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CONTACT DISCONTINUITY FOR A VISCOUS RADIATIVE AND REACTIVE GAS WITH LARGE INITIAL PERTURBATION*

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Dedicated to Professor Ling Hsiao on the occasion of her 80th birthday

Abstract. This paper is concerned with the time-asymptotically nonlinear stability of contact discontinuity to the Cauchy problem of a one-dimensional viscous radiative and reactive gas with large initial perturbation. Our main idea is to use the smallness of the strength of the contact discontinuity to control the possible growth of its solutions induced by the nonlinearity of the system and interactions between the solutions and the contact discontinuity. The key point in our analysis is to obtain the uniform positive lower and upper bounds on the specific volume and the absolute temperature.

Key words. Viscous radiative and reactive gas, contact discontinuity, nonlinear stability, large initial perturbation.

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1. Introduction and main result. This paper is concerned with the large-time behavior of global smooth large-amplitude solutions to the Cauchy problem for the 1D viscous radiative and reactive gas. The model consists of the following equations in the Lagrangian coordinates corresponding to the conservation laws of the mass, the momentum and the energy coupling with a reaction-diffusion equation (cf. [2, 21, 34])

$$
\begin{aligned}
v_t - u_x &= 0,\\
u_t + p(v,\theta)_x &= \left(\frac{\mu u_x}{v}\right)_x,\\
\left(e+\frac{u^2}{2}\right)_t + (up(v,\theta))_x &= \left(\frac{\mu u u_x}{v}\right)_x + \left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_x + \lambda\phi z,\\
z_t &= \left(\frac{d z_x}{v^2}\right)_x - \phi z.
\end{aligned}
\tag{1.1}
$$

Here $x \in \mathbb{R}$ is the Lagrangian space variable, $t \in \mathbb{R}^+$ the time variable and the primary dependent variables are the specific volume $v = v(t,x)$, the velocity $u = u(t,x)$, the absolute temperature $\theta = \theta(t,x)$ and the mass fraction of the reactant $z = z(t,x)$. The positive constants d and λ are the species diffusion coefficient and the difference in the heat between the reactant and the product, respectively. The reaction rate function $\phi = \phi(\theta)$ is defined, from the Arrhenius law [3, 34], by

$$
\phi(\theta) = K\theta^{\beta}\exp\left(-\frac{A}{\theta}\right), \tag{1.2}
$$

where positive constants K and A are the coefficients of the rates of the reactant and the activation energy, respectively, and β is a non-negative number.

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We treat the radiation as a continuous field and consider both the wave and photonic effect. Assume that the high-temperature radiation is at thermal equilibrium with the fluid. Then the pressure p and the specific internal energy e consist of a linear term in θ corresponding to the perfect polytropic contribution and a fourth-order radiative part due to the Stefan-Boltzmann radiative law [29, 34]:

$$p(v,\theta) = \frac{R\theta}{v} + \frac{a\theta^4}{3}, \quad e(v,\theta) = C_v\theta + av\theta^4, \tag{1.3}$$

where the positive constants R, C_v and a are the perfect gas constant, the specific heat and the radiation constant, respectively.

As in [21, 34, 35], we also assume that the bulk viscosity μ is a positive constant and the thermal conductivity $\kappa = \kappa(v,\theta)$ takes the form

$$\kappa(v,\theta) = \kappa_1 + \kappa_2 v\theta^b \tag{1.4}$$

with κ_1, κ_2 and b being some positive constants. Moreover, the initial data is given by

$$(v(t,x), u(t,x), \theta(t,x), z(t,x))|_{t=0} = (v_0(x), u_0(x), \theta_0(x), z_0(x)) \tag{1.5}$$

for $x \in \mathbb{R}$, which is assumed to satisfy the following far-field condition:

$$\lim_{|x|\to\infty} (v_0(x), u_0(x), \theta_0(x), z_0(x)) = (v_\pm, u_\pm, \theta_\pm, 0), \tag{1.6}$$

where $v_\pm > 0$, $u_\pm$ and $\theta_\pm > 0$ are constants.

Many results have been obtained recently on the global solvability and the precise description of the large time behavior of the global solutions constructed to the intial/initial-boundary value problems to the system (1.1), (1.2), (1.3), (1.4). To the best of our knowledge, the results for the corresponding initial-boundary value problems in bounded interval are quite complete, cf. [2, 13, 14, 31, 34, 35] and the references therein. For the Cauchy problem (1.1), (1.2), (1.3), (1.4), (1.5), (1.6), when the far-fields $(v_\pm, u_\pm, \theta_\pm)$ of the initial data $(v_0(x), u_0(x), \theta_0(x))$ are equal, i.e. $(v_-, u_-, \theta_-) = (v_+, u_+, \theta_+)$, its global solvability and the time asymptotically nonlinear stability of the trivial equilibrium state (v_-, u_-, θ_-) are obtained in [21]. We note that, however, that for the case when the far fields $(v_\pm, u_\pm, \theta_\pm)$ of the initial data $(v_0(x), u_0(x), \theta_0(x))$ are not equal, i.e., $(v_-, u_-, \theta_-) \neq (v_+, u_+, \theta_+)$, the asymptotics of the global solutions should be nontrivial and are expected to be determined by the unique global entropy solution $(V^r(x,t), U^r(x,t), \Theta^r(x,t), Z^r(t,x))$ of the following resulting Riemann problem of the corresponding compressible Euler equations

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v,\theta)_x &= 0, \\ \left(e + \frac{u^2}{2}\right)_t + (up(v,\theta))_x &= 0, \\ z_t &= 0 \end{aligned} \tag{1.7}$$

with Riemann data

$$\begin{aligned} (v(t,x), u(t,x), \theta(t,x), z(t,x))|_{t=0} &= (v_0^r(x), u_0^r(x), \theta_0^r(x), z_0^r(x)) \\ &= \begin{cases} (v_-, u_-, \theta_-, 0), & x < 0, \\ (v_+, u_+, \theta_+, 0), & x > 0. \end{cases} \end{aligned} \tag{1.8}$$

The case when the unique global entropy solution $(V^r(x,t), U^r(x,t), \Theta^r(x,t), Z^r(t,x))$ to the Riemann problem (1.7), (1.8) consists of only rarefaction waves is studied in [5] and the main purpose of this paper focuses on the case when the unique global entropy solution $(V^r(x,t), U^r(x,t), \Theta^r(x,t), Z^r(t,x))$ is a contact discontinuity. For such a case, we have, cf. [32]

$$u_- = u_+, \quad p_- = \frac{R\theta_-}{v_-} + \frac{a\theta_-^4}{3} = p_+ = \frac{R\theta_+}{v_+} + \frac{a\theta_+^4}{3}$$

and the unique contact discontinuity solution $\big(V^{CD}(t,x), U^{CD}(t,x), \Theta^{CD}(t,x), Z^{CD}(t,x)\big)$ of the Riemann problem (1.7),(1.8) takes the form

$$\begin{aligned}&\big(V^{CD}(t,x), U^{CD}(t,x), \Theta^{CD}(t,x), Z^{CD}(t,x)\big)\\ &= \begin{cases} (v_-, u_-, \theta_-, 0), & x<0,\ t>0\\ (v_+, u_+, \theta_+, 0), & x>0,\ t>0.\end{cases}\end{aligned} \tag{1.9}$$

To study the time asymptotically nonlinear stability of the contact discontinuity $\big(V^{CD}(t,x), U^{CD}(t,x), \Theta^{CD}(t,x), Z^{CD}(t,x)\big)$, as in [10, 12], we first construct its smooth approximation, i.e. the so-called "viscous contact wave" $(V(t,x), U(t,x), \Theta(t,x))$ as follows.

Without loss of generality, we assume in the rest of this paper that $u_- = u_+ = 0$, if we set

$$\frac{R\Theta}{V} + \frac{a\Theta^4}{3} = p_+, \tag{1.10}$$

from which one can get that

$$V = \frac{R\Theta}{p_+ - \frac{1}{3}a\Theta^4} \tag{1.11}$$

provided that

$$p_+ - \frac{1}{3}a\Theta^4 > 0. \tag{1.12}$$

Moreover, (1.10) indicates that the leading part of the energy equation $(1.1)_3$ is

$$E_t + p_+ U_x = \left(\kappa(V,\Theta)\frac{\Theta_x}{V}\right)_x, \tag{1.13}$$

where $E(V,\Theta) = C_v\Theta + aV\Theta^4$.

By $(1.1)_1$ and (1.10), (1.13) can be written as

$$\left[\frac{\partial E(V,\Theta)}{\partial\Theta} + \left(\frac{\partial E(V,\Theta)}{\partial V} + p_+\right)\frac{\partial V(\Theta)}{\partial\Theta}\right]\Theta_t = \left(\kappa(V,\Theta)\frac{\left(p_+ - \frac{a\Theta^4}{3}\right)}{R\Theta}\Theta_x\right)_x.$$

Since $\kappa(V,\Theta) = \kappa_1 + \kappa_2 V\Theta^b > 0$ and noticing that

$$\begin{aligned}\frac{\partial E(V,\Theta)}{\partial\Theta} &= C_v + 4aV\Theta^3 > 0,\\ \frac{\partial E(V,\Theta)}{\partial V} &= a\Theta^4 > 0,\\ \frac{\partial V}{\partial\Theta} &= \frac{\frac{R}{V} + \frac{4}{3}a\Theta^3}{\frac{R\Theta}{V^2}} > 0,\end{aligned}$$

if we set

$$A(\Theta)=\frac{\partial E(V,\Theta)}{\partial\Theta}+\left(\frac{\partial E(V,\Theta)}{\partial V}+p_+\right)\frac{\partial V(\Theta)}{\partial\Theta},\quad B(\Theta)=\frac{\left(p_+-\frac{a\Theta^4}{3}\right)\kappa(\Theta)}{R\Theta},$$

then we can get that

$$A(\Theta)\Theta_t=(B(\Theta)\Theta_x)_x\,.$$

If we further set

$$\frac{\mathrm{d}H(\Theta)}{\mathrm{d}\Theta}=A(\Theta),\quad \Lambda=H(\Theta),$$

since $A(\Theta)$ and $\kappa(\Theta)$ are smooth enough, so is $H(\Theta)$ and Λ is well defined and (1.13) can be written as

$$\Lambda_t=\left(\frac{B\left(H^{-1}(\Lambda)\right)}{H'(\Lambda)}\Lambda_x\right)_x,\quad \Lambda(t,\pm\infty)=H(\theta_\pm). \tag{1.14}$$

It is easy to see that if (1.12) holds true, one can get that $B(\Theta)>0$, $H'(\Theta)>0$, $\Theta=H^{-1}(\Lambda)>0$, then one can get from [4, 6] that the two-point boundary value problem (1.14) admits a self-similar solution $\Lambda(t,x)=\Lambda\left(\frac{x}{\sqrt{1+t}}\right)$ which is unique up to a shift. Furthermore, $\Lambda(\zeta)$ is monotone, increasing if $H(\theta_+)>H(\theta_-)$ and decreasing if $H(\theta_-)>H(\theta_+)$.

As to the condition (1.12), noticing that $p_-=p_+$ together with the fact that $\Theta(\zeta)$ is monotonic, one can easily deduce from the continuation argument that

$$p_+-\frac{1}{3}a\Theta^4\geq\min\left\{\frac{R\theta_-}{v_-},\frac{R\theta_+}{v_+}\right\}>0,$$

which means that (1.12) holds true.

Consequently, the two-point boundary value problem (1.14) admits a self-similiar solution $\Lambda(t,x)=\Lambda\left(\frac{x}{\sqrt{1+t}}\right)$ which is unique up to a shift. Moreover, one can get that there exists a positive constant C depending only on $\theta_\pm$ such that Λ satisfies

$$\begin{aligned}&(1+t)\left|\Lambda_{xx}\left(\frac{x}{\sqrt{1+t}}\right)\right|+(1+t)^{\frac{1}{2}}\left|\Lambda_x\left(\frac{x}{\sqrt{1+t}}\right)\right|+\left|\Lambda\left(\frac{x}{\sqrt{1+t}}\right)-H(\theta_\pm)\right|\\&\leq O(1)\delta e^{-\frac{Cx^2}{1+t}},\quad \pm x\geq0.\end{aligned} \tag{1.15}$$

Here $\delta:=|\theta_--\theta_+|$ denotes the strength of the contact discontinuity and $O(1)$ is some generic positive constant independent of δ.

Since (1.12) holds true, $\Theta\left(\frac{x}{\sqrt{1+t}}\right)\in[\min\{\theta_\pm\},\max\{\theta_\pm\}]$ and $H'(\Theta)$ is continuous, (1.15) leads to

$$\begin{aligned}&(1+t)\left|\Theta_{xx}\left(\frac{x}{\sqrt{1+t}}\right)\right|+(1+t)^{\frac{1}{2}}\left|\Theta_x\left(\frac{x}{\sqrt{1+t}}\right)\right|+\left|\Theta\left(\frac{x}{\sqrt{1+t}}\right)-\theta_\pm\right|\\&\leq O(1)\delta e^{-\frac{C_1x^2}{1+t}},\quad \pm x\geq0,\end{aligned} \tag{1.16}$$

where $O(1)$ is some positive constant depending only on $\theta_\pm$.

Once $\Theta(t,x)$ is determined, the viscous contact wave profile $(V(t,x), U(t,x), \Theta(t,x))$ is then defined as follows:

$$V = \frac{R\Theta}{p_+ - \frac{a\Theta^4}{3}}, \quad U_x = V_t. \tag{1.17}$$

As shown in [10], the above constructed viscous contact wave $(V(t,x), U(t,x), \Theta(t,x))$ is a nice approximation of the contact discontinuity $\big(V^{CD}(t,x), U^{CD}(t,x), \Theta^{CD}(t,x), Z^{CD}(t,x)\big)$ and it is easy to see that the viscous contact wave $(V(t,x), U(t,x), \Theta(t,x), Z(t,x))$ solves (1.1) time asymptotically, that is

$$\begin{aligned} V_t - U_x &= 0, \\ U_t + P(V,\Theta)_x &= \mu \left(\frac{U_x}{V}\right)_x + R_1, \\ \left(C_v\Theta + aV\Theta^4\right)_t + P(V,\Theta)U_x &= \left(\frac{\kappa(V,\Theta)\Theta_x}{V}\right)_x + \mu\frac{U_x^2}{V} + R_2, \\ Z &= 0, \end{aligned} \tag{1.18}$$

where

$$R_1 = U_t - \mu\left(\frac{U_x}{V}\right)_x, \quad R_2 = -\mu\frac{U_x^2}{V}. \tag{1.19}$$

With the above preparations in hand, our main result can be summarized as follows:

Theorem 1.1. *Suppose that*

- *The parameters b and β are assumed to satisfy*

$$b > \frac{17}{4}, \quad 0 \le \beta < b + 6;$$

- *There exist positive constants $0 < \underline{V} < 1$, $\overline{V} > 1$, $0 < \underline{\Theta} < 1$, $\overline{\Theta} > 1$ such that $2\underline{V} \le v_0(x),\ V(t,x) \le \frac{1}{2}\overline{V}$, $2\underline{\Theta} \le \theta_0(x), \Theta(t,x) \le \frac{1}{2}\overline{\Theta}$, $\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}$, and*

$$\begin{aligned} (v_0(x) - V(0,x), u_0(x) - U(0,x), \theta_0(x) - \Theta(0,x), z_0(x)) &\in H^1(\mathbb{R}), \\ \partial_{xx}(u_0(x) - U(0,x)) \in L^2(\mathbb{R}), \quad z_0(x) &\in L^1(\mathbb{R}), \\ 0 \le z_0(x) \le 1, \quad \forall x &\in \mathbb{R}. \end{aligned}$$

Then for any $m_0 > 0$, there exists a sufficiently small positive constant δ_0, which depends only on $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ and m_0, such that if $0 < \delta < \delta_0$ and $\|(v_0(x) - V(0,x), u_0(x) - U(0,x), \theta_0(x) - \Theta(0,x), z_0(x))\|_{H^1(\mathbb{R})} \le m_0$, the Cauchy problem (1.1)-(1.6) admits a unique global solution $(v(t,x), u(t,x), \theta(t,x), z(t,x))$ which satisfies

$$\begin{aligned} \overline{C}_1^{-1} &\le v(t,x) \le \overline{C}_1, \\ \overline{C}_2^{-1} &\le \theta(t,x) \le \overline{C}_2, \\ 0 &\le z(t,x) \le 1 \end{aligned}$$

for all $(t,x)\in[0,\infty)\times\mathbb{R}$ *and*

$$
\begin{aligned}
&\|(v-V,u-U,\theta-\Theta,z)(t)\|^2_{H^1(\mathbb{R})}\\
&+\int_0^\infty\left(\|\partial_x(v-V)(\tau)\|^2_{L^2(\mathbb{R})}+\left\|\left(\partial_x(u-U),\partial_x(\theta-\Theta),\partial_x z\right)(\tau)\right\|^2_{H^1(\mathbb{R})}\right)d\tau\\
\le{}&\overline{C}_3
\end{aligned}
$$

holds for $t\ge 0$. *Here* $\overline{C}_1$, $\overline{C}_2$ *and* $\overline{C}_3$ *are some uniform positive constants which depend only on* $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ *and* m_0.

