nabla F|_2\big). \tag{2.1}$$

Second, we consider that

$$\begin{aligned} \|F\|_1 =& |\nabla F|_2 + |F|_2 \leq |\nabla F|_2 + |F - \overline{F}|_2 + \overline{F}|\Omega|^{\frac{1}{2}} \\ \leq& C\big(|\sqrt{\rho}F|_2 + (1 + |\rho|_2)|\nabla F|_2\big), \end{aligned} \tag{2.2}$$

then according to (2.1)-(2.2) and the classical Sobolev embedding theorem, one gets

$$|F|_6 \leq C\|F\|_1 \leq C\big(|\sqrt{\rho}F|_2 + (1 + |\rho|_2)|\nabla F|_2\big).$$

□

These two lemmas will be used to control $|f|_6$ during the a priori estimates. Next, we give the regularity estimate for elliptic operator (see [1]).

Lemma 2.3. *Assume that* (1.6) *holds, let Ω be a bounded smooth domain and set $l \in (1, +\infty)$.*

- *For the Dirichlet boundary value problem*

$$-\alpha\Delta U - (\alpha + \beta)\nabla \mathrm{div} U = F, \quad U|_{\partial\Omega} = 0,$$

if $F \in H^{-1}(\Omega)$, then there exists a unique weak solution $U \in H_0^1(\Omega)$, and for $F \in W^{k,l}(\Omega)$, one has

$$\|U\|_{W^{2+k,l}(\Omega)} \leq C\|F\|_{W^{k,l}(\Omega)}, \tag{2.3}$$

where constant C depends only on α, β, l and Ω.

- *For the Neumann boundary value problem*

$$\Delta U = F, \quad \nabla U \cdot n|_{\partial\Omega} = 0,$$

if in addition $\int F = 0$, then there exists a weak solution $U \in H^1(\Omega)$, and (2.3) *also holds.*

We will frequently use Lemma 2.3 in the regularity estimates.

Finally, using Aubin-Lions Lemma, one has,

LEMMA 2.4 ([33]). *Let X_0, X and X_1 be three Banach spaces satisfying $X_0 \subset X \subset X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1.*

I) Let f be bounded in $L^p([0,T];X_0)$ with $1 \le p < +\infty$, and $\frac{\partial f}{\partial t}$ be bounded in $L^1([0,T];X_1)$. Then f is relatively compact in $L^p([0,T];X)$.

II) Let f be bounded in $L^\infty([0,T];X_0)$ and $\frac{\partial f}{\partial t}$ be bounded in $L^p([0,T];X_1)$ with $p > 1$. Then f is relatively compact in $C([0,T];X)$.

This lemma will be used in proving the time and space continuity of the solution.

3. Proof of Theorem 1.1. The goal of this section is to prove the local existence of strong solutions to the IBVP of (1.1) with (1.10)-(1.11). We first reformulate our problem into a new form.

3.1. Reformulation. With the help of (1.13), the IBVP of (1.1) with (1.10)-(1.11) can be rewritten as

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ \rho\psi_t + \rho u\cdot\nabla\psi + a_1\rho\psi\operatorname{div}u - a_2\psi^{\frac{b}{b+1}}\Delta\psi = a_3\psi^{\frac{b}{b+1}}Q(u), \\ \rho u_t + \rho u\cdot\nabla u + \nabla P + Lu = 0, \\ (\rho,u,\psi)|_{t=0} = (\rho_0(x),u_0(x),\psi_0(x)) = \big(\rho_0(x),u_0(x),\theta_0^{b+1}(x)\big), \quad x\in\Omega, \\ u(t,x)|_{\partial\Omega} = 0, \quad \nabla\psi(t,x)\cdot n|_{\partial\Omega} = 0, \quad t\ge 0. \end{cases} \tag{3.1}$$

In order to prove Theorem 1.1, our first step is to establish the following existence result for the reformulated problem (3.1).

THEOREM 3.1. *Let the initial data (ρ_0,u_0,ψ_0) satisfy the following regularities:*

$$\rho_0 \ge 0, \quad \rho_0 \in H^1\cap W^{1,q}, \quad u_0 \in D_0^1\cap D^2, \quad \psi_0\in D^1\cap D^2, \quad \psi_0 \ge \underline{\psi} = \underline{\theta}^{b+1}, \tag{3.2}$$

for $q\in(3,6]$ and the initial layer compatibility conditions

$$\begin{cases} Lu_0 + \nabla\Big(R\rho_0\psi_0^{\frac{1}{b+1}}\Big) = \rho_0^{\frac12}g_1, \\ -\dfrac{1}{c_v}\Big(\dfrac{\nu}{b+1}\Delta\psi_0 + Q(u_0)\Big) = \rho_0^{\frac12}g_2, \end{cases} \tag{3.3}$$

for some $(g_1,g_2)\in L^2$. Then there exists a positive time $T_*>0$, such that the IBVP (3.1) admits a unique strong solution (ρ,u,ψ) on $[0,T_*]\times\Omega$ under the condition

$$|V| \le \epsilon_0, \tag{3.4}$$

where $\epsilon_0 = \epsilon_0(\rho_0,u_0,\psi_0,\alpha,\beta,\nu,R,c_v,b)$ is a small constant. Moreover, (ρ,u,ψ) satisfy

$$\begin{aligned} &0\le\rho\in C([0,T_*];H^1\cap W^{1,q}), \quad (u,\psi)\in C([0,T_*];D^1\cap D^2)\cap L^2([0,T_*];D^{2,q}), \\ &(u_t,\psi_t)\in L^2([0,T_*];D^1), \ (\sqrt{\rho}u_t,\sqrt{\rho}\psi_t)\in L^\infty([0,T_*];L^2). \end{aligned} \tag{3.5}$$

We will prove this existence theorem in the Subsections 3.2-3.5, and at the end of this section, we will show that this theorem indeed implies Theorem 1.1.

3.2. Linearization. Now we consider the following linearized problem:

$$\begin{cases} \rho_t + \mathrm{div}(\rho w) = 0, \\ \rho\psi_t + \rho w\cdot\nabla\psi + a_1\rho\psi\mathrm{div}w - a_2\phi^{\frac{b}{b+1}}\triangle\psi = a_3\phi^{\frac{b}{b+1}}Q(w), \\ \rho u_t + \rho w\cdot\nabla u + \nabla P + Lu = 0, \\ (\rho,u,\psi)|_{t=0} = (\rho_0^\delta(x), u_0(x), \psi_0(x)), \quad x\in\Omega, \\ u(t,x)|_{\partial\Omega} = 0, \quad \nabla\psi(t,x)\cdot n|_{\partial\Omega} = 0, \quad t\geq 0, \end{cases} \tag{3.6}$$

where $\rho_0^\delta = \rho_0 + \delta$ for some constant $\delta > 0$, $w(t,x)\in\mathbb{R}^3$ is a known vector, and $\phi(t,x)$ is a known function. Assume that $w|_{\partial\Omega} = 0$, $\nabla\phi\cdot n|_{\partial\Omega} = 0$, and

$$\begin{aligned} &(w,\phi)\in C([0,T];H^2)\cap L^2([0,T];W^{2,q}), \quad (w_t,\phi_t)\in L^2([0,T];H^1), \\ &\phi \geq \frac{1}{2}\underline{\psi} > 0, \quad (w(t=0,x),\phi(t=0,x)) = (u_0(x),\psi_0(x)), \quad \text{for} \ \ x\in\Omega. \end{aligned} \tag{3.7}$$

We have the following global existence of the unique strong solution $(\rho^\delta, u^\delta, \psi^\delta)$ to (3.6)-(3.7) for every $\delta > 0$ using standard arguments whose details will be omitted.

Lemma 3.2. *Assume that* (ρ_0, u_0, ψ_0) *satisfies* (3.2)-(3.3). *Then for every* $\delta > 0$, *there exists a unique strong solution* $(\rho^\delta, u^\delta, \psi^\delta)$ *to IBVP* (3.6)-(3.7) *satisfying*

$$\rho^\delta \in C([0,T];W^{1,q}), \quad (u^\delta,\psi^\delta)\in C([0,T];H^2)\cap L^2([0,T];W^{2,q}), \tag{3.8}$$

and $\rho^\delta \geq \underline{\delta}$ *for some positive constant* $\underline{\delta} > 0$.

3.3. A priori estimates independent of δ**.** In this subsection, we will get some a priori estimates , independent of δ, for the solution $(\rho^\delta, u^\delta, \psi^\delta)$ obtained in Lemma 3.2.

We first fix a positive constant c_0 such that

$$2 + \|\rho_0\|_{W^{1,q}} + \|(u_0,\psi_0)\|_2 + |(g_1,g_2)|_2 \leq c_0. \tag{3.9}$$

Then we rewrite (3.3) into

$$\begin{cases} Lu_0 + \nabla\Big(R\rho_0^\delta\psi_0^{\frac{1}{b+1}}\Big) = \sqrt{\rho_0^\delta}g_1^\delta, \\ -\dfrac{1}{c_v}\Big(\dfrac{\nu}{b+1}\Delta\psi_0 + Q(u_0)\Big) = \sqrt{\rho_0^\delta}g_2^\delta, \end{cases} \tag{3.10}$$

where

$$g_1^\delta = \left(\frac{\rho_0}{\rho_0^\delta}\right)^{\frac{1}{2}} g_1 + R\delta\frac{\nabla\psi_0^{\frac{1}{b+1}}}{(\rho_0^\delta)^{\frac{1}{2}}}, \quad g_2^\delta = \left(\frac{\rho_0}{\rho_0^\delta}\right)^{\frac{1}{2}} g_2.$$

Then according to (3.9), for any $\delta > 0$ small enough, one has

$$1 + \|\rho_0^\delta\|_{W^{1,q}} + \|(u_0,\psi_0)\|_2 + |(g_1^\delta,g_2^\delta)|_2 \leq c_0. \tag{3.11}$$

Second, for w and ϕ, let positive constants c_i $(i = 1, 2, 3, 4, 5)$ satisfy

$$\begin{aligned}&\sup_{0\le t\le T^*}\|w(t)\|_1^2+\int_0^{T^*}\Big(|w|^2_{D^{2,q}}+\|w_t\|_1^2\Big)\mathrm{d}t\le c_1^2,\quad \sup_{0\le t\le T^*}|w(t)|^2_{D^2}\le c_2^2,\\ &\sup_{0\le t\le T^*}\|\phi(t)\|_1^2\le c_3^2,\quad \int_0^{T^*}\Big(|\phi|^2_{D^{2,q}}+\|\phi_t\|_1^2\Big)\mathrm{d}t\le c_4^2,\quad \sup_{0\le t\le T^*}|\phi(t)|^2_{D^2}\le c_5^2,\end{aligned} \tag{3.12}$$

for some time $T^* \in (0, T)$, where constants c_i $(i = 1, 2, 3, 4, 5)$ satisfy

$$2 < c_0 \le c_1 \le c_2 \le c_3 \le c_4 \le c_5.$$

Constants c_i $(i = 1, 2, 3, 4, 5)$ and T^* will be determined later and they depend only on c_0 and the fixed constants α, β, ν, q, R, c_v, b, $|\Omega|$ and T (see (3.53)).

In the rest of this section, we denote $C \ge 1$ as a generic constant depending only on fixed constants α, β, ν, q, R, c_v, b, $|\Omega|$ and T. We also denote $a_4 = a_2\left(\frac{1}{2}\underline{\psi}\right)^{\frac{b}{b+1}}$. For simplicity, in the rest of Subsection 3.3, we will denote $(\rho^\delta, u^\delta, \psi^\delta)$ by (ρ, u, ψ) if there is no confusion.

Now we start with the a priori estimates for ρ.

Lemma 3.3. *For* $0 \le t \le T_1 = \min\{T^*, (1 + c_2^2)^{-1}\}$, *it holds that*

$$1 + |\rho|^2_\infty + \|\rho\|^2_{W^{1,q}} \le Cc_0^2, \quad |\rho_t|_q \le Cc_0c_2.$$

Proof. From the standard energy estimate for the transport equation, one has

$$\|\rho\|_{W^{1,q}} \le \|\rho_0^\delta\|_{W^{1,q}} \exp\left(\int_0^t \|\nabla w\|_{W^{1,q}}\right) \le c_0 \exp(c_2 t + c_1 t^{\frac{1}{2}}) \le Cc_0, \tag{3.13}$$

for $0 \le t \le T_1 = \min\{T^*, (1 + c_2^2)^{-1}\}$, where we have used the fact $w \cdot n|_{\partial\Omega} = 0$.

For the term ρ_t, from the continuity equation $(3.6)_1$, it arrives at

$$|\rho_t|_q \le C(|\rho|_\infty|\nabla w|_q + |w|_\infty|\nabla\rho|_q) \le Cc_0\|w\|_2 \le Cc_0c_2.$$

Combining with the Sobolev embedding inequality: $|\rho|_\infty \le \|\rho\|_{W^{1,q}}$ with $q \in (3, 6]$ in Ω, we complete the proof of this lemma. □

Next we will study the evolution of the initial vacuum domain. We denote $V_{R_0} \subset \Omega$ a neighborhood containing the initial vacuum region:

$$V \subset V_{R_0} = \{x \in \Omega | \mathrm{dist}(x, V) \le R_0\},$$

where $R_0 > 0$ is a sufficiently small constant. We first give the following lemma about the positive lower bounds of ρ and ψ in $\Omega\backslash V_{R_0}$ and Ω, respectively.

Lemma 3.4. *For* $R_0 \ll 1$, *there exists a constant* $a_{R_0} > 0$ *and a time* $T_2 = \{T_1, (\ln 2)^2(Cc_2)^{-2}, R_0(3c_2)^{-1}\}$ *such that*

$$\rho(t, x) \ge a_{R_0} + \frac{1}{2}\delta > 0, \quad \forall\ (t, x) \in [0, T_2] \times (\Omega\backslash V_{R_0}), \tag{3.14}$$

and

$$\psi(t, x) \ge \frac{1}{2}\underline{\psi}, \quad \forall\ (t, x)\ \in \big[0, T_2\big] \times \Omega, \tag{3.15}$$

where a_{R_0} and T_2 are both independent of δ.

Proof. We divide the proof into two steps.

Step 1. *Proof of* (3.14). We first denote $X \in C([0,T_1]\times[0,T_1]\times\Omega)$ as the solution to the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}X(t;0,x_0)=w\big(t,X(t;0,x_0)\big), \quad 0\le t\le T_1; \\ X(0;0,x_0)=x_0, \quad x_0\in\Omega. \end{cases} \tag{3.16}$$

Second, let $A(t,R_0)$ and $B(t,R_0)$ be closed regions that are the images of V_{R_0} and $\Omega\backslash V_{R_0}$ respectively under the flow map (3.16), that is

$$\begin{cases} A(t,R_0)=\{X(t;0,x_0)|x_0\in V_{R_0}\}, \\ B(t,R_0)=\{X(t;0,x_0)|x_0\in \Omega\backslash V_{R_0}\}. \end{cases}$$

Due to the definition of V and the continuity of ρ_0, it is obvious to see that for any sufficiently small $R'>0$, there exists a constant $a_{R'}$ independent of δ such that

$$\rho_0^\delta(x)\ge a_{R'}+\delta>0, \quad \forall\, x\in\Omega\backslash V_{R'}. \tag{3.17}$$

From the continuity equation $(3.6)_1$, one has

$$\rho(t,x)=\rho_0^\delta(X(0;0,x_0))\exp\Big(-\int_0^t \mathrm{div}w\big(s;X(s;0,x_0)\big)\Big). \tag{3.18}$$

It is easy to show that

$$\int_0^t |\mathrm{div}w\big(t,X(t;0,x_0)\big)|\le\int_0^t|\nabla w|_\infty\le c_2t^{1/2}\le\ln 2, \tag{3.19}$$

for $0\le t\le T'=\min\{T_1,(\ln 2)^2(Cc_2)^{-2}\}$.

It follows from (3.17) and (3.19) that for $0\le t\le T'$,

$$\rho(t,x)\ge\frac{1}{2}(a_{R'}+\delta)>0, \quad \forall\, x\in B(t,R'). \tag{3.20}$$

From the ODE problem (3.16), one has

$$|X(0;0,x_0)-x|=|X(0;0,x_0)-X(t;0,x_0)|\le\int_0^t|w\big(\tau,X(\tau;0,x_0)\big)|\mathrm{d}\tau\le c_2t\le R'/2,$$

for all $(t,x)\in[0,T'']\times\Omega$, and $T''=\min\{T',R'(2c_2)^{-1}\}$, which means,

$$\Omega\backslash V_{3R'/2}\subset B(t,R'). \tag{3.21}$$

Thus we can choose

$$R_0=\frac{3}{2}R', \quad a_{R_0}=\frac{1}{2}a_{R'} \quad \text{and} \quad T_2=\min\Big\{T_1,(\ln 2)^2(Cc_2)^{-2},R_0(3c_2)^{-1}\Big\}.$$

Step 2. *Proof of* (3.15). From equation $(3.6)_2$, it is easy to have

$$\rho\psi_t+\rho w\cdot\nabla\psi-a_2\phi^{\frac{b}{b+1}}\triangle\psi\ge -a_1\rho\psi\mathrm{div}w, \tag{3.22}$$

where we have used the fact that $\phi^{\frac{b}{b+1}}Q(w) \geq 0$. We define

$$T''' = \inf \left\{ \ t \in (0, T] \ \middle| \ \psi(t, x) = 0, \quad \text{for} \quad \text{some} \quad x \in \Omega \right\}.$$

From Lemma 3.2, we know that $\rho \geq \underline{\delta} > 0$. Thus (3.22) implies that

$$\psi_t + w \cdot \nabla \psi - \frac{a_2}{\rho} \phi^{\frac{b}{b+1}} \triangle \psi \geq -a_1 \psi |\text{div} w|_\infty \ \text{for} \ \ (t, x) \ \in \ [0, T'''] \times \Omega. \tag{3.23}$$

Denote

$$\psi^* = \psi \exp \left(a_1 \int_0^t |\text{div} w(s, x)|_\infty \right),$$

then along the curve $X(t; 0, x_0)$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t} \psi^* - \frac{a_2}{\rho} \phi^{\frac{b}{b+1}} \triangle \psi^* \geq 0. \tag{3.24}$$

Then, from $\psi_0(x) \geq \underline{\psi}$ and the classical minimum principle, one has

$$\psi(t, x) \geq \inf_{x \in \Omega} \psi_0(x) \exp \left(- a_1 \int_0^t |\text{div} w(s, x)|_\infty \right) > 0, \tag{3.25}$$

for $t \in [0, T''']$, which contradicts with the definition of T'''. Thus (3.25) holds for $t \in [0, T_1]$. Moreover, one has (3.15). □

Next we give the a priori estimates for ψ.

LEMMA 3.5. *For $0 \leq t \leq T_3 = \min\{T_2, (1 + Cc_5)^{-4K}\}$, and R_0 satisfies $|V_{R_0}| \leq \big(a_4/\big(20Cc_5^K\big)\big)^3$ with a constant $K \geq 9$, it holds that*

$$|\sqrt{\rho}\psi|_2^2 + \|\psi\|_1^2 \leq Cc_0^3, \quad |\sqrt{\rho}\psi_t|_2^2 + \int_0^t |\psi_s|_{D^1}^2 \leq M, \quad \int_0^t |\psi|_{D^{2,q}}^2 \leq M, \quad |\psi|_{D^2} \leq M c_5^{\frac{b}{b+1}},$$

where $M = Cc_2^{10} \exp(Cc_1^2 c_3^6)$.

Proof. We divide the proof into three steps.

Step 1. *First order estimates of ψ.* Multiplying $(3.6)_2$ by ψ and integrating over Ω, one can obtain

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int \rho |\psi|^2 + a_2 \int \phi^{\frac{b}{b+1}} |\nabla \psi|^2 \\
= &\int \Big(- a_2 \nabla \phi^{\frac{b}{b+1}} \cdot \nabla \psi \psi + a_1 \rho \psi^2 \text{div} w + a_3 \phi^{\frac{b}{b+1}} Q(w) \psi \Big) \\
\leq &C \Big(|\rho|_\infty^{\frac{1}{2}} |\sqrt{\rho} \psi|_2 + |\phi|_\infty^{\frac{b}{b+1}} |\nabla w|_2 \Big) |\nabla w|_3 |\psi|_6 - a_2 \int \nabla \phi^{\frac{b}{b+1}} \cdot \nabla \psi \psi \\
\leq &\frac{a_4}{10} |\nabla \psi|_2^2 + \Big(\frac{a_4}{20 c_0^2} + C c_2^5 \Big) |\sqrt{\rho} \psi|_2^2 + C c_5^8 - a_2 \int \nabla \phi^{\frac{b}{b+1}} \cdot \nabla \psi \psi,
\end{aligned} \tag{3.26}$$

where we have used the Poincaré type inequality for ψ in Lemma 2.2 and (3.7).

For the rest term in the right-hand side of (3.26), it follows from Lemmas 3.3-3.4 that

$$\begin{aligned}
&-a_2\int \nabla\phi^{\frac{b}{b+1}}\cdot\nabla\psi\psi\\
&\leq C\int|\nabla\phi\cdot\nabla\psi\psi| = C\Big(\int_{V_{R_0}}+\int_{\Omega\setminus V_{R_0}}\Big)|\nabla\phi\cdot\nabla\psi\psi|\\
&\leq C\Big(|\psi|_6|V_{R_0}|^{\frac{1}{6}}+|\sqrt{\rho}\psi|_2^{\frac{1}{2}}|\sqrt{\rho}\psi|_6^{\frac{1}{2}}\Big)|\nabla\psi|_2|\nabla\phi|_6\\
&\leq \frac{a_4}{10}|\nabla\psi|_2^2+C\Big(\big(c_5^2|V_{R_0}|^{\frac{1}{3}}+c_5^7\big)|\sqrt{\rho}\psi|_2^2+c_5^4|V_{R_0}|^{\frac{1}{3}}|\nabla\psi|_2^2\Big),
\end{aligned}\tag{3.27}$$

which, along with (3.26), from the Gronwall's inequality, implies

$$|\sqrt{\rho}\psi|_2^2+\int_0^t|\nabla\psi|_2^2\leq C(c_0^3+c_5^8t)\exp(Cc_5^7t)\leq Cc_0^3,\tag{3.28}$$

for $0\leq t\leq T_3'=\min\{T_2,(1+c_5^8)^{-1}\}$ and R_0 satisfying $|V_{R_0}|\leq\big(a_4/(10Cc_5^4)\big)^3$.

