METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 28, No. 2, pp. 173–194, June 2021 004

GLOBAL EXISTENCE, EXPONENTIAL DECAY AND FINITE TIME BLOW-UP FOR A CLASS OF FINITELY DEGENERATE COUPLED PARABOLIC SYSTEMS*

HUA CHEN†, JING WANG‡, AND HUIYANG XU§

Dedicated to the 80^{th} *birthday of Professor Ling Hsiao*

Abstract. In this paper, we study the initial-boundary value problem for finitely degenerate coupled parabolic systems. Using potential well method, we first prove the invariance of some sets and vacuum isolating of solutions. Then, by the Galerkin method, we obtain the global existence and finite time blow-up of solutions with low initial energy and critical initial energy and discuss the asymptotic behavior of the solutions.

Key words. Finitely degenerate coupled parabolic systems, global existence, blow-up, asymptotic behavior.

Mathematics Subject Classification. Primary 35K65, 35K51, 35K58; Secondary 35B44.

1. Introduction and main result. In this paper, we consider the initial-boundary value problem of the following finitely degenerate nonlinear parabolic systems with power type source terms

$$\begin{cases} u_t - \triangle_X u = (|u|^{2p} + |v|^{p+1}|u|^{p-1})u, & x \in \Omega,\ t > 0, \\ v_t - \triangle_X v = (|v|^{2p} + |u|^{p+1}|v|^{p-1})v, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ v(x,0) = v_0(x), & x \in \Omega, \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T], \end{cases} \tag{1.1}$$

where Ω is a bounded open domain such that $\Omega \subset\subset \Omega'$, Ω' is an open domain of $\mathbb{R}^n$ $(n \geq 2)$, the system of vector fields $X = (X_1, X_2, \cdots, X_m)$ satisfies the Hörmander's condition (see [20]) in Ω' with $X_j^* = -X_j$, $\triangle_X := \sum_{j=1}^m X_j^2$ is a subelliptic operator. $1 < p < \frac{2}{\nu-2}$, ν is the generalized Métivier index of X on Ω. $\partial\Omega$ is C^∞ and non-characteristic for the system of vector fields X.

More precisely, we say that X satisfies the Hörmander's condition in Ω' if X together with their commutators

$$X_J = [X_{j_1}, [X_{j_2}, \cdots [X_{j_{k-1}}, X_{j_k}] \cdots]],\ 1 \leq j_i \leq m,$$

up to a certain fixed length $|J| = k \leq Q$, span the tangent space at each point of Ω'. Here $Q > 1$ is called the Hörmander index of X in Ω', which is defined as the smallest positive integer for the Hörmander's condition above being satisfied. In this

*Received April 27, 2020; accepted for publication August 7, 2020. This work is supported by the NSFC under the grands 11631011, 1213000191 and 12101194.

†Corresponding author. School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China (chenhua@whu.edu.cn).

‡School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China (whuwangjing@whu.edu.cn).

§School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang 471023, China (xuhuiyang@whu.edu.cn).

case, we call that X is finite degenerate vector fields and the operator $\triangle_X$ is finitely degenerate. Such kinds of degenerate operators arise from both physical applications and mathematical problems, for example, see [23, 25].

Before we state our results, we would like to remark on the classical Laplace operator $\triangle$. When $X = (\partial_{x_1}, \cdots, \partial_{x_n})$, $\triangle_X$ reduces to the standard Laplacian. In this case, for treating the initial boundary value problem of semilinear heat equations

$$u_t - \triangle u = f(u), \quad x \in \Omega,\ t > 0, \tag{1.2}$$

where $f(u)$ is a polynomial such as $|u|^{p-1}$ or $|u|^p$ with $p > 1$, Payne and Sattinger [31] established the potential well method, and showed the global existence and the finite time blow-up of the weak solutions. Furthermore, Liu and Zhao [26] improved the method, and obtained a threshold result of global existence and nonexistence of solutions and obtained the vacuum isolating of solutions.

For semilinear reaction diffusion systems, Galaktionov et al. [15, 16] studied the initial boundary value problem of

$$\begin{cases} u_t - \triangle u^{\nu+1} = v^p, \\ v_t - \triangle v^{\mu+1} = u^q, \end{cases} \tag{1.3}$$

and showed its local and global existence of the solution. Here μ, ν, p, q are positive constants. Later, Escobedo and Herrero [10, 11] considered the Cauchy problem and initial boundary value problem of

$$\begin{cases} u_t - \triangle u = v^p, \\ v_t - \triangle v = u^q. \end{cases} \tag{1.4}$$

They obtained that the solutions exist globally in time or blow up in finite time under suitable assumptions on p and q. One can refer to [24, 35] for more results on (1.4).

Souplet and Tayachi in [36] studied the positive blowing-up solutions of the semilinear parabolic system

$$\begin{cases} u_t - \triangle u = u^r + v^p, \\ v_t - \triangle v = v^s + u^q. \end{cases} \tag{1.5}$$

Based on a continuity argument, Rossi and Souplet [33] studied simultaneous and nonsimultaneous blow-up for solutions of the system (1.5) with Dirichlet boundary conditions.

On the other hand, Escobedo and Levine [12] considered the long-time behavior of nonnegative solutions of the system

$$\begin{cases} u_t - \triangle u = u^\alpha v^p, \\ v_t - \triangle v = u^q v^\beta. \end{cases} \tag{1.6}$$

They proved Fujita-type global existence and global nonexistence results for the initial value problem of (1.6) analogous to the classical result of Fujita and others for $u_t = \triangle u + u^p$, $u(x,0) = u_0(x) \geq 0$. For more related results concerning the critical global existence exponent and the critical Fujita exponent for (1.6), one can see [1, 9, 32, 37, 42, 44] for details.

Recently, Xu, Lian and Niu [38] considered the following initial boundary value problem of

$$\begin{cases} u_t - \triangle u = (|u|^{2p} + |v|^{p+1}|u|^{p-1})u, \\ v_t - \triangle v = (|v|^{2p} + |u|^{p+1}|v|^{p-1})v. \end{cases} \tag{1.7}$$

They obtained global existence, long time behavior and finite time blow up of the solutions at the low initial energy level and critical energy level. For the high energy level, they established the comparison principle for parabolic system, and obtained the global existence and finite time blow up of the solutions.

In the finitely degeneration case, by using the corresponding Sobolev embedding theorem and Poincaré inequality on the finitely degeneration Sobolev spaces, Chen and Xu [8] proved the existence theorem of global solutions with exponential decay, and showed the blow-up in finite time for the solutions of the parabolic problem

$$\begin{cases} u_t - \triangle_X u = |u|^{q-1}u, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.8}$$

where $1 < q < \frac{\nu+2}{\nu-2}$.

In this paper, by the properties of $\triangle_X$ we show the global existence, decay and finite time blow-up of the solutions for problem (1.1). Note that the Hörmander condition permits us to define a metric $\rho(x,y)$ associated with X on Ω. By the induced geometry theory of this metric, which is usually called sub-Riemannian geometry [18] or Carnot-Carathéodory geometry [28], the operator $\triangle_X$ possesses most properties similar to the Laplacian, i.e. the precise estimation of Green's kernel, the Poincaré and Harnack inequalities, etc. Many results are obtained in [3, 4, 5, 14, 19, 21, 22, 23, 27, 30, 34].

For our purpose, we introduce the following definitions. For $n \geq 2$, the systems of real smooth vector fields $X = (X_1, X_2, \cdots, X_m)$ are defined on an open domain Ω' in $\mathbb{R}^n$. Then we introduce the following function space [41]:

$$H_X^1(\Omega') = \left\{u \in L^2(\Omega') \mid X_j u \in L^2(\Omega').\ j = 1, \cdots, m\right\},$$

Then $H_X^1(\Omega')$ is a Hilbert space endowed with norm

$$\|u\|^2_{H_X^1(\Omega')} = \|u\|^2_{L^2(\Omega')} + \|Xu\|^2_{L^2(\Omega')},$$

where $\|Xu\|^2_{L^2(\Omega')} = \sum_{j=1}^m \|X_j u\|^2_{L^2(\Omega')}$. Let $H^1_{X,0}(\Omega)$ be a subspace defined as a closure of $C_0^\infty(\Omega)$ in $H_X^1(\Omega')$. Then $H^1_{X,0}(\Omega)$ is also a Hilbert space.

Definition 1.1 (Weak solution). *A function $(u,v) = (u(x,t), v(x,t))$ is called a weak solution of problem* (1.1) *on $\Omega \times [0,T)$, if $(u,v) \in L^\infty(0,T; H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega))$ with $(u_t, v_t) \in L^2(0,T; L^2(\Omega) \times L^2(\Omega))$ satisfies the problem* (1.1) *in the distribution sense, i.e.*

(1)

$$(u_t, w') + (Xu, Xw') = ((|u|^{2p} + |v|^{p+1}|u|^{p-1})u, w'), \quad \forall\ w' \in H^1_{X,0}(\Omega),\ t \in (0,T), \tag{1.9}$$

$$(v_t, w'') + (Xv, Xw'') = ((|v|^{2p} + |u|^{p+1}|v|^{p-1})v, w''), \quad \forall\ w'' \in H^1_{X,0}(\Omega),\ t \in (0,T), \tag{1.10}$$

(2) $u(x,0) = u_0(x) \in H^1_{X,0}(\Omega)$, $v(x,0) = v_0(x) \in H^1_{X,0}(\Omega)$,
(3) *for all* $0 < t < T$, *we have*

$$\int_0^t \left(\|u_s\|_2^2 + \|v_s\|_2^2\right) ds + J(u,v) \le J(u_0,v_0). \tag{1.11}$$

Definition 1.2 (Maximal existence time). *Let (u,v) be a weak solution of* (1.1). *We define the maximal existence time T of $(u(t),v(t))$ as follows:*
(1) *If (u,v) exists for all $0 \le t < \infty$, then $T = +\infty$.*
(2) *If there exists a $t_0 \in (0,\infty)$ such that $(u(t),v(t))$ exists for $0 \le t < t_0$, but doesn't exist at $t = t_0$, then $T = t_0$.*

Definition 1.3 (Finite time blow-up). *Let (u,v) be a weak solution of* (1.1). *We call that (u,v) blows up in finite time if the maximal existence time T is finite and*

$$\lim_{t\to T^-} \int_0^t \left(\|u(\cdot,\tau)\|_2^2 + \|v(\cdot,\tau)\|_2^2\right) d\tau = +\infty. \tag{1.12}$$