Moreover, it holds that

$$
\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}\left\{\left|\Big(v(t,x)-V(t,x),u(t,x)-U(t,x),\theta(t,x)-\Theta(t,x),z(t,x)\Big)\right|\right\}=0.
$$

Now we outline the main ideas used to yield our main result. The key point in our analysis is to deduce the desired uniform positive lower and upper bounds on the specific volume $v(t,x)$ and the absolute temperature $\theta(t,x)$ and the main difficulties lie in how to control the possible growth of the solutions $(v(t,x),u(t,x),\theta(t,x),z(t,x))$ induced by the nonlinearity of the equations and the interaction between the viscous contact wave $(V(t,x),U(t,x),\Theta(t,x))$ and the solutions $(v(t,x),u(t,x),\theta(t,x),z(t,x))$.

Our strategy to overcome the above difficulties can be summarized as in the following:

- We use the smallness of δ, the strength of the contact wave $(V^{CD}(t,x),U^{CD}(t,x),\Theta^{CD}(t,x))$, to estimate the possible growth of the solutions $(v(t,x),u(t,x),\theta(t,x),z(t,x))$ induced by the nonlinearities of equations under our consideration and the interactions between the viscous contact wave $(V(t,x),U(t,x),\Theta(t,x))$ and the solutions $(v(t,x),u(t,x),\theta(t,x),z(t,x))$;
- The specific volume $v(t,x)$ is shown to be uniformly bounded from below and above with respect to both the space variable and the time variable through a delicate analysis based on the basic energy type estimate and the cut-off technique introduced in [15] for 1D compressible Navier-Stokes equations for ideal polytropic gases and used in [21] for a viscous radiative and reactive gas;
- Motivated by [19, 21], we introduce the auxiliary functions $X(t)$, $Y(t)$, and $Z(t)$ (cf. (4.1)) to deduce the upper bound of $\theta(t,x)$. Compared with the analysis in [21], we do not need to introduce auxiliary function "$W(t)$" (cf. (2.51) and (2.70) in [21]).

Before concluding this section, we notice that there has been extensive literature on the stability analysis of basic wave patterns to the Cauchy problem of the one-dimensional compressible Navier-Stokes equations with positive constant transport coefficients, we refer to [17, 22, 25, 26, 33] for the viscous shock waves, [1, 18, 23, 27, 28, 30] for the rarefaction waves, [7, 10, 12, 24] for the viscous contact waves, and [8, 9, 11] for certain superpositions of the above three types of basic wave patterns. For the corresponding time asymptotically stability of the above three types of basic wave patterns and/or their suitable superpositions with large initial perturbation, we refer to [1, 7, 11, 18, 28, 30, 33] and the references cited therein.

The rest of the paper is arranged as follows: In Section 2, we give some properties of the viscous contact wave and then deduce some basic energy type estimates. In Section 3, we derive the uniform-in-time positive lower and upper bounds of the specific volume $v(t,x)$. And in Section 4, we obtain the uniform-in-time upper bound of the absolute temperature $\theta(t,x)$. Then the higher order energy type estimates will be given in Section 5. Furthermore, a local-in-time positive lower bound on the absolute temperature $\theta(t,x)$ and proof Theorem 1.1 are given in Section 6.

NOTATIONS. Throughout the rest of this paper, C, $O(1)$, C_i, $\overline{C}_i$ or D_i for $i\in\mathbb{N}$ will be used to denote a generic positive constant which is independent of t, δ and x but may depends on $v_\pm$, $u_\pm$, $\theta_\pm$, $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ and m_0. $C_i(\cdot,\cdot)(i\in\mathbb{Z}_+)$ stands for some generic constants depending only on the quantities listed in the parentheses and ϵ represents some small positive constant. Note that all these constants may vary from line to line.

For two functions $f(x)$ and $g(x)$, $f(x)\sim g(x)$ as $x\to x_0$ means that there exists a generic positive constant $C>0$ which is independent of t, δ and x but may depend on $v_\pm$, $u_\pm$, $\theta_\pm$, $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ and m_0 such that $C^{-1}f(x)\leq g(x)\leq Cf(x)$ in a neighborhood of x_0. $B\lesssim B'$ means that there is a generic positive constant $C>0$ independent of t, δ and x such that $B\leq CB'$, while $B\sim B'$ means that $B\lesssim B'$ and $B'\lesssim B$. $H^l(\mathbb{R})(l\geq 0)$ denotes the usual Sobolev space with standard norm $\|\cdot\|_l$, and $\|\cdot\|_0=\|\cdot\|$ will be used to denote the usual L^2-norm. For $1\leq p\leq+\infty$, $f(x)\in L^p(\mathbb{R})$, $\|f\|_{L^p}=(\int_{\mathbb{R}}|f(x)|^p\mathrm{d}x)^{\frac{1}{p}}$. It is easy to see that $\|f\|_{L^2}=\|\cdot\|$. Finally, $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_\infty$ are used to denoted $\|\cdot\|_{L^\infty(\mathbb{R})}$ and $\|\cdot\|_{L^\infty([0,t]\times\mathbb{R})}$ respectively.

2. Preliminaries. For the properties of the viscous contact wave $(V(t,x), U(t,x),\Theta(t,x),Z(t,x))$ defined by (1.17), we have

LEMMA 2.1. *Assume that* $\delta=|\theta_+-\theta_-|$. *Then there exists a positive constant* C_1 *depending only on* $\theta_\pm$ *such that the viscous contact wave* $(V(t,x),U(t,x),\Theta(t,x),Z(t,x))$ *defined in* (1.17) *has the following properties:*

$$\begin{aligned}|V(t,x)-v_\pm|+|\Theta(t,x)-\theta_\pm|\leq O(1)\delta e^{-\frac{C_1x^2}{1+t}},\quad \pm x\geq 0,\\ |V_t(t,x)|+|\Theta_t(t,x)|\leq O(1)(1+t)^{-1}\delta e^{-\frac{C_1x^2}{1+t}},\\ \left|\partial_x^kV(t,x)\right|+\left|\partial_x^{k-1}U(t,x)\right|+\left|\partial_x^k\Theta(t,x)\right|\leq O(1)\delta(1+t)^{-\frac{k}{2}}e^{-\frac{C_1x^2}{1+t}},\quad k\geq 1.\end{aligned}\tag{2.1}$$

Consequently, one can deduce from (2.1) *and the assumptions imposed in Theorem 1.1 that*

$$\underline{V}\leq V(t,x)\leq\overline{V},\quad \underline{\Theta}\leq\Theta(t,x)\leq\overline{\Theta},\tag{2.2}$$

$$|R_1|\leq O(1)\delta(1+t)^{-\frac{3}{2}}e^{-\frac{C_1x^2}{1+t}},\quad |R_2|\leq O(1)\delta(1+t)^{-2}e^{-\frac{C_1x^2}{1+t}}.\tag{2.3}$$

The following lemma will play important roles to obtain the basic energy type estimates, whose proofs can be found in [8].

LEMMA 2.2. *For* $0<T\leq\infty$, *if we assume* $h(t,x)$ *satisfies*

$$h(t,x)\in L^\infty(0,T;L^2(\mathbb{R})),\ h_x(t,x)\in L^2(0,T;L^2(\mathbb{R})),\ h_t(t,x)\in L^2(0,T;H^{-1}(\mathbb{R})),$$

then

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}}h^2w^2dxdt\\ &\leq 4\pi\|h(0)\|^2+4\pi\alpha^{-1}\int_0^T\|h_x(t)\|^2dt+8\alpha\int_0^T\langle h_t(t),h(t)g(t)^2\rangle_{H^{-1}\times H^1}dt\end{aligned}\tag{2.4}$$

holds for each $\alpha > 0$. Here

$$w(t,x) = (1+t)^{-\frac{1}{2}} e^{-\frac{\alpha x^2}{1+t}}, \quad g(t,x) = \int_{-\infty}^{x} w(y,t)dy. \tag{2.5}$$

If we define the perturbation $(\varphi(t,x), \psi(t,x), \zeta(t,x))$ by

$$(\varphi(t,x), \psi(t,x), \zeta(t,x)) = (v(t,x) - V(t,x), u(t,x) - U(t,x), \theta(t,x) - \Theta(t,x)), \tag{2.6}$$

then by combining (1.18) with (1.1),(1.2), one can find that the perturbation $(\varphi(t,x), \psi(t,x), \zeta(t,x), z(t,x))$ solves

$$\begin{aligned} \varphi_t - \psi_x &= 0, \\ \psi_t + (p - p_+)_x &= \mu \left(\frac{u_x}{v} - \frac{U_x}{V} \right)_x - R_1, \\ \left(C_v \zeta + av\theta^4 - aV\Theta^4\right)_t + (pu_x - p_+ U_x) &= \left(\frac{\kappa(v,\theta)\theta_x}{v} - \frac{\kappa(V,\Theta)\Theta_x}{V} \right)_x \\ &\quad + \mu \left(\frac{u_x^2}{v} - \frac{U_x^2}{V} \right) - R_2 + \lambda \phi z, \\ z_t &= \left(\frac{d z_x}{v^2} \right)_x - \phi z, \\ (\varphi(t,x), \psi(t,x), \zeta(t,x), z(t,x))\,|_{t=0} &= (\varphi_0(x), \psi_0(x), \zeta_0(x), z_0(x)), \quad x \in \mathbb{R}. \end{aligned} \tag{2.7}$$

Here $R_i (i = 1,2)$ are defined by (1.19).

Suppose that $(\varphi(t,x), \psi(t,x), \zeta(t,x))$ is a suitably regular solution of the Cauchy problem (2.7) defined on $[0,T] \times \mathbb{R}$ for some $T > 0$, then we can get from Lemma 2.2 that

LEMMA 2.3. *Let $w(t,x)$ be defined by* (2.5) *and suppose that $(\varphi(t,x), \psi(t,x), \zeta(t,x))$ is a suitably regular solution of the Cauchy problem* (2.7) *defined on* $[0,T] \times \mathbb{R}$ *satisfying*

$$\begin{gathered} M_1^{-1} \le v(t,x) \le M_1, \ M_2^{-1} \le \theta(t,x) \le M_2, \ |u(t,x)| \le M_3, \quad \forall (t,x) \in [0,T) \times \mathbb{R}, \\ M := \sup_{\tau \in [0,t)} \|\varphi(\tau)\| + \sup_{\tau \in [0,t)} \|\psi(\tau)\| + \sup_{\tau \in [0,t)} \|\zeta(\tau)\|, \end{gathered}$$

then for $\alpha \in (0, \frac{C_1}{4}]$ with C_1 being the positive constant appeared in the estimate (2.1), *there exists a positive constant C, which depends only on α, such that the following estimate holds for $0 \le t \le T$*

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} \left(\varphi^2 + \psi^2 + \zeta^2\right) w^2 dx ds \\ &\lesssim M_1^9 M_2^{3b+16} M^6 + M_1^7 M_2^{b+13} M_3 M^2 \int_0^T \int_{\mathbb{R}} \phi z dx dt \\ &\quad + M_1^9 M_2^{3b+16} M_3 M^6 \int_0^T \int_{\mathbb{R}} \left(\varphi_x^2 + \psi_x^2 + \zeta_x^2\right) dx dt. \end{aligned} \tag{2.8}$$

Proof. For the corresponding nonlinear stability result with small initial perturbation for 1D compressible Navier-Stokes equations, such a lemma is proved in [8].

Since our main purpose here is to consider the nonlinear stability result with large initial perturbation, we need to show the dependence of the right hand side of the estimate (2.8) on the solution $(\varphi(t,x),\psi(t,x),\zeta(t,x))$.

To prove (2.8), it is easy to see that

$$p(v,\theta)-p_+=v^{-1}\left(\left(R-\frac{C_v}{3}\right)\zeta+\frac{1}{3}\xi-p_+\varphi\right),$$

where $\xi:=C_v\zeta+\left(av\theta^4-aV\Theta^4\right)$.

Denoting by

$$f=\int_{-\infty}^{x}w^2(t,y)dy,$$

one can get that

$$\|f(\cdot,t)\|_{L^\infty}\le(1+t)^{-\frac{1}{2}},\quad \|f_t(\cdot,t)\|_{L^\infty}\le(1+t)^{-\frac{3}{2}}. \tag{2.9}$$

Multiplying $(2.7)_2$ by $v^2(p-p_+)f$, integrating the resulting equation over $\mathbb{R}$ leads to

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}}\left[v(p-p_+)\right]^2w^2dx\\
=&\left(\int_{\mathbb{R}}\psi v^2(p-p_+)fdx\right)_t-\int_{\mathbb{R}}vv_t\psi(p-p_+)fdx-\int_{\mathbb{R}}v\psi f\left[v(p-p_+)\right]_tdx\\
&-\int_{\mathbb{R}}v^2\psi(p-p_+)f_tdx-\int_{\mathbb{R}}(p-p_+)^2vv_xfdx+\mu\int_{\mathbb{R}}\left(\frac{u_x}{v}-\frac{U_x}{V}\right)\left[v^2(p-p_+)f\right]_xdx\\
&+\int_{\mathbb{R}}R_1v^2(p-p_+)fdx \qquad (2.10)\\
:=&\left(\int_{\mathbb{R}}\psi v^2(p-p_+)fdx\right)_t+\sum_{i=1}^{6}K_i.
\end{aligned}$$

Equation $(2.7)_3$ tells us that

$$\begin{aligned}
K_2=&-\left(R-\frac{C_v}{3}\right)\int_{\mathbb{R}}\psi vf\zeta_tdx+\frac{1}{3}\int_{\mathbb{R}}\psi vf(p-p_+)u_xdx\\
&+\frac{1}{3}\int_{\mathbb{R}}\left(\frac{\kappa(v,\theta)\theta_x}{v}-\frac{\kappa(V,\Theta)\Theta_x}{V}\right)(\psi vf)_xdx\\
&-\frac{\mu}{3}\int_{\mathbb{R}}\psi vf\left(\frac{u_x^2}{v}-\frac{U_x^2}{V}\right)dx+\frac{1}{3}\int_{\mathbb{R}}R_2\psi vfdx-\frac{1}{3}\int_{\mathbb{R}}\lambda\phi z\psi vfdx \qquad (2.11)\\
&-\frac{2}{3}\int_{\mathbb{R}}p_+v_xf\psi^2dx-\frac{2}{3}\int_{\mathbb{R}}p_+v\psi^2w^2dx\\
:=&-\left(R-\frac{C_v}{3}\right)\int_{\mathbb{R}}\psi vf\zeta_tdx+\sum_{i=1}^{7}K_2^i.
\end{aligned}$$