Step 2. *Second order estimates of ψ*. Differentiating $(3.6)_2$ with respect to t, multiplying by ψ_t and integrating over Ω, one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|\psi_t|^2+a_2\int\phi^{\frac{b}{b+1}}|\nabla\psi_t|^2\\
=&-a_2\int\nabla\phi^{\frac{b}{b+1}}\cdot\nabla\psi_t\psi_t+a_2\int\phi_t^{\frac{b}{b+1}}\triangle\psi\psi_t+a_3\int\phi_t^{\frac{b}{b+1}}Q(w)\psi_t\\
&+\int\Big(a_3\phi^{\frac{b}{b+1}}Q(w)_t\psi_t+\rho_t w\cdot\nabla\psi\psi_t+a_1\rho_t\psi\mathrm{div}w\psi_t\Big)\\
&-\int\rho(2w\cdot\nabla\psi_t+w_t\cdot\nabla\psi+a_1\psi\mathrm{div}w_t+a_1\psi_t\mathrm{div}w)\psi_t\ =:\sum_{i=1}^5\mathrm{R}_i.
\end{aligned}\tag{3.29}$$

Now we estimate R_1-R_5 one by one. It follows from Lemmas 3.3-3.4, 2.1-2.2 and Young's inequality that

$$\begin{aligned}
\mathrm{R}_1=&-a_2\int\nabla\phi^{\frac{b}{b+1}}\cdot\nabla\psi_t\psi_t\leq C\Big(\int_{V_{R_0}}+\int_{\Omega\setminus V_{R_0}}\Big)|\nabla\phi\cdot\nabla\psi_t\psi_t|\\
\leq& C\Big(|\psi_t|_6|V_{R_0}|^{\frac{1}{6}}+|\sqrt{\rho}\psi_t|_2^{\frac{1}{2}}|\sqrt{\rho}\psi_t|_6^{\frac{1}{2}}\Big)|\nabla\phi|_6|\nabla\psi_t|_2\\
\leq&\frac{a_4}{20}|\nabla\psi_t|_2^2+C\Big(\big(c_5^2|V_{R_0}|^{\frac{1}{3}}+c_5^7\big)|\sqrt{\rho}\psi_t|_2^2+c_5^4|V_{R_0}|^{\frac{1}{3}}|\nabla\psi_t|_2^2\Big),\\
\mathrm{R}_2+\mathrm{R}_3\leq& C\Big(\int_{V_{R_0}}+\int_{\Omega\setminus V_{R_0}}\Big)|\phi_t|\big(|\triangle\psi|+|\nabla w|^2\big)|\psi_t|\\
\leq& C\big(|\phi_t|_6|V_{R_0}|^{\frac{1}{6}}+|\sqrt{\rho}\psi_t|_2^{\frac{1}{2}}|\sqrt{\rho}\psi_t|_6^{\frac{1}{2}}\big)(|\triangle\psi|_2+|\nabla w|_3|\nabla w|_6)|\phi_t|_6\\
\leq&\frac{a_4}{20}|\nabla\psi_t|_2^2+\big(C|V_{R_0}|^{\frac{1}{3}}(|\triangle\psi|_2^2+c_2^6)+\eta\big)|\phi_t|_6^2\\
&+\Big(\frac{a_4}{20c_0^2}+C\eta^{-2}(c_0^3|\triangle\psi|_2^4+c_2^{11})\Big)|\sqrt{\rho}\psi_t|_2^2,
\end{aligned}$$

$$\begin{aligned}
\mathrm{R}_4 =& \int \Big(a_3\phi^{\frac{b}{b+1}}Q(w)_t\psi_t + \rho_t w\cdot\nabla\psi\psi_t + a_1\rho_t\psi\mathrm{div}w\psi_t\Big)\\
\leq& C(|\phi|_6|\nabla w_t|_2|\nabla w|_6 + |\rho_t|_3|w|_\infty|\nabla\psi|_2 + |\rho_t|_2|\psi|_6|\nabla w|_6)|\psi_t|_6\\
\leq& \frac{a_4}{20c_0^2}(|\sqrt{\rho}\psi_t|_2^2 + c_0^2|\nabla\psi_t|_2^2) + C(c_3^6|\nabla w_t|_2^2 + c_2^8|\nabla\psi|_2^2 + c_2^{11}),\\
\mathrm{R}_5 =& -\int \rho(2w\cdot\nabla\psi_t + w_t\cdot\nabla\psi + a_1\psi\mathrm{div}w_t + a_1\psi_t\mathrm{div}w)\psi_t\\
\leq& C\Big(\big(|w_t|_6|\nabla\psi|_2 + |\nabla w_t|_2|\psi|_6\big)|\sqrt{\rho}\psi_t|_2^{\frac{1}{2}}|\sqrt{\rho}\psi_t|_6^{\frac{1}{2}}\\
&+ (|w|_\infty|\nabla\psi_t|_2 + |\nabla w|_3|\psi_t|_6)|\sqrt{\rho}\psi_t|_2\Big)|\rho|_\infty^{\frac{1}{2}}\\
\leq& \frac{a_4}{20}|\nabla\psi_t|_2^2 + C\Big(|\nabla w_t|_2^2 + \big(c_0^{11} + c_2^5 + \frac{a_4}{20c_0^2} + c_0^9|\nabla\psi|_2^4\big)|\sqrt{\rho}\psi_t|_2^2 + c_2^{10}|\nabla\psi|_2^2\Big).
\end{aligned} \tag{3.30}$$

Then denoting $\Gamma(t) = 1+|\nabla\psi|_2^2+|\sqrt{\rho}\psi_t|_2^2$, it follows from the estimates of $\mathrm{R}_1-\mathrm{R}_5$ that,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\sqrt{\rho}\psi_t|_2^2 + a_4|\nabla\psi_t|_2^2\\
\leq& \Big(\frac{a_4}{4} + Cc_5^4|V_{R_0}|^{\frac{1}{3}}\Big)|\nabla\psi_t|_2^2 + \big(C|V_{R_0}|^{\frac{1}{3}}(|\triangle\psi|_2^2 + c_2^6) + \eta\big)|\phi_t|_6^2 + Cc_3^6|\nabla w_t|_2^2\\
&+ C\big(c_5^{11} + c_5^2|V_{R_0}|^{\frac{1}{3}} + \eta^{-2}c_2^{11} + c_0^9|\nabla\psi|_2^4 + \eta^{-2}c_0^3|\triangle\psi|_2^4\big)\Gamma(t),
\end{aligned} \tag{3.31}$$

where $\eta > 0$ is a constant, and we have used Lemma 2.2.

Now we need to consider the term $|\Delta\psi|_2$. From equation $(3.6)_2$, Lemmas 2.1-2.3, one has

$$\begin{aligned}
|\psi|_{D^2} \leq& C\Big(|\rho|_\infty^{\frac{1}{2}}|\sqrt{\rho}\psi_t|_2 + |\rho|_\infty|w|_\infty|\nabla\psi|_2 + |\rho|_6|\psi|_6|\mathrm{div}w|_6 + c_5^{\frac{b}{b+1}}|\nabla w|_6|\nabla w|_3\Big)\\
\leq& C\Big(c_2^3|\nabla\psi|_2 + c_2^3|\sqrt{\rho}\psi_t|_2 + c_2^2c_5^{\frac{b}{b+1}}\Big).
\end{aligned} \tag{3.32}$$

On the other hand, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\psi(t)|_2^2 \leq C|\nabla\psi|_2|\nabla\psi_t|_2 \leq \frac{a_4}{20}|\nabla\psi_t|_2^2 + C|\nabla\psi|_2^2. \tag{3.33}$$

Let R_0 be sufficiently small such that $|V_{R_0}| \leq \big(a_4/(20Cc_5^K)\big)^3$ for a sufficiently large constant $K \geq 9$, and let $\eta = c_5^{-K}$, then from (3.31) and (3.32)-(3.33) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(t) + a_4|\nabla\psi_t|_2^2 \leq Cc_3^6|\nabla w_t|_2^2\Gamma(t) + C\big(c_5^{2K+15} + c_5^{6-K}|\phi_t|_6^2\big)\Gamma^3(t). \tag{3.34}$$

Denote $H(t) = \Gamma(t)\exp\Big(-\int_0^t Cc_3^6|\nabla w_s|_2^2\Big)$, then from (3.34), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}H(t) \leq C\big(c_5^{2K+15} + c_5^{6-K}|\phi_t|_6^2\big)H^3(t). \tag{3.35}$$

Next we need to solve this inequality. From $(3.6)_2$, one has

$$|\sqrt{\rho}\psi_t|_2^2 \leq |\rho|_\infty\|\nabla w\|_1^2|\nabla\psi|_2^2 + \int|\Upsilon(t)|^2/\rho, \tag{3.36}$$

where $\Upsilon(0)=-\psi_0^{\frac{b}{b+1}}\big(a_2\triangle\psi_0+a_3Q(u_0)\big)$. Via Lemma 3.2, one can get

$$\lim_{t\mapsto 0}\int\Big(\frac{|\Upsilon(t)|^2}{\rho}-\frac{|\Upsilon(0)|^2}{\rho_0}\Big)\leq \lim_{t\mapsto 0}\Big(\frac{1}{\underline{\delta}}\int|\Upsilon(t)-\Upsilon(0)|^2+\frac{1}{\delta\underline{\delta}}|\rho(t)-\rho_0|_\infty\int|\Upsilon(0)|^2\Big)=0.$$

According to the compatibility conditions in (3.3) and equation $(3.6)_2$, one has

$$\limsup_{\tau\to 0}|\sqrt{\rho}\psi_t(\tau)|_2^2\leq|\rho_0|_\infty\|\nabla w_0\|_1^2|\nabla\psi_0|_2^2+|g_2|_2^2\leq Cc_0^5, \tag{3.37}$$

which implies that

$$\limsup_{\tau\to 0}H(\tau)\leq Cc_0^5.$$

Now integrating (3.35) over $[\tau,t]$ for any $\tau\in(0,t)$, letting $\tau\to 0$ and solving the resulting inequality, one has

$$H(t)\leq Cc_0^5+\frac{Cc_0^5}{\big(1-Cc_0^5(c_5^{2K+15}t+c_5^{8-K})\big)^{\frac{1}{2}}}\leq Cc_0^5,$$

when $0<t\leq T_3^{''}=\min\big\{T_2,(1+Cc_5)^{-4K}\big\}$ for constant K sufficiently large. Then one obtains

$$\Gamma(t)\leq Cc_0^5\exp(Cc_1^2c_3^6),\quad \text{for}\quad 0\leq t\leq T_3^{''}. \tag{3.38}$$

Therefore, from (3.34) and (3.38), one has

$$\Gamma(t)+a_4\int_0^t|\psi_s|_{D^1}^2\leq Cc_0^5\exp(Cc_1^2c_3^6),\quad \text{for}\quad 0\leq t\leq T_3^{''}. \tag{3.39}$$

From (3.32), one has

$$|\psi|_{D^2}\leq C\Big(c_2^2(|\nabla\psi|_2+|\sqrt{\rho}\psi_t|_2)+c_3^3c_5^{\frac{b}{b+1}}\Big)\leq Cc_2^5c_5^{\frac{b}{b+1}}\exp(Cc_1^2c_3^6). \tag{3.40}$$

For the term $|\psi|_{D^{2,q}}$, similarly, via Lemma 2.3 and $(3.6)_2$, one has

$$\int_0^t|\psi|_{D^{2,q}}^2\leq C\int_0^t\big|\rho\psi_s+\rho w\cdot\nabla\psi+a_1\rho\psi\text{div}w-a_3\phi^{\frac{b}{b+1}}Q(w)\big|_q^2\leq M+Cc_2^4c_5^{\frac{2b}{b+1}}, \tag{3.41}$$

for $0\leq t\leq T_3^{''}$ and $M=Cc_2^{10}\exp(Cc_1^2c_3^6)$.

According to $P=R\rho\psi^{\frac{1}{b+1}}$ and (3.15), for $0\leq t\leq T_3=T_3^{''}$, it arrives at

$$|\nabla P|_2\leq M,\quad |\nabla P|_q\leq Mc_5^{\frac{b}{b+1}},\quad |P_t|_2\leq M. \tag{3.42}$$

Step 3. *Improved estimate of* $|\nabla\psi|_2$. Multiplying $(3.6)_2$ by ψ_t and integrating over Ω, one has

$$\begin{aligned}
&\frac{a_2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\phi^{\frac{b}{b+1}}|\nabla\psi|^2+\int\rho|\psi_t|^2\\
=&-\int\Big(\frac{a_2}{2}\phi_t^{\frac{b}{b+1}}|\nabla\psi|^2+a_2\nabla\phi^{\frac{b}{b+1}}\cdot\nabla\psi\psi_t+\rho w\cdot\nabla\psi\psi_t+a_1\rho\psi\text{div}w\psi_t\\
&\qquad-a_3\phi^{\frac{b}{b+1}}Q(w)\psi_t\Big)\\
\leq& C\Big(|\phi_t|_6|\nabla\psi|_2|\nabla\psi|_3+|\nabla\phi|_3|\nabla\psi|_2|\psi_t|_6+(|\nabla\psi|_2|w|_\infty+|\psi|_6|\nabla w|_3)|\sqrt{\rho}\psi_t|_2|\rho|_\infty^{\frac{1}{2}}\\
&\quad+|\phi|_\infty^{\frac{b}{b+1}}|\nabla w|_2|\nabla w|_3|\psi_t|_6\Big)\\
\leq&\frac{1}{2}|\sqrt{\rho}\psi_t|_2^2+\eta(|\phi_t|_6^2+|\psi_t|_6^2)+C\big(c_2^5+\eta^{-1}c_3c_5+\eta^{-1}M\big)|\nabla\psi|_2^2,
\end{aligned} \tag{3.43}$$

where $\eta = M^{-1}$. Then integrating (3.43) over $(0,t)$, one has

$$|\nabla\psi|_2^2 + \int_0^t |\sqrt{\rho}\psi_s|_2^2 \leq Cc_0^3, \quad \text{for} \quad 0 \leq t \leq T_3. \tag{3.44}$$

□

Now we give the a priori estimates for the velocity u.

LEMMA 3.6. *For* $0 \leq t \leq T_4 = \min\{T_3, (1+Mc_5^2)^{-2}\}$, *one has*

$$\|u\|_1^2 + |\sqrt{\rho}u_t|_2^2 + \int_0^t |u_s|_{D^1}^2 \leq Cc_0^5, \ \int_0^t |u|_{D^{2,q}}^2 \leq Cc_0^7, \quad |u|_{D^2}^2 \leq Cc_1^{13}.$$

Proof. Differentiating $(3.6)_3$ with respect to t, multiplying by u_t and integrating over Ω, one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u_t|^2 + \int \big((\alpha+\beta)|\mathrm{div}u_t|^2 + \alpha|\nabla u_t|^2\big) \\
= & -\int \big(2\rho w\cdot\nabla u_t\cdot u_t + \rho w_t\cdot\nabla u\cdot u_t + \rho_t w\cdot\nabla u\cdot u_t - \ P_t\mathrm{div}u_t\big) \\
\leq & C\Big(\big(|w|_\infty|\nabla u_t|_2 + |w_t|_6|\nabla u|_3\big)|\rho|_\infty^{\frac{1}{2}}|\sqrt{\rho}u_t|_2 + \big(|\rho_t|_3|w|_\infty|\nabla u|_6 + |P_t|_2\big)|\nabla u_t|_2\Big) \\
\leq & \frac{\alpha}{8}|\nabla u_t|_2^2 + (Cc_2^3 + \eta|\nabla w_t|_2^2)|\sqrt{\rho}u_t|_2^2 + M + C(c_0\eta^{-1} + c_2^6)\|\nabla u\|_1^2.
\end{aligned} \tag{3.45}$$

For $|u|_{D^2}$, due to $(3.6)_3$ and Lemma 2.3, one obtains

$$\begin{aligned}
|u|_{D^2} \leq & C\Big(|\rho|_\infty^{\frac{1}{2}}|\sqrt{\rho}u_t|_2 + |\rho|_2|w|_6|\nabla u|_2^{\frac{1}{2}}|\nabla u|_3^{\frac{1}{2}} + |\rho|_\infty|\nabla\psi|_2 + |\psi|_6|\nabla\rho|_3\Big) \\
\leq & \frac{1}{2}|u|_{D^2} + C\Big(c_1^4(|\sqrt{\rho}u_t|_2 + |\nabla u|_2) + c_0^{\frac{5}{2}}\Big).
\end{aligned} \tag{3.46}$$

On the other hand, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla u|_2^2 \leq 2|\nabla u|_2|\nabla u_t|_2 \leq \frac{\alpha}{20}|\nabla u_t|_2^2 + C|\nabla u|_2^2, \tag{3.47}$$

which, along with (3.45)-(3.46), implies that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(|\sqrt{\rho}u_t|_2^2 + |\nabla u|_2^2\big) + \frac{\alpha}{2}|\nabla u_t|_2^2 \\
\leq & M + \eta^{-1}c_0^6 + C(c_2^6 + \eta|\nabla w_t|_2^2 + \eta^{-1}c_1^9)\big(|\sqrt{\rho}u_t|_2^2 + |\nabla u|_2^2\big).
\end{aligned} \tag{3.48}$$

Similarly to the proof of (3.37), via (3.3) and equations $(3.6)_2$, one has

$$\limsup_{\tau\to 0}|\sqrt{\rho}u_t(\tau)|_2^2 \leq |\rho_0|_\infty\|\nabla w_0\|_1^2|\nabla u_0|_2^2 + |g_1|_2^2 \leq Cc_0^5. \tag{3.49}$$

Then from the Gronwall's inequality and (3.49), via letting $\eta = c_2^{-2}$, one gets

$$\begin{aligned}
&|\nabla u|_2^2 + |\sqrt{\rho}u_t|_2^2 + \int_0^t |u_s|_{D^1}^2 \\
\leq & \big(Cc_0^5 + Mt\big)\exp\Big(\int_0^t \big(c_2^{11} + c_2^{-2}|\nabla w_s|_2^2\big)\Big) \leq Cc_0^5,
\end{aligned} \tag{3.50}$$

for $0 \le t \le T_3$, which, together with (3.46), implies that

$$|u|_{D^2} \le Cc_1^{6\frac{1}{2}}, \quad \int_0^t |u|_{D^{2,q}}^2 \le \int_0^t \left(|\rho u_t + \rho w \cdot \nabla u + \nabla P|_q^2\right) \le Cc_0^7,$$

for $0 \le t \le T_4 = \min\left\{T_3, (1 + Mc_5^2)^{-1}\right\}$. □

Then based on Lemmas 3.3-3.6, when $0 \le t \le T_4$, for R_0 satisfying

$$|V_{R_0}| \le \left(a_4/(20Cc_5^K)\right)^3, \tag{3.51}$$

wirh a sufficiently large constant $K \ge 9$, the following a priori estimates hold

$$\begin{aligned} |\rho|_\infty^2 + \|\rho\|_{W^{1,q}}^2 \le Cc_0^2, \quad |\rho_t|_q \le Cc_0c_2, \quad \psi \ge \frac{1}{2}\underline{\psi}, \quad \|\psi\|_1^2 \le Cc_0^3, \\ |\sqrt{\rho}\psi_t|_2^2 + \int_0^t (|\psi_s|_{D^1}^2 + |\psi|_{D^{2,q}}^2) \le Mc_5^{\frac{2b}{b+1}}, \quad |\psi|_{D^2}^2 \le Mc_5^{\frac{2b}{b+1}}, \\ \|u\|_1^2 + |\sqrt{\rho}u_t|_2^2 + \int_0^t \left(|u|_{D^{2,q}}^2 + |u_s|_{D^1}^2\right) \le Cc_0^7, \quad |u|_{D^2}^2 \le Cc_1^{13}. \end{aligned} \tag{3.52}$$

Therefore, we can define the constants c_i $(i = 1, 2, 3, 4, 5)$ and T^* by

$$\begin{aligned} & c_1 = C^{\frac{1}{2}} c_0^{\frac{7}{2}}, \quad c_2 = C^{\frac{1}{2}} c_1^{\frac{13}{2}} = C^{\frac{15}{4}} c_0^{\frac{91}{4}}, \quad c_3 = c_2 = C^{\frac{15}{4}} c_0^{\frac{91}{4}}, \\ & c_4 = c_5 = M^{\frac{b+1}{2}} = \left(C^{\frac{77}{2}} c_0^{\frac{455}{2}} \exp\left(C^{\frac{49}{2}} c_0^{\frac{287}{2}}\right)\right)^{\frac{b+1}{2}}, \\ & T^* = \min\left\{T, \frac{1}{12R_0c_2}, \frac{1}{(1+Cc_5)^{4K}}, \left(1 + C^{\frac{77}{2}} c_0^{\frac{455}{2}} \exp\left(C^{\frac{49}{2}} c_0^{\frac{287}{2}}\right) c_5^2\right)^{-1}\right\}, \ K \ge 9. \end{aligned} \tag{3.53}$$

Then one has

$$\begin{aligned} \sup_{0\le t\le T^*} \|u(t)\|_1^2 + \operatorname{ess}\sup_{0\le t\le T^*} |\sqrt{\rho}u_t(t)|_2^2 + \int_0^{T^*} \left(|u(t)|_{D^{2,q}}^2 + |u_t(t)|_{D^1}^2\right) \mathrm{d}t \le c_1^2, \\ \sup_{0\le t\le T^*} |u(t)|_{D^2}^2 \le c_2^2, \quad \sup_{0\le t\le T^*} |\sqrt{\rho}\psi(t)|_2^2 + \sup_{0\le t\le T^*} \|\psi(t)\|_1^2 \le c_3^2, \\ \operatorname{ess}\sup_{0\le t\le T^*} |\sqrt{\rho}\psi_t(t)|_2^2 + \int_0^{T^*} \left(|\psi_t(t)|_{D^1}^2 + |\psi(t)|_{D^{2,q}}^2\right) \mathrm{d}t \le c_4^2, \quad \sup_{0\le t\le T^*} |\psi(t)|_{D^2}^2 \le c_5^2, \\ \psi(t,x) \ge \frac{1}{2}\underline{\psi}, \quad \sup_{0\le t\le T^*} \left(|\rho(t)|_\infty^2 + \|\rho(t)\|_{W^{1,q}}^2 + |\rho_t(t)|_q\right) \le c_2^2. \end{aligned} \tag{3.54}$$

Moreover, for sufficiently small $R_0 > 0$ satisfying (3.51), one can obtain

$$\rho(t,x) \ge a_{R_0} + \frac{1}{2}\delta > 0, \quad \forall\ (t,x) \in [0,T^*] \times (\Omega \backslash V_{R_0}), \tag{3.55}$$

where a_{R_0} is a positive constant independent of δ.