Next, we introduce the potential well

$$W = \left\{(u,v) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega) |\ I(u,v) > 0, J(u,v) < d\right\} \cup \{0\},$$

and

$$V = \left\{(u,v) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega) |\ I(u,v) < 0, J(u,v) < d\right\},$$

where

$$J(u,v) = \frac{1}{2}\left(\|Xu\|_2^2 + \|Xv\|_2^2\right) - \frac{1}{2(p+1)}\left(\|u\|_{2p+2}^{2p+2} + 2\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}\right),$$
$$I(u,v) = \left(\|Xu\|_2^2 + \|Xv\|_2^2\right) - \left(\|u\|_{2p+2}^{2p+2} + 2\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}\right),$$
$$d = \inf_{(u,v)\in H^1_{X,0}(\Omega)\times H^1_{X,0}(\Omega)} \{\sup_{\lambda,\eta\ge 0} J(\lambda u, \eta v) |\ \|Xu\|_2^2 + \|Xv\|_2^2 \ne 0\}.$$

It is obvious that

$$J(u,v) = \frac{p}{2(p+1)}\left(\|Xu\|_2^2 + \|Xv\|_2^2\right) + \frac{1}{2(p+1)} I(u,v). \tag{1.13}$$

The main results of this paper are as follows:

Theorem 1.1. *Assume that the initial data $(u_0,v_0) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega)$, $0 < J(u_0,v_0) \le d$ and $I(u_0,v_0) \ge 0$, then problem* (1.1) *admits a global weak solution $(u,v) \in L^\infty(0,\infty; H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega))$ with $(u_t,v_t) \in L^2(0,\infty; L^2(\Omega) \times L^2(\Omega))$. Furthermore,*

- *if $J(u_0,v_0) < d$, then $u \in W$ for $t \in [0,+\infty)$, and there exists a constant $\alpha > 0$ such that*

$$\|u\|_2^2 + \|v\|_2^2 \le \left(\|u_0\|_2^2 + \|v_0\|_2^2\right) e^{-\alpha t},\ \forall\, t \ge 0, \tag{1.14}$$

- *if $J(u_0, v_0) = d$ and $I(u_0, v_0) > 0$, then $u \in \bar{W}$ for $t \in [0, +\infty)$, and for any sufficiently small number $\epsilon \in (0, d)$, there exist positive constants t_ϵ, C and β such that*

$$\|u\|_2^2 + \|v\|_2^2 \leq Ce^{-\beta t}, \ \forall \ t \geq t_\epsilon, \tag{1.15}$$

Theorem 1.2. *Assume that the initial data $(u_0, v_0) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega)$, $J(u_0, v_0) \leq d$ and $I(u_0, v_0) < 0$. Then the weak solution $(u(t), v(t))$ of problem* (1.1) *blows up in finite time, i.e. there exists $T > 0$ such that*

$$\lim_{t \to T^-} \int_0^t (\|u\|_2^2 + \|v\|_2^2) ds = +\infty. \tag{1.16}$$

Remark 1.1. *Under the initial data $(u_0, v_0) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega)$ with $J(u_0, v_0) \leq d$, the sign of $I(u_0, v_0)$ gives a key criteria whether problem* (1.1) *admits a global weak solution.*

Remark 1.2. *We shall discuss the problems in degenerate case for high initial energy case $J(u_0, v_0) > d$ in a forthcoming paper. On this aspect, one can see [17, 39] for single equation, or see [38] for system in non-degenerate case.*

The outline of this paper will be as follows. In section 2, we recall two definitions and some known properties on the weighted Sobolev space $H^1_{X,0}(\Omega)$. In section 3, we introduce a family of potential wells and discuss the invariance of some sets and vacuum isolating under the solution flow of (1.1). In section 4, we give the proof of Theorem 1.1. Finally, the proof of Theorem 1.2 will be given in section 5.

Throughout this article, for simplicity, we denote $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$, $\|\cdot\|_p^2 = \|\cdot\|^2_{L^p(\Omega)}$, $1 \leq p \leq \infty$, $\|\cdot\| = \|\cdot\|_2 = \|\cdot\|_{L^2(\Omega)}$, $(u, v) = (u(t), v(t)) = (u(x, t), v(x, t))$, $I(u, v) = I(u(t), v(t)) = I(u(x, t), v(x, t))$, $J(u, v) = J(u(t), v(t)) = J(u(x, t), v(x, t))$ and $T = T(u, v)$ is the maximal existence time of $(u(x, t), v(x, t))$, and $(\cdot, \cdot)_2$ means the inner product $(\cdot, \cdot)_{L^2(\Omega)}$.

2. Preliminaries.

First, we introduce two definitions.

Definition 2.1 (Métivier condition, cf. [29]). *Assume that the system of vector fields X satisfies the Hömander's condition in Ω' with Hömander index Q. Let $V_j(x)$ $(1 \leq j \leq Q)$, spanned by all commutators of $X_1, X_2, \cdots, X_m$ length $\leq j$, be the subspaces of the tangent space at each $x \in \Omega'$. If $\mu_j = \dim V_j(x)$ is constant in a neighborhood of each $x \in \bar{\Omega} \subset \Omega'$, then we say that X satisfies Métivier condition on Ω. The Métivier index*

$$\mu := \sum_{j=1}^{Q} j(\mu_j - \mu_{j-1}), \qquad \mu_0 := 0,$$

is also called the Hausdorff dimension or homogeneous dimension of Ω related to the subelliptic metric induced by X.

The Métivier condition is an important condition on the study of finitely degenerate elliptic operator. However, there exist a lot of vector fields which do not satisfy the Métivier's condition, for example, Grushin type vector fields. Thus, we need to introduce the following

Definition 2.2 (Generalized Métivier index, cf. [6, 7]). *In Definition 2.1, set*

$$\mu(x) = \sum_{j=1}^{Q} j\big(\mu_j(x) - \mu_{j-1}(x)\big), \qquad \mu_0(x) := 0, \tag{2.1}$$

where $\mu_j(x)$ is the dimension of $V_j(x)$ for $x \in \Omega'$. Then, for $\Omega \subset\subset \Omega'$ we define

$$\nu = \max_{x\in\bar{\Omega}} \mu(x) \tag{2.2}$$

as the generalized Métivier index of Ω, which is also called the non-isotropic dimension of Ω related to X (cf. [43]). Here $\mu(x)$ is also called pointwise homogeneous dimension or non-isotropic dimension at x.

If the Métivier's condition is satisfied, then $\nu = \mu$.

Next, we introduce some known properties of $H^1_{X,0}(\Omega)$.

Proposition 2.1 (Weighted Poincaré inequality, cf. [40]). *Assume that the system of vector fields X satisfies Hörmander's condition on Ω, $\partial\Omega$ is C^∞ smooth and non-characteristic for X. Then the first eigenvalue λ_1 of the operator $-\triangle_X$ is strictly positive and there holds*

$$\lambda_1 \|u\|^2_{L^2(\Omega)} \le \|Xu\|^2_{L^2(\Omega)}, \qquad \forall\ u \in H^1_{X,0}(\Omega). \tag{2.3}$$

By Proposition 2.1, we can use $\|Xu\|_{L^2(\Omega)} = \left(\sum_{j=1}^m \|X_j u\|^2_{L^2(\Omega)}\right)^{1/2}$ as the an equivalent of $H^1_{X,0}(\Omega)$.

Proposition 2.2 (See [7]). *Assume that the system of vector fields X satisfies Hörmander's condition on Ω, $\partial\Omega$ is C^∞ smooth and non-characteristic for X. Then the Dirichlet eigenvalue problem*

$$\begin{cases} -\triangle_X u = \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{2.4}$$

has a sequence of discrete eigenvalues $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k \le \cdots$, and $\lambda_k \to +\infty$ as $k \to +\infty$. Moreover, the corresponding eigenfunctions $\{\varphi_k\}_{k\ge1}$ constitute an orthonormal basis of the Sobolev space $H^1_{X,0}(\Omega)$ (or an orthonormal basis of $L^2(\Omega)$).

Proposition 2.3 (Weighted Sobolev embedding theorem, cf. [43]). *If the system of vector fields X satisfies Hörmander's condition on Ω with Hörmander index $Q > 1$, $\partial\Omega$ is C^∞ smooth and non-characteristic for X. Then for any $u \in C^\infty(\bar{\Omega})$, we have*

$$\|u\|_{L^{P^*}(\Omega)} \le C(\|Xu\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}),$$

where C is a positive constant, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{\nu}$ for $p \in [1,\nu), \nu \ge n + Q - 1 > 2$ is generalized Métivier index of X on Ω.

Remark 2.1. *For $p \in (1,\nu), q \in (1,p^*)$, similar to the classical Sobolev compactly embedding (cf.[13]), one can prove that the embedding $H^{1,p}_X(\Omega) \hookrightarrow L^q(\Omega)$ is compact.*

3. Potential wells. First, we discuss the monotonicity of the map $\lambda \mapsto J(\lambda u, \lambda v)$.

LEMMA 3.1. *For any* $(u,v) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega)$, $\|Xu\|_2^2 + \|Xv\|_2^2 \neq 0$, *there hold*

(1) $\lim\limits_{\lambda \to 0} J(\lambda u, \lambda v) = 0$, $\lim\limits_{\lambda \to +\infty} J(\lambda u, \lambda v) = -\infty$;

(2) *For* $\lambda \in (0, \infty)$, *there exists a unique* $\lambda' = \lambda'(u,v)$ *such that* $\frac{d}{d\lambda} J(\lambda u, \lambda v)|_{\lambda = \lambda'} = 0$;

(3) $J(\lambda u, \lambda v)$ *is strictly increasing on* $\lambda \in (0, \lambda']$, *strictly decreasing on* $\lambda \in [\lambda', +\infty)$ *and takes the maximum at* $\lambda = \lambda'$;

(4)

$$I(\lambda u, \lambda v) = \lambda \frac{dJ(\lambda u, \lambda v)}{d\lambda} \begin{cases} > 0, & \lambda \in (0, \lambda'), \\ = 0, & \lambda = \lambda', \\ < 0, & \lambda \in (\lambda', +\infty), \end{cases}$$

where

$$\lambda' = \left(\frac{\|Xu\|_2^2 + \|Xv\|_2^2}{\|u\|_{2p+2}^{2p+2} + 2\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}} \right)^{\frac{1}{2p}}.$$