Using Hölder inequality and Young inequality, we can deduce from Lemma 2.1

and (2.9) that

$$|K_1| + |K_2^1| \le CM\|\psi_x\|^2 + CM(1+t)^{-\frac{5}{3}}, \tag{2.12}$$

$$|K_2^2| \le CM_1^4 M_2^b (1+t)^{-\frac{1}{2}} \left(\|\psi_x\|^2 + \|\zeta_x\|^2 + \|\varphi_x\|^2\right) + CM_1^4 M_2^b M^2 (1+t)^{-\frac{3}{2}}, \tag{2.13}$$

$$|K_2^3| \le CM_1^2 M_3 \|\psi_x\|^2 + CM_1^2 M_3 M^2 (1+t)^{-\frac{3}{2}}, \tag{2.14}$$

$$|K_2^4| \le CM_1 M_2 M^2 (1+t)^{-\frac{3}{2}}, \tag{2.15}$$

$$|K_2^5| \le CM_1 M_3 \int_{\mathbb{R}} \phi z dx, \tag{2.16}$$

$$|K_2^6| \le C\delta \int_{\mathbb{R}} \psi^2 w^2 dx + CM\left(\|\psi_x\|^2 + \|\varphi_x\|^2\right) + CM^4(1+t)^{-2}, \tag{2.17}$$

$$|K_2^7 + I_2^6| \le -\frac{1}{2}\int_{\mathbb{R}} p_+ v\psi^2 w^2 dx + CM\left(\|\psi_x\|^2 + \|\varphi_x\|^2\right) + CM^4(1+t)^{-2}, \tag{2.18}$$

$$|K_3| \le CM_1^3 M_2^8 M^2 (1+t)^{-\frac{3}{2}}, \tag{2.19}$$

$$|K_4| \le CM_1^3 M_2^8 \delta \int_{\mathbb{R}} (\zeta^2 + \varphi^2) w^2 dx + CM_1^3 M_2^8 \left(\|\zeta_x\|^2 + \|\varphi_x\|^2\right) + CM_1^3 M_2^8 M^6 (1+t)^{-2}, \tag{2.20}$$

$$|K_5| \le CM_1^3 M_2^4 \left(\|\zeta_x\|^2 + \|\psi_x\|^2 + \|\varphi_x\|^2\right) + CM_1^3 M_2^4 M^2 (1+t)^{-2}, \tag{2.21}$$

$$|K_6| \le CM_1^2 M_2^4 M^2 (1+t)^{-\frac{3}{2}}. \tag{2.22}$$

Combining (2.10), (2.11), (2.12)-(2.22), we can get that

$$\begin{aligned} &\frac{1}{2}\int_0^T \int_{\mathbb{R}} \left\{ \left[\left(R - \frac{C_v}{3}\right)\zeta + \frac{C_v}{3}\xi - p_+\varphi \right]^2 + p_+ v\psi^2 \right\} w^2 dx \\ \le& -\left(R - \frac{C_v}{3}\right)\int_{\mathbb{R}} \psi v f \zeta_t dx + CM_1^3 M_2^{b+8} M^3 \int_0^T \left(\|\zeta_x\|^2 + \|\psi_x\|^2 + \|\varphi_x\|^2\right) dt \\ &+ CM_1 M_3 \int_0^T \int_{\mathbb{R}} \phi z dx dt + CM_1^4 M_2^{b+8} M^6. \end{aligned} \tag{2.23}$$

Similarly, if we take $h = C_v\xi + p_+\varphi$ in Lemma 2.2, we can deduce from $(2.7)_3$ that

$$\begin{aligned} \langle h_t, hg^2 \rangle_{H^{-1}\times H^1} + J_1 \le& C\delta M_1^4 M_2^{b+5}(\delta+\eta)\int_{\mathbb{R}} (\zeta^2+\varphi^2) w^2 dx \\ &+ \frac{C}{\eta} M_1^4 M_2^{b+5} M_3 \left(\|\zeta_x\|^2 + \|\psi_x\|^2 + \|\varphi_x\|^2\right) \\ &+ CM_1^2 M_2^4 \int_{\mathbb{R}} \phi z dx + CM_1^3 M_2^8 (1+t)^{-\frac{3}{2}}, \end{aligned} \tag{2.24}$$

where $\eta > 0$ can be chosen sufficiently small and

$$J_1 = -\int_{\mathbb{R}} h(p - p_+)\psi_x g^2 dx.$$

Since $\varphi_t = \psi_x$, one can deduce that

$$
\begin{aligned}
-3J_1 &= \left\{ \int_{\mathbb{R}} v^{-1} h g^2 \varphi \left[h + (3R - C_v)\zeta - 2p_+\varphi \right] dx \right\}_t \\
&\quad - \frac{1}{2\alpha} \int_{\mathbb{R}} v^{-1} h g \varphi \left[h + (3R - C_v)\zeta - 2p_+\varphi \right] w_x dx \\
&\quad + \frac{1}{3} \int_{\mathbb{R}} v^{-2} u_x g^2 \varphi \left[(h + (3R - C_v)\zeta - 4p_+\varphi)(h - (3R - C_v)\zeta + 4p_+\varphi) + 2p_+\varphi h \right] dx \\
&\quad + \int_{\mathbb{R}} \left(\frac{\kappa(v,\theta)\theta_x}{v} - \frac{\kappa(V,\Theta)\Theta_x}{V} \right) \left\{ v^{-1} g^2 \varphi \left[2h + (3R - C_v)\zeta - 4p_+\varphi \right] \right\}_x dx \qquad (2.25) \\
&\quad - \mu \int_{\mathbb{R}} \left(\frac{u_x^2}{v} - \frac{U_x^2}{V} \right) v^{-1} g^2 \varphi \left[2h + (3R - C_v)\zeta - 4p_+\varphi \right] dx \\
&\quad + \int_{\mathbb{R}} R_2 v^{-1} g^2 \varphi \left[2h + (3R - C_v)\zeta - 4p_+\varphi \right] dx \\
&\quad - \int_{\mathbb{R}} \lambda \phi z v^{-1} g^2 \varphi \left[2h + (3R - C_v)\zeta - 4p_+\varphi \right] dx - (3R - C_v) \int_{\mathbb{R}} v^{-1} g^2 h \zeta_t dx \\
&:= \left\{ \int_{\mathbb{R}} v^{-1} h g^2 \varphi \left[h + (3R - C_v)\zeta - 2p_+\varphi \right] dx \right\}_t - (3R - C_v) \int_{\mathbb{R}} v^{-1} g^2 h \zeta_t dx + \sum_{i=1}^{6} J_1^i.
\end{aligned}
$$

Similarly, one can get that

$$
|J_1^1| \le CM_1^3 M_2^8 \left(\|\zeta_x\|^2 + \|\varphi_x\|^2 \right) CM_1^3 M_2^8 M^{-\frac{10}{3}} (1+t)^{-\frac{4}{3}}, \qquad (2.26)
$$

$$
|J_1^2| \le CM_1^4 M_2^8 M^2 \left(\|\zeta_x\|^2 + \|\psi_x\|^2 + \|\varphi_x\|^2 \right) CM_1^4 M_2^8 M^2 (1+t)^{-\frac{3}{2}}, \qquad (2.27)
$$

$$
|J_1^3| \le CM_1^6 M_2^{2b+8} \left(\|\zeta_x\|^2 + \|\varphi_x\|^2 \right) + CM_1^6 M_2^{2b+8} (1+t)^{-\frac{4}{3}}, \qquad (2.28)
$$

$$
|J_1^4| \le CM_1^4 M_2^4 M_3 \left(\|\psi_x\|^2 + \|U_x\|^2 \right), \qquad (2.29)
$$

$$
|J_1^5| \le CM_1^3 M_2^4 (1+t)^{-\frac{3}{2}}, \qquad (2.30)
$$

$$
|J_1^6| \le CM_1^3 M_2^4 \int_{\mathbb{R}} \phi z dx. \qquad (2.31)
$$

Combining (2.24), (2.25), (2.26)-(2.31), we can finally get that

$$
\begin{aligned}
\int_0^T \int_{\mathbb{R}} (C_v \xi + p_+ \varphi)^2 w^2 &\le \left(R - \frac{C_v}{3} \right) \int_{\mathbb{R}} v^{-1} g^2 h \zeta_t dx \\
&\quad + CM_1^4 M_2^{2b+8} M_3 \int_0^T \left(\|\zeta_x\|^2 + \|\psi_x\|^2 + \|\varphi_x\|^2 \right) dt \qquad (2.32) \\
&\quad + CM_1^3 M_2^4 \int_0^T \int_{\mathbb{R}} \phi z dx dt + CM_1^6 M_2^{2b+8}.
\end{aligned}
$$

Combining (2.32) and (2.23), one can deduce

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \left(\varphi^2 + \psi^2 + \zeta^2 \right) w^2 dx ds \\
&\le M_1^9 M_2^{3b+16} M^6 + CM_1^3 M_2^8 \left| \int_0^T \int_{\mathbb{R}} \left(\psi v f + v^{-1} g^2 h \right) \zeta_t dx dt \right| \qquad (2.33) \\
&\quad + M_1^7 M_2^{b+13} M_3 M^2 \int_0^T \int_{\mathbb{R}} \phi z dx dt + M_1^9 M_2^{3b+16} M_3 M^6 \int_0^T \int_{\mathbb{R}} \left(\varphi_x^2 + \psi_x^2 + \zeta_x^2 \right) dx dt.
\end{aligned}
$$

Noticing that ζ_t satisfies

$$\begin{aligned}
&\left(C_v + av\theta^3\right)\zeta_t + \left[\left(R - \frac{4C_v}{3}\right)\zeta + \frac{4C_v}{3}\xi - p_+\varphi\right] v^{-1}u_x \\
&+(p_+ + a\Theta^4)U_x + 4a[\varphi\Theta^3 + \zeta(\theta^2 + \theta\Theta + \Theta^2)]\Theta_t \qquad (2.34)\\
&= \left(\frac{\kappa(v,\theta)\theta_x}{v} - \frac{\kappa(V,\Theta)\Theta_x}{V}\right)_x + \mu\left(\frac{u_x^2}{v} - \frac{U_x^2}{V}\right) - R_2 + \lambda\phi z,
\end{aligned}$$

from (2.34) and by employing similar method, the second term in the right hand side of (2.33) can be estimated as follows

$$\begin{aligned}
&\left|\int_0^T \int_{\mathbb{R}} \left(\psi v f + v^{-1}g^2 h\right)\zeta_t dxdt\right| \\
&\leq CM_1^6 M_2^{b+12} M^6 + CM_1^5 M_2^{b+12} M^3 \int_0^T \left(\|\zeta_x\|^2 + \|\psi_x\|^2 + \|\varphi_x\|^2\right) dt \qquad (2.35)\\
&+CM_1^3 M_2^4 M_3 \int_0^T \int_{\mathbb{R}} \phi z dxdt.
\end{aligned}$$

Putting the above estimates (2.33) and (2.35) together, we can then complete the proof of Lemma 2.3. □

To prove Theorem 1.1, for some positive constants $0 < T \leq +\infty, M_i > 0 (i = 1,2,3)$, we first define $X\left([0,T]; M_1, M_2, M_3\right)$, the set of functions for which we seek the solution $(\varphi(t,x), \psi(t,x), \zeta(t,x), z(t,x))$ of the Cauchy problem (2.7), as follows

$$X\left([0,T]; M_1, M_2, M_3\right) := \left\{ (\varphi(t,x), \psi(t,x), \zeta(t,x), z(t,x)) \;\middle|\; \begin{array}{c}
(\varphi(t,x), \psi(t,x), \zeta(t,x)) \in C\left(0,T; H^1(\mathbb{R})\right), \\
(\psi_x(t,x), \zeta_x(t,x), z_x(t,x)) \in L^2\left(0,T; H^1(\mathbb{R})\right), \\
\psi_{xx}(t,x) \in L^2(\mathbb{R}), \\
\varphi_x(t,x) \in L^2\left(0,T; L^2(\mathbb{R})\right), \\
M_1^{-1} \leq V(t,x) + \varphi(t,x) \leq M_1, \forall (t,x) \in [0,T]\times\mathbb{R}, \\
M_2^{-1} \leq \Theta(t,x) + \zeta(t,x) \leq M_2, \forall (t,x) \in [0,T]\times\mathbb{R}, \\
|U(t,x) + \psi(t,x)| \leq M_3, \forall (t,x) \in [0,T]\times\mathbb{R}, \\
z(t,x) \in C\left([0,T]; H^1(\mathbb{R}) \cap L^1(\mathbb{R})\right), \\
0 \leq z(t,x) \leq 1.
\end{array} \right\}.$$

For the local solvability of the Cauchy problem (2.7), we can get that (cf. [32])

Proposition 2.4 (Local solvability result). *Let the initial data $(v_0(x), u_0(x), \theta_0(x), z_0(x))$ satisfy the conditions stated in Theorem 1.1, then there exists a sufficiently small positive constant $t_1 > 0$, which depends only on $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ and m_0, such that the Cauchy problem (2.7) admits a unique solution $(\varphi(t,x), \psi(t,x), \zeta(t,x), z(t,x)) \in X\left([0,t_1]; \max\{\underline{V}^{-1}, \overline{V}\}, \max\{\underline{\Theta}^{-1}, \overline{\Theta}\}, 2(\|U(0,x)\|_{L^\infty} + m_0)\right)$ and satisfies*

$$\left\|\left(\varphi(t), \psi(t), \zeta(t), z(t)\right)\right\|_1 \leq 2\left\|(v_0(x), u_0(x), \theta_0(x), z_0(x))\right\|_1$$

for all $t \in [0,t_1]$.

Suppose now that the local solution $(\varphi(t,x), \psi(t,x), \zeta(t,x), z(t,x))$ constructed in Proposition 2.4 has been extended to the time interval $[0,T]$ $(T \geq t_1)$ and satisfies $(\varphi(t,x), \psi(t,x), \zeta(t,x), z(t,x)) \in X\left([0,T]; M_1, M_2, M_3\right)$ for some positive constants M_i $(i = 1,2,3)$ satisfying $M_1 \geq \max\{\underline{V}^{-1}, \overline{V}\}, M_2 \geq \max\{\underline{V}^{-1}, \overline{V}\}$ and $M_3 \geq 2(\|U(0,x)\|_{L^\infty} + m_0)$ (without loss of generality, we can assume that $M_i \geq 1 (i = 1,2,3)$ in the rest of this paper), we now turn to deduce the following a priori estimates on $(\varphi(t,x), \psi(t,x), \zeta(t,x), z(t,x))$.

Proposition 2.5 (A priori estimates). *Let $(\varphi(t,x), \psi(t,x), \zeta(t,x), z(t,x)) \in X([0,T]; M_1, M_2, M_3)$ be a solution to the Cauchy problem (2.7) defined on the strip $\Pi_T = [0,T] \times \mathbb{R}$, then if $\delta > 0$ is assumed to be sufficiently small such that*

$$M_1^{100} M_2^{100(b+1)} M_3 \left(1 + M^6\right) \delta \ll 1$$

holds with $M := \sup_{t \in [0,T]} \|(\varphi(t), \psi(t), \zeta(t))\|$, then we can deduce that

$$\begin{aligned} & D_1^{-1} \leq v(t,x) \leq D_1, \quad (t,x) \in [0,T] \times \mathbb{R}, \\ & \theta(t,x) \leq D_2, \quad (t,x) \in [0,T] \times \mathbb{R}, \\ & \theta(t,x) \geq \frac{D_3 \min_{x \in \mathbb{R}} \{\theta(s,x)\}}{1 + (t-s) \min_{x \in \mathbb{R}} \{\theta(s,x)\}}, \quad 0 \leq s \leq t \leq T, x \in \mathbb{R} \end{aligned} \tag{2.36}$$

and the following estimate

$$\| (\varphi(t), \psi(t), \zeta(t), z(t)) \|_1^2 + \int_0^T \left(\|\varphi_x(s)\|^2 + \|(\psi_x(s), \zeta_x(s), z_x(s))\|_1^2 \right) ds \leq C_4 \tag{2.37}$$

holds for all $0 \leq t \leq T$. Here $D_i (i = 1,2,3)$ and C_4 are some uniform positive constants which depends only on $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ and m_0.