3.4. Passing limit from non-vacuum to vacuum. In this subsection, we will give the existence of the strong solution with vacuum to our linear problem:

$$\begin{cases} \rho_t + \mathrm{div}(\rho w) = 0, \\ \rho\psi_t + \rho w\cdot\nabla\psi + a_1\rho\psi\mathrm{div}w - a_2\phi^{\frac{b}{b+1}}\triangle\psi = a_3\phi^{\frac{b}{b+1}}Q(w), \\ \rho u_t + \rho w\cdot\nabla u + \nabla P + Lu = 0, \\ (\rho,u,\psi)|_{t=0} = (\rho_0(x),u_0(x),\psi_0(x)), \quad x\in\Omega, \\ u(t,x)|_{\partial\Omega} = 0, \quad \nabla\psi(t,x)\cdot n|_{\partial\Omega} = 0, \quad t\geq 0. \end{cases} \tag{3.56}$$

LEMMA 3.7. *Assume that* (3.2)-(3.3) *hold. Then there exists a unique strong solution* (ρ,u,ψ) *on* $[0,T^*]\times\Omega$ *to IBVP* (3.56) *satisfying*

$$\begin{aligned} &\rho\in C([0,T^*];W^{1,q}), \quad (u,\psi)\in C([0,T^*];H^2)\cap L^2([0,T^*];W^{2,q}), \\ &\psi\geq\frac{1}{2}\underline{\psi}, \quad (\sqrt{\rho}u_t,\sqrt{\rho}\psi_t)\in L^\infty([0,T^*];L^2), \quad (u_t,\psi_t)\in L^2([0,T^*];H^1). \end{aligned} \tag{3.57}$$

Moreover, the a priori estimates (3.54) *also hold for our solution* (ρ,u,ψ)*, and for sufficiently small* $R_0>0$ *there exists a constant* a_{R_0} *independent of* δ *such that*

$$\rho(t,x)\geq a_{R_0}>0, \quad \forall\ (t,x)\in[0,T^*]\times(\Omega\backslash V_{R_0}). \tag{3.58}$$

Proof. We divide the proof into three steps.

Step 1. Existence. For $\rho_0^\delta=\rho_0+\delta$ with $\delta\in(0,1)$, from Lemma 3.2, there exists a unique strong solution $(\rho^\delta,u^\delta,\psi^\delta)$ on $[0,T^*]\times\Omega$ satisfying (3.54)-(3.55), where the constants c_1-c_5, C, R_0, T^* and a_{R_0} are independent of δ. Then there exists a subsequence of solutions (still noted) $(\rho^\delta,u^\delta,\psi^\delta)$ converging to a limit (ρ,u,ψ) in weak or weak* sense:

$$\begin{aligned} \rho^\delta &\overset{*}{\rightharpoonup} \rho \quad \text{weakly* in } L^\infty([0,T^*];W^{1,q}(\Omega)), \\ (u^\delta,\psi^\delta) &\overset{*}{\rightharpoonup} (u,\psi) \quad \text{weakly* in } L^\infty([0,T^*];H^2(\Omega)), \\ (u_t^\delta,\psi_t^\delta) &\rightharpoonup (u_t,\psi_t) \quad \text{weakly in } L^2([0,T^*];H^1(\Omega)). \end{aligned} \tag{3.59}$$

Moreover, due to the compactness property in [33], there exists a subsequence of solutions $(\rho^\delta,u^\delta,\psi^\delta)$ satisfying:

$$(\rho^\delta,u^\delta,\psi^\delta)\to(\rho,u,\psi) \text{ in } C([0,T^*];H^1(\mathbb{K})), \tag{3.60}$$

where $\mathbb{K}$ is any compact subset of Ω.

From the lower semi-continuity of norms, we know that (ρ,u,ψ) also satisfies the estimates (3.54)-(3.55). Then it is easy to show that (ρ,u,ψ) is a weak solution in the sense of distribution and satisfies:

$$\begin{aligned} &\rho\in L^\infty([0,T^*];W^{1,q}), \ \rho_t\in L^\infty([0,T^*];L^q), \\ &(u,\psi)\in L^\infty([0,T^*];H^2)\cap L^2([0,T^*];W^{2,q}), \quad \psi\geq\frac{1}{2}\underline{\psi}, \\ &(\sqrt{\rho}u_t,\sqrt{\rho}\psi_t)\in L^\infty([0,T^*];L^2), \quad (u_t,\psi_t)\in L^2([0,T^*];H^1). \end{aligned} \tag{3.61}$$

Step 2. Uniqueness. Let (ρ_1, u_1, ψ_1) and (ρ_2, u_2, ψ_2) be two solutions obtained above. Then $\rho_1 = \rho_2$ can be obtained by the same method used in Lemma 3.2. Let $\overline{\psi} = \psi_1 - \psi_2$. It follows from equation $(3.56)_2$ that

$$\rho\overline{\psi}_t + \rho w \cdot \nabla\overline{\psi} + a_1\rho\overline{\psi}\text{div}w - a_2\phi^{\frac{b}{b+1}}\triangle\overline{\psi} = 0.$$

Then multiplying the above equations by $\overline{\psi}$ and integrating it over Ω, one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\sqrt{\rho}\overline{\psi}|_2^2 + a_4|\nabla\overline{\psi}|_2^2 \leq C|\overline{\psi}|_2^2, \tag{3.62}$$

which, along with $\nabla\overline{\psi} \cdot n|_{\partial\Omega} = 0$, immediately means that $\psi_1 = \psi_2$. Via the similar argument, we can show that $u_1 = u_2$.

Step 3. Time-continuity. The continuity of ρ can be obtained via the same method as in Lemma 3.2. Similarly, from (3.61), one has

$$(u, \psi) \in C([0, T^*]; H^1) \cap C([0, T^*]; D^2 - \text{weak}).$$

From equations (3.56) and (3.61), one can get

$$(\rho u_t, \rho\psi_t) \in L^2([0, T^*]; L^2), \text{ and } \big((\rho u_t)_t, (\rho\psi_t)_t\big) \in L^2([0, T^*]; H^{-1}).$$

Thus from Lemma 2.4, we have $(\rho u_t, \rho\psi_t) \in C([0, T^*]; L^2)$. Due to equations $(3.6)_2$-$(3.6)_3$ and Lemma 2.3, we have $(u, \psi) \in C([0, T^*]; D^2)$. □

3.5. Strong convergence in L^2 space. In this subsection, we will give the proof for Theorem 3.1 based on some classical iteration scheme. Let us assume as in Subsection 3.3:

$$2 + \|\rho_0\|_{W^{1,q}} + \|(u_0, \psi_0)\|_2 + |(g_1, g_2)|_2 \leq c_0.$$

Next, let (u^0, ψ^0) be the solutions to the following linear problems

$$\begin{cases} u_t^0 - \Delta u^0 = 0; \quad u^0(0) = u_0 \quad \text{in } \ \Omega; \quad u^0|_{\partial\Omega} = 0, \\ \psi_t^0 - \Delta\psi^0 = 0; \quad \psi^0(0) = \psi_0 \quad \text{in } \ \Omega; \quad \nabla\psi^0 \cdot n|_{\partial\Omega} = 0. \end{cases} \tag{3.63}$$

Then we can choose a time $T^{**} \in (0, T^*)$ such that (u^0, ψ^0) satisfies (3.12).

Proof. We divide the proof into three steps.

Step 1. Existence. Let $(w, \phi) = (u^0, \psi^0)$. We can get (ρ^1, u^1, ψ^1) as a strong solution to (3.56). Then we construct approximate solutions (ρ^k, u^k, ψ^k) inductively as follows: assume (u^{k-1}, ψ^{k-1}) is defined for $k \geq 1$, and let (ρ^k, u^k, ψ^k) be the solution to (3.56) with (w, ϕ) replaced by (u^{k-1}, ψ^{k-1}) as following:

$$\begin{cases} \rho_t^k + \text{div}(\rho^k u^{k-1}) = 0, \\ \rho^k\psi_t^k + \rho^k u^{k-1} \cdot \nabla\psi^k + a_1\rho^k\psi^k\text{div}u^{k-1} = (\psi^{k-1})^{\frac{b}{b+1}}\big(a_2\Delta\psi^k + a_3Q(u^{k-1})\big), \\ \rho^k u_t^k + \rho^k u^{k-1} \cdot \nabla u^k + \nabla P^k + Lu^k = 0, \\ (\rho^k, u^k, \psi^k)|_{t=0} = \big(\rho_0(x), u_0(x), \psi_0(x)\big), \quad x \in \Omega, \\ u^k(t, x)|_{\partial\Omega} = 0, \quad \nabla\psi^k(t, x) \cdot n|_{\partial\Omega} = 0, \quad t \geq 0, \end{cases} \tag{3.64}$$

where $P^k = R\rho^k(\psi^k)^{\frac{1}{b+1}}$. Then from Subsection 3.4, we know that the solution sequences (ρ^k, u^k, ψ^k) also satisfy the a priori estimates (3.54) and (3.58).

Next, we show that (ρ^k, u^k, ψ^k) converges to a limit in a strong sense. Denote

$$\overline{\rho}^{k+1} = \rho^{k+1} - \rho^k,\ \overline{u}^{k+1} = u^{k+1} - u^k,\ \overline{\psi}^{k+1} = \psi^{k+1} - \psi^k,$$

then one has

$$\begin{cases} \overline{\rho}_t^{k+1} + \text{div}(\overline{\rho}^{k+1}u^k) + \text{div}(\rho^k\overline{u}^k) = 0, \\ \rho^{k+1}\overline{\psi}_t^{k+1} + \rho^{k+1}u^k\cdot\nabla\overline{\psi}^{k+1} - a_2(\psi^k)^{\frac{b}{b+1}}\triangle\overline{\psi}^{k+1} \\ \quad = a_2\Big((\psi^k)^{\frac{b}{b+1}} - (\psi^{k-1})^{\frac{b}{b+1}}\Big)\triangle\psi^k + a_3\Big((\psi^k)^{\frac{b}{b+1}} - (\psi^{k-1})^{\frac{b}{b+1}}\Big)Q(u^k) \\ \quad\quad + a_3(\psi^{k-1})^{\frac{b}{b+1}}\big(Q(u^k) - Q(u^{k-1})\big) - \overline{\rho}^{k+1}\big(\psi_t^k + u^{k-1}\cdot\nabla\psi^k\big) \\ \quad\quad - a_1\overline{\rho}^{k+1}\psi^k\text{div}u^{k-1} - \rho^{k+1}\Big(\overline{u}^k\cdot\nabla\psi^k + a_1\overline{\psi}^{k+1}\text{div}u^k + a_1\psi^k\text{div}\overline{u}^k\Big), \\ \rho^{k+1}\overline{u}_t^{k+1} + \rho^{k+1}u^k\cdot\nabla\overline{u}^{k+1} + L\overline{u}^{k+1} = \overline{\rho}^{k+1}(-u_t^k - u^{k-1}\cdot\nabla u^k) \\ \quad\quad - \rho^{k+1}\overline{u}^k\cdot\nabla u^k - R\nabla\Big(\rho^{k+1}\big((\psi^{k+1})^{\frac{1}{b+1}} - (\psi^k)^{\frac{1}{b+1}}\big) + \overline{\rho}^{k+1}(\psi^k)^{\frac{1}{b+1}}\Big). \end{cases} \tag{3.65}$$

First, multiplying $(3.65)_1$ by $\overline{\rho}^{k+1}$ and integrating it over Ω, for $0 < \eta \leq \frac{1}{10}$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}|\overline{\rho}^{k+1}|_2^2 \leq C\Big(|u^k|_{D^{2,q}}^2 + \eta^{-1} + 1\Big)|\overline{\rho}^{k+1}|_2^2 + \eta|\nabla\overline{u}^k|_2^2. \tag{3.66}$$

Second, multiplying $(3.65)_2$ by $\overline{\psi}^{k+1}$ and integrating over Ω, one has

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\sqrt{\rho^{k+1}}\overline{\psi}^{k+1}|_2^2 + a_2\int(\psi^k)^{\frac{b}{b+1}}|\nabla\overline{\psi}^{k+1}|^2 \\ =& - a_2\int\nabla(\psi^k)^{\frac{b}{b+1}}\cdot\nabla\overline{\psi}^{k+1}\overline{\psi}^{k+1} + a_2\int\Big((\psi^k)^{\frac{b}{b+1}} - (\psi^{k-1})^{\frac{b}{b+1}}\Big)\triangle\psi^k\overline{\psi}^{k+1} \\ &+ a_3\int\Big((\psi^k)^{\frac{b}{b+1}} - (\psi^{k-1})^{\frac{b}{b+1}}\Big)Q(u^k)\overline{\psi}^{k+1} + a_3\int(\psi^{k-1})^{\frac{b}{b+1}}\big(Q(u^k) - Q(u^{k-1})\big)\overline{\psi}^{k+1} \\ &- \int\overline{\rho}^{k+1}\big(\psi_t^k + u^{k-1}\cdot\nabla\psi^k + a_1\psi^k\text{div}u^{k-1}\big)\overline{\psi}^{k+1} \\ &- \int\rho^{k+1}\big(\overline{u}^k\cdot\nabla\psi^k + a_1\overline{\psi}^{k+1}\text{div}u^k + a_1\psi^k\text{div}\overline{u}^k\big)\overline{\psi}^{k+1} =: \sum_{i=6}^{11}\mathrm{R}_i. \end{aligned}$$

According to Hölder's inequality, Lemma 2.1 and Young's inequality, one has

$$\begin{aligned} \mathrm{R}_6 =& - a_2\int\nabla(\psi^k)^{\frac{b}{b+1}}\cdot\nabla\overline{\psi}^{k+1}\overline{\psi}^{k+1} \\ \leq& C\Big(\int_{V_{R_0}} + \int_{\Omega\setminus V_{R_0}}\Big)|\nabla\psi^k||\nabla\overline{\psi}^{k+1}||\overline{\psi}^{k+1}| \\ \leq& C\Big(|\overline{\psi}^{k+1}|_6|V_{R_0}|^{\frac{1}{6}} + |\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^{\frac{1}{2}}|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_6^{\frac{1}{2}}\Big)|\nabla\psi^k|_6|\nabla\overline{\psi}^{k+1}|_2, \\ \mathrm{R}_7 + \mathrm{R}_8 =& \int\Big((\psi^k)^{\frac{b}{b+1}} - (\psi^{k-1})^{\frac{b}{b+1}}\Big)\big(a_2\triangle\psi^k + a_3Q(u^k)\big)\overline{\psi}^{k+1} \\ \leq& C\Big(\int_{V_{R_0}} + \int_{\Omega\setminus V_{R_0}}\Big)(|\triangle\psi^k| + |\nabla u^k|^2)|\overline{\psi}^k||\overline{\psi}^{k+1}| \\ \leq& C\Big(|\overline{\psi}^{k+1}|_6|V_{R_0}|^{\frac{1}{6}} + |\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^{\frac{1}{2}}|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_6^{\frac{1}{2}}\Big)(|\triangle\psi^k|_2 + |\nabla u^k|_3|\nabla u^k|_6)|\overline{\psi}^k|_6, \end{aligned}$$

and

$$\begin{aligned}
\mathrm{R}_9 =& a_3 \int (\psi^{k-1})^{\frac{b}{b+1}} \Big(Q(u^k) - Q(u^{k-1})\Big)\overline{\psi}^{k+1}\\
\leq& C|\psi^{k-1}|_\infty^{\frac{b}{b+1}}|\nabla u^k + \nabla u^{k-1}|_3|\nabla \overline{u}^k|_2|\overline{\psi}^{k+1}|_6,\\
\mathrm{R}_{10} =& -\int \overline{\rho}^{k+1}\Big(\psi_t^k + u^{k-1}\cdot\nabla\psi^k + a_1\psi^k \mathrm{div} u^{k-1}\Big)\overline{\psi}^{k+1}\\
\leq& C\big(|\psi_t^k|_3 + \|\psi^k\|_2\|u^{k-1}\|_2\big)|\overline{\rho}^{k+1}|_2|\overline{\psi}^{k+1}|_6,\\
\mathrm{R}_{11} =& -\int \rho^{k+1}\Big(\overline{u}^k\cdot\nabla\psi^k + a_1\overline{\psi}^{k+1}\mathrm{div} u^k + a_1\psi^k\mathrm{div}\overline{u}^k\Big)\overline{\psi}^{k+1}\\
\leq& C\big(|\rho^{k+1}|_\infty^{\frac{1}{2}}|\overline{u}^k|_6\|\nabla\psi^k\|_1 + |\mathrm{div} u^k|_{W^{1,q}} + |\rho^{k+1}|_\infty^{\frac{1}{2}}|\nabla\overline{u}^k|_2|\psi^k|_\infty\big)|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^2.
\end{aligned}$$

Then combining the estimates for R_i $(i = 6, ..., 11)$, for $t \in [0, T^{**}]$, one has

$$\begin{cases}
\frac{\mathrm{d}}{\mathrm{d}t}|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^2 + a_2|(\psi^k)^{\frac{b}{2(b+1)}}\nabla\overline{\psi}^{k+1}|_2^2\\
\leq E_{1\eta}^k(t)|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^2 + E_2^k(t)|\overline{\rho}^{k+1}|_2^2 + \eta(|\nabla\overline{\psi}^k|_2^2 + |\sqrt{\rho}^k\overline{\psi}^k|_2^2)\\
\quad + \Big(\frac{a_4}{2} + C\eta^{-1}c_5^6|V_{R_0}|^{\frac{1}{3}})|\nabla\overline{\psi}^{k+1}|_2^2 + C|\nabla\overline{u}^k|_2^2,\\
E_{1\eta}^k(t) = C\Big(1 + \eta^{-2} + |u^k|_{D^{2,q}} + \eta^{-1}|V_{R_0}|^{\frac{1}{3}}\Big), \quad E_2^k(t) = C(1 + |\psi_t^k|_3^2).
\end{cases} \tag{3.67}$$

Finally, similar as before, multiplying $(3.65)_3$ by $\overline{u}^{k+1}$ and integrating it over Ω, one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\sqrt{\rho}^{k+1}\overline{u}^{k+1}|_2^2 + \frac{1}{2}\Big(\alpha|\nabla\overline{u}^{k+1}|_2^2 + (\alpha+\beta)|\mathrm{div}\overline{u}^{k+1}|_2^2\Big)\\
\leq& C\big(1+\eta^{-1}\big)(|\sqrt{\rho}^{k+1}\overline{u}^{k+1}|_2^2 + |\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}|_2^2) + F^k(t)|\overline{\rho}^{k+1}|_2^2 + \eta|\nabla\overline{u}^k|_2^2,
\end{aligned} \tag{3.68}$$

where the term $F^k(t) = C(1 + |u_t^k|_3^2)$.

Now, let $\epsilon > 0$ be a sufficiently small constant and denote

$$\Lambda^{k+1}(T^{**}, \eta, \epsilon) = \sup_{0\leq t\leq T^{**}}\Big(|\overline{\rho}^{k+1}(t)|_2^2 + \epsilon|\sqrt{\rho}^{k+1}\overline{\psi}^{k+1}(t)|_2^2 + |\sqrt{\rho}^{k+1}\overline{u}^{k+1}(t)|_2^2\Big),$$

then let $|V_{R_0}|^{\frac{1}{3}} \leq a_4\eta(Cc_5^6)^{-1}$, from (3.66)-(3.68), by the Gronwall's inequality, it yields

$$\begin{aligned}
&\Lambda^{k+1}(T^{**}, \eta, \epsilon) + \int_0^{T^{**}}\Big(\frac{a_4}{4}\epsilon|\nabla\overline{\psi}^{k+1}|_2^2 + \frac{\alpha}{2}|\nabla\overline{u}^{k+1}|_2^2\Big)\\
\leq& \int_0^{T^{**}} G_{\eta,\epsilon}^k\Lambda^{k+1}(s,\eta,\epsilon) + \int_0^{T^{**}}\Big(\eta\epsilon|\nabla\overline{\psi}^k|_2^2 + \eta\epsilon|\sqrt{\rho}^k\overline{\psi}^k|_2^2 + (\eta + C\epsilon)|\nabla\overline{u}^k|_2^2\Big),
\end{aligned}$$

for some $G_{\eta,\epsilon}^k$ such that

$$\int_0^t G_{\eta,\epsilon}^k(s) \leq C\big(1 + \epsilon + \eta^{-2}t + \eta^{-1}|V_{R_0}|^{\frac{1}{3}}t\big) = f(C, t, \epsilon, \eta, R_0), \quad \text{for} \quad 0 \leq t \leq T^{**}.$$

Then from the Gronwall's inequality, one has

$$\begin{aligned}
&\Lambda^{k+1}(T^{**}, \eta, \epsilon) + \int_0^{T^{**}}\Big(\frac{a_4}{4}\epsilon|\nabla\overline{\psi}^{k+1}|_2^2 + \frac{\alpha}{2}|\nabla\overline{u}^{k+1}|_2^2\Big)\\
\leq& \Big(\eta\epsilon T^{**}\sup_{0\leq t\leq T^{**}}|\sqrt{\rho}^k\overline{\psi}^k(t)|_2^2 + \int_0^{T^{**}}\Big(\eta\epsilon|\nabla\overline{\psi}^k|_2^2 + (\eta + C\epsilon)|\nabla\overline{u}^k|_2^2\Big)\Big)\exp f(C, t, \epsilon, \eta, R_0).
\end{aligned}$$

First, one can choose $0 < \epsilon = \epsilon_0 < 1$ small enough such that

$$(1+C)\epsilon_0 \exp(C + C\epsilon_0) \leq \min\Big\{\frac{\alpha}{32}, \frac{1}{32}\Big\};$$

second, one can choose $0 < \eta = \eta_0$ small enough such that

$$(1+C)(\eta_0 + \eta_0\epsilon_0)\exp\big(C + C\epsilon_0\big) \leq \min\Big\{\frac{a_4\epsilon_0}{32}, \frac{\alpha}{32}, \frac{\epsilon_0}{32}, \frac{1}{32}\Big\};$$

third, one can choose $T^{**} = T_*$ small enough such that

$$(1 + \eta_0\epsilon_0 T_*)\exp\big(C\eta_0^{-2}T_*\big) \leq 2;$$

at last, one can choose R_0 sufficiently small such that

$$\exp\big(C\eta_0^{-1}|V_{R_0}|^{\frac{1}{3}}T_*\big) \leq 2.$$

So, when $\Lambda^{k+1} = \Lambda^{k+1}(T_*, \eta_0, \epsilon_0)$, one has

$$\sum_{k=1}^{\infty}\Big(\frac{1}{2}\Lambda^{k+1} + \int_0^{T_*}\Big(\frac{a_4\epsilon_0}{8}|\nabla\overline{\psi}^{k+1}|_2^2 + \frac{\alpha}{8}|\nabla\overline{u}^{k+1}|_2^2\Big)\Big) \leq C < +\infty.$$

Thus one can obtain that the full consequence (ρ^k, u^k, ψ^k) converges to a limit (ρ, u, ψ) in the following strong sense:

$$\rho^k \to \rho \text{ in } L^\infty\big([0,T_*]; L^2\big),\ (u^k, \psi^k) \to (u, \psi) \text{ in } L^2\big([0,T_*]; D^1\big). \tag{3.69}$$

Due to the local uniform estimates (3.54) and (3.58), and the strong convergence in (3.69), it is easy to show that (ρ, u, ψ) is a weak solution in the sense of distribution. Via the lower semi-continuity of norms, we also have that (ρ, u, ψ) satisfies the regularities in (3.61).