Proof. (1) The conclusion follows directly from

$$J(\lambda u, \lambda v) = \frac{\lambda^2}{2} \left(\|Xu\|_2^2 + \|Xv\|_2^2 \right) - \frac{\lambda^{2p+2}}{2(p+1)} \left(\|u\|_{2p+2}^{2p+2} + 2\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2} \right).$$

(2) By (2.4),we have

$$\frac{d}{d\lambda} J(\lambda u, \lambda v) = \left(\|Xu\|_2^2 + \|Xv\|_2^2 \right) - \lambda^{2p+1} \left(\|u\|_{2p+2}^{2p+2} + 2\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2} \right). \quad (3.1)$$

Hence for $(u,v) \neq (0,0)$, there exists a unique

$$\lambda' = \left(\frac{\|Xu\|_2^2 + \|Xv\|_2^2}{\|u\|_{2p+2}^{2p+2} + 2\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}} \right)^{\frac{1}{2p}}.$$

such that $\frac{d}{d\lambda} J(\lambda u, \lambda v) = 0$. Moreover,

$$\frac{d}{d\lambda} J(\lambda u, \lambda v) > 0 \quad if \quad 0 < \lambda < \lambda',$$
$$\frac{d}{d\lambda} J(\lambda u, \lambda v) < 0 \quad if \quad \lambda' < \lambda < \infty,$$

i.e., $J(\lambda u, \lambda v)$ is increasing on $0 < \lambda \leq \lambda'$, decreasing on $\lambda' \leq \lambda < \infty$ and takes the maximum at $\lambda = \lambda'$.

(3) The conclusion follows directly from

$$\frac{d}{d\lambda} J(\lambda u, \lambda v) = \lambda^{-1} I(\lambda u, \lambda v).$$

□

Next, we introduce the Nehari manifold

$$\mathcal{N} = \left\{(u,v) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega) |\ I(u,v) = 0, \|Xu\|_2^2 + \|Xv\|_2^2 \neq 0\right\}, \tag{3.2}$$

we see from Lemma 3.1 that $d > 0$, and the potential well depth d is also characterized by

$$d = \inf_{(u,v)\in\mathcal{N}} J(u,v), \tag{3.3}$$

For $\delta > 0$, we further define the depth of a family of potential wells

$$d(\delta) = \inf_{(u,v)\in\mathcal{N}_\delta} J(u,v), \tag{3.4}$$

where

$$\mathcal{N}_\delta = \left\{(u,v) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega) |\ I_\delta(u,v) = 0, \|Xu\|_2^2 + \|Xv\|_2^2 \neq 0\right\},$$

$$I_\delta(u,v) = \delta\left(\|Xu\|_2^2 + \|Xv\|_2^2\right) - \left(\|u\|_{2p+2}^{2p+2} + 2\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}\right), \tag{3.5}$$

Now, we are ready to show the relationships between $\|Xu\|_2^2 + \|Xv\|_2^2$ and the sign of $I_\delta(u,v)$ as follows.

Lemma 3.2. *For $(u,v) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega)$ and $r(\delta) = \left(\frac{\delta}{2C_X^{2p+2}}\right)^{\frac{1}{p}}$, where $C_X = \sup\limits_{u\in H^1_{X,0}(\Omega)\setminus\{0\}} \frac{\|u\|_{2p+2}}{\|Xu\|_2}$, there hold*

(1) *if $0 < \|Xu\|_2^2 + \|Xv\|_2^2 < r(\delta)$, then $I_\delta(u,v) > 0$;*
In particular, if $0 < \|Xu\|_2^2 + \|Xv\|_2^2 < r(1)$, then $I(u,v) > 0$.
(2) *if $I_\delta(u,v) < 0$, then $\|Xu\|_2^2 + \|Xv\|_2^2 > r(\delta)$,*
In particular, if $I(u,v) < 0$, then $\|Xu\|_2^2 + \|Xv\|_2^2 > r(1)$.
(3) *if $I_\delta(u,v) = 0$ and $\|Xu\|_2^2 + \|Xv\|_2^2 \neq 0$, then $\|Xu\|_2^2 + \|Xv\|_2^2 > r(\delta)$.*
In particular, if $I(u,v) = 0$, then $\|Xu\|_2^2 + \|Xv\|_2^2 \geq r(1)$.
(4) *if $I_\delta(u,v) = 0$ and $\|Xu\|_2^2 + \|Xv\|_2^2 \neq 0$, then*

$$J(u,v)\begin{cases} > 0, & 0 < \delta < p+1, \\ = 0, & \delta = p+1, \\ < 0, & \delta > p+1. \end{cases}$$

Proof. (1) From $0 < \|Xu\|_2^2 + \|Xv\|_2^2 < r(\delta)$, we have

$$\begin{aligned}
&\|u\|_{2p+2}^{2p+2} + 2\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2} \\
&\quad\leq \|u\|_{2p+2}^{2p+2} + 2\|u\|_{2p+2}^{p+1}\|v\|_{2p+2}^{p+1} + \|v\|_{2p+2}^{2p+2} \\
&\quad\leq C_X^{2p+2}\left(\|Xu\|_2^{2p+2} + 2\|Xu\|_2^{p+1}\cdot\|Xv\|_2^{p+1} + \|Xv\|_2^{2p+2}\right) \\
&\quad\leq 2C_X^{2p+2}\left(\|Xu\|_2^{2p+2} + \|Xv\|_2^{2p+2}\right) \\
&\quad\leq 2C_X^{2p+2}\left(\|Xu\|_2^2 + \|Xv\|_2^2\right)^p\cdot\left(\|Xu\|_2^2 + \|Xv\|_2^2\right) \\
&\quad< 2C_X^{2p+2}r(\delta)^p\left(\|Xu\|_2^2 + \|Xv\|_2^2\right) \\
&\quad= \delta\left(\|Xu\|_2^2 + \|Xv\|_2^2\right).
\end{aligned}$$

By the definitions of $I_\delta(u,v)$, Lemma 3.2 (1) follows.

(2) When $I_\delta(u,v)<0$, we see that $\|Xu\|_2^2+\|Xv\|_2^2\neq 0$ and

$$\begin{aligned}\delta\left(\|Xu\|_2^2+\|Xv\|_2^2\right) &< \|u\|_{2p+2}^{2p+2}+2\|uv\|_{p+1}^{p+1}+\|v\|_{2p+2}^{2p+2}\\ &\le 2C_X^{2p+2}\left(\|Xu\|_2^2+\|Xv\|_2^2\right)^p\cdot\left(\|Xu\|_2^2+\|Xv\|_2^2\right).\end{aligned}$$

Thus, Lemma 3.2 (2) holds.

(3) if $I_\delta(u,v)=0$ and $\|Xu\|_2^2+\|Xv\|_2^2\neq 0$, we have

$$\begin{aligned}\delta\left(\|Xu\|_2^2+\|Xv\|_2^2\right) &= \|u\|_{2p+2}^{2p+2}+2\|uv\|_{p+1}^{p+1}+\|v\|_{2p+2}^{2p+2}\\ &\le 2C_X^{2p+2}\left(\|Xu\|_2^2+\|Xv\|_2^2\right)^p\cdot\left(\|Xu\|_2^2+\|Xv\|_2^2\right).\end{aligned}$$

Then Lemma 3.2 (3) follows.

(4) Lemma 3.2 (4) follows from Lemma 3.2 (3) and

$$J(u,v)=\left(\frac{1}{2}-\frac{\delta}{2(p+1)}\right)\left(\|Xu\|_2^2+\|Xv\|_2^2\right)+\frac{1}{2(p+1)}I_\delta(u,v). \tag{3.6}$$

□

LEMMA 3.3. *The continuous function $d(\delta)$ of δ satisfies the following properties*

(1) $d(\delta)\ge a(\delta)r(\delta)$ *for* $\delta\in(0,p+1)$, *where* $a(\delta)=\frac{1}{2}-\frac{\delta}{2(p+1)}$,

(2) $\lim_{\delta\to 0}d(\delta)=0$, $d(p+1)=0$ *and* $d(\delta)<0$ *for* $\delta\in(p+1,\infty)$,

(3) $d(\delta)$ *is strictly increasing on* $(0,1]$, *decreasing on* $[1,p+1]$ *and takes the maximum* $d=d(1)$ *at* $\delta=1$.

Proof. (1) If $(u,v)\in\mathcal{N}_\delta$, i.e. $I_\delta(u,v)=0$ and $\|Xu\|_2^2+\|Xv\|_2^2\neq 0$, then by Lemma 3.2 (3) there holds $\|Xu\|_2^2+\|Xv\|_2^2\ge r(\delta)$. Hence, from (3.4) and

$$\begin{aligned}J(u,v)&=\left(\frac{1}{2}-\frac{\delta}{2(p+1)}\right)\left(\|Xu\|_2^2+\|Xv\|_2^2\right)+\frac{1}{2(p+1)}I_\delta(u,v)\\ &=a(\delta)\left(\|Xu\|_2^2+\|Xv\|_2^2\right)\\ &\ge a(\delta)r(\delta),\end{aligned}$$

we obtain $d(\delta)\ge a(\delta)r(\delta)$.

(2) For any $(u,v)\in H^1_{X,0}(\Omega)\times H^1_{X,0}(\Omega)$ and $\|Xu\|_2^2+\|Xv\|_2^2\neq 0$, we define $\lambda=\lambda(\delta)$ by

$$\delta\left(\|X(\lambda u)\|_2^2+\|X(\lambda v)\|_2^2\right)=\|\lambda u\|_{2p+2}^{2p+2}+2\|\lambda u\cdot\lambda v\|_{p+1}^{p+1}+\|\lambda v\|_{2p+2}^{2p+2}, \tag{3.7}$$

i.e.

$$\delta\left(\|Xu\|_2^2+\|Xv\|_2^2\right)=\lambda^{2p}\left(\|u\|_{2p+2}^{2p+2}+2\|uv\|_{p+1}^{p+1}+\|v\|_{2p+2}^{2p+2}\right),$$

Hence, for any $\delta>0$ there exists a unique

$$\lambda(\delta)=\left(\frac{\delta(\|Xu\|_2^2+\|Xv\|_2^2)}{\|u\|_{2p+2}^{2p+2}+2\|uv\|_{p+1}^{p+1}+\|v\|_{2p+2}^{2p+2}}\right)^{\frac{1}{2p}} \tag{3.8}$$

satisfying (3.5), which implies that $(\lambda u,\lambda v)\in\mathcal{N}_\delta$. We have

$$\lim_{\delta\to 0}\lambda(\delta)=0.$$

Easily we get

$$\lim_{\delta\to 0} J\left(\lambda(\delta)u, \lambda(\delta)v\right) = \lim_{\lambda\to 0} J(\lambda u, \lambda v) = 0,$$

$$\lim_{\delta\to 0} d(\delta) = 0.$$

By (3.4) and Lemma 3.2 (4), Lemma 3.3 (2) follows.

