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EXISTENCE OF RADIALLY SYMMETRIC STATIONARY SOLUTIONS FOR THE COMPRESSIBLE NAVIER-STOKES EQUATION*

ITSUKO HASHIMOTO† AND AKITAKA MATSUMURA‡

Dedicated to Professor Ling Hsiao on the occasion of her 80th birthday

Abstract. The present paper is concerned with the existence of radially symmetric stationary solutions for exterior problems in $\mathbb{R}^n (n \geq 2)$ to the compressible Navier-Stokes equation, describing the motion of viscous barotropic gas without external forces, where boundary and far field data are prescribed. For both inflow and outflow problems, the existence of a unique radially stationary solution is shown in a suitably small neighborhood of the far field state. The estimates of algebraic decay rate toward the far field state are also obtained. Furthermore, it is shown that the boundary layer of the density appears as the velocity data tend to zero in the inflow problem, but not in the outflow problem.

Key words. Compressible Navier-Stokes equation, stationary solution, radially symmetric solution.

Mathematics Subject Classification. Primary 35Q30; Secondary 76N10.

1. Introduction and Main theorem. In the present paper, we consider the compressible Navier-Stokes equation which describes a barotropic motion of viscous gas in the exterior domain Ω to a ball in $\mathbb{R}^n$ $(n \geq 2)$:

$$\begin{cases} \rho_t + \operatorname{div}(\rho U) = 0, \\ (\rho U)_t + \operatorname{div}(\rho U \otimes U) + \nabla p = \nu \triangle U + (\nu + \lambda)\nabla(\operatorname{div} U), \quad t > 0,\ x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega = \{x \in \mathbb{R}^n\,(n \geq 2); |x| > r_0\}$ (r_0 is a positive constant), $\rho = \rho(t,x) > 0$ is the mass density, $U = (u_1(t,x), \cdots, u_n(t,x))$ is the fluid velocity, and $p = p(\rho)$ is the pressure given by a smooth function of ρ satisfying $p'(\rho) > 0$ $(\rho > 0)$. Furthermore, ν and λ are the shear and second viscosity coefficients respectively, which are assumed to be constants satisfying $\nu > 0, 2\nu + n\lambda \geq 0$. In this paper, we focus our attention on the radially symmetric solutions, which have the form

$$\rho(t,x) = \rho(t,r), \quad U(t,x) = \frac{x}{r}\, u(t,r), \quad r = |x|, \tag{1.2}$$

where $u(t,r)$ is a scalar function. By plugging (1.2) to (1.1), we can rewrite (1.1) as in the form

$$\begin{cases} (r^{n-1}\rho)_t + (r^{n-1}\rho u)_r = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_r + (n-1)\dfrac{\rho u^2}{r} = \mu\Big(\dfrac{(r^{n-1}u)_r}{r^{n-1}}\Big)_r, \quad t > 0,\ r > r_0, \end{cases} \tag{1.3}$$

where $\mu = 2\nu + \lambda > 0$. Now, we consider the initial boundary value problems to (1.3) under the initial condition

$$(\rho, u)(0,r) = (\rho_0, u_0)(r), \quad r > r_0, \tag{1.4}$$

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†Kanazawa University, Osaka City University, Japan (itsukohashimoto@hotmail.com). Supported by JSPS Grant No. 17K14227.
‡Osaka university, Japan (akitaka@math.sci.osaka-u.ac.jp).

the far field condition

$$\lim_{r\to\infty}(\rho,u)(t,r) = (\rho_+,u_+), \quad t>0, \tag{1.5}$$

and also the following two types of boundary conditions depending on the sign of the velocity on the boundary

$$\begin{cases} (\rho,u)(t,r_0) = (\rho_-,u_-), \quad t>0, \quad (u_->0), \\ u(t,r_0) = u_-, \quad t>0, \qquad\qquad\quad (u_-\le 0), \end{cases} \tag{1.6}$$