Once Proposition 2.5 is proved, the unique local solution $(\varphi(t,x), \psi(t,x), \zeta(t,x), z(t,x))$ constructed in Proposition 2.4 can be extended step by step to a global one by using the continuation argument designed in [21]. Thus to prove our main result Theorem 1.1, we only need to prove Proposition 2.5 and the following lemma is devoted to deducing the basic energy type estimate.

Lemma 2.6. *Under the assumptions listed in Proposition 2.5, then if $\delta > 0$ chosen sufficiently small such that*

$$C M_1^{100} M_2^{100b+100} M_3 \left(1 + M^6\right) \delta < \frac{1}{2},$$

where C is a fixed positive constant whose precise range can be specified in the proof of this lemma, we can get that the following estimates

$$0 \leq z(t,x) \leq 1, \tag{2.38}$$

$$\int_{\mathbb{R}} z(t,x) dx + \int_0^t \int_{\mathbb{R}} \phi(s,x) z(s,x) dx ds \leq C_5, \tag{2.39}$$

$$\int_{\mathbb{R}} z^2(t,x) dx + \int_0^t \int_{\mathbb{R}} \left(\frac{d}{v^2} z_x^2 + \phi z^2 \right) (s,x) dx ds \leq C_5, \tag{2.40}$$

and

$$\int_{\mathbb{R}} \eta(t,x) dx + \int_0^t \int_{\mathbb{R}} \left(\frac{\psi_x^2}{\theta v} + \frac{\kappa(v,\theta) \zeta_x^2}{\theta^2 v} + \frac{\phi z}{\theta} \right) (s,x) dx ds \leq C_5 \tag{2.41}$$

hold for all $(t,x)\in[0,T]\times\mathbb{R}$. *Here*

$$\eta=\frac{1}{2}\psi^2+R\Theta\Phi\left(\frac{v}{V}\right)+C_v\Theta\Phi\left(\frac{\theta}{\Theta}\right)+\frac{a}{3}v\zeta^2\left(3\theta^2+2\theta\Theta+\Theta^2\right),$$
$$\Phi(x)=x-1-\ln x,\quad x>0$$

and C_5 *is some positive constant depending only on* $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ *and* m_0, *but independent of* M_1, M_2 *and* M_3.

Proof. The estimate (2.38) can be obtained by applying the maximum principle to $(1.1)_4$, while the estimates (2.39) and (2.40) are standard L^1 and L^2-estimates for the scalar parabolic equation $(1.1)_4$. To prove (2.41), we can get by multiplying $(2.7)_1$ by $R\Theta\left(\frac{1}{v}-\frac{1}{V}\right)$, $(2.7)_2$ by ψ and $(2.7)_3$ by $\frac{\zeta}{\theta}$ and then adding the resulting identities together that

$$\left(\frac{1}{2}\psi^2+R\Theta\Phi\left(\frac{v}{V}\right)+C_v\Theta\Phi\left(\frac{\theta}{\Theta}\right)\right)_t+\frac{\mu}{v}\psi_x^2=Q+L_x-\psi R_1-\frac{\zeta}{\theta}R_2+\frac{\lambda\phi z\zeta}{\theta},\tag{2.42}$$

where $R_i(i=1,2)$ are given by (1.19), L and Q are defined as follows:

$$L=\mu\left(\frac{u_x}{v}-\frac{U_x}{V}\right)\psi-(p-p_+)\psi+\frac{\zeta}{\theta}\left(\frac{\kappa(v,\theta)\theta_x}{v}-\frac{\kappa(V,\Theta)\Theta_x}{V}\right),\tag{2.43}$$

$$\begin{aligned}Q=&-R\Theta\left(\frac{1}{v}-\frac{1}{V}\right)\psi_x+R\Theta_t\Phi\left(\frac{v}{V}\right)-\frac{R\Theta}{V^2v}V_t\varphi^2\\&-\frac{\zeta}{\theta}\left(av\theta^4-aV\Theta^4\right)_t-\frac{\zeta}{\theta}\left(pu_x-p_+U_x\right)\\&+\frac{C_v}{\Theta\theta}\Theta_t\zeta^2-C_v\Theta_t\Phi\left(\frac{\theta}{\Theta}\right)+(p-p_+)\psi_x-\mu\psi_xU_x\left(\frac{1}{v}-\frac{1}{V}\right)\\&-\frac{\theta\zeta_x-\theta_x\zeta}{\theta^2}\left(\frac{\kappa(v,\theta)\theta_x}{v}-\frac{\kappa(V,\Theta)\Theta_x}{V}\right)-\frac{\mu\zeta}{\theta}\left(\frac{u_x^2}{v}-\frac{U_x^2}{V}\right).\end{aligned}\tag{2.44}$$

Now we turn to control those terms related to the terms in the right hand side of (2.42). To this end, since the solution $(\varphi(t,x),\psi(t,x),\zeta(t,x),z(t,x))$ of the Cauchy problem (2.7) is assumed to belong to $X([0,T];M_1,M_2,M_3)$, we can first get that

$$\Phi\left(\frac{v}{V}\right)\leq CM_1^2\varphi^2,\quad\Phi\left(\frac{\theta}{\Theta}\right)\leq CM_2^2\zeta^2.$$

Thus we can get from Lemma 2.1 that

$$\begin{aligned}&\left|\int_0^t\int_{\mathbb{R}}\left(R\Theta_t\Phi\left(\frac{v}{V}\right)-\frac{R\Theta}{V^2v}V_t\varphi^2+\frac{C_v}{\Theta\theta}\Theta_t\zeta^2-C_v\Theta_t\Phi\left(\frac{\theta}{\Theta}\right)\right)dxds\right|\\&\leq CM_1^3M_2^2\delta\int_0^t\int_{\mathbb{R}}(1+s)^{-1}e^{-\frac{C_1x^2}{1+s}}\left(|\zeta(s,x)|^2+|\varphi(s,x)|^2\right)dxds.\end{aligned}\tag{2.45}$$

and

$$\left|\int_0^t\int_{\mathbb{R}}\left[\mu\psi_xU_x\left(\frac{1}{v}-\frac{1}{V}\right)\right]dxds\right|\tag{2.46}$$
$$\leq\frac{1}{8}\int_0^t\int_{\mathbb{R}}\frac{\mu\Theta}{v\theta}\psi_x^2dxds+CM_1^3M_2^2\delta\int_0^t\int_{\mathbb{R}}(1+s)^{-1}e^{-\frac{C_1x^2}{1+s}}|\varphi|^2dxds+CM_1^3M_2^2\delta.\tag{2.47}$$

On the other hand, noticing

$$-\frac{\mu\zeta}{\theta}\left(\frac{u_x^2}{v}-\frac{U_x^2}{V}\right)=\frac{\mu\zeta}{v\theta}\psi_x^2+\frac{2\mu\zeta\psi_xU_x}{v\theta}-\frac{\mu U_x^2\zeta\varphi}{\theta Vv},$$

$$\begin{aligned}&-\frac{\theta\zeta_x-\theta_x\zeta}{\theta^2}\left(\frac{\kappa(v,\theta)\theta_x}{v}-\frac{\kappa(V,\Theta)\Theta_x}{V}\right)\\ =&-\frac{\kappa(v,\theta)\Theta}{v\theta^2}\zeta_x^2-\frac{\kappa(v,\theta)\Theta_x\zeta\zeta_x}{v\theta^2}+\frac{\kappa(v,\theta)\Theta_x}{\theta Vv}\zeta_x\varphi\\ &-\frac{\kappa(v,\theta)-\kappa(V,\Theta)}{v\theta}\zeta_x\Theta_x-\frac{\Theta_x\zeta(\Theta_x+\zeta_x)}{\theta^2}\left(\frac{\kappa(v,\theta)}{v}-\frac{\kappa(V,\Theta)}{V}\right),\end{aligned}$$

and

$$\begin{aligned}&-R\Theta\left(\frac{1}{v}-\frac{1}{V}\right)\psi_x-\frac{\zeta}{\theta}\left(av\theta^4-aV\Theta^4\right)_t-\frac{\zeta}{\theta}\left(pu_x-p_+U_x\right)+(p-p_+)\psi_x\\ =&-\left(\frac{a}{3}v\zeta^2\left(3\theta^2+2\theta\Theta+\Theta^2\right)\right)_t-\frac{4av}{3\theta}\Theta_t\zeta^2\left(3\Theta^3+2\Theta\theta+\theta^2\right)-\frac{RU_x}{v\theta}\zeta^2+\frac{R\Theta U_x}{\theta Vv}\zeta\varphi\\ &-\frac{a\zeta}{\theta}\left(V_t\left(\theta^4-\Theta^4\right)+4\Theta^3\Theta_t\varphi\right),\end{aligned}$$

we can get from the Cauchy inequality and Lemma 2.1 that

$$\begin{aligned}&\left|\int_0^t\int_{\mathbb{R}}\left(\frac{2\mu\zeta\psi_xU_x}{v\theta}-\frac{\mu U_x^2\zeta\varphi}{\theta Vv}\right)dxds\right|\\ \le&\frac{1}{8}\int_0^t\int_{\mathbb{R}}\frac{\mu\Theta}{v\theta}\psi_x^2dxds+CM_1^2M_2^2\delta\int_0^t\int_{\mathbb{R}}(1+s)^{-1}e^{-\frac{C_1x^2}{1+s}}\left(|\zeta|^2+|\varphi|^2\right)dxds\\ &+CM_1^2M_2^2\delta,\end{aligned}\tag{2.48}$$

$$\begin{aligned}&\left|\int_0^t\int_{\mathbb{R}}\left(-\frac{\kappa(v,\theta)\Theta_x\zeta\zeta_x}{v\theta^2}+\frac{\kappa(v,\theta)\Theta_x}{\theta Vv}\zeta_x\varphi-\frac{\kappa(v,\theta)-\kappa(V,\Theta)}{v\theta}\zeta_x\Theta_x\right)dxds\right|\\ \le&\frac{1}{8}\int_0^t\int_{\mathbb{R}}\frac{\kappa(v,\theta)\Theta}{v\theta^2}\zeta_x^2dxds+CM_1^4M_2^{3b+3}\delta\int_0^t\int_{\mathbb{R}}(1+s)^{-1}e^{-\frac{C_1x^2}{1+s}}\left(|\zeta|^2+|\varphi|^2\right)dxds\\ &+CM_1^4M_2^{3b+3}\delta,\end{aligned}\tag{2.49}$$

$$\begin{aligned}&\left|\int_0^t\int_{\mathbb{R}}\frac{\Theta_x\zeta(\Theta_x+\zeta_x)}{\theta^2}\left(\frac{\kappa(v,\theta)}{v}-\frac{\kappa(V,\Theta)}{V}\right)dxds\right|\\ \le&\frac{1}{8}\int_0^t\int_{\mathbb{R}}\frac{\kappa(v,\theta)\Theta}{v\theta^2}\zeta_x^2dxds+CM_1^8M_2^{2b+3}\delta\int_0^t\int_{\mathbb{R}}(1+s)^{-1}e^{-\frac{C_1x^2}{1+s}}\left(|\zeta|^2+|\varphi|^2\right)dxds\\ &+CM_1^8M_2^{2b+3}\delta,\end{aligned}\tag{2.50}$$

$$\begin{aligned}&\left|\int_0^t\int_{\mathbb{R}}\left(-\frac{4av}{3\theta}\Theta_t\zeta^2\left(3\Theta^3+2\Theta\theta+\theta^2\right)-\frac{RU_x}{v\theta}\zeta^2+\frac{R\Theta U_x}{\theta Vv}\zeta\varphi\right)\right|\\ \le&CM_1^2M_3\delta\int_0^t\int_{\mathbb{R}}(1+s)^{-1}e^{-\frac{C_1x^2}{1+s}}\left(|\zeta|^2+|\varphi|^2\right)dxds,\end{aligned}\tag{2.51}$$

and

$$\begin{aligned} &\left|\int_0^t\int_{\mathbb{R}}\left(\frac{a\zeta}{\theta}\left(V_t\left(\theta^4-\Theta^4\right)+4\Theta^3\Theta_t\varphi\right)\right)\right| \\ &\leq CM_2^4\delta\int_0^t\int_{\mathbb{R}}(1+s)^{-1}e^{-\frac{C_1x^2}{1+s}}\left(|\zeta|^2+|\varphi|^2\right)dxds. \end{aligned} \tag{2.52}$$

With the above estimates in hand, if we integrate (2.42) with respect to t and x over $[0,t]\times\mathbb{R}$, we can obtain by (2.39), Lemma 2.3 and the estimates (2.45)-(2.52) that

$$\begin{aligned} &\int_{\mathbb{R}}\eta(t,x)dx+\int_0^t\int_{\mathbb{R}}\left(\frac{\psi_x^2}{v\theta}+\frac{\kappa(v,\theta)}{v\theta^2}\zeta_x^2+\frac{\phi z}{\theta}\right)dxds \\ &\leq O(1)+CM_1^{11}M_2^{4b+7}\delta\left(1+\int_0^t\int_{\mathbb{R}}(1+s)^{-1}e^{-\frac{C_1x^2}{1+s}}\left(|\zeta|^2+|\varphi|^2\right)dxds\right) \qquad (2.53) \\ &\leq O(1)+CM_1^{20}M_2^{7b+23}M_3\left(1+M^6\right)\delta\left(1+\int_0^t\int_{\mathbb{R}}\left(\frac{\kappa(v,\theta)}{v\theta^2}\zeta_x^2+\frac{\theta\varphi_x^2}{v^3}\right)dxds\right). \end{aligned}$$

If $\delta>0$ is chosen sufficiently small such that

$$CM_1^{100}M_2^{100b+100}M_3\left(1+M^6\right)\delta<\frac{1}{2}, \tag{2.54}$$

we can get from (2.53) that

$$\begin{aligned} &\int_{\mathbb{R}}\eta(t,x)dx+\int_0^t\int_{\mathbb{R}}\left(\frac{\psi_x^2}{v\theta}+\frac{\kappa(v,\theta)}{v\theta^2}\zeta_x^2+\frac{\phi z}{\theta}\right)dxds \\ &\leq O(1)+CM_1^{20}M_2^{7b+23}M_3M^6\delta\int_0^t\int_{\mathbb{R}}\frac{\theta\varphi_x^2}{v^3}dxds. \end{aligned} \tag{2.55}$$

Now we turn to estimate the last term in the right hand side of (2.55). For this purpose, one has by multiplying $(2.7)_2$ by $\frac{\varphi_x}{v}$ that

$$\begin{aligned} &\left[\frac{\mu}{2}\left(\frac{\varphi_x}{v}\right)^2-\frac{\varphi_x\psi}{v}\right]_t+\frac{R\theta\varphi_x^2}{v^3}+\left(\frac{\psi\psi_x}{v}\right)_x \\ &=\left[\frac{\psi_x^2}{v}+\frac{p_\theta(v,\theta)\varphi_x\zeta_x}{v}\right] \\ &\quad+\left\{\frac{\left(p_v(v,\theta)-p_V(V,\Theta)\right)V_x\varphi_x}{v}+\frac{\left(p_\theta(v,\theta)-p_\Theta(V,\Theta)\right)\Theta_x\varphi_x}{v}\right\} \\ &\quad+\left[\frac{\mu\varphi_xU_{xx}}{v}\left(\frac{1}{v}-\frac{1}{V}\right)+\frac{\mu\varphi_xV_xU_x}{v}\left(\frac{1}{V^2}-\frac{1}{v^2}\right)\right]+\frac{\mu\varphi_x\psi_xV_x}{vV^2}+\frac{R_1\varphi_x}{v}. \end{aligned} \tag{2.56}$$