Step 2. Uniqueness. Let (ρ_1, u_1, ψ_1) and (ρ_2, u_2, ψ_2) be two strong solutions to the IBVP (3.1) satisfying the regularity (3.61). We denote that

$$\overline{\rho} = \rho_1 - \rho_2, \quad \overline{u} = u_1 - u_2, \quad \overline{\psi} = \psi_1 - \psi_2.$$

Similarly to the derivations of (3.66)-(3.68), let

$$\Lambda(t) = |\overline{\rho}|_2^2 + \overline{C}|\sqrt{\rho_1}\overline{u}|_2^2 + |\sqrt{\rho_1}\overline{\psi}|_2^2,$$

where $\overline{C} > 0$ is a sufficiently large constant, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(t) + \frac{1}{2}\overline{C}\alpha|\nabla\overline{u}|_2^2 + |\nabla\overline{\psi}|_2^2 \leq \Psi(t)\Lambda(t), \quad \text{with} \quad \int_0^t \Psi(s) \leq C \quad \text{for} \quad t \in [0, T_*].$$

Then from the Gronwall's inequality and $\overline{u}\cdot n = 0, \quad \nabla\overline{\psi}\cdot n|_{\partial\Omega} = 0$, we deduce that $\overline{\rho} = \overline{u} = \overline{\psi} = 0$.

Step 3. Time-continuity. The time-continuity can be obtained by the same method as in the proof of Lemma 3.7. Here we omit the details. □

3.6. Proof of Theorem 1.1. Now we give the proof for Theorem 1.1.

Proof. First, from (1.12), the IBVP of (1.1) with (1.10)-(1.11) can be written into

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ \rho u_t + \rho u \cdot \nabla u + \nabla P + Lu = 0, \\ \rho\theta_t + \rho u \cdot \nabla\theta + \dfrac{1}{c_v}\big(R\rho\theta \operatorname{div} u - \nu \operatorname{div}(\theta^b \nabla\theta)\big) = \dfrac{1}{c_v}Q(u), \\ (\rho, u, \theta)|_{t=0} = \big(\rho_0(x), u_0(x), \theta_0(x)\big), \quad x \in \Omega, \\ u(t,x)|_{\partial\Omega} = 0, \quad \nabla\theta(t,x)\cdot n|_{\partial\Omega} = 0, \quad t \geq 0. \end{cases} \tag{3.70}$$

Second, from Theorem 3.1, we know that the IBVP (3.1) has a unique strong solution $(\rho, u, \psi^{\frac{1}{b+1}}) \in \Phi$ on $[0, T_*] \times \Omega$, which immediately implies that $(\rho, u, \theta) = (\rho, u, \psi^{\frac{1}{b+1}})$ is our desired unique strong solution to (3.70), and thus the IBVP of (1.1) with (1.10)-(1.11). □

4. Proof of Theorem 1.2. In this section, we aim at proving the local existence of classical solutions to the IBVP of (1.1) with (1.10)-(1.11). As we did in the Section 3, we first establish the following existence result to the reformulated IBVP (3.1).

Theorem 4.1. *Let the initial data (ρ_0, u_0, ψ_0) satisfy the following regularities*

$$\rho_0 \geq 0, \quad \rho_0 \in H^3, \quad u_0 \in D_0^1 \cap D^3, \quad \psi_0 \in D^1 \cap D^3, \quad \psi_0 \geq \underline{\psi} = \underline{\theta}^{b+1}, \tag{4.1}$$

and the initial layer compatibility conditions

$$\begin{cases} Lu_0 + \nabla\Big(R\rho_0\psi_0^{\frac{1}{b+1}}\Big) = \rho_0 g_1, \\ -\dfrac{1}{c_v}\Big(\dfrac{\nu}{b+1}\Delta\psi_0 + Q(u_0)\Big) = \rho_0 g_2, \end{cases} \tag{4.2}$$

for some $g_1 \in D^1$ with $\big(\rho_0^{\frac{1}{2}} g_1, \rho_0^{\frac{1}{2}} g_2\big) \in L^2$. Then there exists a positive time $T_* > 0$, such that the IBVP (3.1) admits a unique classical solution (ρ, u, ψ) on $(0, T_*] \times \Omega$ under the condition

$$|V| \leq \epsilon_0,$$

where $\epsilon_0 = \epsilon_0(\rho_0, u_0, \psi_0, \alpha, \beta, \nu, R, c_v, b)$ is a small constant.

One can see that Theorem 4.1 indeed implies Theorem 1.2, therefore, it remains to prove Theorem 4.1, which will be carried out in the subsequent part. As a preparation, we claim that

Lemma 4.2. *For some $T > 0$, if (ρ, u, ψ) is a strong solution to the IBVP (3.1) on $[0, T] \times \Omega$ and belongs to the following solution class Ξ:*

$$\begin{aligned} \Xi = \Big\{(\rho, u, \psi) | \rho \in L^\infty([0,T]; H^3), \quad & \rho_t \in L^\infty([0,T]; H^2), \\ & (u, \psi) \in L^\infty([0,T]; D^1 \cap D^2), \quad u \in L^\infty([0,T]; D^3), \\ & u_t \in L^\infty([0,T]; D^1) \cap L^2([0,T]; D^2), \quad t^{\frac{1}{2}} u \in L^\infty([0,T]; D^4), \\ & t^{\frac{1}{2}} u_t \in L^\infty([0,T]; D^2) \cap L^2([0,T]; D^3), \quad t^{\frac{1}{2}} \psi \in L^\infty([0,T]; D^3), \\ & t^{\frac{1}{2}} \psi_t \in L^\infty([0,T]; D^1), \quad t\psi_t \in L^\infty([0,T]; D^2) \cap L^2([0,T]; D^3), \\ & t\psi \in L^\infty([0,T]; D^4), \quad (t^{\frac{3}{4}} u_t, t^{\frac{3}{2}} \psi_t) \in L^\infty([0,T]; D^3), \\ & (tu_{tt}, t^{\frac{3}{2}} \psi_{tt}) \in L^\infty([0,T]; D^1) \cap L^2([0,T]; D^2)\Big\}, \end{aligned} \tag{4.3}$$

then (ρ, u, ψ) is a classical solution to the IBVP (3.1) on $(0, T] \times \Omega$.

Proof. First, one knows that (see [5] for details)

$$\rho \in C([0,T]; H^3), \quad \rho_t \in C([0,T]; H^2),$$

combining with classical Sobolev embedding inequality, it implies

$$\rho \in C([0,T]; C^1), \quad \rho_t \in C([0,T] \times \Omega). \tag{4.4}$$

Second, according to (4.3), one knows that

$$\begin{aligned}
&u \in L^\infty([0,T]; D^1 \cap D^2 \cap D^3), \quad t^{\frac{1}{2}} u \in L^\infty([0,T]; D^4),\\
&u_t \in L^\infty([0,T]; D^1) \cap L^2([0,T]; D^2), \ t^{\frac{1}{2}} u_t \in L^\infty([0,T]; D^2) \cap L^2([0,T]; D^3),
\end{aligned}$$

with the help of Lemma 2.4, one deduces

$$t^{\frac{1}{2}} u \in C([0,T]; D^1 \cap D^2 \cap D^3),$$

this, together with Sobolev embedding, implies that

$$t^{\frac{1}{2}} u \in C([0,T]; C^1). \tag{4.5}$$

Similarly, we have

$$t\psi \in C([0,T]; C^1). \tag{4.6}$$

Third, we know from (4.3) that u satisfies

$$\begin{aligned}
&u_t \in L^\infty([0,T]; D^1), \quad t^{\frac{1}{2}} u_t \in L^\infty([0,T]; D^2),\\
&t^{\frac{3}{4}} u_t \in L^\infty([0,T]; D^3), \quad tu_{tt} \in L^\infty([0,T]; D^1) \cap L^2([0,T]; D^2),
\end{aligned}$$

with the help of Lemma 2.4, one deduces

$$tu_t \in C([0,T]; D^1 \cap D^2),$$

this, together with the Sobolev embedding, implies that

$$tu_t \in C([0,T] \times \Omega). \tag{4.7}$$

Similarly, we have

$$t^{\frac{3}{2}} \psi_t \in C([0,T] \times \Omega). \tag{4.8}$$

At last, from the momentum and energy equations, one has

$$\begin{aligned}
Lu &= -\rho u_t - \rho u \cdot \nabla u - \nabla\Big(R\rho\psi^{\frac{1}{b+1}}\Big) =: F_1,\\
a_2 \psi^{\frac{b}{b+1}} \Delta\psi &= \rho\psi_t + \rho u \cdot \nabla\psi + a_1 \rho\psi \mathrm{div} u - a_3 \psi^{\frac{b}{b+1}} Q(u) =: F_2,
\end{aligned}$$

according to (4.5)-(4.8), we get $(tF_1, t^{\frac{3}{2}} F_2) \in C([0,T] \times \Omega)$, thus

$$(t\nabla^2 u, t^{\frac{3}{2}} \nabla^2 \psi) \in C([0,T] \times \Omega). \tag{4.9}$$

Combining (4.4)-(4.9), the proof of this lemma is completed. □

Now, we give the proof of Theorem 4.1. In Section 3, we have proved the existence of strong solutions to the IBVP (3.1). In order to prove that the solution is classical, we need to show the approximate solutions (ρ^k, u^k, ψ^k) belong to a more regular space Ξ. With the help of (3.64) and the estimates in Section 3, we first have the following uniform estimates of (ρ^k, u^k, ψ^k) up to second-order.

LEMMA 4.3. *Under the assumptions of Theorem 4.1, there exists a constant* $T_* > 0$ *independent of* k, *for any* $k \geq 1$, *a.e.* $t \in [0, T_*]$ *and* $q \in (3, 6]$, *such that*

$$\begin{aligned}
&\rho^k(t,x) \geq a_{R_0} > 0, \quad \forall (t,x) \in [0,T_*] \times (\Omega \backslash V_{R_0}),\\
&|\rho^k|_\infty + \|\rho^k\|_{H^1 \cap W^{1,q}} + \|\rho^k_t\|_{L^2 \cap L^q} \leq C', \quad |u^k|_\infty + \|\nabla u^k\|_1 + \|\nabla \psi^k\|_1 \leq C',\\
&|\sqrt{\rho^k} u^k|_2 + |\sqrt{\rho^k} u^k_t|_2 + \|\nabla^2 u^k\|_{L^2 L^q} + \|\nabla u^k_t\|_{L^2 L^2} \leq C',\\
&|\sqrt{\rho^k} \psi^k|_2 + |\sqrt{\rho^k} \psi^k_t|_2 + \|\nabla^2 \psi^k\|_{L^2 L^q} + \|\nabla \psi^k_t\|_{L^2 L^2} \leq C',\\
&\psi^k(t,x) \geq \frac{1}{2}\underline{\psi}, \quad \forall (t,x) \in [0,T_*] \times \Omega,
\end{aligned} \tag{4.10}$$

where C' *denotes a generic constant which is independent of* k *and depends only* $\alpha, \beta, \nu, R, b, q, a_{R_0}, T_*, |\Omega|$ *and the initial data.*

Hereinafter, C denotes a generic constant depending only on $\alpha, \beta, \nu, c_v, R, b, |\Omega|$. Based on Lemma 4.3, we start with the estimates of (ρ^k, u^k, ψ^k) until the third-order derivatives in the following lemma, which helps us to prove the continuity of u_t and ∇u.

LEMMA 4.4. *Under the assumptions of Theorem 4.1, it holds that*

$$\begin{aligned}
\|\rho^k\|_3 + \|\rho^k_t\|_2 + |\nabla u^k_t|_2 + |\nabla^3 u^k|_2 + |\rho^k_{tt}|_2 \leq& C',\\
\int_0^t \left(|\sqrt{\rho^k} u^k_{ss}|_2^2 + |\nabla^2 u^k_s|_2^2 + |\nabla^4 u^k|_2^2 + |\nabla^3 \psi^k|_2^2 + |\nabla \rho^k_{ss}|_2^2\right) \leq& C',
\end{aligned}$$

for any $k \geq 1$ *and a.e.* $t \in [0, T_*]$.

Proof. We divide the proof into five steps.

Step 1. *Estimates of* $\|\nabla^2 \rho^k\|_{L^\infty L^2}$. Taking the second-order derivatives of x to $(3.64)_1$, multiplying by $\nabla^2 \rho^k$ and integrating over Ω, according to the Cauchy inequality, Sobolev inequality, interpolation inequality and Lemma 4.3, one has

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|\nabla^2 \rho^k|_2^2 &\leq C\big(|\nabla u^{k-1}|_\infty |\nabla^2 \rho^k|_2 + |\nabla \rho^k|_3 |\nabla^2 u^{k-1}|_6 + |\rho|_\infty |\nabla^3 u^{k-1}|_2\big)|\nabla^2 \rho^k|_2\\
&\leq C'\big((|\nabla u^{k-1}|^2_{W^{1,4}} + 1)|\nabla^2 \rho^k|_2^2 + |\nabla^3 u^{k-1}|_2^2\big).
\end{aligned} \tag{4.11}$$

For the last term in (4.11), with the help of $(3.64)_3$ and Lemmas 2.3, 4.3, we have

$$\begin{aligned}
|\nabla^3 u^k|_2^2 &\leq C\big|\nabla(\rho^k u^k_t + \rho^k u^{k-1} \cdot \nabla u^k + \nabla P^k)\big|_2^2\\
&\leq C'\big(|\nabla u^k_t|_2^2 + |\nabla^2 \rho^k|_2^2 + 1\big).
\end{aligned} \tag{4.12}$$

Thus one has

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla^2 \rho^k|_2^2 \leq C'\big(|\nabla u^{k-1}|^2_{W^{1,4}} + |\nabla u^{k-1}_t|_2^2 + 1\big)\big(|\nabla^2 \rho^k|_2^2 + |\nabla^2 \rho^{k-1}|_2^2 + 1\big), \tag{4.13}$$

which, by the Gronwall's inequality, for any given $N \in \mathbb{Z}_+$, implies

$$\max_{1\le k\le N} |\nabla^2 \rho^k|_2 \le C'. \tag{4.14}$$

Step 2. *Estimates of* $\|\nabla \rho^k_t\|_{L^\infty L^2}$ *and* $\|\rho^k_{tt}\|_{L^2 L^2}$. First, with the help of the Cauchy inequality, Sobolev inequality, Lemma 4.3 and the estimates in ***Step 1***, one has

$$\|\rho^k_t\|_1 = |\rho^k_t|_2 + |\nabla \rho^k_t|_2 \le |\rho^k_t|_2 + |\nabla \mathrm{div}(\rho^k u^{k-1})|_2 \le C'. \tag{4.15}$$

Similarly, from $(3.64)_1$ and (4.15), we immediately have

$$|\rho^k_{tt}|_2 \le |\mathrm{div}(\rho^k u^{k-1})_t|_2 \le C'\big(|\nabla u^{k-1}_t|_2 + 1\big), \tag{4.16}$$

combining (4.16) with Lemma 4.3, we complete the proof of ***Step 2***.

Step 3. *Estimates of* $\|\nabla u^k_t\|_{L^\infty L^2}$ *and* $\|\sqrt{\rho^k} u^k_{tt}\|_{L^2L^2}$. Differentiating $(3.64)_3$ with respect to t, multiplying by u^k_{tt} and integrating over Ω, one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \big(\alpha|\nabla u^k_t|^2 + (\alpha+\beta)|\mathrm{div} u^k_t|^2\big) + \int \rho^k |u^k_{tt}|^2 \\
=& -\int \nabla P^k_t \cdot u^k_{tt} - \int \rho^k_t \big(u^k_t + u^{k-1}\cdot \nabla u^k\big)\cdot u^k_{tt} \\
&- \int \rho^k \big(u^{k-1}_t \cdot \nabla u^k + u^{k-1}\cdot \nabla u^k_t\big)\cdot u^k_{tt} \;=: \sum_{i=1}^{3} \mathrm{I}_i.
\end{aligned} \tag{4.17}$$

Now, we estimate the right-hand side of (4.17) term by term. It follows from integrating by parts that

$$\begin{aligned}
\mathrm{I}_1 =& -\int \nabla P^k_t \cdot u^k_{tt} = \int P^k_t \mathrm{div} u^k_{tt} = \frac{\mathrm{d}}{\mathrm{d}t}\int P^k_t \cdot \mathrm{div} u^k_t - \int P^k_{tt}\mathrm{div} u^k_t \\
=& \frac{\mathrm{d}}{\mathrm{d}t}\int \Big(P^k_t \cdot \mathrm{div} u^k_t - \frac{1}{4\alpha+2\beta}(P^k_t)^2\Big) - \frac{1}{2\alpha+\beta}\int P^k_{tt} G_t,
\end{aligned} \tag{4.18}$$

where we have used

$$G = (2\alpha+\beta)\mathrm{div} u^k - P^k, \quad \mathrm{div} u^k_t = \frac{1}{2\alpha+\beta}(G+P^k)_t.$$

According to $(3.64)_2$, we find that P^k satisfies

$$P^k_t + \mathrm{div}(P^k u^{k-1}) + RP^k \mathrm{div} u^{k-1} = RQ(u^{k-1}) + R\nu \Delta \psi^k, \tag{4.19}$$

differentiating (4.19) with respect to t, we have

$$P^k_{tt} = -\mathrm{div}(P^k u^{k-1})_t - R(P^k \mathrm{div} u^{k-1})_t + RQ(u^{k-1})_t + R\nu\Delta\psi^k_t. \tag{4.20}$$

Thus, the last term in (4.18) can be controlled as

$$\begin{aligned}
&-\frac{1}{2\alpha+\beta}\int P^k_{tt} G_t \\
\le& C\Big(|(\rho^k(\psi^k)^{\frac{1}{b+1}} u^{k-1})_t|_2 |\nabla G_t|_2 + \big(|\rho^k_t|_3 |\psi^k|_\infty^{\frac{1}{b+1}} + |\rho^k|_6 |\psi^k_t|_6\big)|\mathrm{div} u^{k-1}|_2 |G_t|_6 \\
&+ |\rho^k|_\infty |\psi^k|_\infty^{\frac{1}{b+1}} |\mathrm{div} u^{k-1}_t|_2 |G_t|_2 + |\nabla G_t|_2 |\nabla \psi^k_t|_2 + |\nabla u^{k-1}|_3 |\nabla u^{k-1}_t|_2 |G_t|_6\Big) \\
\le& C'\big(|\nabla G_t|_2 + |G_t|_2\big)\big(|\nabla u^{k-1}_t|_2 + |\nabla \psi^k_t|_2 + 1\big).
\end{aligned} \tag{4.21}$$

Now we need to consider the terms $|\nabla G_t|_2$ and $|G_t|_2$. Since G satisfies

$$\Delta G = \mathrm{div}\big(\rho^k u_t^k + \rho^k u^{k-1}\cdot\nabla u^k\big). \tag{4.22}$$

thus with the help of (4.22) and Lemma 2.3, one has

$$\begin{aligned}|\nabla G_t|_2 &\le C\big|\big(\rho^k u_t^k + \rho^k u^{k-1}\cdot\nabla u^k\big)_t\big|_2\\ &\le C'\big(|\nabla u_t^k|_2 + |\nabla u_t^{k-1}|_2 + |\sqrt{\rho^k}u_{tt}^k|_2 + 1\big).\end{aligned} \tag{4.23}$$

Moreover, according to the definition of G, one has

$$|G_t|_2 \le C'\big(|\nabla u_t^k|_2 + 1\big). \tag{4.24}$$

Putting (4.23)-(4.24) into (4.21), it arrives at

$$-\frac{1}{2\alpha+\beta}\int P_{tt}^k G_t \le \frac{1}{4}|\sqrt{\rho^k}u_{tt}^k|_2^2 + C'\big(|\nabla u_t^k|_2^2 + |\nabla u_t^{k-1}|_2^2 + |\nabla\psi_t^k|_2^2 + 1\big). \tag{4.25}$$

Thus, combining (4.25) with (4.18) implies

$$\begin{aligned}\mathrm{I}_1 \le\ & \frac{\mathrm{d}}{\mathrm{d}t}\int\Big(P_t^k\mathrm{div}u_t^k - \frac{1}{4\alpha+2\beta}(P_t^k)^2\Big) + \frac{1}{4}|\sqrt{\rho^k}u_{tt}^k|_2^2\\ &+ C'\big(|\nabla u_t^k|_2^2 + |\nabla u_t^{k-1}|_2^2 + |\nabla\psi_t^k|_2^2 + 1\big).\end{aligned} \tag{4.26}$$

Next, for I_2 and I_3, by the Cauchy inequality and Lemma 4.3, one has

$$\begin{aligned}\mathrm{I}_2 =\ & -\int \rho_t^k\big(u_t^k + u^{k-1}\cdot\nabla u^k\big)\cdot u_{tt}^k\\ =\ & -\frac{\mathrm{d}}{\mathrm{d}t}\int\rho_t^k\Big(\frac{1}{2}|u_t^k|^2 + u^{k-1}\cdot\nabla u^k\cdot u_t^k\Big) + \frac{1}{2}\int\rho_{tt}^k|u_t^k|^2\\ &+\int\big(\rho_{tt}^k u^{k-1}\cdot\nabla u^k + \rho_t^k u_t^{k-1}\cdot\nabla u^k + \rho_t^k u^{k-1}\cdot\nabla u_t^k\big)\cdot u_t^k\\ \le\ & -\frac{\mathrm{d}}{\mathrm{d}t}\int\rho_t^k\Big(\frac{1}{2}|u_t^k|^2 + u^{k-1}\cdot\nabla u^k\cdot u_t^k\Big) + \int|\rho_t^k u^{k-1} + \rho^k u_t^{k-1}||\nabla u_t^k||u_t^k|\\ &+\Big(\big(|\rho_{tt}^k|_2|\nabla u^k|_3 + |\rho_t^k|_3|\nabla u_t^k|_2\big)|u^{k-1}|_\infty + |\rho_t^k|_2|u_t^{k-1}|_6|\nabla u^k|_6\Big)|u_t^k|_6\\ \le\ & -\frac{\mathrm{d}}{\mathrm{d}t}\int\rho_t^k\Big(\frac{1}{2}|u_t^k|^2 + u^{k-1}\cdot\nabla u^k\cdot u_t^k\Big) + C'\big(|\rho_{tt}^k|_2^2 + |\nabla u_t^k|_2^3 + |\nabla u_t^{k-1}|_2^2 + 1\big),\\ \mathrm{I}_3 =\ & -\int\rho^k\big(u_t^{k-1}\cdot\nabla u^k + u^{k-1}\cdot\nabla u_t^k\big)\cdot u_{tt}^k\\ \le\ & \big(|u_t^{k-1}|_6|\nabla u^k|_3 + |u^{k-1}|_\infty|\nabla u_t^k|_2\big)|\sqrt{\rho^k}|_\infty|\sqrt{\rho}^k u_{tt}^k|_2\\ \le\ & \frac{1}{4}|\sqrt{\rho}^k u_{tt}^k|_2^2 + C'\big(|\nabla u_t^k|_2^2 + |\nabla u_t^{k-1}|_2^2\big).\end{aligned} \tag{4.27}$$