where $\rho_\pm > 0, u_\pm$ are given constants. The case $u_- > 0$ is known as "inflow problem", the case $u_- = 0$ as "impermeable wall problem", and the case $u_- < 0$ as "outflow problem". For these initial boundary value problems, when the space dimension is one ($n = 1$), there have been many results on the existence of time-global solutions and their asymptotic behaviors toward various nonlinear waves depending on the far field and boundary conditions, for example, toward stationary waves, rarefaction waves, viscous shock waves, and even their composite waves (cf. [3],[5],[6],,, etc.). On the other hand, when the problems are multi-dimensional ($n \ge 2$), there seem no results except the case $u_\pm = 0$ studied by Jiang [2] and Nakamura-Nishibata-Yanagi [7]. They study more general compressible Navier-Stokes equation, describing the motion of viscous polytropic ideal gas, Jiang [2] first showed the global asymptotic stability of the constant states, and later Nakamura-Nishibata-Yanagi [7] extended the results to the case with external potential forces. In this paper, as the first step to study the multi-dimensional problems in more general cases $u_- \neq 0$ or $u_+ \neq 0$, we shall show the existence of the stationary solution in a suitably small neighborhood of the far field state, by using the similar arguments as in Germain-Iwabuchi [1] where self-similar solutions of the equation for viscous polytropic ideal gas are studied in the whole space $\mathbb{R}^n$. The asymptotic stability will be studied in forthcoming papers.

The stationary problem corresponding to the problem (1.3)-(1.6) is written as

$$\begin{cases} (r^{n-1}\rho u)_r = 0, \\ \rho u u_r + p(\rho)_r = \mu\left(\dfrac{(r^{n-1}u)_r}{r^{n-1}}\right)_r, \qquad r>r_0, \\ \lim_{r\to\infty}(\rho,u)(r) = (\rho_+,u_+), \\ (\rho,u)(r_0) = (\rho_-,u_-) \;\; (u_->0), \quad u(r_0)=u_- \;\; (u_-\le 0). \end{cases} \tag{1.7}$$

From the first equation in (1.7), we easily see it holds

$$r^{n-1}\rho(r)u(r) = \epsilon, \qquad r \ge r_0, \tag{1.8}$$

for some constant ϵ, and it also holds from the boundary conditions that

$$\epsilon = r_0^{n-1}\rho_- u_- \quad (u_->0), \qquad \epsilon = r_0^{n-1}\rho(r_0)u_- \quad (u_-\le 0), \tag{1.9}$$

where note that in the case $u_- \le 0$, ϵ includes the unknown $\rho(r_0)$ which should be determined later. The formula (1.8) implies that if $n \ge 2$,

$$u_+ = \lim_{r\to\infty} u(r) = \lim_{r\to\infty}\frac{\epsilon}{r^{n-1}\rho_+} = 0.$$

Hence, we need to assume $u_+ = 0$ for the existence of multi-dimensional stationary solutions. Now we are ready to state the main result in the present paper.

THEOREM 1.1. *Let $n \geq 2$ and $u_+ = 0$. Then, for any $\rho_+ > 0$, there exist positive constants ϵ_0 and C satisfying the following:*

(I) *Let $u_- > 0$. If $|u_-| + |\rho_- - \rho_+| \leq \epsilon_0$, there exists a unique smooth solution (ρ, u) of the problem* (1.7) *satisfying*

$$\begin{aligned} &|\rho(r) - \rho_+| \leq Cr^{-(n-1)}(|u_-|^2 + |\rho_- - \rho_+|), \\ &C^{-1}r^{-(n-1)}|u_-| \leq |u(r)| \leq Cr^{-(n-1)}|u_-|, \qquad r \geq r_0. \end{aligned} \tag{1.10}$$

Furthermore, for any positive constant h, there exists a positive constant C_h such that it holds

$$\sup_{r \geq r_0 + h} |\rho(r) - \rho_+| \leq C_h |u_-|^2. \tag{1.11}$$

(II) *Let $u_- \leq 0$. If $|u_-| \leq \epsilon_0$, there exists a unique smooth solution (ρ, u) of the problem* (1.7) *satisfying*

$$\begin{aligned} &|\rho(r) - \rho_+| \leq Cr^{-2(n-1)}|u_-|^2, \\ &C^{-1}r^{-(n-1)}|u_-| \leq |u(r)| \leq Cr^{-(n-1)}|u_-|, \qquad r \geq r_0. \end{aligned} \tag{1.12}$$

REMARK 1. *When $u_- = 0$, by* (1.7)-(1.9), *we easily see the solution is the trivial constant state $(\rho, u) \equiv (\rho_+, 0)$. Therefore, when $u_- > 0$ and $\rho_- \neq \rho_+$, the estimate* (1.11) *shows a boundary layer for mass density does appear as $u_- \to +0$.*