Integrating (2.56) with respect to t and x over $(0,t)\times\mathbb{R}$ yields

$$\begin{aligned}
&\left\|\left(\frac{\varphi_x}{v}\right)(t)\right\|^2+\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}dxds\\
&\lesssim 1+\|\psi(t)\|^2+\underbrace{\int_0^t\int_{\mathbb{R}}\left[\frac{\psi_x^2}{v}+\frac{p_\theta(v,\theta)\varphi_x\zeta_x}{v}\right]dxds}_{I_1}\\
&\quad+\underbrace{\int_0^t\int_{\mathbb{R}}\left\{\frac{\left(p_v(v,\theta)-p_V(V,\Theta)\right)V_x\varphi_x}{v}+\frac{\left(p_\theta(v,\theta)-p_\Theta(V,\Theta)\right)\Theta_x\varphi_x}{v}\right\}dxds}_{I_2}\\
&\quad+\underbrace{\int_0^t\int_{\mathbb{R}}\left[\frac{\mu\varphi_xU_{xx}}{v}\left(\frac{1}{v}-\frac{1}{V}\right)+\frac{\mu\varphi_xV_xU_x}{v}\left(\frac{1}{V^2}-\frac{1}{v^2}\right)\right]dxds}_{I_3} \qquad (2.57)\\
&\quad+\underbrace{\int_0^t\int_{\mathbb{R}}\left(\frac{\mu\varphi_x\psi_xV_x}{vV^2}+\frac{R_1\varphi_x}{v}\right)dxds}_{I_4}.
\end{aligned}$$

For the estimate of $I_j (j=1,2,3,4)$, we can get from the fact that $(\varphi(t,x),\psi(t,x),\zeta(t,x),z(t,x))\in X([0,T];M_1,M_2,M_3)$, Lemma 2.1 and Cauchy's inequality that

$$\begin{aligned}
I_1&\le\varepsilon\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}dxds \qquad (2.58)\\
&\quad+C\left(\|\theta\|_\infty\int_0^t\int_{\mathbb{R}}\frac{\mu\Theta\psi_x^2}{v\theta}dxds+\left\|\frac{\theta^2p_\theta^2(v,\theta)}{\kappa(v,\theta)p_v(v,\theta)}\right\|_\infty\int_0^t\int_{\mathbb{R}}\frac{\kappa(v,\theta)\Theta\zeta_x^2}{v\theta^2}dxds\right)\\
&\le\varepsilon\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}dxds+C(\varepsilon)CM_1^3M_2^{b+8}\int_0^t\int_{\mathbb{R}}\left(\frac{\mu\Theta\psi_x^2}{v\theta}+\frac{\kappa(v,\theta)\Theta\zeta_x^2}{v\theta^2}\right)dxds,
\end{aligned}$$

$$\begin{aligned}
I_2&\le\varepsilon\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}dxds+C(\varepsilon)CM_1^9M_2^5 \qquad (2.59)\\
&\quad\times\int_0^t\int_{\mathbb{R}}\left[|V_x|^2\left(|\varphi|^2+|\zeta|^2\right)+|\Theta_x|^2\left(|\varphi|^2+|\zeta|^2\right)\right]dxds\\
&\le\varepsilon\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}dxds+C(\varepsilon)CM_1^9M_2^5\delta\int_0^t\int_{\mathbb{R}}(1+s)^{-1}e^{-\frac{C_1x^2}{1+s}}|(\varphi,\zeta)|^2dxds,
\end{aligned}$$

$$\begin{aligned}
I_3&\le\varepsilon\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}dxds+C(\varepsilon)CM_1^9M_2\int_0^t\int_{\mathbb{R}}\left(|U_{xx}|^2+|U_xV_x|^2\right)|\varphi|^2dxds\\
&\le\varepsilon\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}dxds+C(\varepsilon)CM_1^9M_2\delta\int_0^t\int_{\mathbb{R}}(1+s)^{-2}e^{-\frac{C_1x^2}{1+s}}|\varphi|^2dxds, \qquad (2.60)
\end{aligned}$$

and

$$\begin{aligned}
I_4&\le\varepsilon\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}dxds+C(\varepsilon)CM_1^6M_2\int_0^t\int_{\mathbb{R}}\left(\frac{\mu\Theta\psi_x^2}{v\theta}|V_x|^2+|R_1|^2\right)dxds\\
&\le\varepsilon\int_0^t\int_{\mathbb{R}}\frac{R\theta\varphi_x^2}{v^3}dxds+C(\varepsilon)CM_1^6M_2\delta\left(1+\int_0^t\int_{\mathbb{R}}\frac{\mu\Theta\psi_x^2}{v\theta}dxds\right). \qquad (2.61)
\end{aligned}$$

Substituting (2.58)-(2.61) into (2.57), we can derive that

$$\begin{aligned} &\left\|\left(\frac{\varphi_x}{v}\right)(t)\right\|^2+\int_0^t\int_{\mathbb{R}}\frac{\theta\varphi_x^2}{v^3}dxds \\ &\leq O(1)+CM_1^9M_2^5\delta\int_0^t\int_{\mathbb{R}}(1+s)^{-1}e^{-\frac{C_1x^2}{1+s}}|(\varphi,\zeta)|^2dxds \\ &+CM_1^6M_2^{b+8}(1+\delta)\int_0^t\int_{\mathbb{R}}\left(\frac{\psi_x^2}{v\theta}+\frac{\kappa(v,\theta)\zeta_x^2}{v\theta^2}\right)dxds. \end{aligned} \tag{2.62}$$

Noticing that Lemma 2.3 tells us that

$$\begin{aligned} &CM_1^9M_2^5\delta\int_0^t\int_{\mathbb{R}}(1+s)^{-1}e^{-\frac{C_1x^2}{1+s}}|(\varphi(s,x),\zeta(s,x))|^2dxds \\ &\leq CM_1^{21}M_2^{4b+23}M_3\left(1+M^6\right)\delta\left(1+\int_0^t\int_{\mathbb{R}}\left(\frac{\theta\varphi_x^2}{v^3}+\frac{\kappa(v,\theta)\zeta_x^2}{v\theta^2}\right)dxds\right), \end{aligned} \tag{2.63}$$

if we assume that $\delta>0$ is sufficiently small such that (2.54) holds, then we can get from (2.62) and (2.63) that

$$\begin{aligned} &\left\|\left(\frac{\varphi_x}{v}\right)(t)\right\|^2+\int_0^t\int_{\mathbb{R}}\frac{\theta\varphi_x^2}{v^3}dxds \\ &\leq O(1)+CM_1^{21}M_2^{4b+24}M_3\left(1+M^6\right)(1+\delta)\int_0^t\int_{\mathbb{R}}\left(\frac{\psi_x^2}{v\theta}+\frac{\kappa(v,\theta)\zeta_x^2}{v\theta^2}\right)dxds. \end{aligned} \tag{2.64}$$

(2.55) together with (2.64) imply that

$$\begin{aligned} &\int_{\mathbb{R}}\eta(t,x)dx+\int_0^t\int_{\mathbb{R}}\left(\frac{\psi_x^2}{v\theta}+\frac{\kappa(v,\theta)}{v\theta^2}\zeta_x^2+\frac{\phi z}{\theta}\right)dxds \\ &\leq O(1)+CM_1^{32}M_2^{8b+31}M_3\left(1+M^6\right)\delta\left(1+\int_0^t\int_{\mathbb{R}}\left(\frac{\psi_x^2}{v\theta}+\frac{\kappa(v,\theta)\zeta_x^2}{v\theta^2}\right)dxds\right). \end{aligned} \tag{2.65}$$

From (2.65), one can deduce (2.41) immediately if we take $\delta>0$ sufficiently small such that (2.54) holds. This completes the proof of Lemma 2.6. □

3. Uniform positive bounds for the specific volume. In this section, we want to derive the uniform-in-time pointwise bounds for the specific volume $v(t,x)$. To this end, we first have the following result whose proof is similar to the proof of Lemma 3.1 in [5] and thus we omit the details for brevity.

LEMMA 3.1. *Let α_1,α_2 be the two positive roots of the equation $y-\log y-1=C_5$ with C_5 being the positive constant appeared in* (2.6). *Then for each $k\in\mathbb{Z}$ and $t\geq 0$, we can get that*

$$\alpha_1\leq\int_k^{k+1}\tilde{v}(x,t)dx,\quad \int_k^{k+1}\tilde{\theta}(x,t)dx\leq\alpha_2. \tag{3.1}$$

Consequently, each $t\geq 0$, there exist $a_k(t),b_k(t)\in[k,k+1]$ such that

$$\alpha_1\leq\tilde{v}(t,a_k(t)),\quad \tilde{\theta}(t,b_k(t))\leq\alpha_2. \tag{3.2}$$

where $\tilde{v}(t,x):=\frac{v(t,x)}{V(t,x)},\tilde{\theta}(t,x):=\frac{\theta(t,x)}{\Theta(t,x)}$.

The next lemma is concerned with a rough estimate on $\theta(t,x)$ in terms of the entropy dissipation rate functional $V(t)=\int_{\mathbb{R}}\left(\frac{\psi_x^2}{v\theta}+\frac{\kappa(v,\theta)\zeta_x^2}{v\theta^2}\right)(t,x)dx$.

LEMMA 3.2. *For* $0\leq m\leq\frac{b+1}{2}, x\in\mathbb{R}$ *(without loss of generality, we can assume that* $x\in\Omega_k=(-k-1,k+1)$ *for some* $k\in\mathbb{Z}$*), we can deduce that*

$$|\theta^m(t,x)-\theta^m\left(t,b_k(t)\right)|\lesssim V^{\frac{1}{2}}(t)+1 \tag{3.3}$$

holds for $0\leq t\leq T$ *and consequently*

$$|\theta(t,x)|^{2m}\lesssim 1+V(t),\quad x\in\overline{\Omega}_k,\quad 0\leq t\leq T \tag{3.4}$$

provided that $\delta>0$ *is chosen sufficiently small such that* $M_2^{\frac{b-1}{2}}\delta^2\lesssim 1$.

Proof. From (1.4), we have

$$\begin{aligned}
&|\theta^m(t,x)-\theta^m\left(t,b_k(t)\right)|\\
&\lesssim\int_{\Omega_k}\left|\theta^{m-1}(t,x)\left(\Theta_x(t,x)+\zeta_x(t,x)\right)\right|dx\\
&\lesssim\left(\int_{\Omega_k}\frac{v\theta^{2m}}{1+v\theta^b}dx\right)^{\frac{1}{2}}\left(\int_{\Omega_k}\frac{\kappa(v,\theta)\zeta_x^2}{v\theta^2}dx\right)^{\frac{1}{2}}+\int_{\Omega_k}\theta^{m-1}(t,x)\left|\Theta_x(t,x)\right|dx\\
&\lesssim V^{\frac{1}{2}}(t)+M_2^{\frac{b-1}{2}}\delta^2\\
&\lesssim V^{\frac{1}{2}}(t)+1.
\end{aligned}$$

Here we have used the assumption $0\leq m\leq\frac{b+1}{2}$, boundedness of Ω_k, (3.1) and the assumption that $\delta>0$ is sufficiently small such that $M_2^{\frac{b-1}{2}}\delta^2\lesssim 1$. □

Motivated by the work of [15] and [20], our next lemma is concerned with a local representation of $v(t,x)$ by using the following cut-off function $\varphi\in W^{1,\infty}(\mathbb{R})$:

$$\varphi(x)=\begin{cases}1, & x\leq k+1,\\ k+2-x, & k+1\leq x\leq k+2,\\ 0, & x\geq k+2.\end{cases} \tag{3.5}$$

LEMMA 3.3. *Under the assumptions stated in Proposition 2.5, we have for each* $0\leq t\leq T$ *that*

$$v(t,x)=B(t,x)Q(t)+\frac{1}{\mu}\int_0^t\frac{B(t,x)Q(t)v(\tau,x)p(\tau,x)}{B(\tau,x)Q(\tau)}d\tau,\quad x\in\overline{\Omega}_k. \tag{3.6}$$

Here

$$\begin{aligned}
B(t,x)&:=v_0(x)\exp\left\{\frac{1}{\mu}\int_x^\infty\left(u_0(y)-u(t,y)\right)\varphi(y)dy\right\},\\
Q(t)&:=\exp\left\{\frac{1}{\mu}\int_0^t\int_{k+1}^{k+2}\sigma(\tau,y)\right\},\\
\sigma&:=-p(v,\theta)+\frac{\mu u_x}{v}.
\end{aligned} \tag{3.7}$$

With the above presentation in hand, we can derive uniform-in-time pointwise bounds of $v(t,x)$ by repeating the argument used in [5, 21], hence we omit the proof for brevity.

LEMMA 3.4. *Under the conditions of the Proposition 2.5. Then there exists a positive constant C_2 which depends only on $\underline{V}, \overline{V}, \underline{\Theta}, \overline{\Theta}$ and m_0 such that*

$$C_2^{-1} \leq v(t,x) \leq C_2, \quad \forall (t,x) \in [0,T] \times \mathbb{R}. \tag{3.8}$$

With the estimate (3.8) on the uniform positive lower and upper bounds on the specific volume $v(t,x)$ in hand, we now control the term $\|\varphi_x(t)\|^2$ in terms of $\|\theta\|_\infty$. Such an estimate will be frequently used later on.

LEMMA 3.5. *Under the conditions of Proposition 2.5, then for $0 \leq t \leq T$, we can get that*

$$\|\varphi_x(t)\|^2 + \int_0^t \int_{\mathbb{R}} \theta \varphi_x^2 dx ds \lesssim 1 + \|\theta\|_\infty^{\max\{1,(7-b)_+\}} \tag{3.9}$$

provided that $\delta > 0$ is chosen sufficiently small.

Proof. Now the estimate (3.8) tells us that the specific volume $v(t,x)$ is uniformly bounded from below and above, then the term I_1 in (2.57) can be re-estimated as in the following:

$$\begin{aligned} I_1 &\leq \varepsilon \int_0^t \int_{\mathbb{R}} \frac{R\theta\varphi_x^2}{v^3} dx ds + C(\varepsilon) \left(\|\theta\|_\infty + \left\| \frac{\theta^2 p_\theta^2(v,\theta)}{\kappa(v,\theta) p_v(v,\theta)} \right\|_\infty \right) \\ &\leq \varepsilon \int_0^t \int_{\mathbb{R}} \frac{R\theta\varphi_x^2}{v^3} dx ds + C(\varepsilon) \left(1 + \|\theta\|_\infty + \|\theta\|_\infty^{(7-b)_+} \right). \end{aligned} \tag{3.10}$$

Here $(7-b)_+ := \max\{0, 7-b\}$.

With (3.10) in hand, we can get (3.9) by combining the estimate (2.8) obtained in Lemma 2.3, the estimate (3.8), (2.57), (2.59), (2.60) and (2.61). This completes the proof of Lemma 3.5. □

Next, we derive an estimate on the term $\int_0^t \|\psi_{xx}(s)\|^2 ds$ in terms of $\|\theta\|_\infty$, which plays a key role in deriving the upper bound of $\theta(t,x)$.