Thus combining (4.26)-(4.27) and (4.17) implies

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\big(\alpha|\nabla u_t^k|^2 + (\alpha+\beta)|\mathrm{div}u_t^k|^2\big) + \frac{1}{2}\int\rho^k|u_{tt}^k|^2\\ \le\ & \frac{\mathrm{d}}{\mathrm{d}t}\int\Big(-\frac{1}{2}\rho_t^k|u_t^k|^2 - \rho_t^k u^{k-1}\cdot\nabla u^k\cdot u_t^k + P_t^k\mathrm{div}u_t^k - \frac{1}{4\alpha+2\beta}(P_t^k)^2\Big)\\ &+ C'\big(|\nabla u_t^k|_2^3 + |\nabla u_t^{k-1}|_2^2 + |\nabla\psi_t^k|_2^2 + |\rho_{tt}^k|_2^2 + 1\big).\end{aligned} \tag{4.28}$$

At last, denoting

$$A_1(t) := \int \Big(\alpha|\nabla u_t^k|^2 + (\alpha+\beta)|\mathrm{div} u_t^k|^2 + \rho_t^k|u_t^k|^2 + 2\rho_t^k u^{k-1}\cdot\nabla u^k\cdot u_t^k \\ - 2P_t^k \mathrm{div} u_t^k + \frac{1}{2\alpha+\beta}(P_t^k)^2\Big),$$

then it is easy to get

$$\frac{\alpha}{2}|\nabla u_t^k|_2^2 - C' \le A_1(t) \le C'\big(|\nabla u_t^k|_2^2 + 1\big). \tag{4.29}$$

Similarly to the proof of (3.37) and (3.49), via (4.2) and equations $(3.64)_3$, one has

$$\limsup_{\tau\to 0}|\nabla u_t(\tau)|_2^2 \le \|\nabla u_0\|_2^2|\nabla u_0|_2^2 + |g_1|_{D^1}^2 \le C', \tag{4.30}$$

thus $|A_1(0)| \le C'$. Then integrating (4.28) over $[0,t]$, by the Gronwall's inequality, with the help of (4.29), Lemma 4.3 and the estimates in ***Step 2***, we immediately have

$$|\nabla u_t^k|_2^2 + \int_0^t |\sqrt{\rho^k}u_{ss}^k|_2^2 \le C'. \tag{4.31}$$

Step 4. *Estimates of* $\|\rho_{tt}^k\|_{L^\infty L^2}, \|\nabla^3 u^k\|_{L^\infty L^2}, \|\nabla^2 u_t^k\|_{L^2L^2}$. First, combining (4.12), (4.14), (4.16) and (4.31) implies

$$|\nabla^3 u^k|_2 + |\rho_{tt}^k|_2 \le C'.$$

Then with the help of $(3.64)_3$ and Lemmas 2.3, 4.3, one has

$$\begin{aligned}|\nabla^2 u_t^k|_2 &\le C\big|\big(\rho^k u_t^k + \rho^k u^{k-1}\cdot\nabla u^k + \nabla P^k\big)_t\big|_2\\ &\le C'\big(|\sqrt{\rho^k}u_{tt}^k|_2 + |\nabla u_t^k|_2 + |\nabla u_t^{k-1}|_2 + |\nabla\psi_t^k|_2 + 1\big),\end{aligned} \tag{4.32}$$

this, together with (4.31), gives

$$\int_0^t |\nabla^2 u_s^k|_2^2 \le C'. \tag{4.33}$$

Step 5. *Estimates of* $\|\nabla^3\rho^k\|_{L^\infty L^2}, \|\nabla^2\rho_t^k\|_{L^\infty L^2}, \|\nabla\rho_{tt}^k\|_{L^2L^2}, \|\nabla^4 u^k\|_{L^2L^2}$*, and* $\|\nabla^3\psi^k\|_{L^2L^2}$. First, taking the third-order derivatives of x to$(3.64)_1$, multiplying by $\nabla^3\rho^k$ and integrating over Ω, one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla^3\rho^k|_2^2 \le\ & C\big(|\nabla u^{k-1}|_\infty|\nabla^3\rho^k|_2^2 + |\nabla\rho^k|_6|\nabla^3 u^{k-1}|_3|\nabla^3\rho^k|_2\\ &+ |\nabla^2 u^{k-1}|_6|\nabla^2\rho^k|_3|\nabla^3\rho^k|_2 + |\nabla^4 u^{k-1}|_2|\nabla^3\rho^k|_2\big)\\ \le\ & C'\big((|\nabla u^{k-1}|_{W^{1,4}}+1)|\nabla^3\rho^k|_2^2 + |\nabla^4 u^{k-1}|_2^2 + 1\big).\end{aligned} \tag{4.34}$$

For the last term in (4.34), with the help of $(3.64)_3$ and Lemma 2.3, one has

$$\begin{aligned}|\nabla^4 u^k|_2 &\le C\big|\nabla^2\big(\rho^k u_t^k + \rho^k u^{k-1}\cdot\nabla u^k + \nabla P^k\big)\big|_2\\ &\le C'\big(|\nabla^3\rho^k|_2 + |\nabla^2 u_t^k|_2 + |\nabla^3\psi^k|_2 + 1\big),\end{aligned} \tag{4.35}$$

where $|\nabla^3\psi^k|_2$ can be controlled by

$$\begin{aligned}|\nabla^3\psi^k|_2 \leq & C\Big|\nabla\Big(\big(\rho^k\psi_t^k+\rho^k u^{k-1}\cdot\nabla\psi^k+a_1\rho^k\psi^k\mathrm{div}u^{k-1}\big)\big(\psi^{k-1}\big)^{\frac{-b}{b+1}}\Big)\Big|_2\\ &+C\big|\nabla Q(u^{k-1})\big|_2\\ \leq & C'\big(|\nabla\psi_t^k|_2+1\big).\end{aligned} \tag{4.36}$$

Substituted (4.35)-(4.36) back to (4.34), it arrives at

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla^3\rho^k|_2^2 \leq C'\Big(\big(|\nabla u^{k-1}|_{W^{1,4}}+1\big)|\nabla^3\rho^k|_2^2+|\nabla^3\rho^{k-1}|_2^2+|\nabla^2 u_t^{k-1}|_2+|\nabla\psi_t^{k-1}|_2+1\Big),$$

this, combining with the Gronwall's inequality, for any given $N\in\mathbb{Z}_+$, gives

$$\max_{1\leq k\leq N}|\nabla^3\rho^k|_2\leq C', \tag{4.37}$$

where we have used Lemma 4.3 and (4.33). Moreover, (4.35)-(4.37) imply

$$\int_0^t\big(|\nabla^4 u^k|_2^2+|\nabla^3\psi^k|_2^2\big)\leq C'.$$

Second, according to $(3.64)_1$, with the help of the Cauchy inequality, Lemma 4.3, (4.33) and (4.37), one has

$$|\nabla^2\rho_t^k|_2=\big|\nabla^2\big(\mathrm{div}(\rho^k u^{k-1})\big)\big|_2\leq C', \tag{4.38}$$

and

$$|\nabla\rho_{tt}^k|_2=\big|\nabla\mathrm{div}(\rho^k u^{k-1})_t\big|_2\leq C'\big(|\nabla^2\rho_t^k|_2+|\nabla^2 u_t^{k-1}|_2+1\big). \tag{4.39}$$

Combining (4.39) with (4.38) and (4.33), it arrives at

$$\int_0^t|\nabla\rho_{ss}^k|_2^2\leq C',$$

this ends the proof of ***Step 5***. We complete the proof of Lemma 4.4. □

Based on Lemma 4.4, in order to prove the continuity of the first-order derivative of ψ, we need to show $t^{\frac{1}{2}}\psi_t^k\in L^\infty([0,T_*];D^1)$. Unlike the case of constant heat conductivity, there is strong coupling between u^k and ψ^k, which requires more attention to deal with. We first claim that $\|t^{\frac{1}{2}}\nabla u_{tt}^k\|_{L^2L^2}$ is controlled by $\|t\nabla\psi_{tt}^k\|_{L^2L^2}$ in the following lemma.

LEMMA 4.5. *Under the assumptions of Theorem 4.1, it holds that*

$$t\big(|\nabla\psi_t^k|_2^2+|\sqrt{\rho^k}u_{tt}^k|_2^2\big)+\int_0^t s\big(|\sqrt{\rho^k}\psi_{ss}^k|_2^2+|\nabla u_{ss}^k|_2^2\big)\leq\eta\int_0^t s^2|\nabla\psi_{ss}^k|_2^2+C',$$

for any $k\geq 1$ and a.e. $t\in[0,T_*]$. Hereinafter $\eta>0$ stands for some constant that is sufficiently small and may differ from line to line.

Proof. We divide the proof into two steps.

Step 1. *Estimates of* u^k. Differentiating $(3.64)_3$ with respect to t twice, multiplying by u^k_{tt} and integrating over Ω, one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho^k|u^k_{tt}|^2+\int \big(\alpha|\nabla u^k_{tt}|^2+(\alpha+\beta)|\mathrm{div}u^k_{tt}|^2\big)\\
=&-2\int \rho^k_t|u^k_{tt}|^2-\int \rho^k_{tt}u^k_t\cdot u^k_{tt}+\int P^k_{tt}\mathrm{div}u^k_{tt}\\
&-\int \big(\rho^k_{tt}u^{k-1}\cdot\nabla u^k+2\rho^k_t u^{k-1}_t\cdot\nabla u^k+2\rho^k_t u^{k-1}\cdot\nabla u^k_t\big)\cdot u^k_{tt}\\
&-\int \rho^k\big(2u^{k-1}_t\cdot\nabla u^k_t+u^{k-1}_{tt}\cdot\nabla u^k\big)\cdot u^k_{tt} \;=:\sum_{i=1}^{5}\mathrm{II}_i. \qquad (4.40)
\end{aligned}$$

Now, we estimate the terms on the right-hand side of (4.40) one by one. First, we have

$$\begin{aligned}
\mathrm{II}_1=&-2\int \rho^k_t|u^k_{tt}|^2=-4\int \rho^k u^{k-1}\cdot\nabla u^k_{tt}\cdot u^k_{tt}\leq \frac{\alpha}{16}|\nabla u^k_{tt}|_2^2+C'|\sqrt{\rho^k}u^k_{tt}|_2^2,\\
\mathrm{II}_2=&-\int \rho^k_{tt}u^k_t\cdot u^k_{tt}=-\int \big(\rho^k u^{k-1}\big)_t\cdot\nabla\big(u^k_t\cdot u^k_{tt}\big)\\
\leq&\Big(|\rho^k_t|_3|u^{k-1}|_\infty\big(|\nabla u^k_t|_2|u^k_{tt}|_6+|u^k_t|_6|\nabla u^k_{tt}|_2\big)\\
&+|\sqrt{\rho^k}u^k_{tt}|_2|\sqrt{\rho^k}_\infty|\nabla u^k_t|_3|u^{k-1}_t|_6+|\rho^k u^k_t|_3|\nabla u^k_{tt}|_2|u^{k-1}_t|_6\Big)\\
\leq&\frac{\alpha}{16}|\nabla u^k_{tt}|_2^2+C'\big(|\nabla^2 u^k_t|_2+|\sqrt{\rho^k}u^k_{tt}|_2^2+1\big), \qquad (4.41)
\end{aligned}$$

where we have used $|\nabla u^k_t|_3\leq|\nabla u^k_t|_2^{\frac{1}{2}}\|\nabla u^k_t\|_1^{\frac{1}{2}}$ and Lemmas 4.3-4.4.

Noticing that

$$|P^k_{tt}|_2^2=\big|\big(R\rho^k(\psi^k)^{\frac{1}{\delta+1}}\big)_{tt}\big|_2^2\leq C'\big(|\rho^k_{tt}|_2^2+|\sqrt{\rho^k}\psi^k_{tt}|_2^2+|\nabla\psi^k_t|_2^3+1\big), \qquad (4.42)$$

we have

$$\begin{aligned}
\mathrm{II}_3=\int P^k_{tt}\mathrm{div}u^k_{tt}\leq&\frac{\alpha}{16}|\nabla u^k_{tt}|_2^2+C'|P^k_{tt}|_2^2\\
\leq&\frac{\alpha}{16}|\nabla u^k_{tt}|_2^2+C'\big(|\rho^k_{tt}|_2^2+|\sqrt{\rho^k}\psi^k_{tt}|_2^2+|\nabla\psi^k_t|_2^3+1\big). \qquad (4.43)
\end{aligned}$$

For the rest terms, we have

$$\begin{aligned}
\mathrm{II}_4=&-\int \big(\rho^k_{tt}u^{k-1}\cdot\nabla u^k+2\rho^k_t u^{k-1}_t\cdot\nabla u^k+2\rho^k_t u^{k-1}\cdot\nabla u^k_t\big)\cdot u^k_{tt}\\
\leq&\big(|\rho^k_{tt}|_2|u^{k-1}|_\infty|\nabla u^k|_3+2|\rho^k_t|_2|u^{k-1}_t|_6|\nabla u^k|_6+2|\rho^k_t|_3|u^{k-1}|_\infty|\nabla u^k_t|_2\big)|u^k_{tt}|_6\\
\leq&\frac{\alpha}{16}|\nabla u^k_{tt}|_2^2+C'\big(|\rho^k_{tt}|_2^2+1\big),\\
\mathrm{II}_5=&-\int \rho^k\big(2u^{k-1}_t\cdot\nabla u^k_t+u^{k-1}_{tt}\cdot\nabla u^k\big)\cdot u^k_{tt}\\
\leq&\big(2|u^{k-1}_t|_6|\nabla u^k_t|_3+|u^{k-1}_{tt}|_6|\nabla u^k|_3\big)|\sqrt{\rho^k}u^k_{tt}|_2|\sqrt{\rho^k}|_\infty\\
\leq&\frac{\alpha}{16}|\nabla u^{k-1}_{tt}|_2^2+C'\big(|\nabla^2 u^k_t|_2+|\sqrt{\rho^k}u^k_{tt}|_2^2\big). \qquad (4.44)
\end{aligned}$$

Substituting (4.41)-(4.44) back to (4.40), it arrives at

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\sqrt{\rho^k}u_{tt}^k|_2^2+\frac{3\alpha}{4}|\nabla u_{tt}^k|_2^2\\ \leq&\frac{\alpha}{16}|\nabla u_{tt}^{k-1}|_2^2+C'\big(|\rho_{tt}^k|_2^2+|\nabla\psi_t^k|_2^3+|\nabla^2u_t^k|_2+|\sqrt{\rho^k}\psi_{tt}^k|_2^2+|\sqrt{\rho^k}u_{tt}^k|_2^2+1\big).\end{aligned}\tag{4.45}$$

Multiplying (4.45) with t and integrating over $[0,t]$, for any given $N\in\mathbb{Z}_+$, one has

$$\begin{aligned}&\max_{0\leq k\leq N}\Big(t|\sqrt{\rho^k}u_{tt}^k|_2^2+\int_0^t s|\nabla u_{ss}^k|_2^2\Big)\\ \leq&\max_{0\leq k\leq N}C'\int_0^t s\big(|\sqrt{\rho^k}\psi_{ss}^k|_2^2+|\nabla\psi_s^k|_2^3\big)+C'.\end{aligned}\tag{4.46}$$

It remains to control the remaining terms on the right-hand side of (4.46), which are generated from the nonlinear term $P^k=R\rho^k(\psi^k)^{\frac{1}{b+1}}$. We mention that the second term on the right-hand side of (4.46) will not appear for the case of constant heat conductivity. Now, we turn to the estimates of ψ^k.

Step 2. *Estimates of* ψ^k. Differentiating $(3.64)_2$ with respect to t, multiplying by ψ_{tt}^k and integrating over Ω, one has

$$\begin{aligned}&\frac{a_2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\psi^{k-1})^{\frac{b}{b+1}}|\nabla\psi_t^k|^2+\int\rho^k|\psi_{tt}^k|^2\\ =&\frac{a_2}{2}\int\Big((\psi^{k-1})^{\frac{b}{b+1}}\Big)_t|\nabla\psi_t^k|^2-\frac{ba_2}{b+1}\int(\psi^{k-1})^{\frac{-1}{b+1}}\big(\nabla\psi^{k-1}\cdot\nabla\psi_t^k-\psi_t^{k-1}\Delta\psi^k\big)\psi_{tt}^k\\ &+a_3\int\Big((\psi^{k-1})^{\frac{b}{b+1}}\Big)_tQ(u^{k-1})\psi_{tt}^k+a_3\int(\psi^{k-1})^{\frac{b}{b+1}}Q(u^{k-1})_t\psi_{tt}^k-\int\rho_t^k\psi_t^k\psi_{tt}^k\\ &-\int\big(\rho^ku^{k-1}\cdot\nabla\psi^k+a_1\rho^k\psi^k\mathrm{div}u^{k-1}\big)_t\psi_{tt}^k\ =:\sum_{i=6}^{11}\mathrm{II}_i.\end{aligned}\tag{4.47}$$

For the additional terms II_6-II_9 compared with the case of constant heat conductivity, it follows from Lemmas 4.3-4.4 and the Cauchy inequality that

$$\begin{aligned}\mathrm{II}_6=&\frac{a_2}{2}\int\Big((\psi^{k-1})^{\frac{b}{b+1}}\Big)_t|\nabla\psi_t^k|^2\leq C'|\psi_t^{k-1}|_3|\nabla\psi_t^k|_2|\nabla\psi_t^k|_6\\ \leq&C'|\psi_t^{k-1}|_3|\nabla\psi_t^k|_2\big(|\nabla\psi_t^k|_2+|\nabla^2\psi_t^k|_2\big)\\ \leq&C'|\psi_t^{k-1}|_3|\nabla\psi_t^k|_2\big(|\sqrt{\rho^k}\psi_{tt}^k|_2+|\nabla\psi_t^k|_2+|\nabla\psi_t^{k-1}|_2+|\psi_t^{k-1}|_6|\nabla^2\psi^k|_3+1\big)\\ \leq&\frac{1}{8}|\sqrt{\rho^k}\psi_{tt}^k|_2^2+C'\big(|\nabla\psi_t^k|_2^2+1\big)\big(|\nabla\psi_t^{k-1}|_2^2+1\big),\end{aligned}\tag{4.48}$$

where we have used $|\nabla\psi_t^{k-1}|_3\leq|\nabla\psi_t^{k-1}|_2^{\frac{1}{2}}\|\nabla\psi_t^{k-1}\|_1^{\frac{1}{2}}$ and the following estimate

$$\begin{aligned}|\nabla^2\psi_t^k|_2\leq&C'\big|\big(\rho^k\psi_t^k+\rho^ku^{k-1}\cdot\nabla\psi^k+a_1\rho^k\psi^k\mathrm{div}u^{k-1}-a_3(\psi^{k-1})^{\frac{b}{b+1}}Q(u^{k-1})\big)_t\big|_2\\ &+C'\big|(\psi^{k-1})_t^{\frac{b}{b+1}}\Delta\psi^k\big|_2\\ \leq&C'\big(|\sqrt{\rho^k}\psi_{tt}^k|_2+|\nabla\psi_t^k|_2+|\nabla\psi_t^{k-1}|_2+|\psi_t^{k-1}|_6|\nabla^2\psi^k|_3+1\big).\end{aligned}\tag{4.49}$$

For II_7-II_9, we have

$$
\begin{aligned}
\mathrm{II}_7 =& -\frac{ba_2}{b+1}\int (\psi^{k-1})^{\frac{-1}{b+1}}\big(\nabla\psi^{k-1}\cdot\nabla\psi^k_t-\psi^{k-1}_t\Delta\psi^k\big)\psi^k_{tt}\\
\le& C'\int\big(|\nabla\psi^{k-1}\cdot\nabla\psi^k_t\psi^k_{tt}|+|\psi^{k-1}_t\Delta\psi^k\psi^k_{tt}|\big)\\
\le& C'\big(|\nabla\psi^{k-1}|_3|\nabla\psi^k_t|_2+|\psi^{k-1}_t|_3|\nabla^2\psi^k|_2\big)|\psi^k_{tt}|_6\\
\le& C'\big(|\nabla\psi^k_t|_2+|\nabla\psi^{k-1}_t|_2+1\big)|\psi^k_{tt}|_6,\\
\mathrm{II}_8+\mathrm{II}_9 =& a_3\int\Big(\big((\psi^{k-1})^{\frac{b}{b+1}}\big)_tQ(u^{k-1})+(\psi^{k-1})^{\frac{b}{b+1}}Q(u^{k-1})_t\Big)\psi^k_{tt}\\
\le& C'\int\Big(|\psi^{k-1}_t||\nabla u^{k-1}|+(\psi^{k-1})^{\frac{b}{b+1}}|\nabla u^{k-1}_t|\Big)|\nabla u^{k-1}||\psi^k_{tt}|\\
\le& C'\big(|\psi^{k-1}_t|_6|\nabla u^{k-1}|_2+|\psi^{k-1}|_6|\nabla u^{k-1}_t|_2\big)|\nabla u^{k-1}|_6|\psi^k_{tt}|_6\\
\le& C'\big(|\nabla\psi^{k-1}_t|_2+1\big)|\psi^k_{tt}|_6.
\end{aligned}\tag{4.50}
$$

For the rest terms, we have

$$
\begin{aligned}
\mathrm{II}_{10} =& -\int\rho^k_t\psi^k_t\psi^k_{tt}=\int\big(\nabla\rho^k\cdot u^{k-1}+\rho^k\mathrm{div}u^{k-1}\big)\psi^k_t\psi^k_{tt}\\
\le& \big(|\nabla\rho^k|_2|u^{k-1}|_6|\psi^k_{tt}|_6+|\sqrt{\rho^k}|_\infty|\sqrt{\rho^k}\psi^k_{tt}|_2|\nabla u^{k-1}|_3\big)|\psi^k_t|_6\\
\le& C'\big(|\nabla\psi^k_t|_2+1\big)\big(|\psi^k_{tt}|_6+|\sqrt{\rho^k}\psi^k_{tt}|_2\big),\\
\mathrm{II}_{11} =& -\int\big(\rho^ku^{k-1}\cdot\nabla\psi^k+a_1\rho^k\psi^k\mathrm{div}u^{k-1}\big)_t\psi^k_{tt}\\
\le& C\Big(|\rho^k_t|_2\big(|u^{k-1}|_6|\nabla\psi^k|_6+|\psi^k|_6|\nabla u^{k-1}|_6\big)|\psi^k_{tt}|_6+\big(|u^{k-1}_t|_6|\nabla\psi^k|_3\\
&+|u^{k-1}|_\infty|\nabla\psi^k_t|_2+|\psi^k|_\infty|\nabla u^{k-1}_t|_2+|\psi^k_t|_6|\nabla u^{k-1}|_3\big)|\sqrt{\rho^k}|_\infty|\sqrt{\rho^k}\psi^k_{tt}|_2\Big)\\
\le& C'\Big(|\psi^k_{tt}|_6+\big(|\nabla\psi^k_t|_2+1\big)|\sqrt{\rho^k}\psi^k_{tt}|_2\Big).
\end{aligned}\tag{4.51}
$$

By Lemma 2.2 and Lemma 4.3, one has

$$
|\psi^k_{tt}|_6\le C'\big(|\nabla\psi^k_{tt}|_2+|\sqrt{\rho^k}\psi^k_{tt}|_2\big).\tag{4.52}
$$

Substituting (4.48)-(4.52) back to (4.47), together with the Young's inequality, one has

$$
\begin{aligned}
&\frac{a_2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\psi^{k-1})^{\frac{b}{b+1}}|\nabla\psi^k_t|^2+\frac12\int\rho^k|\psi^k_{tt}|^2\\
\le& C'\Big(|\nabla\psi^k_{tt}|_2\big(|\nabla\psi^k_t|_2+|\nabla\psi^{k-1}_t|_2+1\big)+\big(|\nabla\psi^{k-1}_t|_2^2+1\big)\big(|\nabla\psi^k_t|_2^2+1\big)\Big).
\end{aligned}\tag{4.53}
$$

We notice that, unlike the estimates of $|\nabla u^k_t|_2$ in Lemma 4.4, due to extra term II_7 that comes from the nonlinear elliptic term $(\psi^{k-1})^{\frac{b}{b+1}}\Delta\psi^k$, the appearance of $|\nabla\psi^k_{tt}|_2$ is inevitable. In addition, $|\nabla\psi^k_{tt}|_2$ is two order higher than the term $|\nabla\psi^k_t|_2$ on the left-hand side of (4.47). In order to close the whole estimates in future development, our main ingredient is to refine the estimates, provide the smallness to the coefficient of $|\nabla\psi^k_{tt}|_2$ and use the time wighted estimates.