REMARK 2. *When $n = 1$ and $u_+ = 0$, by* (1.7)-(1.9), *we also easily see the solution has to be the trivial constant state $(\rho, u) \equiv (\rho_+, 0)$, which, in particular, implies $u_- = 0$. The multi-dimensional results in* Theorem 1.1 *shows that even for $u_- \neq 0$, a non-trivial solution does exist. On the other hand, in the one-dimensional problem, it is well known that even for $u_+ \neq 0$, there are various cases where non-trivial solutions do exist (cf.*[5]*). These facts show a sharp difference between the multi-dimensional and the one-dimensional problems.*

REMARK 3. *It is interesting to compare the results in* Theorem 1.1 *with that for the multi-dimensional Burgers equation investigated in* [3], *where the corresponding stationary problem for the radial velocity u is formulated as*

$$\begin{cases} uu_r = \mu(\dfrac{(r^{n-1}u)_r}{r^{n-1}})_r, \quad r > r_0, \\ u(r_0) = u_-, \ \lim\limits_{r \to \infty} u(r) = u_+. \end{cases} \tag{1.13}$$

In the case $n \geq 2$ and $u_+ = 0$, it is shown that if and only if $u_- \leq 2\mu(n-2)/r_0$, the solution of (1.13) *exists and is exactly given by the formula*

$$u(r) = \frac{u_-}{r\big(\frac{1}{r_0} - \frac{u_-}{2\mu}\log(r/r_0)\big)} \quad (n = 2), \tag{1.14}$$

and

$$u(r) = \frac{u_-}{(1 - \frac{r_0 u_-}{2\mu(n-2)})(r/r_0)^{n-1} + \frac{r_0 u_-}{2\mu(n-2)}(r/r_0)} \quad (n \geq 3). \tag{1.15}$$

Although the results in Theorem 1.1 *for* $n \geq 3$ *seem well consistent with the formula* (1.15)*, the case* $n = 2$ *shows a sharp difference, that is, the solution of* (1.7) *exists even for* $u_- > 0$ *(no solution of* (1.13) *for* $u_- > 0$*) and the decay rate toward the far field state is algebraic (logarithmic as* (1.14) *shows for* (1.13)*).*

2. Preliminary. In this section, we reformulate the problem (1.7). First, we assume $u_- \neq 0$ in what follows, because if $u_- = 0$, that is, $\epsilon = 0$, the unique solution of (1.7) easily turns out to be the trivial constant state $(\rho, u) \equiv (\rho_+, 0)$, which satisfies the estimate (1.13) in Theorem 1.1. We also assume $r_0 = 1$ without loss of generality. In fact, if not, we may introduce the new variable $\tilde{r} = r/r_0$ and the viscosity coefficient $\tilde{\mu} = \mu r_0$, and then write $\tilde{r}$ and $\tilde{\mu}$ as r and μ again. Next, introduce the specific volume v by $v = 1/\rho$ (accordingly, denote $v_\pm$ by $1/\rho_\pm$). Then, by (1.8) and (1.9), the velocity u is given in terms of v as

$$u(r) = \frac{\epsilon}{r^{n-1}} v(r), \quad r \geq 1, \tag{2.1}$$

where $\epsilon = u_-/v_-$ $(u_- > 0)$, and $\epsilon = u_-/v(1)$ $(u_- \leq 0)$. Substituting (2.1) into the second equation of (1.7), we have

$$\frac{\epsilon^2}{r^{n-1}} \left(\frac{v}{r^{n-1}} \right)_r + \tilde{p}(v)_r = \epsilon\mu \left(\frac{v_r}{r^{n-1}} \right)_r, \tag{2.2}$$

where $\tilde{p}(v) := p(1/v)$, and it holds $\tilde{p}'(v) < 0\,(v > 0)$ by the assumption on $p(v)$. Now, we further introduce a new unknown function η, as the deviation of v from the far field state v_+, by

$$\eta(r) = v(r) - v_+, \qquad r \geq 1. \tag{2.3}$$

Plugging (2.3) into (2.2), we have

$$\epsilon\mu \left(\frac{\eta_r}{r^{n-1}} \right)_r = \tilde{p}(v_+ + \eta)_r + \frac{\epsilon^2 v_+}{2} \left(\frac{1}{r^{2(n-1)}} \right)_r + \frac{\epsilon^2}{r^{n-1}} \left(\frac{\eta}{r^{n-1}} \right)_r, \tag{2.4}$$