(3) It is enough to prove that for any $0 < \delta_1 < \delta_2 < 1$ or $1 < \delta_2 < \delta_1 < p+1$ and for any $(u,v) \in \mathcal{N}_{\delta_2}$, there exists a $(u_1, v_1) \in \mathcal{N}_{\delta_1}$ and a constant $\varepsilon(\delta_1, \delta_2) > 0$ such that $J(u_1, v_1) < J(u,v) - \varepsilon(\delta_1, \delta_2)$.

In fact, for above (u,v), we can define $\lambda(\delta)$ by (3.8), then $I_\delta(\lambda(\delta)u, \lambda(\delta)v) = 0$ and $\lambda(\delta_2) = 1$. Let $h(\lambda) = J(\lambda u, \lambda v)$, we get

$$\begin{aligned}\frac{d}{d\lambda}h(\lambda) &= \frac{1}{\lambda}I(\lambda u, \lambda v)\\ &= \frac{1}{\lambda}\Big((1-\delta)\left(\|X(\lambda u)\|_2^2 + \|X(\lambda v)\|_2^2\right) + I_\delta(\lambda u, \lambda v)\Big)\\ &= (1-\delta)\lambda\left(\|Xu\|_2^2 + \|Xv\|_2^2\right).\end{aligned}$$

Taking $(u_1, v_1) = \lambda(\delta_1)(u,v)$, then $(u_1, v_1) \in \mathcal{N}_{\delta_1}$. If $0 < \delta_1 < \delta_2 < 1$, as $\lambda(\delta)$ is increasing in δ, we have

$$\begin{aligned}J(u,v) - J(u_1, v_1) &= h(1) - h(\lambda(\delta_1))\\ &= h(\lambda(\delta_2)) - h(\lambda(\delta_1))\\ &= \int_{\lambda(\delta_1)}^{\lambda(\delta_2)} (1-\delta)\lambda\left(\|Xu\|_2^2 + \|Xv\|_2^2\right) d\lambda\\ &\geq (1-\delta_2)\lambda(\delta_1)\left(\lambda(\delta_2) - \lambda(\delta_1)\right) r(\delta_2)\\ &\triangleq \varepsilon(\delta_1, \delta_2)\\ &> 0.\end{aligned}$$

If $1 < \delta_2 < \delta_1 < p+1$, we have

$$\begin{aligned}J(u,v) - J(u_1, v_1) &= h(1) - h(\lambda(\delta_1))\\ &= \int_{\lambda(\delta_1)}^{\lambda(\delta_2)} (1-\delta)\lambda\left(\|Xu\|_2^2 + \|Xv\|_2^2\right) d\lambda\\ &\geq (\delta_2 - 1)\lambda(\delta_2)\left(\lambda(\delta_1) - \lambda(\delta_2)\right) r(\delta_2)\\ &\triangleq \varepsilon(\delta_1, \delta_2)\\ &> 0.\end{aligned}$$

Therefore, the conclusion of Lemma 3.3 (4) is proved. □

LEMMA 3.4. *For* $(u,v) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega)$*, if* $J(u,v) \leq d(\delta), \delta \in (0, p+1)$*, then we have*

(1) *If* $I_\delta(u,v) > 0$*, then* $\|Xu\|_2^2 + \|Xv\|_2^2 < \frac{d(\delta)}{a(\delta)}$*;*

(2) *If* $\|Xu\|_2^2 + \|Xv\|_2^2 > \frac{d(\delta)}{a(\delta)}$*, then* $I_\delta(u,v) < 0$*,*

(3) *If* $I_\delta(u,v) = 0$*, then* $\|Xu\|_2^2 + \|Xv\|_2^2 \leq \frac{d(\delta)}{a(\delta)}$*.*

Proof. For $0 < \delta < p+1$, by (3.6) and $J(u,v) \le d(\delta)$, there holds

$$a(\delta)(\|Xu\|_2^2 + \|Xv\|_2^2) + \frac{1}{2(p+1)} I_\delta(u,v) = J(u,v) \le d(\delta).$$

thus Lemma 3.4 follows. □

Now, for $\delta \in (0, p+1)$ we define a family of potential wells

$$W_\delta = \left\{(u,v) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega) |\ I_\delta(u,v) > 0, J(u,v) < d(\delta)\right\} \cup \{0\},$$

and its outsider

$$V_\delta = \left\{(u,v) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega) |\ I_\delta(u,v) < 0, J(u,v) < d(\delta)\right\}.$$

By Lemma 3.3 we can obtain the following result

Lemma 3.5.
(1) *If $0 < \delta' < \delta'' \le 1$, then $W_{\delta'} \subset W_{\delta''}$,*
(2) *If $1 < \delta'' < \delta' < p+1$, then $V_{\delta'} \subset V_{\delta''}$,*

For $0 < \delta < p+1$, we define

$$B_\delta = \left\{(u,v) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega) |\ \|Xu\|_2^2 + \|Xv\|_2^2 < r(\delta)\right\},$$
$$\bar{B}_\delta = B_\delta \cup \partial B_\delta = \left\{(u,v) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega) |\ \|Xu\|_2^2 + \|Xv\|_2^2 \le r(\delta)\right\},$$
$$B^c_\delta = \left\{(u,v) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega) |\ \|Xu\|_2^2 + \|Xv\|_2^2 > r(\delta)\right\}.$$

Lemma 3.6. *For $0 < \delta < p+1$, there hold*

$$B_{r_1(\delta)} \subset W_\delta \subset B_{r_2(\delta)},$$

$$V_\delta \subset B^c_\delta,$$

where

$$B_{r_1(\delta)} = \left\{(u,v) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega) |\ \|Xu\|_2^2 + \|Xv\|_2^2 < \min\{r(\delta), 2d(\delta)\}\right\},$$
$$B_{r_2(\delta)} = \left\{(u,v) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega) |\ \|Xu\|_2^2 + \|Xv\|_2^2 < \frac{d(\delta)}{a(\delta)}\right\}.$$

Proof. For any $(u,v) \in B_{r_1(\delta)}$, we see from $\|Xu\|_2^2 + \|Xv\|_2^2 < r(\delta)$ and Lemma 3.2 (1) that either $\|Xu\|_2^2 + \|Xv\|_2^2 = 0$ or $I_\delta(u,v) > 0$ occurs. Moreover, it follows from $J(u,v) \le \frac{1}{2}(\|Xu\|_2^2 + \|Xv\|_2^2)$ and $\|Xu\|_2^2 + \|Xv\|_2^2 < 2d(\delta)$ that $J(u,v) < d(\delta)$, thus $(u,v) \in W_\delta$, i.e. $B_{r_1(\delta)} \subset W_\delta$. The remainder of the lemma follow Lemma 3.2 and Lemma 3.4. □

Lemma 3.7. *Let $0 < J(u,v) < d$ for some $(u,v) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega)$, and δ_1, δ_2 are two roots of the equation $d(\delta) = J(u,v)$ with $\delta_1 < 1 < \delta_2$. Then the sign of $I_\delta(u,v)$ is unchangeable for $\delta \in (\delta_1, \delta_2)$.*

Proof. First, the inequality $J(u,v) > 0$ implies $\|Xu\|_2^2 + \|Xv\|_2^2 \ne 0$. If the sign of $I_\delta(u,v)$ is changed on $\delta \in (\delta_1, \delta_2)$, then there exists a $\bar{\delta} \in (\delta_1, \delta_2)$ such that $I_{\bar{\delta}}(u,v) = 0$. Therefore by (3.4) we have $J(u,v) \ge d(\bar{\delta})$, which is contradictive with $J(u,v) = d(\delta_1) = d(\delta_2) < d(\bar{\delta})$. □

Next, we discuss the invariance of some sets and vacuum isolating behavior of solutions for problem (1.1).

Proposition 3.1. *Suppose that* $(u_0, v_0) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega)$*,* $\mu \in (0, d)$ *and* δ_1*,* δ_2 *are two roots of the equation* $d(\delta) = \mu$ *with* $\delta_1 < 1 < \delta_2$*. Furthermore, let* (u, v) *be a weak solution of problem* (1.1) *with* $0 < J(u_0, v_0) \le \mu$*, then, for any* $\delta \in (\delta_1, \delta_2)$ *and* $t \in [0, T)$*, there hold*

(1) *if* $I(u_0, v_0) > 0$*, we have* $(u, v) \in W_\delta$*,*

(2) *if* $I(u_0, v_0) < 0$*, we have* $(u, v) \in V_\delta$*.*

Proof. (1) *First, we show* $(u_0, v_0) \in W_\delta$ *for* $\delta \in (\delta_1, \delta_2)$*. If* $J(u_0, v_0) \le \mu$ *and* $I(u_0, v_0) > 0$*, then by Lemma 3.7, we have* $J(u_0, v_0) < d(\delta)$ *and* $I_\delta(u_0, v_0) > 0$*. i.e.* $(u_0, v_0) \in W_\delta$ *for* $\delta \in (\delta_1, \delta_2)$*.*