Now, multiplying (4.53) by t and integrating over $[0,t]$, combining with Young's inequality and the Gronwall's inequality, it arrives that

$$t|\nabla\psi_t^k|_2^2+\int_0^t s|\sqrt{\rho^k}\psi_{ss}^k|_2^2\le\eta\int_0^t s^2|\nabla\psi_{ss}^k|_2^2+C',\tag{4.54}$$

where $\eta>0$ is a sufficiently small constant, and here C' depends on η, but we still denote as C' without confusion.

Multiplying (4.54) by $2C'$ and combining (4.46), it arrives at

$$\begin{aligned}&t\big(|\nabla\psi_t^k|_2^2+|\sqrt{\rho^k}u_{tt}^k|_2^2\big)+\int_0^t s\big(|\sqrt{\rho^k}\psi_{ss}^k|_2^2+|\nabla u_{ss}^k|_2^2\big)\\ \le{}& C'\int_0^t s|\nabla\psi_s^k|_2^3+\eta\int_0^t s^2|\nabla\psi_{ss}^k|_2^2+C'\le C't|\nabla\psi_t^k|_2+\eta\int_0^t s^2|\nabla\psi_{ss}^k|_2^2+C'\\ \le{}&\frac{1}{4}t|\nabla\psi_t^k|_2^2+\eta\int_0^t s^2|\nabla\psi_{ss}^k|_2^2+C',\end{aligned}\tag{4.55}$$

thus we immediately have

$$t\big(|\nabla\psi_t^k|_2^2+|\sqrt{\rho^k}u_{tt}^k|_2^2\big)+\int_0^t s\big(|\sqrt{\rho^k}\psi_{ss}^k|_2^2+|\nabla u_{ss}^k|_2^2\big)\le\eta\int_0^t s^2|\nabla\psi_{ss}^k|_2^2+C',$$

where we denote $\eta C'$ as η for simplicity. This completes the proof of this lemma. □

Based on Lemma 4.5, we also have the following estimates.

LEMMA 4.6. *Under the assumptions of Theorem 4.1, it holds that*

$$t\big(|\nabla^2u_t^k|_2^2+|\nabla^4u^k|_2+|\nabla^3\psi^k|_2^2\big)+\int_0^t s\big(|\nabla^2\psi_s^k|_2^2+|\nabla^4\psi^k|_2^2\big)\le\eta\int_0^t s^2|\nabla\psi_{ss}^k|_2^2+C',$$

for any $k\ge1$ and a.e. $t\in[0,T_*]$.

Proof. First, according to (4.32) and Lemma 4.4, one has

$$|\nabla^2u_t^k|_2\le C'\big(|\sqrt{\rho^k}u_{tt}^k|_2+|\nabla\psi_t^k|_2+1\big),$$

which together with Lemma 4.5 implies

$$t|\nabla^2u_t^k|_2^2\le\eta\int_0^t s^2|\nabla\psi_{ss}^k|_2^2+C'.$$

Second, with the help of (4.35), (4.49) and (4.36), one has

$$t|\nabla^4u^k|_2^2+t|\nabla^3\psi^k|_2^2\le\eta\int_0^t s^2|\nabla\psi_{ss}^k|_2^2+C',$$

and

$$\int_0^t s|\nabla^2\psi_s^k|_2^2\le\eta\int_0^t s^2|\nabla\psi_{ss}^k|_2^2+C'.\tag{4.56}$$

At last, by $(3.64)_2$ and Lemma 2.3, we get

$$\begin{aligned}|\nabla^4\psi^k|_2\le C'\Big(&\big|\nabla^2\big(\rho^k\psi_t^k+\rho^ku^{k-1}\cdot\nabla\psi^k+a_1\rho^k\psi^k\mathrm{div}u^{k-1}-a_3(\psi^{k-1})^{\frac{b}{b+1}}Q(u^{k-1})\big)\big|_2\\ &+\big|\nabla^2\big((\psi^{k-1})^{\frac{b}{b+1}}\big)\Delta\psi^k\big|_2+\big|\,|\nabla^3\psi^k|\,|\nabla\psi^{k-1}|\,\big|_2\Big),\end{aligned}$$

which implies

$$|\nabla^4\psi^k|_2 \leq C'\big(|\nabla^2\psi^k_t|_2+|\nabla\psi^k_t|_2+|\nabla^3\psi^k|_2+|\nabla^3\psi^{k-1}|_2+1\big), \tag{4.57}$$

where we have used the Cauchy inequality, Sobolev inequality and Lemmas 4.3-4.5. This together with (4.56) implies

$$\int_0^t s|\nabla^4\psi^k|_2^2 \leq \eta\int_0^t s^2|\nabla\psi^k_{ss}|_2^2+C'.$$

The proof of this lemma is completed. □

Now, we are ready to show $t\psi^k_{tt}\in L^2([0,T_*];D^1)$ in the following lemma, which plays an important role in closing the estimates in Lemmas 4.5-4.6.

Lemma 4.7. *Under the assumptions of Theorem 4.1, it holds that*

$$t^2|\sqrt{\rho^k}\psi^k_{tt}|_2^2+\int_0^t s^2|\nabla\psi^k_{ss}|_2^2 \leq C',$$

for any $k\geq 1$ and a.e. $t\in[0,T_*]$.

Proof. First, differentiating $(3.64)_2$ with respect to t twice, multiplying by ψ^k_{tt} and integrating over Ω, one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho^k(\psi^k_{tt})^2+a_2\int(\psi^{k-1})^{\frac{b}{b+1}}(\nabla\psi^k_{tt})^2\\
=&-a_2\int\nabla\psi^k_{tt}\cdot\nabla\big(\psi^{k-1}\big)^{\frac{b}{b+1}}\psi^k_{tt}+\int\Big(\big(\psi^{k-1}\big)^{\frac{b}{b+1}}\Big)_{tt}\big(a_2\Delta\psi^k+a_3Q(u^{k-1})\big)\psi^k_{tt}\\
&+2\int\Big(\big(\psi^{k-1}\big)^{\frac{b}{b+1}}\Big)_t\big(a_2\Delta\psi^k_t+a_3Q(u^{k-1})_t\big)\psi^k_{tt}\\
&+a_3\int(\psi^{k-1})^{\frac{b}{b+1}}Q(u^{k-1})_{tt}\psi^k_{tt}-\int\Big(\rho^k_{tt}\psi^k_t+\frac{3}{2}\rho^k_t\psi^k_{tt}\Big)\psi^k_{tt}\\
&-\int(\rho^k u^{k-1}\cdot\nabla\psi^k+a_1\rho^k\psi^k\mathrm{div}u^{k-1})_{tt}\psi^k_{tt}=:\sum_{i=1}^{6}\mathrm{III}_i.
\end{aligned}\tag{4.58}$$

Then, by using Lemmas 4.3-4.6, we first deal with the extra nonlinear terms as follows.

$$\begin{aligned}
\mathrm{III}_1=&-a_2\int\nabla\psi^k_{tt}\cdot\nabla\big(\psi^{k-1}\big)^{\frac{b}{b+1}}\psi^k_{tt}\leq C'\int|\nabla\psi^k_{tt}||\nabla\psi^{k-1}||\psi^k_{tt}|\\
=&C'\Big(\int_{V_{R_0}}+\int_{\Omega\setminus V_{R_0}}\Big)|\nabla\psi^k_{tt}||\nabla\psi^{k-1}||\psi^k_{tt}|\\
\leq& C'\big(|V_{R_0}|^{\frac{1}{6}}|\psi^k_{tt}|_6+|\sqrt{\rho^k}\psi^k_{tt}|_3\big)|\nabla\psi^k_{tt}|_2|\nabla\psi^{k-1}|_6\\
\leq&\Big(C'|V_{R_0}|^{\frac{1}{6}}+\frac{a_4}{16}\Big)|\nabla\psi^k_{tt}|_2^2+C'|\sqrt{\rho^k}\psi^k_{tt}|_2^2,
\end{aligned}\tag{4.59}$$

$$
\begin{aligned}
\mathrm{III}_2 =& \int \Big(\big(\psi^{k-1}\big)^{\frac{b}{b+1}} \Big)_{tt} \big(a_2 \Delta \psi^k + a_3 Q(u^{k-1})\big)\psi^k_{tt} \\
\leq & C'\Big(\int_{V_{R_0}} + \int_{\Omega \setminus V_{R_0}} \Big) \big(|\nabla^2 \psi^k| + |\nabla u^{k-1}|^2\big)|\psi^{k-1}_{tt}||\psi^k_{tt}| \\
& + C' \int \big(|\nabla^2 \psi^k| + |\nabla u^{k-1}|^2\big)|\psi^{k-1}_t|^2 |\psi^k_{tt}| \\
\leq & C'\big(|\nabla^2 \psi^k|_2 + |\nabla u^{k-1}|^2_4\big)\Big(\big(|V_{R_0}|^{\frac{1}{6}}|\psi^k_{tt}|_6 + |\sqrt{\rho^k}\psi^k_{tt}|_3\big)|\psi^{k-1}_{tt}|_6 + |\psi^{k-1}_t|^2_6 |\psi^k_{tt}|_6\Big) \\
\leq & \frac{a_4}{16}|\nabla \psi^k_{tt}|^2_2 + C'\big(|V_{R_0}|^{\frac{1}{3}}|\nabla \psi^{k-1}_{tt}|^2_2 + |\sqrt{\rho^k}\psi^k_{tt}|^2_2 + |\sqrt{\rho^{k-1}}\psi^{k-1}_{tt}|^2_2 + |\nabla \psi^{k-1}_t|^4_2\big), \\
\mathrm{III}_3 =& 2\int \Big(\big(\psi^{k-1}\big)^{\frac{b}{b+1}} \Big)_{t} \big(a_2 \Delta \psi^k_t + a_3 Q(u^{k-1})_t\big)\psi^k_{tt} \\
\leq & C' \int \big(|\psi^{k-1}_t \Delta \psi^k_t| + |\psi^{k-1}_t||\nabla u^{k-1}_t||\nabla u^{k-1}|\big)|\psi^k_{tt}| \\
\leq & C'\big(|\psi^{k-1}_t \Delta \psi^k_t|_{\frac{6}{5}} + |\nabla u^{k-1}_t|_2 |\nabla u^{k-1}|_6 \psi^{k-1}_t|_6\big)|\psi^k_{tt}|_6 \\
\leq & \frac{a_4}{16}|\nabla \psi^k_{tt}|^2_2 + C'\big((|\nabla \psi^{k-1}_t|^2_2 + 1)|\sqrt{\rho^k}\psi^k_{tt}|^2_2 + |\nabla \psi^k_t|^4_2 + |\nabla \psi^{k-1}_t|^4_2 + 1\big), \\
\mathrm{III}_4 =& a_3 \int (\psi^{k-1})^{\frac{b}{b+1}} Q(u^{k-1})_{tt}\psi^k_{tt} \\
\leq & a_3 |\psi^{k-1}|^{\frac{b}{b+1}}_\infty \big(|\nabla u^{k-1}_{tt}|_2 |\nabla u^{k-1}|_3 + |\nabla u^{k-1}_t|_3 |\nabla u^{k-1}_t|_2\big)|\psi^k_{tt}|_6 \\
\leq & \frac{a_4}{16}|\nabla \psi^k_{tt}|^2_2 + C'\big(|\nabla u^{k-1}_{tt}|^2_2 + |\nabla^2 u^{k-1}_t|_2 + |\sqrt{\rho^k}\psi^k_{tt}|^2_2 + 1\big),
\end{aligned} \tag{4.60}
$$

where, in the estimate of III_3, we have used

$$
\begin{aligned}
|\psi^{k-1}_t \Delta \psi^k_t|_{\frac{6}{5}} \leq & C'\Big(\big|\psi^{k-1}_t\big(\rho^k \psi^k_t + \rho^k u^{k-1}\cdot \nabla \psi^k + a_1 \rho^k \psi^k \mathrm{div} u^{k-1} \\
& - a_3 (\psi^{k-1})^{\frac{b}{b+1}} Q(u^{k-1})\big)_t\big|_{\frac{6}{5}} + \big|\psi^{k-1}_t (\psi^{k-1})^{\frac{b}{b+1}}_t \Delta \psi^k\big|_{\frac{6}{5}}\Big) \\
\leq & C'\Big(|\psi^{k-1}_t|_3 \big(|\sqrt{\rho^k}\psi^k_{tt}|_2 + |\nabla \psi^k_t|_2 + |\nabla \psi^{k-1}_t|_2\big) + |\psi^{k-1}_t|^2_6 |\nabla^2 \psi^k|_2 + 1\Big). \\
\leq & C'\big(|\psi^{k-1}_t|_3 |\sqrt{\rho^k}\psi^k_{tt}|_2 + |\nabla \psi^k_t|^2_2 + |\nabla \psi^{k-1}_t|^2_2 + 1\big).
\end{aligned}
$$

For the rest terms on the right-hand side of (4.58), we control them as follows.

$$
\begin{aligned}
\mathrm{III}_5 =& -\int \Big(\rho^k_{tt}\psi^k_t \psi^k_{tt} + \frac{3}{2}\rho^k_t (\psi^k_{tt})^2\Big) \\
=& -\int \Big((\rho^k_t u^{k-1} + \rho^k u^{k-1}_t)\cdot (\nabla \psi^k_t \psi^k_{tt} + \psi^k_t \nabla \psi^k_{tt}) + 3\rho^k u^{k-1}\cdot \nabla \psi^k_{tt}\psi^k_{tt}\Big) \\
\leq & \Big(|u^{k-1}|_\infty \big(|\rho^k_t|_3 |\nabla \psi^k_t|_2 |\psi^k_{tt}|_6 + (|\rho^k_t|_3 |\psi^k_t|_6 + 3|\sqrt{\rho^k}|_\infty \sqrt{\rho^k}\psi^k_{tt}|_2)|\nabla \psi^k_{tt}|_2\big) \\
& + |u^{k-1}_t|_6 \big(|\sqrt{\rho^k}|_\infty |\nabla \psi^k_t|_3 |\sqrt{\rho^k}\psi^k_{tt}|_2 + |\rho^k \psi^k_t|_3 |\nabla \psi^k_{tt}|_2\big)\Big) \\
\leq & \frac{a_4}{16}|\nabla \psi^k_{tt}|^2_2 + C'\big(|\nabla \psi^k_t|^2_2 + |\nabla^2 \psi^k_t|^2_2 + |\sqrt{\rho^k}\psi^k_{tt}|^2_2 + 1\big),
\end{aligned}
$$

$$\begin{aligned}
\mathrm{III}_6 = & -\int (\rho^k u^{k-1}\cdot\nabla\psi^k + a_1\rho^k\psi^k \mathrm{div} u^{k-1})_{tt}\psi^k_{tt} \\
\le & C\Big(|\rho^k_{tt}|_2\big(|u^{k-1}|_\infty|\nabla\psi^k|_3 + |\psi^k|_\infty|\nabla u^{k-1}|_3\big) + |\rho^k_t|_3\big(|u^{k-1}_t|_6|\nabla\psi^k|_3 \\
& + |\psi^k_t|_6|\nabla u^{k-1}|_3 + |u^{k-1}|_\infty|\nabla\psi^k_t|_2 + |\psi^k|_\infty|\nabla u^{k-1}_t|_2\big)\Big)|\psi^k_{tt}|_6 \\
& + C\Big(|u^{k-1}_t|_6|\nabla\psi^k_t|_3 + |u^{k-1}_{tt}|_6|\nabla\psi^k|_3 + |\psi^k_{tt}|_6|\nabla u^{k-1}|_3 + |\nabla\psi^k_{tt}|_2|u^{k-1}|_\infty \\
& + |\nabla u^{k-1}_{tt}|_2|\psi^k|_\infty + |\nabla u^{k-1}_t|_3|\psi^k_t|_6\Big)|\sqrt{\rho^k}\psi^k_{tt}|_2|\sqrt{\rho^k}|_\infty \\
\le & \frac{a_4}{16}|\nabla\psi^k_{tt}|_2^2 + C'\big(|\nabla\psi^k_t|_2^2 + |\sqrt{\rho^k}\psi^k_{tt}|_2^2 + |\nabla^2\psi^k_t|_2^2 + |\nabla u^{k-1}_{tt}|_2^2 \\
& + |\nabla^2 u^{k-1}_t|_2(|\nabla\psi^k_t|_2^2+1)+1\big).
\end{aligned} \tag{4.61}$$

Putting (4.59)-(4.61) and (4.49) into (4.58), one has

$$\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\sqrt{\rho^k}\psi^k_{tt}|_2^2 + \frac{5a_4}{8}|\nabla\psi^k_{tt}|_2^2 \\
\le & C'\big(|V_{R_0}|^{\frac{1}{6}}|\nabla\psi^k_{tt}|_2^2 + |V_{R_0}|^{\frac{1}{3}}|\nabla\psi^{k-1}_{tt}|_2^2\big) + C'\Big(|\nabla^2 u^{k-1}_t|_2^2 + |\nabla u^{k-1}_{tt}|_2^2 + |\sqrt{\rho^k}\psi^k_{tt}|_2^2 \\
& + |\sqrt{\rho^{k-1}}\psi^{k-1}_{tt}|_2^2(|\nabla\psi^k_t|_2^2+1) + |\nabla\psi^k_t|_2^4 + |\nabla\psi^{k-1}_t|_2^4 + 1\Big),
\end{aligned} \tag{4.62}$$

taking $|V_{R_0}| \le \min\big\{\big(a_4/32C'\big)^3, \big(a_4/32C'\big)^6\big\}$ in (4.62), multiplying by t^2 and integrating over $[0,t]$, by Lemmas 4.3-4.6 and the Gronwal's inequality, for

$$\eta \le a_4/32C', \tag{4.63}$$

and any given $N \in \mathbb{Z}_+$, one has

$$\max_{1\le k\le N} t^2|\sqrt{\rho^k}\psi^k_{tt}|_2^2 + \max_{1\le k\le N}\int_0^t s^2|\nabla\psi^k_{ss}|_2^2 \le C'. \tag{4.64}$$

We complete the proof of this lemma. □

With the help of Lemma 4.7, we can close the estimates in Lemmas 4.5-4.6:

$$\begin{aligned}
t\big(|\nabla\psi^k_t|_2^2 + |\sqrt{\rho^k}u^k_{tt}|_2^2\big) + \int_0^t s\big(|\sqrt{\rho^k}\psi^k_{ss}|_2^2 + |\nabla u^k_{ss}|_2^2\big) \le C', \\
t\big(|\nabla^2 u^k_t|_2^2 + |\nabla^3\psi^k|_2^2\big) + \int_0^t s\big(|\nabla^2\psi^k_s|_2^2 + |\nabla^4\psi^k|_2^2\big) \le C'.
\end{aligned} \tag{4.65}$$

Moreover, we immediately have the following lemma, which helps us in proving the continuity of ∇u.

LEMMA 4.8. *Under the assumptions of Theorem 4.1, it holds that*

$$t|\nabla\rho^k_{tt}|_2^2 + t^{\frac{3}{4}}|\nabla^3 u^k_t|_2^2 + t^2\big(|\nabla^2\psi^k_t|_2^2 + |\nabla^4\psi^k|_2^2\big) + \int_0^t s\big(|\rho^k_{sss}|_2^2 + |\nabla^3 u^k_s|_2^2\big) \le C',$$

for any $k\ge 1$ and a.e. $t\in[0,T_*]$.

Proof. First, (4.39) and (4.65) imply

$$t|\nabla\rho^k_{tt}|_2^2 \le C'.$$

Second, it follows from $(3.64)_1$ that

$$|\rho^k_{ttt}|_2 = \big|\text{div}\big(\rho^k u^{k-1}\big)_{tt}\big|_2 \leq C'\big(|\nabla \rho^k_{tt}|_2 + |\nabla u^{k-1}_{tt}|_2 + 1\big),$$

where we have used Lemmas 4.4-4.5, and we immediately have

$$\int_0^t s|\rho^k_{sss}|_2^2 \leq C'.$$

Third, according to $(3.64)_3$ and Lemma 2.3, one has

$$\begin{aligned}|\nabla^3 u^k_t|_2 \leq & C'\big|\nabla\big(\rho^k u^k_t + \rho^k u^{k-1}\cdot\nabla u^k + \nabla P^k\big)_t\big|_2\\ \leq & C'\big(|\nabla u^k_{tt}|_2 + |\nabla^2 u^k_t|_2 + |\nabla^2\psi^k_t|_2 + |\nabla \psi^k_t|_2 + |\nabla\psi^k_t|_2^{\frac{3}{2}} + 1\big),\end{aligned} \tag{4.66}$$

where we have used

$$|\nabla^2 P^k_t|_2 = \big|\nabla^2\big(R\rho^k(\psi^k)^{\frac{1}{\delta+1}}\big)_t\big|_2 \leq C'\big(|\nabla^2\psi^k_t|_2 + |\nabla\psi^k_t|_2 + |\nabla\psi^k_t|_2^{\frac{3}{2}} + 1\big). \tag{4.67}$$

Then (4.66) implies

$$\int_0^t s|\nabla^3 u^k_s|_2^2 \leq C'.$$

At last, combining (4.49), (4.57) with Lemma 4.7 and (4.65), implies

$$t^2\big(|\nabla^2\psi^k_t|_2^2 + |\nabla^4\psi^k|_2^2\big) \leq C'.$$

Thus we completes the proof of this lemma. □

In order to deduce $\big(tu^k_{tt}, t^{\frac{3}{2}}\psi^k_{tt}\big) \in L^\infty([0,T_*];D^1)$, which will used to prove the continuity of u_t and ψ_t, our very first step is to prove the following lemma.