where $\epsilon = u_-/v_-$ $(u_- > 0)$, and $\epsilon = u_-/(v_+ + \eta(1))$ $(u_- < 0)$. Under the far field condition $\eta(\infty) = 0$, the equation (2.4) is also equivalent to

$$\left(\epsilon\mu \frac{\eta_r}{r^{n-1}} - \tilde{p}(v_+ + \eta) - \frac{\epsilon^2 v_+}{2r^{2(n-1)}} - \frac{\epsilon^2 \eta}{r^{2(n-1)}} + \epsilon^2 (n-1) \int_r^\infty \frac{\eta(s)}{s^{2n-1}} ds \right)_r = 0, \tag{2.5}$$

which implies that the function in the parenthesis in the left hand side of (2.5) is identically equals to a constant c_0 for $r > 1$. Then, it follows from the far field condition that

$$\lim_{r \to \infty} \frac{\eta_r(r)}{r^{n-1}} = \frac{1}{\epsilon\mu}(c_0 + \tilde{p}(v_+)),$$

which concludes $c_0 = -\tilde{p}(v_+)$, otherwise it contradicts the far field condition again. Thus, we finally have the following reformulated problem in terms of η:

$$\begin{cases} \eta_r = \dfrac{r^{n-1}}{\epsilon\mu}\big(\tilde{p}(v_+ + \eta) - \tilde{p}(v_+)\big) \\ \qquad + \dfrac{\epsilon v_+}{2\mu}\dfrac{1}{r^{n-1}} + \dfrac{\epsilon\eta}{\mu r^{n-1}} - \dfrac{\epsilon(n-1)r^{n-1}}{\mu}\displaystyle\int_r^\infty \dfrac{\eta(s)}{s^{2n-1}}ds, \quad r > 1, \\ \lim\limits_{r\to\infty} \eta(r) = 0, \\ \eta(1) = \eta_- := v_- - v_+ \ (u_- > 0), \quad \textit{no boundary condition } (u_- < 0), \end{cases} \tag{2.6}$$

where $\epsilon = u_-/v_-$ $(u_- > 0)$, and $\epsilon = u_-/(v_+ + \eta(1))$ $(u_- < 0)$. Once the desired solution η of (2.6) is obtained, the velocity u is immediately obtained by (2.1) as

$$u(r) = \frac{u_-(v_+ + \eta(r))}{v_- r^{n-1}} \quad (u_- > 0), \quad u(r) = \frac{u_-(v_+ + \eta(r))}{(v_+ + \eta(1))r^{n-1}} \quad (u_- < 0). \tag{2.7}$$

The theorem for the reformulated problem (2.6) which we need to prove is

THEOREM 2.1. *Let $n \geq 2$. Then, for any $v_+ > 0$, there exist positive constants ϵ_0 and C satisfying the following:*

(I) *Let $u_- > 0$. If $|u_-| + |\eta_-| \leq \epsilon_0$, there exists a unique smooth solution η of the problem* (2.6) *satisfying*

$$|\eta(r)| \leq Cr^{-(n-1)}(|u_-|^2 + |\eta_-|), \qquad r \geq 1. \tag{2.8}$$

Furthermore, for any positive constant h, there exists a positive constant C_h satisfying

$$\sup_{r \geq r_0 + h} |\eta(r)| \leq C_h |u_-|^2. \tag{2.9}$$

(II) *Let $u_- < 0$. If $|u_-| \leq \epsilon_0$, there exists a unique smooth solution η of the problem* (2.6) *satisfying*

$$|\eta(r)| \leq Cr^{-2(n-1)}|u_-|^2, \qquad r \geq 1. \tag{2.10}$$

We can easily see that the main Theorem 1.1 is a direct consequence by Theorem 2.1, the formula (2.7), and the trivial case $u_- = 0$.