Next, we prove $(u(t), v(t)) \in W_\delta$ *for* $\delta \in (\delta_1, \delta_2)$*,* $t \in (0, T)$*. If it is false, then by the continuity of* $I(u, v)$*, there exist* $\delta_0 \in (\delta_1, \delta_2)$ *and* $t_0 \in (0, T)$ *such that* $(u(t_0), v(t_0)) \in \partial W_{\delta_0}$*, which means that either* $I_{\delta_0}(u(t_0), v(t_0)) = 0$*,* $\|Xu(t_0)\|_2 \neq 0$*,* $\|Xv(t_0)\|_2 \neq 0$ *or* $J(u(t_0), v(t_0)) = d(\delta_0)$ *occurs. From* (1.11) *we can see that*

$$\int_0^t \left(\|u_s\|_2^2 + \|v_s\|_2^2\right) ds + J(u, v) \le J(u_0, v_0) < d(\delta), \quad \forall\, 0 \le t < T,\ \delta \in (\delta_1, \delta_2), \tag{3.9}$$

which implies $J(u(t_0), v(t_0)) \neq d(\delta_0)$*, thus* $I_{\delta_0}(u(t_0), v(t_0)) = 0$*,* $\|Xu(t_0)\|_2^2 + \|Xv(t_0)\|_2^2 \neq 0$*, then by* (3.4)*, we get* $J(u(t_0), v(t_0)) \ge d(\delta_0)$*, which is contradictive with* (3.9)*.*

(2) *First, we prove* $(u_0, v_0) \in V_\delta$ *for* $\delta \in (\delta_1, \delta_2)$*. If* $J(u_0, v_0) \le \mu$ *and* $I(u_0, v_0) < 0$*, then by Lemma 3.7, we obtain* $J(u_0, v_0) < d(\delta)$ *and* $I_\delta(u_0, v_0) < 0$*. i.e.* $(u_0, v_0) \in V_\delta$ *for* $\delta \in (\delta_1, \delta_2)$*.*

Next, we show $(u(t), v(t)) \in V_\delta$ *for any* $\delta \in (\delta_1, \delta_2)$ *and* $t \in (0, T)$*. If it is false, there exist* $\delta_0 \in (\delta_1, \delta_2)$ *and* $t_0 \in (0, T)$ *such that* $I_{\delta_0}(u(t), v(t)) < 0$ *for* $t \in [0, t_0)$ *and* $(u(t_0), v(t_0)) \in \partial V_{\delta_0}$*, i.e.* $I_{\delta_0}(u(t_0), v(t_0)) = 0$ *or* $J(u(t_0), v(t_0)) = d(\delta_0)$*. From* (3.9) *we get* $J(u(t_0), v(t_0)) \neq d(\delta_0)$*. Thus* $I_{\delta_0}(u(t_0), v(t_0)) = 0$*, by Lemma 3.2, we have* $\|Xu(t)\|_2^2 + \|Xv(t)\|_2^2 \ge r(\delta_0)$ *for* $t \in [0, t_0]$*. Therefore, by* (3.4) *we have* $J(u(t_0), v(t_0)) \ge d(\delta_0)$*, which is contradictive with* (3.9)*.* □

Proposition 3.2. *Suppose that* $(u_0, v_0) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega)$*,* $0 < J(u_0, v_0) \le \mu < d$*, and* δ_1, δ_2 *are two roots of equation* $d(\delta) = \mu$ *with* $\delta_1 < 1 < \delta_2$*. Then for any* $\delta \in (\delta_1, \delta_2)$*, both sets* W_δ *and* V_δ *are invariant, thus both sets*

$$W_{\delta_1\delta_2} = \bigcup_{\delta_1 < \delta < \delta_2} W_\delta,$$

and

$$V_{\delta_1\delta_2} = \bigcup_{\delta_1 < \delta < \delta_2} V_\delta,$$

are also invariant under the solution flow of (1.1)*.*

Proposition 3.3. *Suppose that* $(u_0, v_0) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega)$*, then all nontrivial solutions of problem* (1.1) *with* $J(u_0, v_0) = 0$ *belong to*

$$\bar{B}^c_{r_0} = \left\{(u, v) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega) |\ \|Xu\|_2^2 + \|Xv\|_2^2 \ge r_0\right\},$$

where $r_0 = \left(\frac{p+1}{2C_X^{2p+2}}\right)^{\frac{1}{p}}$.

Proof. Let (u, v) *be any solution of problem* (1.1) *with* $J(u_0, v_0) = 0$. *From energy inequality* (1.11), *we get* $J(u, v) \le 0$ *for* $t \in [0, T)$. *Then by*

$$\begin{aligned}\frac{1}{2}\left(\|Xu\|_2^2 + \|Xv\|_2^2\right) &\le \frac{1}{2(p+1)}\left(\|u\|_{2p+2}^{2p+2} + 2\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2}\right)\\ &\le \frac{C_X^{2p+2}}{p+1}\cdot\left(\|Xu\|_2^2 + \|Xv\|_2^2\right)^p\cdot\left(\|Xu\|_2^2 + \|Xv\|_2^2\right)\end{aligned} \tag{3.10}$$

we must have either $\|Xu\|_2^2 + \|Xv\|_2^2 = 0$ *or* $\|Xu\|_2^2 + \|Xv\|_2^2 \ge r_0$. *If* $\|Xu_0\|_2^2 + \|Xv_0\|_2^2 = 0$, *we claim* $\|Xu\|_2^2 + \|Xv\|_2^2 \equiv 0$ *for* $t \in [0, T)$, *otherwise, there exists* $t_0 \in (0, T)$ *such that* $0 < \|Xu(t_0)\|_2^2 + \|Xv(t_0)\|_2^2 < r_0$, *which leads to a contradiction. If* $\|Xu_0\|_2^2 + \|Xv_0\|_2^2 \ge r_0$, *by a similar argument we can prove that* $\|Xu\|_2^2 + \|Xv\|_2^2 \ge r_0$ *for* $t \in [0, T)$. □

PROPOSITION 3.4. *Suppose that* $(u_0, v_0) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega)$ *such that either* $J(u_0, v_0) < 0$ *or* $J(u_0, v_0) = 0$, $\|Xu_0\|_2^2 + \|Xv_0\|_2^2 \ne 0$ *holds. Then all weak solutions of problem* (1.1) *belong to* V_δ *for* $\delta \in (0, p+1)$.

Proof. Let (u, v) be any solution of problem (1.1) with initial data (u_0, v_0) as above. From the energy inequality (1.11) we have

$$a(\delta)\left(\|Xu\|_2^2 + \|Xv\|_2^2\right) + \frac{1}{2(p+1)}I_\delta(u, v) = J(u, v) \le J(u_0, v_0), \quad \forall\, \delta \in (0, p+1). \tag{3.11}$$

If $J(u_0, v_0) < 0$, we obtain

$$I_\delta(u, v) < 0,\ J(u, v) < 0 < d(\delta), \quad \forall\, \delta \in (0, p+1).$$

This implies that

$$(u, v) \in V_\delta, \quad \forall\, \delta \in (0, p+1),\ t \in [0, T). \tag{3.12}$$

If $J(u_0, v_0) = 0$ and $\|Xu_0\|_2^2 + \|Xv_0\|_2^2 \ne 0$, we see from Proposition 3.3 that $\|Xu\|_2^2 + \|Xv\|_2^2 \ge r_0$ for $t \in [0, T)$. Again by (3.11) we also get (3.12). Proposition (3.4) is proved. □

COROLLARY 3.1. *Suppose that* $(u_0, v_0) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega)$ *such that either* $J(u_0, v_0) < 0$ *or* $J(u_0, v_0) = 0$, $\|Xu_0\|_2^2 + \|Xv_0\|_2^2 \ne 0$ *holds. Then all weak solutions of problem* (1.1) *belong to* $\bar{B}^c_{p+1}$.

Proof. Let (u, v) be any solution of problem (1.1) with initial data (u_0, v_0) as above. Then it follows from Proposition 3.4 and Lemma 3.2 that

$$\|Xu\|_2^2 + \|Xv\|_2^2 > r(\delta), \quad \forall\, \delta \in (0, p+1),\ t \in [0, T).$$

Let $\delta \to p+1$, we obtain

$$\|Xu\|_2^2 + \|Xv\|_2^2 \ge r(p+1), \quad \forall\, t \in [0, T).$$

Corollary 3.1 is proved. □

Proposition 3.5. *Suppose that* $(u_0, v_0) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega)$, $0 < \mu < d$, *and* δ_1, δ_2 *are two roots of equation* $d(\delta) = \mu$ *with* $\delta_1 < 1 < \delta_2$. *Then for all weak solutions of problem* (1.1) *with* $J(u_0, v_0) \leq \mu$, *there exists a vacuum region*

$$U_\mu = \left\{(u,v) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega) |\ \|Xu\|_2^2 + \|Xv\|_2^2 \neq 0,\ I_\delta(u,v) = 0,\ \delta_1 < \delta < \delta_2\right\},$$

such that no solution of problem (1.1) *belongs to* U_μ. *The vacuum region* U_μ *becomes bigger and bigger when* μ *is decreasing. As the limit case we obtain*

$$U_0 = \left\{(u,v) \in H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega) |\ \|Xu\|_2^2 + \|Xv\|_2^2 \neq 0,\ I_\delta(u,v) = 0,\ 0 < \delta < p+1\right\}.$$

Proof. *Let* (u, v) *be any solution of problem* (1.1) *with* $J(u_0, v_0) \leq \mu$. *We only need to prove that if* $\|Xu\|_2^2 + \|Xv\|_2^2 \neq 0$ *and* $J(u_0, v_0) \leq \mu$, *then for all* $\delta \in (\delta_1, \delta_2)$, $(u(t), v(t)) \notin \mathcal{N}_\delta$, *i.e.* $I_\delta(u(t), v(t)) \neq 0$ *for all* $t \in [0, T)$.

Claim: $I_\delta(u_0, v_0) \neq 0$. *Otherwise, if* $I_\delta(u_0, v_0) = 0$, *then* $J(u_0, v_0) \geq d(\delta) > d(\delta_1) = d(\delta_2)$, *which is contradictive with* $J(u_0, v_0) \leq \mu$.

If it is false, assume that there exists $t' > 0$ *such that* $(u(t'), v(t')) \in U_\mu$, *which implies that there exists* $\delta_0 \in (\delta_1, \delta_2)$ *such that* $(u(t'), v(t')) \in \mathcal{N}_{\delta_0}$. *It follows from* (3.9) *that* $J(u_0, v_0) \geq J(u(t'), v(t')) \geq d(\delta_0) > J(u_0, v_0)$, *which leads to a contradiction.* □

4. Proof of Theorem 1.1.

Proof. We divide the proof into four steps.