LEMMA 4.9. *Under the assumptions of Theorem 4.1, it holds that*

$$t^2|\nabla u^k_{tt}|_2^2 + t^3|\nabla\psi^k_{tt}|_2^2 + \int_0^t \big(s^2|\sqrt{\rho^k}u^k_{sss}|_2^2 + s^3|\sqrt{\rho^k}\psi^k_{sss}|_2^2\big) \leq \eta\int_0^t s^4|\nabla\psi^k_{sss}|_2^2 + C',$$

for any $k \geq 1$ and a.e. $t \in [0,T_*]$.

Proof. We divide the proof into two steps.

Step 1. *Estimates of $\|t\nabla u^k_{tt}\|_{L^\infty L^2}$.* Differentiating $(3.64)_3$ with respect to t twice, multiplying by u^k_{ttt} and integrating over Ω, one has

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\big(\alpha|\nabla u^k_{tt}|^2 + (\alpha+\beta)|\text{div}u^k_{tt}|^2\big) + \int \rho^k|u^k_{ttt}|^2\\ =&-\int \rho^k_{tt}u^k_t\cdot u^k_{ttt} - \int\big(\rho^k_{tt}u^{k-1}\cdot\nabla u^k + 2\rho^k_t u^k_{tt}\big)\cdot u^k_{ttt} - \int 2\rho^k_t(u^{k-1}\cdot\nabla u^k)_t\cdot u^k_{ttt}\\ &-\int\rho^k\big(u^{k-1}\cdot\nabla u^k\big)_{tt}\cdot u^k_{ttt} - \int\nabla P^k_{tt}\cdot u^k_{ttt} \ =: \sum_{i=1}^5 \text{IV}_i.\end{aligned} \tag{4.68}$$

We will deal with the terms on the right-hand side of (4.68) one by one. First

$$\text{IV}_1 = -\int\rho^k_{tt}u^k_t\cdot u^k_{ttt} = -\frac{\mathrm{d}}{\mathrm{d}t}\int\rho^k_{tt}u^k_t\cdot u^k_{tt} + \int\rho^k_{ttt}u^k_t\cdot u^k_{tt} + \int\rho^k_{tt}|u^k_{tt}|^2, \tag{4.69}$$

where the last two terms on the right-hand side of (4.69) can be treated as

$$\begin{aligned}
\int \rho^k_{ttt} u^k_t \cdot u^k_{tt} =& -\int \mathrm{div}\big(\rho^k u^{k-1}\big)_{tt} u^k_t \cdot u^k_{tt} \\
\leq& \int |(\rho^k u^{k-1})_{tt}|(|u^k_t||\nabla u^k_{tt}| + |\nabla u^k_t||u^k_{tt}|) \\
\leq& \Big(\big(|\rho^k|_6 |u^k_t|_6 |\nabla u^k_{tt}|_2 + |\sqrt{\rho^k} u^k_{tt}|_3 |\sqrt{\rho^k}{}_\infty |\nabla u^k_t|_2\big)|u^{k-1}_{tt}|_6 \\
&+ \big(|\rho^k_{tt}|_2 |u^{k-1}|_\infty + 2|\rho^k_t|_3 |u^{k-1}_t|_6\big)\big(|u^k_t|_\infty |\nabla u^k_{tt}|_2 + |\nabla u^k_t|_3 |u^k_{tt}|_6\big)\Big) \\
\leq& C'\big(|\nabla u^k_{tt}|^2_2 + |\nabla u^{k-1}_{tt}|^2_2 + |\nabla^2 u^k_t|^2_2 + |\sqrt{\rho^k} u^k_{tt}|^2_2 + 1\big), \\
\int \rho^k_{tt} |u^k_{tt}|^2 =& -\int \mathrm{div}(\rho^k u^{k-1})_t |u^k_{tt}|^2 \leq \int |\rho^k_t u^{k-1} + \rho^k u^{k-1}_t||\nabla u^k_{tt}||u^k_{tt}| \\
\leq& \big(|\rho^k_t|_3 |u^{k-1}|_\infty |u^k_{tt}|_6 + |\sqrt{\rho^k} u^k_{tt}|_3 |\sqrt{\rho^k}|_\infty |u^{k-1}_t|_6\big)|\nabla u^k_{tt}|_2 \\
\leq& C'\big(|\sqrt{\rho^k} u^k_{tt}|^2_2 + |\nabla u^k_{tt}|^2_2\big),
\end{aligned}$$

substituting the above estimates back to (4.69) implies

$$\begin{aligned}
\mathrm{IV}_1 \leq& -\frac{\mathrm{d}}{\mathrm{d}t} \int \rho^k_{tt} u^k_t \cdot u^k_{tt} \\
&+ C'\big(|\nabla u^k_{tt}|^2_2 + |\nabla u^{k-1}_{tt}|^2_2 + |\nabla^2 u^k_t|^2_2 + |\sqrt{\rho^k} u^k_{tt}|^2_2 + 1\big).
\end{aligned} \tag{4.70}$$

We mention that we avoid the appearance of the higher-order term $|\nabla u^k_{ttt}|_2$ in the estimate of IV_1 by using identical deformation in (4.69). Similarly, for the rest terms, we have

$$\begin{aligned}
\mathrm{IV}_2 =& -\int \big(\rho^k_{tt}(u^{k-1} \cdot \nabla u^k) + 2\rho^k_t u^k_{tt}\big) \cdot u^k_{ttt} \\
=& -\frac{\mathrm{d}}{\mathrm{d}t} \int \Big(\rho^k_{tt}\big(u^{k-1} \cdot \nabla u^k\big) \cdot u^k_{tt} + \rho^k_t |u^k_{tt}|^2\Big) + \int \rho^k_{tt} |u^k_{tt}|^2 \\
&+ \int \rho^k_{ttt}\big(u^{k-1} \cdot \nabla u^k\big) \cdot u^k_{tt} + \int \rho^k_{tt}\big(u^{k-1} \cdot \nabla u^k\big)_t \cdot u^k_{tt} \\
\leq& -\frac{\mathrm{d}}{\mathrm{d}t} \int \Big(\rho^k_{tt}\big(u^{k-1} \cdot \nabla u^k\big) \cdot u^k_{tt} + \rho^k_t |u^k_{tt}|^2\Big) \\
&+ C'\big(|\sqrt{\rho^k} u^k_{tt}|^2_2 + |\nabla u^k_{tt}|^2_2 + |\rho^k_{ttt}|^2_2 + |\nabla^2 u^k_t|_2 + 1\big), \\
\mathrm{IV}_3 + \mathrm{IV}_4 =& -2\int \rho^k_t \big(u^{k-1} \cdot \nabla u^k\big)_t \cdot u^k_{ttt} - \int \rho^k \big(u^{k-1} \cdot \nabla u^k\big)_{tt} \cdot u^k_{ttt} \\
=& -2\frac{\mathrm{d}}{\mathrm{d}t} \int \rho^k_t \big(u^{k-1} \cdot \nabla u^k\big)_t \cdot u^k_{tt} + 2\int \rho^k_{tt}\big(u^{k-1} \cdot \nabla u^k\big)_t \cdot u^k_{tt} \\
&+ 2\int \rho^k_t \big(u^{k-1} \cdot \nabla u^k\big)_{tt} \cdot u^k_{tt} - \int \rho^k \big(u^{k-1} \cdot \nabla u^k\big)_{tt} \cdot u^k_{ttt} \\
\leq& -2\frac{\mathrm{d}}{\mathrm{d}t} \int \rho^k_t \big(u^{k-1} \cdot \nabla u^k\big)_t \cdot u^k_{tt} + \frac{1}{2}|\sqrt{\rho^k} u^k_{ttt}|^2_2 \\
&+ C'\big(|\nabla u^k_{tt}|^2_2 + |\nabla u^{k-1}_{tt}|^2_2 + |\nabla^2 u^k_t|_2 + 1\big), \\
\mathrm{IV}_5 =& -\int \nabla P^k_{tt} \cdot u^k_{ttt} = \int P^k_{tt} \mathrm{div} u^k_{ttt} = \frac{\mathrm{d}}{\mathrm{d}t} \int P^k_{tt} \mathrm{div} u^k_{tt} - \int P^k_{ttt} \mathrm{div} u^k_{tt} \\
\leq& \frac{\mathrm{d}}{\mathrm{d}t} \int P^k_{tt} \mathrm{div} u^k_{tt} + |P^k_{ttt}|_2 |\nabla u^k_{tt}|_2.
\end{aligned} \tag{4.71}$$

Substituting (4.70)-(4.71) back to (4.68) yields that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\alpha|\nabla u_{tt}^k|_2^2+(\alpha+\beta)|\mathrm{div}u_{tt}^k|_2^2\big)+\frac{1}{2}|\sqrt{\rho^k}u_{ttt}^k|_2^2\\
\leq&\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int P_{tt}^k\mathrm{div}u_{tt}^k-\rho_{tt}^k u_t^k\cdot u_{tt}^k-\rho_{tt}^k(u^{k-1}\cdot\nabla u^k)\cdot u_{tt}^k-\rho_t^k|u_{tt}^k|^2\\
&-2\rho_t^k(u^{k-1}\cdot\nabla u^k)_t\cdot u_{tt}^k\bigg)+C'\Big(|\nabla u_{tt}^k|_2^2+|\nabla u_{tt}^{k-1}|_2^2+|\nabla^2 u_t^k|_2^2+|\sqrt{\rho^k}u_{tt}^k|_2^2\\
&+|\rho_{ttt}^k|_2^2+|P_{ttt}^k|_2|\nabla u_{tt}^k|_2+1\Big).
\end{aligned}\tag{4.72}
$$

Next, denoting

$$
\begin{aligned}
A_2(t):=\int\Big(&P_{tt}^k\mathrm{div}u_{tt}^k-\rho_{tt}^k u_t^k\cdot u_{tt}^k-\rho_{tt}^k(u^{k-1}\cdot\nabla u^k)\cdot u_{tt}^k-\rho_t^k|u_{tt}^k|^2\\
&-2\rho_t^k(u^{k-1}\cdot\nabla u^k)_t\cdot u_{tt}^k\Big),
\end{aligned}\tag{4.73}
$$

one has

$$
\begin{aligned}
|A_2(t)|\leq&|P_{tt}^k|_2|\nabla u_{tt}^k|_2+C'\Big(|\sqrt{\rho^k}u_{tt}^k|_2|\nabla u_{tt}^k|_2+|\nabla u_{tt}^k|_2+1\Big)\\
\leq&\frac{\alpha}{4}|\nabla u_{tt}^k|_2^2+C'\big(|\sqrt{\rho^k}u_{tt}^k|_2^2+|P_{tt}^k|_2^2+1\big),
\end{aligned}\tag{4.74}
$$

in which, according to (4.42) and Lemmas 4.4-4.8, $|P_{tt}^k|_2$ satisfies

$$
t^2|P_{tt}^k|_2^2\leq C'.\tag{4.75}
$$

Moreover, the term $|P_{ttt}^k|_2$ in (4.72) can be controlled by

$$
\begin{aligned}
|P_{ttt}^k|_2\leq C'\Big(&|\sqrt{\rho^k}\psi_{ttt}^k|_2+|\rho_{ttt}^k|_2+|\nabla\psi_{tt}^k|_2+|\nabla\psi_t^k|_2^2|\sqrt{\rho^k}\psi_{tt}^k|_2\\
&+|\nabla\psi_t^k|_2^3+|\rho_{tt}^k|_3|\psi_t^k|_6+1\Big),
\end{aligned}\tag{4.76}
$$

with the help of Lemmas 4.3-4.8, we also have

$$
\int_0^t s^3|P_{sss}^k|_2^2\leq C'\int_0^t s^3|\sqrt{\rho^k}\psi_{sss}^k|_2^2+C'.\tag{4.77}
$$

Then substituting (4.76) back to (4.72), multiplying by t^2 and integrating over $[0,t]$, with the help of the Young's inequality, the Gronwall's inequality, (4.74)-(4.75) and (4.77), it arrives at

$$
\begin{aligned}
&\frac{\alpha}{2}t^2|\nabla u_{tt}^k|_2^2+\frac{1}{2}\int_0^t s^2|\sqrt{\rho^k}u_{sss}^k|_2^2\\
\leq&|t^2A_2(t)|+2\int_0^t|sA_2(s)|+C'\int_0^t s^2|P_{sss}^k|_2|\nabla u_{ss}^k|_2+C'\\
\leq&\frac{\alpha}{4}t^2|\nabla u_{tt}^k|_2^2+\eta\int_0^t s^3|\sqrt{\rho^k}\psi_{sss}^k|_2^2+C',
\end{aligned}
$$

which implies

$$
t^2|\nabla u_{tt}^k|_2^2+\int_0^t s^2|\sqrt{\rho^k}u_{sss}^k|_2^2\leq\eta\int_0^t s^3|\sqrt{\rho^k}\psi_{sss}^k|_2^2+C'.\tag{4.78}
$$

It remains to control the right-hand side of (4.78).

Step 2. *Estimates of* $\|t^{\frac{3}{2}}\sqrt{\rho^k}\psi^k_{ttt}\|_{L^2L^2}$ *and* $\|t^{\frac{3}{2}}\nabla\psi^k_{tt}\|_{L^\infty L^2}$. Differentiating $(3.64)_2$ with respect to t twice, multiplying by ψ^k_{ttt} and integrating over Ω, one has

$$\begin{aligned}
&\frac{a_2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\psi^{k-1})^{\frac{b}{b+1}}|\nabla\psi^k_{tt}|^2+\int\rho^k|\psi^k_{ttt}|^2\\
=&\frac{a_2}{2}\int\big((\psi^{k-1})^{\frac{b}{b+1}}\big)_t|\nabla\psi^k_{tt}|^2-a_2\int\nabla\big((\psi^{k-1})^{\frac{b}{b+1}}\big)\cdot\nabla\psi^k_{tt}\psi^k_{ttt}\\
&+\int\Big(a_2\big((\psi^{k-1})^{\frac{b}{b+1}}\big)_{tt}\Delta\psi^k+2a_2\Delta\psi^k_t\big((\psi^{k-1})^{\frac{b}{b+1}}\big)_t\\
&\quad+a_3\big((\psi^{k-1})^{\frac{b}{b+1}}Q(u^{k-1})\big)_{tt}\Big)\psi^k_{ttt}-\int\big(\rho^k_{tt}\psi^k_t+2\rho^k_t\psi^k_{tt}\big)\psi^k_{ttt}\\
&-\int\big(\rho^k u^{k-1}\cdot\nabla\psi^k+a_1\rho^k\psi^k\mathrm{div}u^{k-1}\big)_{tt}\psi^k_{ttt}=:\sum_{i=6}^{10}\mathrm{IV}_i.
\end{aligned}\tag{4.79}$$

Noticing that we first control the extra terms IV_6-IV_8 that come from the nonlinear elliptic term and nonlinear quadratic term as follows

$$\begin{aligned}
\mathrm{IV}_6=&\frac{a_2}{2}\int\big((\psi^{k-1})^{\frac{b}{b+1}}\big)_t|\nabla\psi^k_{tt}|^2\leq C'\int|\psi^{k-1}_t||\nabla\psi^k_{tt}|^2\leq C'|\psi^{k-1}_t|_\infty|\nabla\psi^k_{tt}|_2^2,\\
\mathrm{IV}_7=&-a_2\int\nabla\big((\psi^{k-1})^{\frac{b}{b+1}}\big)\cdot\nabla\psi^k_{tt}\psi^k_{ttt}\leq C'\int|\nabla\psi^{k-1}\cdot\nabla\psi^k_{tt}\psi^k_{ttt}|\\
\leq& C'|\nabla\psi^{k-1}|_3|\nabla\psi^k_{tt}|_2|\psi^k_{ttt}|_6\leq C'|\nabla\psi^k_{tt}|_2|\psi^k_{ttt}|_6,\\
\mathrm{IV}_8=&\int\Big(a_2\big((\psi^{k-1})^{\frac{b}{b+1}}\big)_{tt}\Delta\psi^k+2a_2\Delta\psi^k_t\big((\psi^{k-1})^{\frac{b}{b+1}}\big)_t\\
&\quad+a_3\big((\psi^{k-1})^{\frac{b}{b+1}}Q(u^{k-1})\big)_{tt}\Big)\psi^k_{ttt}\\
\leq& C'\Big(|\nabla\psi^{k-1}_t|_2^2+|\psi^{k-1}_{tt}|_6+|\nabla^2\psi^k_t|_2\big(|\nabla\psi^{k-1}_t|_2+1\big)+|\nabla u^{k-1}_{tt}|_2\\
&+|\nabla^2u^{k-1}_t|_2^{\frac{1}{2}}+1\Big)|\psi^k_{ttt}|_6.
\end{aligned}\tag{4.80}$$

For the rest terms, we control them as follows for simplicity.

$$\begin{aligned}
\mathrm{IV}_9=&-\int\big(\rho^k_{tt}\psi^k_t+2\rho^k_t\psi^k_{tt}\big)\psi^k_{ttt}\\
\leq&\int\Big(\big(|\rho^k_t\mathrm{div}u^{k-1}|+|\rho^k\mathrm{div}u^{k-1}_t|\big)|\psi^k_t|\\
&\quad+2\big(|\rho^k\mathrm{div}u^{k-1}|+|\nabla\rho^k\cdot u^{k-1}|\big)|\psi^k_{tt}|\Big)|\psi^k_{ttt}|\\
\leq&\Big(|\rho^k_t|_2|\nabla u^{k-1}|_6|\psi^k_t|_6+|\rho^k|_6\big(|\nabla u^{k-1}_t|_2+2|\nabla u^{k-1}|_2\big)|\psi^k_t|_6\\
&\quad+2|\nabla\rho^k|_2|u^{k-1}|_6|\psi^k_{tt}|_6\Big)|\psi^k_{ttt}|_6\\
\leq& C'(|\psi^k_{tt}|_6+|\psi^k_t|_6)|\psi^k_{ttt}|_6,\\
\mathrm{IV}_{10}=&-\int\big(\rho^k u^{k-1}\cdot\nabla\psi^k+a_1\rho^k\psi^k\mathrm{div}u^{k-1}\big)_{tt}\psi^k_{ttt}\\
\leq& C'\Big(\big(|\psi^k_{tt}|_6+|u^{k-1}_{tt}|_6\big)|\sqrt{\rho^k}\psi^k_{ttt}|_2\\
&\quad+\big(|\nabla u^{k-1}_{tt}|_2+|\nabla\psi^k_{tt}|_2+|\nabla\psi^k_t|_2+1\big)|\psi^k_{ttt}|_6\Big).
\end{aligned}\tag{4.81}$$

Submitting (4.80)-(4.81) into (4.79), one has

$$\begin{aligned}
&\frac{a_2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\psi^{k-1})^{\frac{b}{b+1}}|\nabla\psi^k_{tt}|^2+\frac{3}{4}\int\rho^k|\psi^k_{ttt}|^2\\
\le\,& C'\Big(|\psi^{k-1}_t|_\infty|\nabla\psi^k_{tt}|_2^2+|\sqrt{\rho^k}\psi^k_{ttt}|_2^2+|\nabla u^{k-1}_{tt}|_2^2\\
&+\big(|\nabla\psi^k_{tt}|_2+|\nabla\psi^{k-1}_{tt}|_2+|\sqrt{\rho^k}\psi^k_{tt}|_2+|\sqrt{\rho^{k-1}}\psi^{k-1}_{tt}|_2+|\nabla u^{k-1}_{tt}|_2+|\nabla^2u^{k-1}_t|_2^{\frac{1}{2}}\\
&+|\nabla\psi^k_t|_2+|\nabla\psi^{k-1}_t|_2^2+|\nabla^2\psi^k_t|_2|\nabla\psi^{k-1}_t|_2+|\nabla^2\psi^k_t|_2+1\big)|\psi^k_{ttt}|_6\Big). \qquad (4.82)
\end{aligned}$$

Noticing that the appearance of the higher-order term $|\psi^k_{ttt}|_6$ is inevitable due to (4.80), similarly to the way we deal with $|\nabla\psi^k_{tt}|_2$ in (4.53), we need to provide the smallness to the coefficient of $|\nabla\psi^k_{tt}|_2$ and carefully use the time wighted estimates.

Now, multiplying both sides of (4.82) by t^3 and integrating over $[0,t]$, with the help of Young's inequality and Lemmas 4.3-4.8, it arrives at

$$t^3|\nabla\psi^k_{tt}|_2^2+\int_0^t s^3|\sqrt{\rho^k}\psi^k_{sss}|_2^2\le\eta\int_0^t s^4|\nabla\psi^k_{sss}|_2^2+C'. \qquad (4.83)$$

This, together with (4.78) implies

$$\begin{aligned}
&t^2|\nabla u^k_{tt}|_2^2+t^3|\nabla\psi^k_{tt}|_2^2+\int_0^t\big(s^2|\sqrt{\rho^k}u^k_{sss}|_2^2+s^3|\sqrt{\rho^k}\psi^k_{sss}|_2^2\big)\\
\le\,&\eta\int_0^t s^4|\nabla\psi^k_{sss}|_2^2+C'. \qquad (4.84)
\end{aligned}$$

The proof of this lemma is completed. □

Now, we need to control the remaining term on the right-hand side of (4.84). As a preparation, we need the estimates of ∇u^k_{ttt}.

Lemma 4.10. *Under the assumptions of Theorem 4.1, it holds that*

$$t^3|\sqrt{\rho^k}u^k_{ttt}|_2^2+\int_0^t s^3|\nabla u^k_{ttt}|_2^2\le\eta\int_0^t s^4|\nabla\psi^k_{sss}|_2^2+C',$$

for any $k\ge1$ *and a.e.* $t\in[0,T_*]$.