LEMMA 3.6. *Under the conditions of Proposition 2.5, then for $0 \leq t \leq T$, we have*

$$\|\psi_x(t)\|^2 + \int_0^t \|\psi_{xx}(\tau)\|^2 d\tau \lesssim 1 + \|\theta\|_\infty^{1+2\max\{1,(7-b)_+\}}. \tag{3.11}$$

Proof. First, multiplying $(2.7)_2$ by $-\psi_{xx}$, we get that

$$\begin{aligned} &\partial_t \left(\frac{\psi_x^2}{2} \right) + \frac{\mu \psi_{xx}^2}{v} - (\psi_t \psi_x)_x \\ &= (p - p_+)_x \psi_{xx} - \frac{\mu \varphi \psi_{xx} U_{xx}}{vV} - \frac{\mu \varphi_x \psi_{xx} U_x}{vV} - R_1 \psi_{xx} \\ &\quad - \frac{\mu \varphi \psi_{xx} U_x V_x}{vV^2} - \frac{\mu \varphi \psi_{xx} U_x (\varphi_x + V_x)}{v^2 V} + \frac{\mu \psi_x \psi_{xx} (\varphi_x + V_x)}{v^2}. \end{aligned} \tag{3.12}$$

Then, we can get by integrating the above identity with respect to t and x over $(0,t)\times\mathbb{R}$ and by employing Cauchy's inequality, Sobolev's inequality, (3.8) and (3.9) that

$$\begin{aligned}
&\int_{\mathbb{R}}\frac{\psi_x^2}{2}dx+\int_0^t\int_{\mathbb{R}}\frac{\mu\psi_{xx}^2}{v}dxds\\
&\leq\epsilon\int_0^t\int_{\mathbb{R}}\frac{\mu\psi_{xx}^2}{v}dxds+C(\epsilon)\int_0^t\int_{\mathbb{R}}\Big[\left(1+\theta^6\right)|\zeta_x|^2+\left|(V_x,\Theta_x)\right|^2|\varphi|^2+|\theta\varphi_x|^2\\
&\quad+\zeta^2|V_x|^2+\left(1+\theta^4\right)|\Theta_x|^2|\zeta|^2+U_{xx}^2\varphi^2+|R_1|^2+U_x^2\varphi_x^2+|U_xV_x|^2\varphi^2\\
&\quad+U_x^2\varphi^2\varphi_x^2+V_x^2\psi_x^2+\varphi_x^2\psi_x^2\Big]dxds\\
&\leq\epsilon\int_0^t\int_{\mathbb{R}}\frac{\mu\psi_{xx}^2}{v}dxds+C(\epsilon)\Big(1+\|\theta\|_\infty^{(8-b)_+}+\|\theta\|_\infty^{1+\max\{1,(7-b)_+\}}\\
&\quad+\int_0^t\|\psi_x\|\,\|\psi_{xx}\|\,\|\varphi_x\|^2ds\Big)\\
&\leq2\epsilon\int_0^t\int_{\mathbb{R}}\frac{\mu\psi_{xx}^2}{v}dxds+C(\epsilon)\Big(1+\|\theta\|_\infty^{1+\max\{1,(7-b)_+\}}\\
&\quad+\left(1+\|\theta\|_\infty^{2\max\{1,(7-b)_+\}}\right)\int_0^t\int_{\mathbb{R}}\frac{\psi_x^2}{\theta}\cdot\theta dxds\Big)\\
&\leq2\epsilon\int_0^t\int_{\mathbb{R}}\frac{\mu\psi_{xx}^2}{v}dxds+C(\epsilon)\left(1+\|\theta\|_\infty^{1+2\max\{1,(7-b)_+\}}\right).
\end{aligned}\tag{3.13}$$

By choosing $\epsilon>0$ small enough, we can get (3.11) from (3.13). The proof of Lemma 3.6 is complete. □

4. Uniform upper bound of the absolute temperature. In this section, we are in a position to derive an estimate on the upper bound of $\theta(t,x)$. For this purpose, motivated by [19, 21], we set

$$\begin{aligned}
X(t)&:=\int_0^t\int_{\mathbb{R}}\left(1+\theta^{b+3}(s,x)\right)\zeta_t^2(s,x)dxds,\\
Y(t)&:=\sup_{s\in(0,t)}\left\{\int_{\mathbb{R}}\left(1+\theta^{2b}(s,x)\right)\zeta_x^2(s,x)\mathrm{d}x\right\},\\
Z(t)&:=\sup_{s\in(0,t)}\left\{\int_{\mathbb{R}}\psi_{xx}^2(s,x)\mathrm{d}x\right\}
\end{aligned}\tag{4.1}$$

and then try to deduce certain estimates among them by making use of the special structure of system (2.7) under our considerations.

Lemma 4.1. *Under the assumptions listed in Proposition 2.5, we have for all* $0\leq t\leq T$ *that*

$$\|\theta(t)\|_{L^\infty}\lesssim1+Y(t)^{\frac{1}{2b+3}},\tag{4.2}$$

$$\sup_{\tau\in(0,t)}\left\{\|\psi_x(\tau)\|^2\right\}\lesssim1+Z(t)^{\frac12},\quad\|\psi_x(t)\|_{L^\infty}\lesssim1+Z(t)^{\frac38}.\tag{4.3}$$

Proof. We assume without loss of generality that $x \in [-k-1, k+1]$ for some $k \in \mathbb{Z}$ and $x \geq b_k(t)$. Then

$$\begin{aligned}
&(\theta(t,x) - \Theta(t,x))^{2b+3} \\
&= (\theta(t,b_k(t)) - \Theta(t,b_k(t)))^{2b+3} + (2b+3)\int_{b_k(t)}^{x} (\theta(t,x) - \Theta(t,x))^{2b+2}\zeta_x(t,y)\mathrm{d}y \\
&\lesssim 1 + \|(\theta-\Theta)(t)\|_{L^\infty}^{\frac{2b+3}{2}}\left[\int_{-k-1}^{k+1}\left(1+\Phi\left(\frac{\theta}{\Theta}\right)\right)\mathrm{d}x\right]^{\frac{1}{2}}\left[\int_{\mathbb{R}}(\theta-\Theta)^{2b}\zeta_x^2\mathrm{d}x\right]^{\frac{1}{2}} \\
&\lesssim 1 + \|(\theta-\Theta)(t)\|_{L^\infty}^{\frac{2b+3}{2}} Y^{\frac{1}{2}}(t),
\end{aligned}$$

from which we can deduce (4.2) by using Cauchy's inequality.

The proof of (4.3) follows easily by employing the Gagliardo–Nirenberg inequality and the Sobolev inequality. This completes the proof of Lemma 4.1. □

Our next lemma is to show that $X(t)$ and $Y(t)$ can be controlled by $Z(t)$.

LEMMA 4.2. *Under the assumptions listed in Proposition 2.5, if we assume further that δ is chosen sufficiently small such that $M_2^{6(b+1)}\delta \lesssim 1$, then we have for $0 \leq t \leq T$ that*

$$X(t) + Y(t) \lesssim 1 + Z(t)^{\lambda_1}, \tag{4.4}$$

where

$$\lambda_1 = \max\left\{\frac{6b+9}{12b+12-8\max\{1,(7-b)_+\}}, \frac{6b+9}{16b+8}\right\}. \tag{4.5}$$

Proof. As in [19, 21], we set

$$\mathbb{K}(v,\theta) = \int_0^\theta \frac{\kappa(v,\xi)}{v}d\xi = \frac{\kappa_1\theta}{v} + \frac{\kappa_2\theta^{b+1}}{b+1}. \tag{4.6}$$

Then, we have from the estimate (3.8) that

$$\begin{aligned}
&\mathbb{K}_t(v,\theta) = \mathbb{K}_v(v,\theta)\psi_x + \mathbb{K}_\theta(v,\theta)\zeta_t + \mathbb{K}_v(v,\theta)U_x + \mathbb{K}_\theta(v,\theta)\Theta_t, \\
&\mathbb{K}_x(v,\theta) = \mathbb{K}_v(v,\theta)\varphi_x + \mathbb{K}_\theta(v,\theta)\zeta_x + \mathbb{K}_v(v,\theta)V_x + \mathbb{K}_\theta(v,\theta)\Theta_x, \\
&\mathbb{K}_{xt}(v,\theta) = (\mathbb{K}_\theta(v,\theta)\zeta_x)_t + [\mathbb{K}_{vv}(v,\theta)(\psi_x+U_x) \\
&\qquad +\mathbb{K}_{v\theta}(v,\theta)(\zeta_t+\Theta_t)]\varphi_x + \mathbb{K}_v(v,\theta)\psi_{xx} \\
&\qquad +[\mathbb{K}_{vv}(v,\theta)(\psi_x+U_x) + \mathbb{K}_{v\theta}(v,\theta)(\zeta_t+\Theta_t)]V_x + \mathbb{K}_v(v,\theta)U_{xx} \\
&\qquad +[\mathbb{K}_{v\theta}(v,\theta)(\psi_x+U_x) + \mathbb{K}_{\theta\theta}(v,\theta)(\zeta_t+\Theta_t)]\Theta_x + \mathbb{K}_\theta(v,\theta)\Theta_{xt}, \\
&|\mathbb{K}_v(v,\theta)| + |\mathbb{K}_{vv}(v,\theta)| \lesssim \theta, \quad |\mathbb{K}_\theta(v,\theta)| \lesssim 1+\theta^b, \\
&|\mathbb{K}_{v\theta}(v,\theta)| \lesssim 1, \quad |\mathbb{K}_{\theta\theta}(v,\theta)| \lesssim \theta^{b-1}.
\end{aligned} \tag{4.7}$$

To simplify the presentation, in the rest of this paper, we use $\mathbb{K}, p, e, P, E$ to denote $\mathbb{K}(v,\theta), p(v,\theta), e(v,\theta), p(V,\Theta), e(V,\Theta)$, respectively. Then multiplying $(2.7)_3$ by $\mathbb{K}_t$ and integrating the resulting identity with respect to t and x over $(0,t)\times\mathbb{R}$, we can get that

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}} e_\theta \mathbb{K}_\theta \zeta_t^2 dxds + \int_0^t\int_{\mathbb{R}} \left(\mathbb{K}_\theta\zeta_x\right)\left(\mathbb{K}_\theta\zeta_x\right)_t dxds - \int_0^t\int_{\mathbb{R}} \mathbb{K}_\theta\mathbb{K}_{v\theta}U_x\zeta_x^2 dxds\\
=&\underbrace{\int_0^t\int_{\mathbb{R}}\left(\mathbb{K}_\theta\mathbb{K}_{\theta\theta}\zeta_t\Theta_x^2+\mathbb{K}_\theta^2\zeta_t\Theta_{xx}+\mathbb{K}_\theta\mathbb{K}_{v\theta}\zeta_t\Theta_xV_x\right)dxds}_{I_5}\\
&+\underbrace{\int_0^t\int_{\mathbb{R}}\left\{\mathbb{K}_\theta\mathbb{K}_{\theta\theta}\zeta_x^2\Theta_t+\mathbb{K}_\theta\mathbb{K}_{v\theta}\zeta_t\varphi_x\Theta_x-\mathbb{K}_\theta\mathbb{K}_{v\theta}\zeta_x\zeta_tV_x-\left(\theta p_\theta-\frac{e_\theta\Theta P_\Theta}{E_\Theta}\right)U_x\mathbb{K}_\theta\zeta_t\right\}dxds}_{I_6}\\
&+\underbrace{\int_0^t\int_{\mathbb{R}}\mathbb{K}_\theta\mathbb{K}_{v\theta}\zeta_x^2\psi_x dxds}_{I_7}-\underbrace{\int_0^t\int_{\mathbb{R}}\mathbb{K}_\theta\mathbb{K}_{v\theta}\zeta_t\varphi_x\zeta_x dxds}_{I_8} \qquad (4.8)\\
&+\underbrace{\int_0^t\int_{\mathbb{R}}\left(\lambda\phi z-\frac{e_\theta}{E_\Theta}\left(\frac{k(V,\Theta)\Theta_x}{V}\right)_x\right)\mathbb{K}_\theta\zeta_t dxds}_{I_9}\\
&-\underbrace{\int_0^t\int_{\mathbb{R}}\theta p_\theta\mathbb{K}_\theta\psi_x\zeta_t dxds}_{I_{10}}+\underbrace{\int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2\mathbb{K}_\theta\zeta_t}{v}dxds}_{I_{11}}.
\end{aligned}
$$

To yield the desired estimate among $X(t), Y(t)$ and $Z(t)$ based on (4.8), we first find from (3.8), (4.1) and (4.7) that the first two terms in the left hand side of (4.8) can be estimated as follows:

$$
\int_0^t\int_{\mathbb{R}} e_\theta\mathbb{K}_\theta\zeta_t^2 dxds \gtrsim \int_0^t\int_{\mathbb{R}}\left(1+a\theta^3\right)\left(1+\theta^b\right)\zeta_t^2 dxds \gtrsim X(t) \qquad (4.9)
$$

and

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}}\left(\mathbb{K}_\theta\zeta_x\right)\left(\mathbb{K}_\theta\zeta_x\right)_t dxds &= \frac{1}{2}\int_{\mathbb{R}}\left|\left(\mathbb{K}_\theta\zeta_x\right)(t,x)\right|^2 dx - \frac{1}{2}\int_{\mathbb{R}}\left|\left(\mathbb{K}_\theta\zeta_x\right)(0,x)\right|^2 dx\\
&\gtrsim Y(t) - C. \qquad (4.10)
\end{aligned}
$$

Now for the terms $I_k (k = 5, 6, \ldots, 11)$ in the right hand side of (4.8), we can get from Lemma 2.1, (2.8) and (4.7) that I_5 can be bounded as

$$
\begin{aligned}
I_5 &\le \epsilon X(t) + C(\epsilon)\int_0^t\int_{\mathbb{R}}\Big[\left(1+\theta^{3b-5}\right)|\Theta_x|^4+\left(1+\theta^{3b-3}\right)|\Theta_{xx}|^2\\
&\quad+\left(1+\theta^{b-3}\right)|\Theta_x|^2|V_x|^2\Big]dxds \qquad (4.11)\\
&\le \epsilon X(t) + C(\epsilon)\left[\left(1+M_2^{3b-5}\right)\delta+\left(1+M_2^{3b-3}\right)\delta^{\frac{1}{2}}+\left(1+M_2^{b-3}\right)\delta\right]\\
&\le \epsilon X(t) + C(\epsilon).
\end{aligned}
$$

Here we have used the fact that $\delta > 0$ is chosen sufficiently small such that

$$
M_2^{\max\{3b-5,6(b-1),b-3\}}\delta \lesssim 1. \qquad (4.12)
$$

For I_6, since

$$
\left|\frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)} - \frac{\Theta P_\Theta(V,\Theta)}{E_\Theta(V,\Theta)}\right| \lesssim |\varphi| + |\zeta| \qquad (4.13)
$$

and by employing Taylor's formula, we have for $0<\omega<1$ that

$$\int_{\mathbb{R}} \zeta^2 dx = 2\int_{\mathbb{R}} \Phi\left(\frac{\theta}{\Theta}\right)(\omega\Theta+(1-\omega)\theta)^2\,dx \lesssim 1+M_2^2, \tag{4.14}$$

we can deduce from Lemma 2.1, (2.41), (3.9), (3.11), (4.7), (4.13) and (4.14) that

$$\begin{aligned} I_6 &\leq \epsilon X(t) + C\|\Theta_t\|_{L^\infty}\int_0^t\int_{\mathbb{R}} \frac{\left(1+\theta^b\right)\Theta|\zeta_x|^2}{v\theta^2}\cdot\left(1+\theta^{b+1}\right)dxds \\ &\quad + C(\epsilon)\int_0^t\int_{\mathbb{R}}\left[\left(1+\theta^{b-3}\right)|\varphi_x|^2|\Theta_x|^2 + \frac{\left(1+\theta^b\right)\Theta|\zeta_x|^2}{v\theta^2}\cdot\theta^2|V_x|^2\right. \\ &\quad \left. + \left(1+\theta^{b+3}\right)\left(\varphi^2+\zeta^2\right)|U_x|^2\right]dxds \\ &\leq \epsilon X(t) + C(\epsilon)\left[\left(1+M_2^{b-3+\max\{1,(7-b)_+\}}\right)\delta + M_2^2\delta^4\right. \\ &\quad \left. + \left(1+M_2^{b+3}\right)\delta + \delta^2\left(1+M_2^{b+1}\right)\right] \\ &\leq \epsilon X(t) + C(\epsilon). \end{aligned} \tag{4.15}$$