Step 1: Global existence for low initial energy $J(u_0, v_0) < d$**.**

First, we can exclude some special cases as follows.

- $0 < J(u_0, v_0) < d, I(u_0, v_0) = 0$. This is contradictive with the definition (3.3) of d.
- $J(u_0, v_0) = 0, I(u_0, v_0) = 0$. It follows from (1.13) that $(u_0, v_0) = 0$, which is a trivial case.
- $J(u_0, v_0) = 0, I(u_0, v_0) > 0$. This is contradictive with (1.13).
- $J(u_0, v_0) < 0, I(u_0, v_0) \geq 0$. This is contradictive with (1.13).

It remains for us to consider the case $0 < J(u_0, v_0) < d, I(u_0, v_0) > 0$. By Proposition 2.2, we choose a sequence of eigenfunctions $\{\phi_j(x)\}_{j\geq 1}$ as an orthogonal basis of $H^1_{X,0}(\Omega)$. Construct the approximate solutions $(u_m(x,t), v_m(x,t))$ of problem (1.1)

$$\begin{cases} u_m(x,t) = \sum\limits_{j=1}^{m} g_{jm}(t)\phi_j(x), & m = 1, 2, \cdots; \\ v_m(x,t) = \sum\limits_{j=1}^{m} h_{jm}(t)\phi_j(x), & m = 1, 2, \cdots; \end{cases}$$

which satisfy

$$\begin{gathered} (u_{mt}, \phi_\tau) + (Xu_m, X\phi_\tau) = \left((|u_m|^{2p} + |v_m|^{p+1}|u_m|^{p-1})u_m, \phi_\tau\right), \\ \tau = 1, 2, ..., m, \end{gathered} \tag{4.1}$$

$$\begin{gathered} (v_{mt}, \phi_\tau) + (Xv_m, X\phi_\tau) = \left((|v_m|^{2p} + |u_m|^{p+1}|v_m|^{p-1})v_m, \phi_\tau\right), \\ \tau = 1, 2, ..., m, \end{gathered} \tag{4.2}$$

and as $m \to \infty$,

$$\begin{cases} u_m(x,0) = \sum\limits_{j=1}^{m} g_{jm}(0)\phi_j(x) \to u_0(x) \quad in\ H^1_{X,0}(\Omega), \\ v_m(x,0) = \sum\limits_{j=1}^{m} h_{jm}(0)\phi_j(x) \to v_0(x) \quad in\ H^1_{X,0}(\Omega), \end{cases} \tag{4.3}$$

Multiplying (4.1) and (4.2) by $g'_{\tau m}(t)$ and $h'_{\tau m}(t)$ respectively, summing for τ, integrating with respect to t from 0 to t and adding these two equations, we obtain

$$\int_0^t \left(\|u_{ms}\|_2^2 + \|v_{ms}\|_2^2\right) ds + J(u_m, v_m) = J(u_m(0), v_m(0)), \quad 0 \le t < \infty, \tag{4.4}$$

It follows from (4.3) that $J(u_m(0), v_m(0)) \to J(u_0, v_0)$, we obtain

$$\int_0^t \left(\|u_{ms}\|_2^2 + \|v_{ms}\|_2^2\right) ds + J(u_m, v_m) < d, \quad 0 \le t < \infty, \tag{4.5}$$

for sufficiently large m.

Following an argument similar to that in the proof of Proposition 3.1(1), and by (4.5) we can deduce $(u_m(t), v_m(t)) \in W$ for sufficiently large m and $0 \le t < \infty$. Thus from (4.5) and (1.13) we have

$$\int_0^t (\|u_{ms}\|_2^2 + \|v_{ms}\|_2^2) ds + \frac{p}{2(p+1)} \left(\|Xu_m\|_2^2 + \|Xv_m\|_2^2\right) < d, \quad 0 \le t < \infty, \tag{4.6}$$

which shows that

$$\|Xu_m\|_2^2 < \frac{2(p+1)}{p} d, \quad \|Xv_m\|_2^2 < \frac{2(p+1)}{p} d,$$

$$\int_0^t \left(\|u_{ms}\|_2^2 + \|v_{ms}\|_2^2\right) ds < d,$$

$$\|u_m\|_{2p+2}^{2p+2} \le C_X^{2p+2} \|Xu_m\|_2^{2p+2} \le C_X^{2p+2} \left(\frac{2(p+1)}{p} d\right)^{p+1},$$

$$\|v_m\|_{2p+2}^{2p+2} \le C_X^{2p+2} \|Xv_m\|_2^{2p+2} \le C_X^{2p+2} \left(\frac{2(p+1)}{p} d\right)^{p+1}.$$

Denote $\xrightarrow{w^*}$, $\xrightarrow{w}$ as the weakly star, weakly convergence respectively. Therefore, the estimates above imply that there exists subsequences, still denoted by $\{u_m\}, \{v_m\}$, such that as $m \to \infty$,

$$\begin{aligned}
&u_m \xrightarrow{w^*} u \ \text{ in } \ L^\infty(0,\infty; H^1_{X,0}(\Omega)) \text{ and a.e. in } \Omega \times [0,+\infty), \\
&v_m \xrightarrow{w^*} v \ \text{ in } \ L^\infty(0,\infty; H^1_{X,0}(\Omega)) \text{ and a.e. in } \Omega \times [0,+\infty), \\
&(|u_m|^{2p} + |v_m|^{p+1}|u_m|^{p-1})u_m \xrightarrow{w^*} (|u|^{2p} + |v|^{p+1}|u|^{p-1})u \ \text{ in } \ L^\infty(0,\infty; L^{\frac{2p+2}{2p+1}}(\Omega)), \\
&(|v_m|^{2p} + |u_m|^{p+1}|v_m|^{p-1})v_m \xrightarrow{w^*} (|v|^{2p} + |u|^{p+1}|v|^{p-1})u \ \text{ in } \ L^\infty(0,\infty; L^{\frac{2p+2}{2p+1}}(\Omega)), \\
&u_{mt} \xrightarrow{w} u_t \ \text{ in } \ L^2(0,\infty; L^2(\Omega)), \\
&v_{mt} \xrightarrow{w} v_t \ \text{ in } \ L^2(0,\infty; L^2(\Omega)).
\end{aligned}$$

Hence in (4.1),(4.2) for fixed τ, letting $m \to \infty$, we have

$$(u_t, \phi_\tau) + (Xu, X\phi_\tau) = \left((|u|^{2p} + |v|^{p+1}|u|^{p-1})u, \phi_\tau\right), \quad \forall\ \tau = 1, 2, \cdots, m,$$

$$(v_t, \phi_\tau) + (Xv, X\phi_\tau) = \left((|v|^{2p} + |u|^{p+1}|v|^{p-1})v, \phi_\tau\right), \quad \forall\ \tau = 1, 2, \cdots, m,$$

Furthermore,

$$(u_t, w') + (Xu, Xw') = \left((|u|^{2p} + |v|^{p+1}|u|^{p-1})u, w'\right), \quad \forall\ w' \in H^1_{X,0}(\Omega),\ t > 0,$$

$$(v_t, w'') + (Xv, Xw'') = \left((|v|^{2p} + |u|^{p+1}|v|^{p-1})v, w''\right), \quad \forall\ w'' \in H^1_{X,0}(\Omega),\ t > 0.$$

Meanwhile, from (4.3) we have $u(x, 0) = u_0(x)$, $v(x, 0) = v_0(x)$ in $H^1_{X,0}(\Omega)$. By density we see that $(u(t), v(t)) \in L^\infty(0, \infty; H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega))$, $(u_t(t), v_t(t)) \in L^2(0, \infty; L^2(\Omega) \times L^2(\Omega))$ and from Proposition 3.1, we have $(u(t), v(t)) \in W$.

Step 2: Decay estimate in the case: $J(u_0, v_0) < d$.

Let (u, v) be a global solution of problem (1.1) with $0 < J(u_0, v_0) < d$ and $I(u_0, v_0) > 0$. Then for $w', w'' \in L^\infty(0, T; H^1_{X,0}(\Omega)) \cap L^2(0, T; L^2(\Omega))$, (1.9) and (1.10) imply that

$$(u_t, w') + (Xu, Xw') = \left((|u|^{2p} + |v|^{p+1}|u|^{p-1})u, w'\right), \tag{4.7}$$

$$(v_t, w'') + (Xv, Xw'') = \left((|v|^{2p} + |u|^{p+1}|v|^{p-1})v, w''\right), \tag{4.8}$$

Taking $w' = u, w'' = v$, and summing (4.7) and (4.8), then we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_2^2 + \|v\|_2^2\right) + I(u, v) = 0, \quad \forall\ t \in [0, \infty). \tag{4.9}$$

By Proposition 3.1, we have $(u(t), v(t)) \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$, $t \in [0, \infty)$, where $\delta_1 < \delta_2$ are two roots of the equation $d(\delta) = J(u_0, v_0)$. This implies that $I_\delta(u, v) \geq 0$ for $\delta \in (\delta_1, \delta_2)$ and $I_{\delta_1}(u, v) \geq 0$ for $t \in [0, T)$. Therefore, by (4.9) we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_2^2 + \|v\|_2^2\right) + (1 - \delta_1)\left(\|Xu\|_2^2 + \|Xv\|_2^2\right) + I_{\delta_1}(u, v) = 0, \quad \forall\ t \in [0, \infty). \tag{4.10}$$

It follows from weighted Poincaré inequality (2.3) that

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_2^2 + \|v\|_2^2\right) + (1 - \delta_1)\lambda_1\left(\|u\|_2^2 + \|v\|_2^2\right) \leq 0, \quad \forall\ t \in [0, \infty).$$

Integrating the above inequality, we obtain

$$\|u\|_2^2 + \|v\|_2^2 \leq \left(\|u_0\|_2^2 + \|v_0\|_2^2\right) - 2(1 - \delta_1)\lambda_1 \int_0^t \left(\|u(s)\|_2^2 + \|v(s)\|_2^2\right) ds, \quad \forall\ t \in [0, \infty).$$

Together with Gronwall inequality shows that

$$\|u\|_2^2 + \|v\|_2^2 \leq \left(\|u_0\|_2^2 + \|v_0\|_2^2\right) e^{-2(1-\delta_1)\lambda_1 t}, \quad \forall\ t \in [0, \infty).$$

Setting $\alpha = 2(1 - \delta_1)\lambda_1 > 0$, we can get the decay estimate (1.14).