Proof. Differentiating $(3.64)_3$ with respect to t three times, multiplying by u^k_{ttt} and integrating over Ω, one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho^k|u^k_{ttt}|^2+\int\big(\alpha|\nabla u^k_{ttt}|^2+(\alpha+\beta)|\mathrm{div}u^k_{ttt}|^2\big)\\
=&-\frac{5}{2}\int\rho^k_t|u^k_{ttt}|^2-\int\big(\rho^k_{ttt}u^k_t+3\rho^k_{tt}u^k_{tt}\big)\cdot u^k_{ttt}\\
&-\int\Big(\rho^k(u^{k-1}\cdot\nabla u^k)_{ttt}+3\rho^k_t(u^{k-1}\cdot\nabla u^k)_{tt}+3\rho^k_{tt}(u^{k-1}\cdot\nabla u^k)_t\Big)\cdot u^k_{ttt}\\
&-\int\rho^k_{ttt}\big(u^{k-1}\cdot\nabla u^k\big)\cdot u^k_{ttt}+\int P^k_{ttt}\mathrm{div}u^k_{ttt}\;=:\sum_{i=1}^5\mathrm{V}_i. \qquad (4.85)
\end{aligned}$$

Based on Lemmas 4.3-4.9, the Cauchy inequality and the Sobolev inequality, we control the terms on the right-hand side of (4.85) as follows.

$$\begin{aligned}
\mathrm{V}_1 =& -\frac{5}{2}\int \rho^k_t |u^k_{ttt}|^2 = \frac{5}{2}\int \mathrm{div}(\rho^k u^{k-1})|u^k_{ttt}|^2\\
&\leq 5\int |\rho^k u^{k-1}\cdot\nabla u^k_{ttt}\cdot u^k_{ttt}| \leq \frac{\alpha}{16}|\nabla u^k_{ttt}|_2^2 + C'|\sqrt{\rho^k}u^k_{ttt}|_2^2,\\
\mathrm{V}_2 =& -\int \left(\rho^k_{ttt}u^k_t + 3\rho^k_{tt}u^k_{tt}\right)\cdot u^k_{ttt}\\
=& \int \Big(\mathrm{div}\big(\rho^k u^{k-1}\big)_{tt}u^k_t + 3\mathrm{div}\big(\rho^k u^{k-1}\big)_t u^k_{tt}\Big)\cdot u^k_{ttt}\\
&\leq C'\big(|\nabla\rho^k_{tt}|_2 + |\nabla u^k_{tt}|_2 + |\nabla u^{k-1}_{tt}|_2 + 1\big)|\nabla u^k_{ttt}|_2\\
&\leq \frac{\alpha}{32}|\nabla u^k_{ttt}|_2^2 + C'\big(|\nabla\rho^k_{tt}|_2^2 + |\nabla u^k_{tt}|_2^2 + |\nabla u^{k-1}_{tt}|_2^2 + 1\big),
\end{aligned} \tag{4.86}$$

$$\begin{aligned}
\mathrm{V}_3 =& -\int \Big(\rho^k\big(u^{k-1}\cdot\nabla u^k\big)_{ttt} + 3\rho^k_t\big(u^{k-1}\cdot\nabla u^k\big)_{tt} + 3\rho^k_{tt}\big(u^{k-1}\cdot\nabla u^k\big)_t\Big)\cdot u^k_{ttt}\\
&\leq \frac{\alpha}{16}\big(|\nabla u^k_{ttt}|_2^2 + |\nabla u^{k-1}_{ttt}|_2^2\big) + C'\Big(|\sqrt{\rho^k}u^k_{ttt}|_2^2\big(|\nabla^2 u^k_t|_2 + |\nabla^2 u^{k-1}_t|_2^2 + 1\big)\\
&\qquad + |\nabla u^{k-1}_{tt}|_2^2 + |\nabla u^k_{tt}|_2^2 + 1\Big),\\
\mathrm{V}_4 + \mathrm{V}_5 =& -\int \Big(\rho^k_{ttt}(u^{k-1}\cdot\nabla u^k)\cdot u^k_{ttt} - P^k_{ttt}\mathrm{div}u^k_{ttt}\Big)\\
&\leq \big(|\rho^k_{ttt}|_2|u^{k-1}|_\infty|\nabla u^k|_3|u^k_{ttt}|_6 + |\nabla u^k_{ttt}|_2|P^k_{ttt}|_2\big)\\
&\leq \frac{\alpha}{16}|\nabla u^k_{ttt}|_2^2 + C'\big(|\rho^k_{ttt}|_2^2 + |P^k_{ttt}|_2^2\big).
\end{aligned} \tag{4.87}$$

Putting (4.86)-(4.87) into (4.85), one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\sqrt{\rho^k}u^k_{ttt}|_2^2 + \frac{9\alpha}{16}|\nabla u^k_{ttt}|_2^2\\
\leq& \frac{\alpha}{16}|\nabla u^{k-1}_{ttt}|_2^2 + C'\Big(|P^k_{ttt}|_2^2 + |\nabla\rho^k_{tt}|_2 + |\rho^k_{ttt}|_2^2 + |\nabla u^k_{tt}|_2^2 + |\nabla u^{k-1}_{tt}|_2^2 + |\nabla^2 u^k_t|_2\\
&\qquad + |\sqrt{\rho^k}u^k_{ttt}|_2^2\big(|\nabla^2 u^{k-1}_t|_2^2 + |\nabla^2 u^k_t|_2 + 1\big)\Big),
\end{aligned} \tag{4.88}$$

multiplying (4.88) by t^3 and integrating the result over $[0,t]$, for any given $N\in\mathbb{Z}_+$, by the Gronwal's inequality, we have

$$\max_{1\leq k\leq N} t^3|\sqrt{\rho^k}u^k_{ttt}|_2^2 + \max_{1\leq k\leq N}\int_0^t s^3|\nabla u^k_{sss}|_2^2 \leq \max_{1\leq k\leq N}\eta\int_0^t s^4|\nabla\psi^k_{sss}|_2^2 + C', \tag{4.89}$$

where we have used (4.77) and (4.84). The proof of this lemma is completed. □

Finally, we are ready to give the estimates of $\nabla\psi^k_{ttt}$ in the following lemma and close the estimates in Lemmas 4.9-4.10.

LEMMA 4.11. *Under the assumptions of Theorem 4.1, it holds that*

$$t^4|\sqrt{\rho^k}\psi^k_{ttt}|_2^2 + \int_0^t s^4|\nabla\psi^k_{sss}|_2^2 \leq C',$$

for any $k \geq 1$ *and a.e.* $t \in [0, T_*]$.

Proof. Differentiating $(3.64)_2$ with respect to t three times, multiplying by ψ^k_{ttt} and integrating over Ω, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho^k|\psi^k_{ttt}|^2 + a_2\int (\psi^{k-1})^{\frac{b}{b+1}}|\nabla\psi^k_{ttt}|^2\\
=\,& a_2\int \Big(-\nabla\big((\psi^{k-1})^{\frac{1}{b+1}}\big)\cdot\nabla\psi^k_{ttt} + 3\big((\psi^{k-1})^{\frac{b}{b+1}}\big)_t\Delta\psi^k_{tt}\Big)\psi^k_{ttt}\\
&+ 3a_2\int \big((\psi^{k-1})^{\frac{b}{b+1}}\big)_{tt}\Delta\psi^k_t\psi^k_{ttt} + \int \big((\psi^{k-1})^{\frac{b}{b+1}}\big)_{ttt}\big(a_2\Delta\psi^k + a_3 Q(u^{k-1})\big)\psi^k_{ttt}\\
&+ 3a_3\int \Big(\big((\psi^{k-1})^{\frac{b}{b+1}}\big)_{tt}Q(u^{k-1})_t + \big((\psi^{k-1})^{\frac{b}{b+1}}\big)_t Q(u^{k-1})_{tt}\Big)\psi^k_{ttt}\\
&+ a_3\int (\psi^{k-1})^{\frac{b}{b+1}}Q(u^{k-1})_{ttt}\psi^k_{ttt} - \int \Big(\frac{5}{2}\rho^k_t(\psi^k_{ttt})^2 + 3\rho^k_{tt}\psi^k_{tt}\psi^k_{ttt}\Big)\\
&- \int \rho^k_{ttt}\psi^k_t\psi^k_{ttt} - \int \big(\rho^k u^{k-1}\cdot\nabla\psi^k + a_1\rho^k\psi^k\mathrm{div}u^{k-1}\big)_{ttt}\psi^k_{ttt} =: \sum_{i=1}^{8}\mathrm{VI}_i. \qquad (4.90)
\end{aligned}
$$

We first consider the extra terms VI_1-VI_5 that caused by non-constant heat conductivity.

$$
\begin{aligned}
\mathrm{VI}_1 =\,& a_2\int \Big(-\nabla\big((\psi^{k-1})^{\frac{1}{b+1}}\big)\cdot\nabla\psi^k_{ttt} + 3\big((\psi^{k-1})^{\frac{b}{b+1}}\big)_t\Delta\psi^k_{tt}\Big)\psi^k_{ttt}\\
\leq\,& C'\Big(\int_{\Omega\setminus V_{R_0}} + \int_{V_{R_0}}\Big)|\nabla\psi^{k-1}\cdot\nabla\psi^k_{ttt}||\psi^k_{ttt}| + \int |\psi^{k-1}_t\Delta\psi^k_{tt}||\psi^k_{ttt}|\\
\leq\,& C'\Big(\big(|\sqrt{\rho^k}\psi^k_{ttt}|_3 + |V_{R_0}|^{\frac{1}{6}}|\psi^k_{ttt}|_6\big)|\nabla\psi^k_{ttt}|_2|\nabla\psi^{k-1}|_6 + |\psi^{k-1}_t|_3|\nabla^2\psi^k_{tt}|_2|\psi^k_{ttt}|_6\Big)\\
\leq\,& \Big(\frac{a_4}{16} + C'|V_{R_0}|^{\frac{1}{6}}\Big)|\nabla\psi^k_{ttt}|^2_2 + C'\big(|\sqrt{\rho^k}\psi^k_{ttt}|^2_2 + |\nabla\psi^{k-1}_t|^2_2|\nabla^2\psi^k_{tt}|^2_2\big),\\
\mathrm{VI}_2 =\,& 3a_2\int \Big((\psi^{k-1})^{\frac{b}{b+1}}\Big)_{tt}\Delta\psi^k_t\psi^k_{ttt} \leq C'\big(|\psi^{k-1}_t|^2_6 + |\psi^{k-1}_{tt}|_6\big)|\nabla^2\psi^k_t|_2|\psi^k_{ttt}|_6\\
\leq\,& \frac{a_4}{32}|\nabla\psi^k_{ttt}|^2_2 + C'\Big(\big(|\nabla\psi^{k-1}_t|^4_2 + |\nabla\psi^{k-1}_{tt}|^2_2 + |\sqrt{\rho^{k-1}}\psi^{k-1}_{tt}|^2_2 + 1\big)|\nabla^2\psi^k_t|^2_2\\
&+ |\sqrt{\rho^k}\psi^k_{ttt}|^2_2\Big),\\
\mathrm{VI}_3 =\,& \int \Big((\psi^{k-1})^{\frac{b}{b+1}}\Big)_{ttt}\big(a_2\Delta\psi^k + a_3 Q(u^{k-1})\big)\psi^k_{ttt}\\
\leq\,& C'\Big(|\psi^{k-1}_{ttt}|_6\big(|V_{R_0}|^{\frac{1}{6}}(|\nabla^2\psi^k|_2 + |\nabla u^{k-1}|_3|\nabla u^{k-1}|_6)|\psi^k_{ttt}|_6\\
&+ |\sqrt{\rho^k}\psi^k_{ttt}|_2(|\nabla^2\psi^k|_3 + |\nabla u^{k-1}|^2_6)\big)\\
&+ \big(|\psi^{k-1}_t|_6|\psi^{k-1}_{tt}|_6(|\nabla^2\psi^k|_2 + |\nabla u^{k-1}|_3|\nabla u^{k-1}|_6)\\
&+ |\psi^{k-1}_t|^3_6(|\nabla^2\psi^k|_3 + |\nabla u^{k-1}|^2_6)\big)|\psi^k_{ttt}|_6\Big)\\
\leq\,& \frac{a_4}{32}|\nabla\psi^k_{ttt}|^2_2 + \Big(\frac{a_4}{64} + C'|V_{R_0}|^{\frac{1}{3}}\Big)\big(|\nabla\psi^{k-1}_{ttt}|^2_2 + |\sqrt{\rho^{k-1}}\psi^{k-1}_{ttt}|^2_2\big)\\
&+ C'\Big(\big(|\sqrt{\rho^k}\psi^k_{ttt}|^2_2 + |\nabla\psi^{k-1}_t|^6_2 + 1\big)\big(|\nabla^3\psi^k|_2 + 1\big)\\
&+ \big(|\nabla\psi^{k-1}_{tt}|^2_2 + |\sqrt{\rho^{k-1}}\psi^{k-1}_{tt}|^2_2\big)\big(|\nabla\psi^{k-1}_t|^2_2 + 1\big)\Big), \qquad (4.91)
\end{aligned}
$$

$$\begin{aligned}
\mathrm{VI}_4 =& 3a_3\int\Big(\big((\psi^{k-1})^{\frac{b}{b+1}}\big)_{tt}Q(u^{k-1})_t+\big((\psi^{k-1})^{\frac{b}{b+1}}\big)_tQ(u^{k-1})_{tt}\Big)\psi^k_{ttt}\\
\le& \frac{a_4}{32}|\nabla\psi^k_{ttt}|_2^2+C'\Big(|\sqrt{\rho^k}\psi^k_{ttt}|_2^2+|\nabla\psi^{k-1}_t|_2^4+|\nabla\psi^{k-1}_{tt}|_2^2+|\sqrt{\rho^{k-1}}\psi^{k-1}_{tt}|_2^2\\
&+\big(|\nabla\psi^{k-1}_t|_2^2+1\big)\big(|\nabla u^{k-1}_{tt}|_2^2+|\nabla^2u^{k-1}_t|_2\big)+1\Big),\\
\mathrm{VI}_5 =& a_3\int(\psi^{k-1})^{\frac{b}{b+1}}Q(u^{k-1})_{ttt}\psi^k_{ttt}\\
\le& \frac{a_4}{32}|\nabla\psi^k_{ttt}|_2^2+C'\big(|\sqrt{\rho^k}\psi^k_{ttt}|_2^2+|\nabla u^{k-1}_{ttt}|_2^2+|\nabla u^{k-1}_{tt}|_2^2|\nabla^2u^{k-1}_t|_2\big).
\end{aligned}\tag{4.92}$$

Second, based on Lemmas 4.3-4.10, we control the rest terms as follows

$$\begin{aligned}
\mathrm{VI}_6 =& -\int\Big(\frac{5}{2}\rho^k_t(\psi^k_{ttt})^2+3\rho^k_{tt}\psi^k_{tt}\psi^k_{ttt}\Big)\\
=& \int\Big(\frac{5}{2}\mathrm{div}(\rho^ku^{k-1})(\psi^k_{ttt})^2+3\mathrm{div}(\rho^ku^{k-1})_t\psi^k_{tt}\psi^k_{ttt}\Big)\\
\le& \frac{a_4}{32}|\nabla\psi^k_{ttt}|_2^2+C'\big(|\sqrt{\rho^k}\psi^k_{ttt}|_2^2+|\nabla\psi^k_{tt}|_2^2+|\sqrt{\rho^k}\psi^k_{tt}|_2^2\big),\\
\mathrm{VI}_7 =& -\int\rho^k_{ttt}\psi^k_t\psi^k_{ttt}=\int\mathrm{div}(\rho^ku^{k-1})_{tt}\psi^k_t\psi^k_{ttt}\\
\le& \frac{a_4}{32}|\nabla\psi^k_{ttt}|_2^2+C'\Big(\big(|\rho^k_{tt}|_3^2+|\nabla u^{k-1}_{tt}|_2^2+1\big)\big(|\nabla\psi^k_t|_2^2+1\big)+|\sqrt{\rho^k}\psi^k_{ttt}|_2^2\Big),\\
\mathrm{VI}_8 =& -\int\big(\rho^ku^{k-1}\cdot\nabla\psi^k+a_1\rho^k\psi^k\mathrm{div}u^{k-1}\big)_{ttt}\psi^k_{ttt}\\
\le& \frac{a_4}{32}|\nabla\psi^k_{ttt}|_2^2+C'\Big(|\sqrt{\rho^k}\psi^k_{ttt}|_2^2+|\rho^k_{ttt}|_2^2+|\nabla\psi^k_{tt}|_2^2+|\sqrt{\rho^k}\psi^k_{tt}|_2^2\\
&+\big(|\rho^k_{tt}|_3^2+|\nabla u^{k-1}_{tt}|_2^2+1\big)\big(|\nabla\psi^k_t|_2^2+1\big)+|\nabla u^{k-1}_{ttt}|_2^2\Big).
\end{aligned}\tag{4.93}$$

Taking $|V_{R_0}|\le\min\big\{\big(a_4/32C'\big)^3,\big(a_4/32C'\big)^6\big\}$, substituting (4.91)-(4.93) back to (4.90), for all $k\in\mathbb{Z}_+$, one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\sqrt{\rho^k}\psi^k_{ttt}|_2^2+\frac{a_4}{2}|\nabla\psi^k_{ttt}|_2^2\\
\le& C'\Big(|\nabla\psi^{k-1}_t|_2^2|\nabla^2\psi^k_{tt}|_2^2+\big(|\nabla\psi^{k-1}_t|_2^4+|\nabla\psi^{k-1}_{tt}|_2^2+|\sqrt{\rho^{k-1}}\psi^{k-1}_{tt}|_2^2+1\big)|\nabla^2\psi^k_t|_2^2\\
&+\big(|\nabla\psi^k_{tt}|_2^2+|\nabla\psi^{k-1}_{tt}|_2^2+|\sqrt{\rho^k}\psi^k_{tt}|_2^2+|\sqrt{\rho^{k-1}}\psi^{k-1}_{tt}|_2^2+|\nabla u^{k-1}_{tt}|_2^2\\
&\quad+|\nabla^2u^{k-1}_t|_2+|\rho^k_{tt}|_3^2\big)\big(|\nabla\psi^k_t|_2^2+|\nabla\psi^{k-1}_t|_2^2+1\big)+|\sqrt{\rho^{k-1}}\psi^{k-1}_{ttt}|_2^2\\
&+|\nabla u^{k-1}_{tt}|_2^2|\nabla^2u^{k-1}_t|_2+\big(|\sqrt{\rho^k}\psi^k_{ttt}|_2^2+|\nabla\psi^{k-1}_t|_2^6+1\big)\big(|\nabla^3\psi^k|_2+1\big)\\
&+\big(|\nabla\psi^k_t|_2^2+|\nabla\psi^{k-1}_t|_2^2+|\nabla^2u^{k-1}_t|_2+1\big)|\nabla u^{k-1}_{tt}|_2^2+|\rho^k_{ttt}|_2^2+|\nabla u^{k-1}_{ttt}|_2^2\Big).
\end{aligned}\tag{4.94}$$

Now it remains to estimate $|\nabla^2\psi^k_{tt}|_2$. According to $(3.64)_2$ and Lemma 2.3, one has

$$\begin{aligned}
|\nabla^2\psi^k_{tt}|_2\le& C'\Big(\big(|\nabla\psi^{k-1}_t|_2+|\nabla^2\psi^{k-1}_t|_2+1\big)|\nabla^2\psi^k_t|_2+|\nabla\psi^{k-1}_t|_2^3+|\nabla\psi^k_t|_2^2\\
&+(|\nabla\psi^{k-1}_{tt}|_2+|\sqrt{\rho^{k-1}}\psi^{k-1}_{tt}|_2)(|\nabla^2\psi^k|_3+1)+|\sqrt{\rho^k}\psi^k_{ttt}|_2\\
&+|\nabla\psi^k_{tt}|_2+|\sqrt{\rho^k}\psi^k_{tt}|_2+|\nabla u^{k-1}_{tt}|_2+|\nabla\rho^k_{tt}|_2+|\nabla^2u^{k-1}_t|_2+1\Big).
\end{aligned}\tag{4.95}$$

Putting (4.95) into (4.94), multiplying by t^4 and integrating over $[0,t]$, it arrives at

$$t^4|\sqrt{\rho^k}\psi^k_{ttt}|_2^2+\int_0^t s^4|\nabla\psi^k_{sss}|_2^2\leq C'\int_0^t s^3|\sqrt{\rho^k}\psi^k_{sss}|_2^2+C'. \tag{4.96}$$

Thus, multiplying (4.84) by $2C'$ and taking summation with (4.96), for $\eta\leq a_4/32C'$, we have

$$t^4|\sqrt{\rho^k}\psi^k_{ttt}|_2^2+\int_0^t s^4|\nabla\psi^k_{sss}|_2^2\leq C'. \tag{4.97}$$

The proof of this lemma is completed. □

With the help of (4.97), we can close the estimates in Lemmas 4.9-4.10:

$$\begin{aligned}t^2|\nabla u^k_{tt}|_2^2+t^3|\nabla\psi^k_{tt}|_2^2+\int_0^t\big(s^2|\sqrt{\rho^k}u^k_{sss}|_2^2+s^3|\sqrt{\rho^k}\psi^k_{sss}|_2^2\big)\leq C',\\ t^3|\sqrt{\rho^k}u^k_{ttt}|_2^2+\int_0^t s^3|\nabla u^k_{sss}|_2^2\leq C'.\end{aligned} \tag{4.98}$$

Moreover, we have

Lemma 4.12. *Under the assumptions of Theorem 4.1, it holds that*

$$t^3|\nabla^3\psi^k_t|_2^2+\int_0^t\big(s^2|\nabla^2u^k_{ss}|_2^2+s^3|\nabla^2\psi^k_{ss}|_2^2\big)\leq C',$$

for any $k\geq 1$ and a.e. $t\in[0,T_*]$.

Proof. First, from Lemma 2.3 and $(3.64)_2$, it deduces

$$\begin{aligned}|\nabla^3\psi^k_t|_2\leq C'\Big(&\big(|\nabla^3\psi^{k-1}|_2+1\big)\big(|\nabla^2\psi^k_t|_2+|\nabla\psi^k_t|_2\big)+|\nabla\psi^k_{tt}|_2\\&+|\nabla^2u^{k-1}_t|_2+|\nabla\psi^{k-1}_t|_2+1\Big),\end{aligned} \tag{4.99}$$

thus

$$t^3|\nabla^3\psi^k_t|_2^2\leq C'.$$

Second, from $(3.64)_3$ and Lemma 2.3, one has

$$\begin{aligned}|\nabla^2u^k_{tt}|_2\leq C'\Big(&|\sqrt{\rho^k}u^k_{ttt}|_2+|\nabla u^k_{tt}|_2+|\nabla u^{k-1}_{tt}|_2+|\rho^k_{tt}|_3+|\nabla\psi^k_{tt}|_2\\&+|\nabla^2\psi^k_t|_2^2+|\nabla\psi^k_t|_2^2+1\Big).\end{aligned}$$

this together with (4.95) and Lemmas 4.4-4.11, we complete the proof of the lemma. □

Finally, according to Lemmas 4.3-4.12, we have proved that the approximate solutions (ρ^k,u^k,ψ^k) belong to the solution class Ξ. Together with the strong convergence in Section 3.5, the lower semi-continuity of the norms and Lemma 4.2, one knows that (ρ^k,u^k,ψ^k) converge to a unique classical solution (ρ,u,ψ) on $(0,T_*]\times\Omega$. Thus we complete the proof of Theorem 4.1. By the continuity and positivity of ψ, Theorem 1.2 follows immediately.

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