3. Inflow problem. In this section, we consider the case $u_- > 0$, that is, inflow problem, and show the result (I) in Theorem 2.1. In this case, recalling $\epsilon = u_-/v_- > 0$, we further rewrite the equation of η in (2.6) as in the form

$$\eta_r - \frac{\tilde{p}'(v_+)}{\mu\epsilon} r^{n-1}\eta = F[\eta], \quad r > 1, \tag{3.1}$$

where

$$F[\eta](r) := \frac{\epsilon v_+}{2\mu}\frac{1}{r^{n-1}} + \frac{\epsilon\eta(r)}{\mu r^{n-1}} - \frac{\epsilon(n-1)r^{n-1}}{\mu}\int_r^\infty \frac{\eta(s)}{s^{2n-1}}\, ds + \frac{r^{n-1}}{\mu\epsilon}N(\eta(r)),$$

$$N(\eta) := \tilde{p}(v_+ + \eta) - \tilde{p}(v_+) - \tilde{p}'(v_+)\eta.$$

Solving the linear differential equation (3.1) in terms of η with the initial data $\eta(1) = \eta_-$ and the inhomogeneous term F, we have, by the Duhamel's principle,

$$\eta(r) = \eta_- e^{-\frac{\kappa}{\epsilon}(r^n-1)} + \int_1^r e^{-\frac{\kappa}{\epsilon}(r^n-s^n)} F[\eta](s)\,ds, \quad r \geq 1, \tag{3.2}$$

where $\kappa := -\tilde{p}'(v_+)/(\mu n) > 0$. Thus, to prove the existence of the solution of (2.6) with the decay rate estimate (2.8), we look for a solution of (3.2) in the Banach space X, with its norm $\|\cdot\|_X$, defined by

$$X = \{\eta \in C([1,\infty));\ \sup_{r\geq 1} |r^{n-1}\eta(r)| < \infty\}, \quad \|\eta\|_X = \sup_{r\geq 1}|r^{n-1}\eta(r)|.$$

To do that, we construct the approximate sequence $\{\eta^{(m)}\}_{m\geq 0}$ by

$$\begin{cases} \eta^{(0)}(r) = \eta_- e^{-\frac{\kappa}{\epsilon}(r^n-1)}, \quad r \geq 1, \\ \eta^{(m+1)}(r) = \eta^{(0)}(r) + \displaystyle\int_1^r e^{-\frac{\kappa}{\epsilon}(r^n-s^n)} F[\eta^{(m)}](s)\,ds \ \ (m \geq 0), \quad r \geq 1. \end{cases} \tag{3.3}$$

To show $\{\eta^{(m)}\}_{m\geq 0}$ is a Cauchy sequence in X for suitably small $|u_-| + |\eta_-|$, we prepare the following lemma.

LEMMA 3.1. (I) *If $\epsilon \leq \frac{n\kappa}{n-1}$, then it holds that*

$$r^{n-1}e^{-\frac{\kappa}{\epsilon}(r^n-1)} \leq 1, \qquad r \geq 1. \tag{3.4}$$

(II) *If $\epsilon \leq \frac{n\kappa}{4(n-1)}$, then there exists a positive constant C which is independent of ϵ such that*

$$r^{n-1}\left|\int_1^r e^{-\frac{\kappa}{\epsilon}(r^n-s^n)} f(s)\,ds\right| \leq C\epsilon\|f\|_X, \qquad r \geq 1, \quad f \in X. \tag{3.5}$$

Proof of Lemma 3.1. First, note that for $1 \leq s \leq r$, it holds

$$\begin{aligned} &r^n - s^n = (r-s)(r^{n-1} + r^{n-2}s + \cdots + s^{n-1}) \geq n(r-s), \\ &e^{-\frac{\kappa}{\epsilon}(r^n-s^n)} \leq e^{-\frac{n\kappa}{\epsilon}(r-s)}. \end{aligned} \tag{3.6}$$

Under the assumption $\epsilon \leq \frac{n\kappa}{n-1}$, due to (3.6), we easily see

$$r^{n-1}e^{-\frac{\kappa}{\epsilon}(r^n-1)} \leq r^{n-1}e^{-\frac{n\kappa}{\epsilon}(r-1)} \leq 1, \qquad r \geq 1, \tag{3.7}$$

which proves (3.4). Next, under the assumption $\epsilon \leq \frac{n\kappa}{4(n-1)}$, by using (3.6) again, we estimate the left hand side of (3.5) as follows:

$$\begin{aligned} &r^{n-1}\left|\int_1^r e^{-\frac{\kappa}{\epsilon}(r^n-s^n)} f(s)\,ds\right| \\ &\leq r^{n-1}\int_1^{\frac{r+1}{2}} e^{-\frac{n\kappa}{\epsilon}(r-s)}|f(s)|\,ds + r^{n-1}\int_{\frac{r+1}{2}}^r e^{-\frac{n\kappa}{\epsilon}(r-s)}|f(s)|\,ds \\ &=: I_1 + I_2; \end{aligned} \tag{3.8}$$

$$\begin{aligned} I_1 &\le r^{n-1} e^{-\frac{n\kappa}{4\epsilon}(r-1)} \int_1^{\frac{r+1}{2}} e^{-\frac{n\kappa}{2\epsilon}(r-s)}\, ds \cdot \sup_{1\le s} |f(s)| \\ &\le \frac{2\epsilon}{n\kappa} \sup_{1\le r}(r^{n-1}|f(r)|) \le C\epsilon \|f\|_X; \end{aligned} \tag{3.9}$$

$$\begin{aligned} I_2 &\le r^{n-1} \int_{\frac{r+1}{2}}^{r} e^{-\frac{n\kappa}{\epsilon}(r-s)}\, ds \cdot \sup_{\frac{r+1}{2}\le s} |f(s)| \\ &\le r^{n-1} \cdot \frac{\epsilon}{n\kappa} \cdot \sup_{\frac{r+1}{2}\le s} s^{-(n-1)} \cdot \sup_{1\le s}(s^{n-1}|f(s)|) \\ &\le \frac{2^{n-1}}{n\kappa}\epsilon \cdot \sup_{1\le r}(r^{n-1}|f(r)|) \le C\epsilon \|f\|_X. \end{aligned} \tag{3.10}$$

Hence, plugging (3.9) and (3.10) to (3.8), we obtain the desired estimate (3.5). Thus, the proof of the Lemma 3.1 is completed.

Now, by using Lemma 3.1, we show the uniform boundedness of $\eta^{(m)}$ $(m \ge 0)$ in X for suitably small $|u_-| + |\eta_-|$. More precisely, we show that for any fixed v_+, there exist positive constants ϵ_0 and C which are independent of u_- and η_- such that if $|u_-| + |\eta_-| \le \epsilon_0$, then there exists a positive constant M satisfying

$$\|\eta^{(m)}\|_X \le M \le C(|u_-|^2 + |\eta_-|), \quad m \ge 0. \tag{3.11}$$

Here and in what follows, we use the letter C and ϵ_0 to denote generic positive constants which are independent of u_- and η_-, but may depend on v_+ and other fixed constants like $\mu, n, \dots, etc$. For the proof, in particular, to use Lemma 3.1, we first assume

$$|\eta_-| = |v_- - v_+| \le \frac{v_+}{2}, \quad u_- \le \frac{n\kappa v_+}{8(n-1)}, \tag{3.12}$$

which assures

$$\frac{v_+}{2} \le v_- \le \frac{3v_+}{2}, \quad \epsilon = \frac{u_-}{v_-} \le \frac{n\kappa}{4(n-1)}. \tag{3.13}$$

Let us show (3.11) by mathematical induction:

<u>Case $m = 0$</u>. Due to (3.13), we have from Lemma 3.1 that

$$|r^{n-1}\eta^{(0)}(r)| = |\eta_-||r^{n-1}e^{-\frac{\kappa}{\epsilon}(r^n-1)}| \le |\eta_-|, \quad r \ge 1.$$

Hence, we ask the constant M to satisfy

$$|\eta_-| \le M, \tag{3.14}$$

so that it holds $\|\eta^{(0)}\|_X \le M$.

<u>Case $m = k+1$ $(k \ge 0)$</u>. Suppose $\|\eta^{(k)}\|_X \le M$. Here we ask the constant M to satisfy another assumption

$$M \le \frac{v_+}{2}, \tag{3.15}$$

which, in particular, implies

$$|\eta^{(k)}(r)| \le \frac{v_+}{2}, \quad r \ge 1. \tag{3.16}$$

Then, by using Lemma 3.1, (3.15),(3.16), and also the Taylor's theorem, we estimate $\eta^{(k+1)}$ defined by (3.3) as follows:

$$\begin{aligned}
r^{n-1}|\eta^{(k+1)}(r)| &\le r^{n-1}|\eta^{(0)}(r)| + r^{n-1}\int_1^r e^{-\frac{\kappa}{\epsilon}(r^n-s^n)}|F[\eta^{(k)}](s)|\,ds \\
&\le |\eta_-| + r^{n-1}\Big|\int_1^r e^{-\frac{\kappa}{\epsilon}(r^n-s^n)}\Big(\frac{\epsilon v_+}{2\mu}\frac{1}{s^{n-1}}\Big)\,ds\Big| \\
&\quad + r^{n-1}\Big|\int_1^r e^{-\frac{\kappa}{\epsilon}(r^n-s^n)}\Big(\frac{\epsilon\eta^{(k)}(s)}{\mu s^{n-1}}\Big)\,ds\Big| \\
&\quad + r^{n-1}\Big|\int_1^r e^{-\frac{\kappa}{\epsilon}(r^n-s^n)}\Big(\frac{\epsilon(n-1)s^{n-1}}{\mu}\int_s^\infty \frac{\eta^{(k)}(\tau)}{\tau^{2n-1}}\,d\tau\Big)\,ds\Big| \\
&\quad + r^{n-1}\Big|\int_1^r e^{-\frac{\kappa}{\epsilon}(r^n-s^n)}\Big(\frac{r^{n-1}}{\mu\epsilon}N(\eta^{(k)}(s))\Big)\,ds\Big| \\
&=: |\eta_-| + I_1 + I_2 + I_3 + I_4;
\end{aligned} \tag{3.17}$$

$$I_1 \le C\epsilon \sup_{r\ge1}|r^{n-1}(\frac{\epsilon v_+}{2\mu}\frac{1}{r^{n-1}})| \le C\epsilon^2\frac{v_+}{2\mu} \le C|u_-|^2; \tag{3.18}$$

$$I_2 \le C\epsilon \sup_{r\ge1}|r^{n-1}(\frac{\epsilon\eta^{(k)}(r)}{\mu r^{n-1}})| \le C\frac{\epsilon^2}{\mu}M \le C|u_-|^2; \tag{3.19}$$

$$I_3 \le \frac{C\epsilon^2(n-1)}{\mu}\sup_{r\ge1}|r^{2(n-1)}\int_r^\infty \frac{s^{n-1}\eta^{(k)}(s)}{s^{3n-2}}\,ds| \le \frac{C\epsilon^2 M}{3\mu} \le C|u_-|^2; \tag{3.20}$$

$$\begin{aligned}
I_4 &\le C\epsilon\sup_{r\ge1}|r^{n-1}(\frac{r^{n-1}}{\mu\epsilon}N(\eta^{(k)}(r)))| \\
&\le \frac{C}{\mu}\sup_{r\ge1}|r^{2(n-1)}(\tilde p(v_+ + \eta^{(k)}(r)) - \tilde p(v_+) - \tilde p'(v_+)\eta^{(k)}(r))| \\
&\le \frac{C}{2\mu}\cdot \sup_{v_+/2\le t\le 3v_+/2}|\tilde p''(t)|\cdot\sup_{r\ge1}(r^{2(n-1)}|\eta^{(k)}(r)|^2) \le CM^2.
\end{aligned} \tag{3.21}$$

Substituting (3.18)-(3.21) into (3.17), we obtain

$$\|\eta^{(k+1)}\|_X \le |\eta_-| + C|u_-|^2 + CM^2. \tag{3.22}$$

Therefore, we further assume

$$|\eta_-| + C|u_-|^2 \le \frac{M}{2}, \qquad CM \le \frac{1}{2}, \tag{3.23}$$

so that (3.22) gives the desired estimate $\|\eta^{(k+1)}\|_X \le M$. By elementary calculations, it is easy to see there exists a positive constant ϵ_0 such that if $|\eta_-| + |u_-| \le \epsilon_0$, all the assumptions (3.12),(3.14),(3.15), and (3.23) hold, and in particular, M can be chosen by

$$M = 2(|\eta_-| + C|u_-|^2),$$

which proves the uniform boundedness of $\eta^{(m)}$ $(m \geq 0)$ in X with the estimate (3.11). Once (3.11) is proved, the proof to show $\{\eta^{(m)}\}_{m\geq 0}$ is a Cauchy sequence in X is very standard. In fact, by using Lemma 3.1, we may estimate