Here $\delta>0$ is chosen sufficiently small such that

$$M_2^{\max\{b-3+\max\{1,(7-b)_+\},b+3\}}\delta \lesssim 1. \tag{4.16}$$

Now for I_7, we can get by combining the estimates (2.41), (4.2), (4.3), and (4.7) that

$$\begin{aligned} I_7 &\lesssim \int_0^t\int_{\mathbb{R}} \frac{\left(1+\theta^b\right)\Theta|\zeta_x|^2}{v\theta^2}\cdot|\psi_x|\theta^2 dxds \\ &\lesssim \left(1+Y(t)^{\frac{2}{2b+3}}\right)\left(1+Z(t)^{\frac{3}{8}}\right) \\ &\leq \epsilon Y(t) + C(\epsilon)\left(1+Z(t)^{\frac{6b+9}{16b+8}}\right). \end{aligned} \tag{4.17}$$

To deal with I_8, we first deduce by employing Sobolev's inequality, the estimates (2.41) and (3.9) that

$$\begin{aligned} I_8 &\leq \epsilon X(t) + C(\epsilon)\int_0^t \left\|\frac{\kappa(v,\theta)\zeta_x}{v}\right\|_{L^\infty}^2\|\varphi_x\|^2 d\tau \\ &\leq \epsilon X(t) + C(\epsilon)\left(1+\|\theta\|_\infty^{\max\{1,(7-b)_+\}}\right)\int_0^t\int_{\mathbb{R}}\left|\frac{\kappa(v,\theta)\zeta_x}{v}\right|\left|\left(\frac{\kappa(v,\theta)\zeta_x}{v}\right)_x\right|dxds \\ &\leq \epsilon X(t) + C(\epsilon)\left(1+\|\theta\|_\infty^{\max\{1,(7-b)_+\}}\right)\left(\int_0^t\int_{\mathbb{R}}\theta^2\kappa(v,\theta)\left|\left(\frac{\kappa(v,\theta)\zeta_x}{v}\right)_x\right|^2dxds\right)^{\frac{1}{2}} \\ &\quad \times\left(\int_0^t\int_{\mathbb{R}}\frac{\kappa(v,\theta)\Theta\zeta_x^2}{v\theta^2}dxds\right)^{\frac{1}{2}} \\ &\leq \epsilon X(t) + C(\epsilon)\left(1+Y(t)^{\frac{\max\{1,(7-b)_+\}}{2b+3}}\right) \\ &\quad \times \underbrace{\left(\int_0^t\int_{\mathbb{R}}\left(1+\theta^{b+2}\right)\left|\left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_x - \left(\frac{\kappa(v,\theta)\Theta_x}{v}\right)_x\right|^2dxds\right)^{\frac{1}{2}}}_{J^{\frac{1}{2}}}. \end{aligned} \tag{4.18}$$

To estimate J, one has from $(2.7)_3$, Lemma 2.1, (2.41), (4.3) and (4.13) that

$$
\begin{aligned}
J \lesssim & \int_0^t \int_{\mathbb{R}} \Bigg[\left(1+\theta^{b+2}\right) e_\theta^2 \zeta_t^2 + \left(1+\theta^{b+2}\right) \theta^2 p_\theta^2 \psi_x^2 \\
& + \left(1+\theta^{b+2}\right) e_\theta^2 \left(\frac{\theta p_\theta}{e_\theta} - \frac{\Theta P_\Theta}{E_\Theta}\right)^2 U_x^2 + \left(1+\theta^{b+2}\right) \psi_x^4 \\
& + \left(1+\theta^{b+2}\right) U_x^4 + \left(1+\theta^{b+2}\right) \phi^2 z^2 + \left(1+\theta^{b+2}\right) \left[\frac{e_\theta}{E_\Theta}\left(\frac{k(V,\Theta)\Theta_x}{V}\right)_x\right]^2 \Bigg] dxds \\
& + \underbrace{\int_0^t \int_{\mathbb{R}} \left(1+\theta^{b+2}\right) \left|\left(\frac{\kappa(v,\theta)\Theta_x}{v}\right)_x\right|^2 dxds}_{J^a} \qquad (4.19) \\
\lesssim & J^a + \int_0^t \int_{\mathbb{R}} \left[\left(1+\theta^{b+5}\right)\zeta_t^2 + \frac{\mu\Theta\psi_x^2}{v\theta}\cdot\left(1+\theta^{b+11}\right) + \left(1+\theta^{b+8}\right)\left(\zeta^2+\varphi^2\right)U_x^2\right] dxds \\
& + \left(1+M_2^{b+2}\right)\delta + \left(1+\|\theta\|_\infty^{b+3}\right)\|\psi_x\|_{L^\infty}^2 \int_0^t\int_{\mathbb{R}} \frac{\mu\Theta\psi_x^2}{v\theta} dxds \\
& + \left(1+M_2^{b+8}\right)\delta^{\frac{4}{3}} + \left(1+\|\theta\|_\infty^{b+\beta+2}\right)\int_0^t\int_{\mathbb{R}} \phi z^2 dxds \\
\lesssim & 1 + X(t)\left(1+Y(t)^{\frac{5}{2b+3}}\right) + Y(t)^{\frac{b+11}{2b+3}} + \left(1+Y(t)^{\frac{b+3}{2b+3}}\right)\left(1+Z(t)^{\frac{3}{4}}\right) \\
& + Y(t)^{\frac{b+\beta+2}{2b+3}} + J^a,
\end{aligned}
$$

and J^a can be further estimated as follows

$$
\begin{aligned}
J^a \lesssim & \int_0^t \int_{\mathbb{R}} \left(1+\theta^{b+2}\right) \Big[\theta^{2b}\varphi_x^2\Theta_x^2 + \theta^{2b}V_x^2\Theta_x^2 + \theta^{2b-2}\zeta_x^2\Theta_x^2 + \theta^{2b-2}\Theta_x^4 + \Theta_{xx}^2 \\
& + \theta^{2b}\Theta_{xx}^2 + \left(1+\theta^{2b}\right)\Theta_x^2\left(\varphi_x^2+V_x^2\right)\Big] dxds \\
\lesssim & \left(1+\|\theta\|_\infty^{3b+2}\right)\int_0^t \|\Theta_x\|_{L^\infty}^2 \|\varphi_x\|^2 d\tau + \left(1+\|\theta\|_\infty^{3b+2}\right)\int_0^t\int_{\mathbb{R}} V_x^2\Theta_x^2 dxds \\
& + \int_0^t\int_{\mathbb{R}} \frac{\left(1+\theta^b\right)\Theta|\zeta_x|^2}{v\theta^2}\cdot\left(1+\theta^{2b+2}\right)\Theta_x^2 dxds \qquad (4.20) \\
& + \left(1+\|\theta\|_\infty^{3b}\right)\int_0^t\int_{\mathbb{R}} \Theta_x^4 dxds + \left(1+\|\theta\|_\infty^{3b+2}\right)\int_0^t\int_{\mathbb{R}} \Theta_{xx}^2 dxds \\
\lesssim & \left(1+M_2^{3b+3}\right)\left(\delta^{\frac{1}{2}}+\delta^4\right) \\
\lesssim & 1.
\end{aligned}
$$

Here we have used the fact that $\delta>0$ is chosen sufficiently small such that

$$
M_2^{6(b+1)}\delta \lesssim 1 \qquad (4.21)
$$

in deducing (4.19) and (4.20).

The combination of (4.19)-(4.20) yields

$$
\begin{aligned}
J \lesssim & 1 + X(t)\left(1+Y(t)^{\frac{5}{2b+3}}\right) + Y(t)^{\frac{b+11}{2b+3}} + Z(t)^{\frac{3}{4}} \qquad (4.22) \\
& + Y(t)^{\frac{b+3}{2b+3}} Z(t)^{\frac{3}{4}} + Y(t)^{\frac{b+\beta+2}{2b+3}}
\end{aligned}
$$

and by inserting (4.22) into (4.18), we can derive that

$$I_8 \leq \epsilon\left(X(t)+Y(t)\right)+C\left(\epsilon\right)\left(1+Z(t)^{\frac{6b+9}{12b+12-8\max\{1,(7-b)_+\}}}\right), \tag{4.23}$$

where we have used the two facts that $b>4$ and $0\leq\beta<\min\{3b+4,5b-8\}$.

As to the term I_9, we have from Lemma 2.1, (2.40) and the assumption $0\leq\beta<b+6$ that

$$\begin{aligned} I_9 &\lesssim \int_0^t\int_{\mathbb{R}}\left[\left(1+\theta^b\right)\phi z+\left(1+\theta^b\right)\left(1+a\theta^3\right)\left|\frac{e_\theta}{E_\Theta}\left(\frac{k(V,\Theta)\Theta_x}{V}\right)_x\right|\right]\left|\zeta_t\right|dxds\\ &\leq \epsilon X(t)+C\left(\epsilon\right)\int_0^t\int_{\mathbb{R}}\left[\left(1+\theta^{b-3}\right)\phi^2z^2+\left(1+\theta^{b+3}\right)\left|\frac{e_\theta}{E_\Theta}\left(\frac{k(V,\Theta)\Theta_x}{V}\right)_x\right|^2\right]dxds\\ &\leq \epsilon X(t)+C\left(\epsilon\right)\left(1+\|\theta\|_\infty^{b+\beta-3}+\delta^{\frac{4}{3}}\left(1+M_2^{b+3}\right)\right) \\ &\leq \epsilon\left(X(t)+Y(t)\right)+C\left(\epsilon\right), \end{aligned} \tag{4.24}$$

while for the term I_{10}, we apply (4.7) and the assumption $b>4$ to know that

$$\begin{aligned} I_{10} &\lesssim \int_0^t\int_{\mathbb{R}}\left(1+\theta^b\right)\theta\left(1+a\theta^3\right)\left|\psi_x\zeta_t\right|dxds\\ &\leq \epsilon X(t)+C\left(\epsilon\right)\int_0^t\int_{\mathbb{R}}\left(1+\theta^{b+6}\right)\frac{\psi_x^2}{\theta}dxds\\ &\leq \epsilon X(t)+C\left(\epsilon\right)\left(1+Y(t)^{\frac{b+6}{2b+3}}\right)\\ &\leq \epsilon\left(X(t)+Y(t)\right)+C\left(\epsilon\right). \end{aligned} \tag{4.25}$$

It suffices to bound the term I_{11}. To this end, we conclude from Lemma 2.1 and (4.7) that

$$\begin{aligned} I_{11} &\leq \epsilon X(t)+C\left(\epsilon\right)\int_0^t\int_{\mathbb{R}}\left(1+\theta^{b-3}\right)\left(\psi_x^4+U_x^4\right)dxds\\ &\leq \epsilon X(t)+C\left(\epsilon\right)\left(1+\int_0^t\int_{\mathbb{R}}\left(1+\theta^{b-3}\right)\psi_x^4dxds\right). \end{aligned} \tag{4.26}$$

For the last term in the right hand side of (4.26), we can derive by using Sobolev's inequality, Lemma 3.6 and (2.41) that

$$\begin{aligned} &\int_0^t\int_{\mathbb{R}}\left(1+\theta^{b-3}\right)\psi_x^4dxds\\ &\lesssim \left(1+\|\theta\|_\infty^{b-3}\right)\int_0^t\|\psi_x\|_{L^\infty}^2\|\psi_x\|^2d\tau\\ &\lesssim \left(1+\|\theta\|_\infty^{b-3}\right)\int_0^t\|\psi_x\|^3\|\psi_{xx}\|d\tau\\ &\lesssim \left(1+\|\theta\|_\infty^{b-2+2\max\{1,(7-b)_+\}}\right)\left(\int_0^t\|\psi_x\|^2d\tau\right)^{\frac{1}{2}}\left(\int_0^t\|\psi_{xx}\|^2d\tau\right)^{\frac{1}{2}}\\ &\lesssim 1+\|\theta\|_\infty^{b-1+3\max\{1,(7-b)_+\}}, \end{aligned} \tag{4.27}$$

and we can get by (4.26), (4.27) and the assumption $b > \frac{17}{4}$ that

$$I_{11} \leq \epsilon \left(X(t) + Y(t)\right) + C\left(\epsilon\right). \tag{4.28}$$

Substituting (4.9), (4.10), (4.11), (4.15), (4.17), (4.23), (4.24), (4.25) and (4.28) into (4.8) and by choosing $\epsilon > 0$ small enough yields (4.4) provided that we choose $\delta > 0$ sufficiently small such that (4.12), (4.16) and (4.21) hold. This completes the proof of Lemma 4.2. □

The following lemma is to show that $Z(t)$ can also be controlled by $X(t)$ and $Y(t)$.

LEMMA 4.3. *Under the conditions stated in Proposition 2.5, then $\delta > 0$ is assumed to be sufficiently small such that*

$$M_2^6 \delta \lesssim 1, \tag{4.29}$$

then we have for all $0 \leq t \leq T$ that

$$Z(t) \lesssim 1 + X(t) + Y(t) + Z(t)^{\lambda_2}. \tag{4.30}$$

Here

$$\lambda_2 = \frac{6b+9}{8b+12-4\max\{1,(7-b)_+\}}. \tag{4.31}$$

Proof. Differentiating $(2.7)_2$ with respect to t and multiplying it by ψ_t, we can obtain

$$\begin{aligned}&\left(\frac{\psi_t^2}{2}\right)_t + \frac{\mu\psi_{xt}^2}{v} + \left[\left((p-P)_t + H_t - \mu\left(\frac{u_x}{v}\right)_t\right)\psi_t\right]_x \\ &= \left[\frac{\mu\left(\psi_x+U_x\right)^2}{v^2} - \frac{\mu U_{xt}}{v} + (p-P)_t + H_t\right]\psi_{xt},\end{aligned} \tag{4.32}$$

where, $H_t = \int^x U_{tt}(t,y)dy$.