Step 3: Global existence for critical initial energy $J(u_0, v_0) = d$.

First $J(u_0, v_0) = d$ implies that $\|Xu_0\|_2^2 + \|Xv_0\|_2^2 \neq 0$. Set $(u_{0m}, v_{0m}) = \eta_m(u_0, v_0)$, where $\eta_m = 1 - \frac{1}{m}$, $m \geq 2$, we consider the initial-boundary value problem

$$\begin{cases} u_{mt} - \triangle_X u_m = \left(|u_m|^{2p} + |v_m|^{p+1}|u_m|^{p-1}\right) u_m, & x \in \Omega,\ t > 0, \\ v_{mt} - \triangle_X v_m = \left(|v_m|^{2p} + |u_m|^{p+1}|v_m|^{p-1}\right) v_m, & x \in \Omega,\ t > 0, \\ u_m(x, 0) = u_{0m}(x), & x \in \Omega, \\ v_m(x, 0) = v_{0m}(x), & x \in \Omega, \\ u_m(x, t) = v_m(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T]. \end{cases} \tag{4.11}$$

By $I(u_0, v_0) \geq 0$ and Lemma 3.1, we obtain

$$\begin{aligned} \lambda' &= \lambda'(u_0, v_0) \geq 1, \\ I(u_{0m}, v_{0m}) &= I(\eta_m u_0, \eta_m v_0) \\ &= \left(\eta_m^2 - \eta_m^{2p+2}\right)\left(\|Xu_0\|_2^2 + \|Xv_0\|_2^2\right) + \eta_m^{2p+2} I(u_0, v_0) > 0, \\ J(u_{0m}, v_{0m}) &= J(\eta_m u_0, \eta_m v_0) < J(u_0, v_0) = d. \end{aligned}$$

It follows from the result of step 1 that, for each m, problem (4.11) admits a global weak solution $(u_m, v_m) \in L^\infty(0, \infty; H^1_{X,0}(\Omega) \times H^1_{X,0}(\Omega))$ with $(u_{mt}, v_{mt}) \in L^2(0, \infty; L^2(\Omega) \times L^2(\Omega))$ and $(u_m, v_m) \in W$ for $0 \leq t < \infty$ satisfying

$$\begin{aligned} (u_{mt}, w') + (Xu_m, Xw') &= \left((|u_m|^{2p} + |v_m|^{p+1}|u_m|^{p-1})u_m, w'\right), \\ &\qquad \forall\ w' \in H^1_{X,0}(\Omega),\ t > 0, \\ (v_{mt}, w'') + (Xv_m, Xw'') &= \left((|v_m|^{2p} + |u_m|^{p+1}|v_m|^{p-1})v_m, w''\right), \\ &\qquad \forall\ w'' \in H^1_{X,0}(\Omega),\ t > 0, \end{aligned}$$

and

$$\int_0^t \left(\|u_{m\tau}\|_2^2 + \|v_{m\tau}\|_2^2\right) d\tau + J(u_m, v_m) \leq J(u_{0m}, v_{0m}) < d, \qquad \forall\ t \in [0, \infty). \tag{4.12}$$

The proof left is similar to that in the step 1.

Step 4: Decay estimate in the case: $J(u_0, v_0) = d$ **and** $I(u_0, v_0) > 0$**.**

Let $(u(t), v(t))$ be any global weak solution of the problem (1.1) with $J(u_0, v_0) = d$ and $I(u_0, v_0) > 0$. We claim that $I(u, v) > 0$ for any $t > 0$. If it is false, then there exists $t_0 > 0$ such that $I(u(t_0), v(t_0)) = 0$, $I(u(t), v(t)) > 0$ for any $t \in [0, t_0)$. Together with the definition of d mean that $J(u(t_0), v(t_0)) \geq d$. Furthermore, by (1.11) we have

$$J(u(t_0), v(t_0)) \leq J(u_0, v_0) - \int_0^{t_0} \left(\|u_\tau\|_2^2 + \|v_\tau\|_2^2\right) d\tau \leq d. \tag{4.13}$$

Then, by above inequality, we obtain $\int_0^{t_0} \left(\|u_\tau\|_2^2 + \|v_\tau\|_2^2\right) d\tau = 0$, i.e. $u_t = 0$ and $v_t = 0$ for $t \in [0, t_0]$, which is contradictory with $I(u_0, v_0) > 0$. Thus $I(u, v) > 0$ for $t \geq 0$.

It follows from (1.11) that for any given sufficiently small number $\varepsilon \in (0, d)$, there exists $t_\varepsilon > 0$ such that

$$0 < d - \varepsilon = J(u(t_\varepsilon), v(t_\varepsilon)) \leq J(u_0, v_0) - \int_0^{t_\varepsilon} \left(\|u_\tau\|_2^2 + \|v_\tau\|_2^2\right) d\tau.$$

Taking $t = t_\varepsilon$ as the first time, similar to step 2, we have the decay estimate (1.15). □

5. Proof of Theorem 1.2.

Proof. We divide the proof into two steps.

Step 1: Blow-up in case of $J(u_0, v_0) < d$.

Let $(u(t), v(t))$ be any weak solution of problem (1.1) with $J(u_0, v_0) < d$ and $I(u_0, v_0) < 0$. We will prove $T < \infty$ by contradiction. Now we define

$$G(t) = \int_0^t \left(\|u\|_2^2 + \|v\|_2^2\right) ds,$$

then

$$\begin{aligned}
\dot{G}(t) &= \|u\|_2^2 + \|v\|_2^2, \\
\ddot{G}(t) &= 2(u_t, u) + 2(v_t, v) \\
&= 2\left[-(\|Xu\|_2^2 + \|Xv\|_2^2) + (\|u\|_{2p+2}^{2p+2} + 2\|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2})\right] \\
&= -2I(u, v),
\end{aligned}$$

By Proposition 3.1, we obtain $\ddot{G}(t) > 0$. By (1.11), (1.13) and (2.3), we can deduce

$$\begin{aligned}
\ddot{G}(t) &= 2p\left(\|Xu\|_2^2 + \|Xv\|_2^2\right) - 4(p+1)J(u, v) \\
&\geq 2p\left(\|Xu\|_2^2 + \|Xv\|_2^2\right) - 4(p+1)J(u_0, v_0) \\
&\quad + 4(p+1)\int_0^t \left(\|u_s\|_2^2 + \|v_s\|_2^2\right) ds \\
&\geq 2p\lambda_1 \dot{G}(t) - 4(p+1)J(u_0, v_0) + 4(p+1)\int_0^t \left(\|u_s\|_2^2 + \|v_s\|_2^2\right) ds.
\end{aligned} \tag{5.14}$$

Note that

$$\begin{aligned}
&\left(\int_0^t ((u_s, u) + (v_s, v)) ds\right)^2 \\
&\qquad = \left(\frac{1}{2}\int_0^t \frac{d}{ds}(\|u\|_2^2 + \|v\|_2^2) ds\right)^2 \\
&\qquad = \frac{1}{4}\left(\dot{G}^2(t) - 2\dot{G}(t)(\|u_0\|_2^2 + \|v_0\|_2^2) + (\|u\|_2^2 + \|v\|_2^2)^2\right).
\end{aligned} \tag{5.15}$$

then

$$\dot{G}^2(t) = 4\left(\int_0^t \Big((u_s, u) + (u_s, u)\Big) ds\right)^2 + 2\dot{G}(t)\left(\|u_0\|_2^2 + \|v_0\|_2^2\right) - \left(\|u\|_2^2 + \|v\|_2^2\right)^2.$$

Combing with (5.14) and Hölder's inequality we can deduce that

$$\begin{aligned}
\ddot{G}(t)G(t) - (p+1)\dot{G}^2(t) &\geq 2p\lambda_1 \dot{G}(t)G(t) - 4(p+1)J(u_0, v_0)G(t) \\
&\qquad - 2(p+1)\dot{G}(t)\left(\|u_0\|_2^2 + \|v_0\|_2^2\right).
\end{aligned} \tag{5.16}$$

Thus we have

(1) If $J(u_0, v_0) \leq 0$, by (5.16), we have

$$\ddot{G}(t)G(t) - (p+1)\dot{G}^2(t) \geq 2\dot{G}(t)\Big(p\lambda_1 G(t) - (p+1)(\|u_0\|_2^2 + \|v_0\|_2^2)\Big). \tag{5.17}$$

We claim that $I(u(t),v(t))<0$ for $t\in(0,+\infty)$. Arguing by contradiction, we suppose that there exists a $t_0>0$ such that $I(u(t_0),v(t_0))=0$ and $I(u(t),v(t))<0$ for $0\le t<t_0$. By Lemma 3.2, we have $\|Xu(t)\|_2^2+\|Xv(t)\|_2^2>r(1)$ for $0\le t\le t_0$. Hence by (3.4), we get $J(u(t_0),v(t_0))\ge d$, which is contradictive with (1.11). Then we get $\ddot{G}(t)>0$ for $t\in[0,\infty)$. It implies that

$$G(t)\ge G(0)+t\dot{G}(0)\ge t\dot{G}(0),\quad \forall\, t\in[0,\infty).$$

Then, for sufficiently large t, there exists

$$G(t)>\frac{p+1}{p\lambda_1}\left(\|u_0\|_2^2+\|v_0\|_2^2\right),$$

which together with (5.17) shows that

$$\ddot{G}(t)G(t)-(p+1)\dot{G}^2(t)>0. \tag{5.18}$$

(2) If $0<J(u_0,v_0)<d$, it follows from Proposition 3.1 that $(u(t),v(t))\in V_\delta$, i.e. $I_\delta(u,v)<0$ for $\delta\in[1,\delta'')$ and $t\in[0,\infty)$, where δ'' is the larger root of equation $d(\delta)=J(u_0,v_0)$. From Lemma 3.2 and the continuity of $I_\delta(u,v)$, we show that $I_{\delta''}(u(t),v(t))\le 0$ and $\|Xu(t)\|_2^2+\|Xv(t)\|_2^2\ge r(\delta'')$ for $t\in[0,+\infty)$, then we obtain

$$\begin{aligned}
\ddot{G}(t)&=-2I(u,v)=2(\delta''-1)\left(\|Xu\|_2^2+\|Xv\|_2^2\right)-2I_{\delta''}(u,v)\\
&\ge 2(\delta''-1)r(\delta'')>0,\quad \forall\, t\ge 0,\\
\dot{G}(t)&=2(\delta''-1)r(\delta'')t+\dot{G}(0)\ge 2(\delta''-1)r(\delta'')t,\quad \forall\, t\ge 0,\\
G(t)&=(\delta''-1)r(\delta'')t^2+G(0)=(\delta''-1)r(\delta'')t^2,\quad \forall\, t\ge 0,
\end{aligned}$$

Therefore, for sufficiently large t, we have

$$\begin{aligned}
&p\lambda_1\dot{G}(t)>4(p+1)J(u_0,v_0),\\
&p\lambda_1 G(t)>2(p+1)\left(\|u_0\|_2^2+\|v_0\|_2^2\right).
\end{aligned}$$

which together with (5.16) prove that (5.18) is still valid for sufficiently large t.