Integrating the above identity (4.32) with respect to t and x over $(0,t)\times\mathbb{R}$, we conclude that

$$\begin{aligned}&\int_{\mathbb{R}} \frac{\psi_t^2}{2}dx + \int_0^t\int_{\mathbb{R}} \frac{\mu\psi_{tx}^2}{v}dxds \\ &= \int_{\mathbb{R}} \frac{\psi_{0t}^2}{2}dx + \underbrace{\int_0^t\int_{\mathbb{R}}\left(\frac{\mu\psi_{tx}\psi_x^2 + \mu\psi_{tx}U_x^2 + 2\mu\psi_{tx}\psi_x U_x}{v^2} - \frac{\mu\psi_{tx}U_{tx}}{v} + H_t\psi_{tx}\right)dxds}_{I_{12}} \\ &\quad + \underbrace{\int_0^t\int_{\mathbb{R}}(p-P)_t\,\psi_{tx}dxds}_{I_{13}}.\end{aligned} \tag{4.33}$$

Now we turn to estimate $I_k (k = 12, 13)$ term by term. To this end, we use (4.29), (2.41) and (3.8) to deduce that

$$\begin{aligned}I_{12} &\leq \epsilon\int_0^t\int_{\mathbb{R}}\frac{\mu\psi_{tx}^2}{v}dxds + C\left(\epsilon\right)\int_0^t\int_{\mathbb{R}}\left(\frac{\psi_x^2}{\theta}\cdot\left(\theta\psi_x^2+\theta U_x^2\right) + U_x^4 + U_{xt}^2 + \left|H_t\right|^2\right)dxds \\ &\leq \epsilon\int_0^t\int_{\mathbb{R}}\frac{\mu\psi_{tx}^2}{v}dxds + C\left(\epsilon\right)\left(\left(1+Y(t)^{\frac{1}{2b+3}}\right)\left(1+Z(t)^{\frac{3}{4}}\right) + \delta + \delta^4 M_2\right)dxds \\ &\leq \epsilon\int_0^t\int_{\mathbb{R}}\frac{\mu\psi_{tx}^2}{v}dxds + C\left(\epsilon\right)\left(1+Y(t)+Z(t)^{\frac{6b+9}{8b+8}}\right).\end{aligned} \tag{4.34}$$

To deal with I_{13}, noticing that

$$\begin{aligned} |(p-P)_t|^2 \lesssim & \left(1+a^2\theta^6\right)\zeta_t^2 + |\Theta_t|^2\left(\varphi^2+\zeta^2\left(1+\theta^4\right)\right) \\ & +\zeta^2\psi_x^2+\zeta^2U_x^2+\psi_x^2+U_x^2\varphi^2, \end{aligned} \tag{4.35}$$

we can get from Lemma 2.1 and (2.41) that

$$\begin{aligned} I_{13} \leq & \epsilon\int_0^t\int_{\mathbb{R}}\frac{\mu\psi_{tx}^2}{v}dxds + C(\epsilon)\int_0^t\int_{\mathbb{R}}|(p-P)_t|^2\,dxds \\ \leq & \epsilon\int_0^t\int_{\mathbb{R}}\frac{\mu\psi_{tx}^2}{v}dxds + C(\epsilon)\Bigg[X(t)+\int_0^t\|\Theta_t\|_{L^\infty}^2\left(\|\varphi\|^2+\left(1+\|\theta\|_{L^\infty}^4\right)\|\zeta\|^2\right)d\tau \\ & +\int_0^t\int_{\mathbb{R}}\frac{\psi_x^2}{\theta}\cdot\left(1+\theta^3\right)dxds+\int_0^t\|U_x\|_{L^\infty}^2\,d\tau\Bigg] \\ \leq & \epsilon\int_0^t\int_{\mathbb{R}}\frac{\mu\psi_{tx}^2}{v}dxds + C(\epsilon)\left(1+X(t)+Y(t)+\delta\left(1+M_2^6\right)\right) \\ \leq & \epsilon\int_0^t\int_{\mathbb{R}}\frac{\mu\psi_{tx}^2}{v}dxds + C(\epsilon)\left(1+X(t)+Y(t)\right). \end{aligned} \tag{4.36}$$

If we choose $\epsilon>0$ small enough, then we can deduce from (4.33)-(4.36) and (3.8) that

$$\int_{\mathbb{R}}\psi_t^2dx+\int_0^t\int_{\mathbb{R}}\psi_{tx}^2dxds \lesssim 1+X(t)+Y(t)+Z(t)^{\frac{6b+9}{8b+8}}. \tag{4.37}$$

Now we are in a position to yield an estimate on $\int_{\mathbb{R}}\psi_{xx}^2dx$. To do so, firstly, $(2.7)_2$ tells us that

$$\begin{aligned} \psi_{xx} = & \frac{\varphi_x\psi_x+\varphi_xU_x+V_x\psi_x+V_xU_x}{v}-U_{xx} \\ & +\frac{v}{\mu}\left[\psi_t+\left(p(v,\theta)-P(V,\Theta)\right)_x+U_t\right]. \end{aligned} \tag{4.38}$$

On the other hand, Lemma 2.1, (4.29), (3.9), (4.2) and (4.14) show that

$$\begin{aligned} & \|(p-P)_x(t)\|^2 \\ \lesssim & \int_{\mathbb{R}}\left[\left(1+a^2\theta^6\right)|\zeta_x|^2+|(V_x,\Theta_x)|^2|\varphi|^2+\left(1+\theta^4\right)|(V_x,\Theta_x)|^2|\zeta|^2+|\theta\varphi_x|^2\right]dx \\ \lesssim & Y(t)+\left(1+\|\theta\|_{L^\infty}^{2\max\{1,(7-B)_+\}}\right)\left(1+\|\theta\|_{L^\infty}^{\max\{1,(7-b)_+\}}\right)+\delta^4\left(1+M_2^6\right) \\ \lesssim & 1+Y(t). \end{aligned} \tag{4.39}$$

Then, we can deduce from Lemma 3.6, (3.9), (4.3), (4.37)-(4.39) and Young's inequality that

$$
\begin{aligned}
\int_{\mathbb{R}} \psi_{xx}^2 dx \lesssim & \int_{\mathbb{R}} \Big(\varphi_x^2\psi_x^2 + \varphi_x^2 U_x^2 + \psi_x^2 V_x^2 + V_x^2 U_x^2 + U_{xx}^2 + \psi_t^2 \\
& + \left|(p(v,\theta) - P(V,\Theta))_x\right|^2 + |U_t|^2\Big)\, dx \\
\lesssim & 1 + X(t) + Y(t) + Z(t)^{\frac{6b+9}{8b+8}} + \|\psi_x\|_{L^\infty}^2 \|\varphi_x\|^2 + \|U_x\|_{L^\infty}^2 \|\varphi_x\|^2 \\
& + \|V_x\|_{L^\infty}^2 \|\psi_x\|^2 + \|V_x\|_{L^\infty}^2 \|U_x\|^2 + \|U_{xx}\|^2 + \|U_t\|^2 \qquad (4.40)\\
\lesssim & 1 + X(t) + Y(t) + Z(t)^{\frac{6b+9}{8b+12-4\max\{1,(7-b)_+\}}} + \delta^4\left(1 + M_2^3\right) \\
& + \left(1 + Y(t)^{\frac{\max\{1,(7-b)_+\}}{2b+3}}\right)\left(1 + Z(t)^{\frac{3}{4}}\right) \\
\lesssim & 1 + X(t) + Y(t) + Z(t)^{\max\left\{\frac{6b+9}{8b+8}, \frac{6b+9}{8b+12-4\max\{1,(7-b)_+\}}\right\}}.
\end{aligned}
$$

Having obtained (4.40), the estimate (4.30) follows by employing the definition of $Z(t)$. This completes the proof of Lemma 4.3. □

From Lemmas 4.1-4.3, if we let $b > \frac{17}{4}$, then one can get that

$$
\begin{aligned}
\lambda_1 &= \max\left\{\frac{6b+9}{12b+12-8\max\{1,(7-b)_+\}}, \frac{6b+9}{16b+8}\right\} < 1, \\
\lambda_2 &= \frac{6b+9}{8b+12-4\max\{1,(7-b)_+\}} < 1,
\end{aligned}
$$

we can derive from the estimates (4.4) and (4.30) that $Y(t) \lesssim 1$. Then the desired upper bound on $\theta(t,x)$ follows from (4.2) immediately. Therefore, we can deduce from Lemmas 2.1-4.3 that

Lemma 4.4. *Under the conditions stated in Proposition 2.5, there exists a positive constant C_2 which depends only on $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ and m_0, such that*

$$
\theta(t,x) \le C_2, \quad \forall (t,x) \in [0,T] \times \mathbb{R}. \tag{4.41}
$$

Moreover, we have for $0 \le t \le T$ that

$$
\begin{aligned}
& \|(\varphi,\psi,\zeta,z,\varphi_x,\psi_x,\psi_t,\zeta_x,\psi_{xx})(t)\|^2 \\
& \quad + \int_0^t \left\|\left(\sqrt{\theta}\varphi_x, \psi_x, \zeta_t, \zeta_x, \psi_{xx}, \psi_{xt}, z_x\right)(\tau)\right\|^2 d\tau \lesssim 1
\end{aligned} \tag{4.42}
$$

and

$$
\int_0^t \|\psi_x(\tau)\|_{L^4(\mathbb{R})}^4 d\tau \lesssim 1, \quad \|\psi_x\|_{L^\infty([0,T]\times\mathbb{R})} \lesssim 1. \tag{4.43}
$$

5. Higher order energy estimates. Now we deduce nice bounds on the terms $\int_0^t \|\zeta_{xx}(\tau)\|^2 d\tau$ and $\|z_x(t)\|^2$. The following lemma is concerned with the estimate on $\int_0^t \|\zeta_{xx}(\tau)\|^2 d\tau$.

Lemma 5.1. *Under the assumptions listed in Proposition 2.5, we have for $0 \le t \le T$ that*

$$
\|\zeta_x(t)\|^2 + \int_0^t \|\zeta_{xx}(\tau)\|^2 d\tau \lesssim 1. \tag{5.1}
$$

Proof. In order to get (5.1), we multiply $(2.7)_3$ by ζ_{xx} to get that

$$
\begin{aligned}
&\partial_t\left(\frac{\zeta_x^2}{2}\right)+\frac{\kappa(v,\theta)\zeta_{xx}^2}{ve_\theta}-(\zeta_t\zeta_x)_x\\
=&\frac{\theta p_\theta\psi_x\zeta_{xx}}{e_\theta}+\left(\frac{\theta p_\theta}{e_\theta}-\frac{\Theta P_\Theta}{E_\Theta}\right)U_x\zeta_{xx}-\frac{\mu u_x^2\zeta_{xx}}{ve_\theta}-\frac{\kappa(v,\theta)\Theta_{xx}\zeta_{xx}}{ve_\theta}-\frac{\lambda\phi z\zeta_{xx}}{e_\theta}\\
&+\left(\frac{\kappa}{v^2e_\theta}-\frac{\kappa_v}{ve_\theta}\right)(\Theta_x+\zeta_x)(V_x+\varphi_x)\zeta_{xx}\\
&-\frac{\kappa_\theta}{ve_\theta}(\zeta_x+\Theta_x)^2\zeta_{xx}+\frac{1}{E_\Theta}\left(\frac{k(V,\Theta)\Theta_x}{V}\right)_x\zeta_{xx}.
\end{aligned}
$$

Integrating the above identity with respect to t and x over $(0,t)\times\mathbb{R}$, one can deduce by using Lemma 2.1, (2.40), (2.41), (3.8), (4.13), Lemma 4.4 and Sobolev's inequality that

$$
\begin{aligned}
&\int_{\mathbb{R}}\frac{\zeta_x^2}{2}dx+\int_0^t\int_{\mathbb{R}}\frac{\kappa(v,\theta)\zeta_{xx}^2}{ve_\theta}dxds\\
\leq\ &\epsilon\int_0^t\int_{\mathbb{R}}\frac{\kappa(v,\theta)\zeta_{xx}^2}{ve_\theta}dxds+C(\epsilon)\int_0^t\int_{\mathbb{R}}\left[\frac{\theta^2p_\theta^2\psi_x^2}{\kappa(v,\theta)e_\theta}+\frac{\left(\varphi^2+\zeta^2\right)U_x^2e_\theta}{1+\theta^b}+\frac{\kappa(v,\theta)\Theta_{xx}^2}{e_\theta}\right.\\
&+\left(\frac{1}{E_\Theta}\left(\frac{k(V,\Theta)\Theta_x}{V}\right)_x\right)^2\frac{e_\theta}{\kappa(v,\theta)}+\frac{\phi^2z^2}{\kappa(v,\theta)e_\theta}+\psi_x^4\\
&\left.+U_x^4+\left(\Theta_x^2+\zeta_x^2\right)\left(\varphi_x^2+V_x^2\right)+\zeta_x^4+\Theta_x^4\right]dxds\\
\leq\ &\epsilon\int_0^t\int_{\mathbb{R}}\frac{\kappa(v,\theta)\zeta_{xx}^2}{ve_\theta}dxds \qquad (5.2)\\
&+C(\epsilon)\left[\int_0^t\left(\|\Theta_x\|_{L^\infty}^2+\|U_x\|_{L^\infty}^2+\|\Theta_x\|_{L^\infty}^2\|V_x\|^2+\|\zeta_x\|\|\zeta_{xx}\|\right)d\tau+\delta\right.\\
&\left.+\int_0^t\int_{\mathbb{R}}\left(\frac{\psi_x^2}{\theta}\cdot\left(\theta+\frac{\theta^3p_\theta^2}{(1+\theta^b)e_\theta}\right)+\Theta_{xx}^2+\left(\frac{1}{E_\Theta}\left(\frac{k(V,\Theta)\Theta_x}{V}\right)_x\right)^2+\phi z^2\right)dxds\right]\\
\leq\ &2\epsilon\int_0^t\int_{\mathbb{R}}\frac{\kappa(v,\theta)\zeta_{xx}^2}{ve_\theta}dxds+C(\epsilon).
\end{aligned}
$$

With (5.2) in hand, the estimate (5.1) follows immediately by choosing $\epsilon>0$ small enough. This completes the proof of Lemma 5.1. □

Next, we turn to derive the desired bound on $\|z_x(t)\|^2$. In fact, we have the following lemma.

LEMMA 5.2. *Under the conditions of Proposition 2.5, we can obtain for any* $0\leq t\leq T$ *that*

$$
\|z_x(t)\|^2+\int_0^t\|z_{xx}(\tau)\|^2d\tau\lesssim 1. \tag{5.3}
$$

Proof. To begin with, we multiply $(2.7)_4$ by z_{xx} to derive

$$
\partial_t\left(\frac{z_x^2}{2}\right)+\frac{dz_{xx}^2}{v^2}-(z_tz_x)_x=\frac{2dv_xz_xz_{xx}}{v^3}+\phi zz_{xx}.
$$

Integrating the above identity with respect to t and x over $(0,t)\times\mathbb{R}$, we can get that

$$\int_{\mathbb{R}}\frac{z_x^2}{2}dx+\int_0^t\int_{\mathbb{R}}\frac{dz_{xx}^2}{v^2}dxds=\int_{\mathbb{R}}\frac{z_{0x}^2}{2}dx+\int_0^t\int_{\mathbb{R}}\left(\frac{2dv_xz_xz_{xx}}{v^3}+\phi zz_{xx}\right)dxds. \quad (5.4)$$

For the last term in the right hand side of (5.4), we can get from Lemma 2.1, (3.8), (3.9), Lemma 4.4 and Sobolev's inequality that

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}}\left(\frac{2dv_xz_xz_{xx}}{v^3}+\phi zz_{xx}\right)dxds\\
&\le\epsilon\int_0^t\int_{\mathbb{R}}\frac{dz_{xx}^2}{v^2}dxds+C(\epsilon)\int_0^t\int_{\mathbb{R}}\left(v_x^2z_x^2+\phi^2z^2\right)dxds\\
&\le\epsilon\int_0^t\int_{\mathbb{R}}\frac{dz_{xx}^2}{v^2}dxds+C(\epsilon)\int_0^t\int_{\mathbb{R}}\left(\left(V_x^2+\varphi_x^2\right)z_x^2+\phi z^2\right)dxds\\
&\le\epsilon\int_0^t\int_{\mathbb{R}}\frac{dz_{xx}^2}{v^2}dxds+C(\epsilon)\left(1+\delta^4+\int_0^t\|z_x(\tau)\|\,\|z_{xx}(\tau)\|\,d\tau\right)\\
&\le 2\epsilon\int_0^t\int_{\mathbb{R}}\frac{dz_{xx}^2}{v^2}dxds+C(\epsilon).
\end{aligned} \quad (5.5)$$

Plugging (5.5) into (5.4) and choosing $\epsilon>0$ small enough, we can obtain (5.3) immediately. □

6. A local-in-time lower bound on the absolute temperature and the proof of Theorem 1.1. The following lemma will give a local-in-time lower bound on $\theta(t,x)$. To this end, we can deduce by repeating the method used in [21] that

LEMMA 6.1. *Under the assumptions stated in Proposition 2.5, then for each $0\le s\le t\le T$ and $x\in\mathbb{R}$, the following estimate*

$$\theta(t,x)\ge\frac{C\min_{x\in\mathbb{R}}\{\theta(s,x)\}}{1+(t-s)\min_{x\in\mathbb{R}}\{\theta(s,x)\}} \quad (6.1)$$

holds for some positive constant C which depends only on $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ and m_0.

With Lemma 2.6, Lemma 3.4, Lemma 3.5, Lemma 3.6, Lemma 4.4, Lemma 5.1, Lemma 5.2 and Lemma 6.1 in hand, if we choose $\delta_0>0$ sufficiently small such that

$$\left(1+M_1^{100}\right)M_2^{100(b+1)}M_3\left(1+M^6\right)\delta_0\le C_6 \quad (6.2)$$

holds for some positive constant C_6 depending only on $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ and m_0, then for $0<\delta<\delta_0$, we can deduce that the results stated in Proposition 2.5 hold. Once we obtained Proposition 2.5, Theorem 1.1 can be proved by using the continuation argument designed in [21] and we omit the details for brevity.

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