Then

$$(G^{-p}(t))'=\frac{-p}{G^{\beta+1}(t)}\dot{G}(t),$$

$$(G^{-p}(t))''=\frac{-p}{G^{\beta+2}(t)}\left(\ddot{G}(t)G(t)-(p+1)\dot{G}^2(t)\right)<0.$$

Thus, there exists a large enough $t'>0$, such that

$$G^{-p}(t)\le G^{-p}(t')\left(1-p\frac{\dot{G}(t')}{G(t')}(t-t')\right),\quad \forall\, t>t',$$

Hence there exists a $T>0$ such that

$$\lim_{t\to T^-}G^{-p}(t)=0,\text{i.e.,}\lim_{t\to T^-}G(t)=+\infty.$$

Step 2: Blow-up for the case: $J(u_0,v_0)=d$.

Let $(u(t), v(t))$ be any solution of problem (1.1) with $J(u_0, v_0) = d$ and $I(u_0, v_0) < 0$. We can deduce that $I(u, v) < 0$ for $t \in [0, T)$, which together with $I(u) = -(u_t, u)_2 - (v_t, v)$ shows that $\|u_t\|_2^2 + \|v_t\|_2^2 > 0$ and $\int_0^t \left(\|u_\tau\|_2^2 + \|v_\tau\|_2^2\right) d\tau$ is increasing for $t \in [0, T)$. By the energy inequality (1.11), we can choose $t'' > 0$ such that

$$0 < J(u(t''), v(t'')) \leq d - \int_0^{t''} \left(\|u_\tau\|_2^2 + \|v_\tau\|_2^2\right) d\tau < d.$$

Then we take $t = t''$ as the initial time and similar to that in the proof of the step 1, we can deduce that the weak solution (u, v) blows up in finite time. □

REFERENCES

[1] X. Bai, *Finite time blow up for a reaction-diffusion system in bounded domain*, Z. Angew. Math. Phys., 65 (2014), pp. 135–138.

[2] J. M. Ball, *Remarks on blow-up and nonexistence theorems for nonlinear evolution equations*, Quart. J. Math. Oxford Ser., 28 (1977), pp. 473–486.

[3] J. M. Bony, *Principe du maximum, inégalité de Harnack et unicité du probléme de Cauchy pourles operateurs elliptiques dégeneres*, Ann. Inst. Fourier, 19 (1969), pp. 277–304.

[4] M. Bramanti, *An Invitation to Hypoelliptic Operators and Hörmander's Vector Fields*, Springer, New York, 2014.

[5] L. Capogna, D. Danielli, and N. Garofalo, *An embedding theorem and the Harnack inequality for nonlinear subelliptic equations*, Comm. Partial Differential Equations, 18 (1993), pp. 1765–1794.

[6] H. Chen, H. G. Chen, J. F. Wang, and N. N. Zhang, *Lower Bounds of Dirichlet Eigenvalues for General Grushin Type Bi-Subelliptic Operators*, Anal. Theory Appl., 35:1 (2019), pp. 66–84.

[7] H. Chen and P. Luo, *Lower bounds of Dirichlet eigenvalues for some degenerate elliptic operators*, Calc. Var. Partial Differ. Equ., 54 (2015), pp. 2831–2852.

[8] H. Chen and H. Y. Xu, *Global existence, exponential decay and blow-up in finite time for a class of finitely degenerate semilinear parabolic equations*, Acta Math. Sci., 39 B (2019), pp. 1290–1308.

[9] H. W. Chen, *Global existence and blow-up for a nonlinear reaction-diffusion system*, J. Math. Anal. Appl., 212 (1997), pp. 481–492.

[10] M. Escobedo and M. A. Herrero, *Boundedness and blow up for a semilinear reaction-diffusion system*, J. Differ Equ., 89 (1991), pp. 176–202.

[11] M. Escobedo and M. A. Herrero, *A semilinear parabolic system in a bounded domain*, Ann. Mat. Pura Appl., 165 (1993), pp. 315–336.

[12] M. Escobedo and H. A. Levine, *Critical blowup and global existence numbers for a weakly coupled system of reaction-diffusion equations*, Arch. Rational Mech. Anal., 129 (1995), pp. 47–100.

[13] L. C. Evans, *Partial differential equations*, Amer. Math. Soc. (2015).

[14] C. Fefferman and D. Phong, *Subelliptic eigenvalue problems*, in: Proceedings of the Conference on Harmonic Analysis in Honor of Antoni Zygmund, Wadsworth Math. Series (1981), pp. 590–606.

[15] V. A. Galaktionov, S. P. Kurdyumov, and A. A. Samarskii, *A parabolic system of quasilinear equations. I,* Differ. Uravn., 19 (1983), pp. 2123–2143.

[16] V. A. Galaktionov, S. P. Kurdyumov, and A. A. Samarskii, *A parabolic system of quasilinear equations. II,* Differ. Uravn., 21 (1985), pp. 1544–1559.

[17] F. Gazzola and T. Weth, *Finite time blow up and global solutions for semilinear parabolic equations with initial data at high energy level*, Differential Integral Equations, 18 (2005), pp. 961-990.

[18] R. Ghezzi and F. Jean, *Hausdorff volume in non equiregular sub-Riemannian manifolds*, Nonlinear Anal., 126 (2015), pp. 345–377.

[19] P. Hajlasz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc., 145 (1998).

[20] L. Hörmander, *Hypoelliptic second order differential operators*, Acta Math., 119 (1967), pp. 147–171.

[21] D. Jerison, *The Poincaré inequality for vector fields satisfying Hörmander's condition*, Duke Math. J., 53 (1986), pp. 503–523.

[22] D. Jerison and A. Sánchez-Calle, *Subelliptic, second order differential operators*, Complex analysis III, 1277 (1987), pp. 46–77.
[23] J. J. Kohn, *Subellipticity of the $\bar{\partial}$-Neumann problem on pseudo-convex domains: sufficient conditions*, Acta Math., 142 (1979), pp. 79–122.
[24] T. A. Kwembe and Z. B. Zhang, *A semilinear parabolic system with generalized Wentzell boundary condition*, Nonlinear Anal., 75 (2012), pp. 3078–3091.
[25] H. Lewy, *An example of a smooth linear partial differential equation without solution*, Ann. Math., 66 (1957), pp. 155–158.
[26] Y. C. Liu and J. S. Zhao, *On potential wells and applications to semilinear hyperbolic equations and parabolic equations*, Nonlinear Anal., 64 (2006), pp. 2665–2687.
[27] D. Morbidelli, *Fractional Sobolev norms and structure of Carnot-Carathéodory balls for Hörmander vector fields*, Studia Mathematica, 139 (2000), pp. 213–244.
[28] R. Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, Amer. Math. Soc., 91 (2002).
[29] G. Métivier, *Fonction spectrale et valeurs propres d'une classe d'opérateurs non elliptiques*, Comm. Partial Differ. Equ., 1 (1976), pp. 467–519.
[30] A. Nagel, E. M. Stein, and S. Wainger, *Balls and metrics defined by vector fields I, basic properties*, Acta Math., 155 (1985), pp. 103–147.
[31] L. E. Payne and D. H. Sattinger, *Saddle points and instability of nonlinear hyperbolic equations*, Isr. J. Math., 22 (1975), pp. 273–303.
[32] F. Quiros and J. D. Rossi, *Non-simultaneous blow-up in a semilinear parabolic system*, Z. Angew. Math. Phys., 52 (2001), pp. 342–346.
[33] J. D. Rossi and P. Souplet, *Coexistence of simultaneous and nonsimultaneous blow-up in a semilinear parabolic system*, Differ. Integral Equ., 18 (2005), pp. 405–418.
[34] L. P. Rothschild and E. M. Stein, *Hypoelliptic diferential operators and nilpotent groups*, Acta Math., 137 (1976), pp. 247–320.
[35] S. Sato, *Life span of solutions with large initial data for a semilinear parabolic system*, J. Math. Anal. Appl., 380 (2011), pp. 632–641.
[36] P. Souplet and S. Tayachi, *Optimal condition for non-simultaneous blow-up in a reaction-diffusion system*, J. Math. Soc. Japan, 56 (2004), pp. 571–584.
[37] M. X. Wang, *Global existence and finite time blow up for a reaction-diffusion system*, Z. Angew. Math. Phys., 51 (2000), pp. 160–167.
[38] R. Z. Xu, W. Lian and Y. Niu, *Global well-posedness of coupled parabolic systems*, Sci. China Math., 63 (2020), pp. 321–356.
[39] R. Z. Xu and J. Su,' *Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations*, Journal of Functional Analysis, 264 (2013), pp. 2732-2763.
[40] C. J. Xu, *Semilinear subelliptic equations and Sobolev inequality for vector fields satisfying Hörmander's condition*, Chinese J. Contemp. Math., 15 (1994), pp. 185–192.
[41] C. J. Xu, *Subelliptic variational problems*, Bull. Soc. Math. France, 118 (1990), pp. 147–169.
[42] X. Xu and Z. Ye, *Life span of solutions with large initial data for a class of coupled parabolic systems*, Z. Angew. Math. Phys., 64 (2013), pp. 705–717.
[43] P. L. Yung, *A sharp subelliptic Sobolev embedding theorem with weights*, Bull. London Math. Soc., 47 (2015), pp. 396–406.
[44] H. Zou, *Blow-up rates for semi-linear reaction-diffusion systems*, J. Differential Equations, 257 (2014), pp. 843–867.