$$\begin{aligned}
\|\eta^{(m+1)} - \eta^{(m)}\|_X &= \sup_{r\geq 1} |r^{n-1} \int_1^r e^{-\frac{\kappa}{\epsilon}(r^n - s^n)} (F[\eta^{(m)}] - F[\eta^{(m-1)}])(s)\, ds| \\
&\leq C\epsilon \|F[\eta^{(m)}] - F[\eta^{(m-1)}]\|_X \\
&\leq C(|\eta_-| + |u_-|^2)\|\eta^{(m)} - \eta^{(m-1)}\|_X, \quad m \geq 1,
\end{aligned}$$

in the same way as in (3.18)-(3.21), and taking ϵ_0 suitably small again if needed, we can show

$$\|\eta^{(m+1)} - \eta^{(m)}\|_X \leq \frac{1}{2}\|\eta^{(m)} - \eta^{(m-1)}\|_X, \quad m \geq 1,$$

which proves that $\{\eta^{(m)}\}_{m\geq 0}$ is a Cauchy sequence in X. Thus, as the limit, the solution η of (2.6) with the desired estimate (2.8) is obtained. The arguments on the regularity and uniqueness of the solution are also very standard, so we omit them.

Finally, we show the estimate (2.9), that is, the existence of the boundary layer for the density. We turn back to the equation in (2.6), and again rewrite it as

$$\eta_r - \frac{a(\eta) r^{n-1}}{\epsilon}\eta = q[\eta], \quad r > 1, \tag{3.24}$$

where

$$\begin{aligned}
a(\eta) &:= \frac{\tilde{p}(v_+ + \eta) - \tilde{p}(v_+)}{\eta\mu}, \\
q[\eta](r) &:= \frac{\epsilon v_+}{2\mu}\frac{1}{r^{n-1}} + \frac{\epsilon\eta(r)}{\mu r^{n-1}} - \frac{r^{n-1}\epsilon(n-1)}{\mu}\int_r^\infty \frac{\eta(s)}{s^{2n-1}}\, ds.
\end{aligned}$$

Here we note that we already know the existence of the solution η of (3.24) with the estimate (2.8). Solving the equation (3.24) in terms of η with the initial data $\eta(1) = \eta_-$, we have, by the Duhamel's principle,

$$\eta(r) = e^{\int_1^r \frac{a(\eta(s))}{\epsilon} s^{n-1} ds}\eta_- + \int_1^r e^{\int_\tau^r \frac{a(\eta(s))}{\epsilon} s^{n-1} ds} q[\eta](\tau)\, d\tau, \quad r \geq 1. \tag{3.25}$$

By the estimate (2.8) and the assumption $\tilde{p}'(v) < 0$ $(v > 0)$, it is easy to see that there exist positive constants δ and C satisfying

$$-a(\eta(r)) \geq \delta, \quad |q[\eta](r)| \leq C\epsilon, \qquad r \geq 1. \tag{3.26}$$

Therefore, for any positive constant h, if $r \geq 1 + h$, it follows from (3.25) and (3.26) that

$$\begin{aligned}|\eta(r)| &\le e^{-\frac{\delta}{\epsilon}\int_1^r s^{n-1}ds}|\eta_-| + \int_1^r e^{-\frac{\delta}{\epsilon}\int_\tau^r s^{n-1}ds}|q[\eta](\tau)|\,d\tau \\ &\le e^{-\frac{\delta}{\epsilon}(r-1)}|\eta_-| + C\epsilon\int_1^r e^{-\frac{\delta}{\epsilon}(r-\tau)}\,d\tau \\ &\le e^{-\frac{\delta}{\epsilon}h}|\eta_-| + \frac{C}{\delta}\epsilon^2 \le \frac{|\eta_-|}{\delta^2h^2}\epsilon^2 + \frac{C}{\delta}\epsilon^2 \le C_h|u_-|^2,\end{aligned}$$

which proves the desired estimate (2.9). Thus, the proof for the result (I) in Theorem 2.1 is completed.

4. Outflow problem. In this section, we consider the case $u_- < 0$, that is, outflow problem, and show the result (II) in Theorem 2.1. In this case, recalling $\epsilon = u_-/(v_+ + \eta(1)) < 0$, we again rewrite the equation of η in (2.6) as in the form

$$\eta_r - \frac{v_+\tilde{p}'(v_+)}{\mu u_-}r^{n-1}\eta = G[\eta], \quad r > 1, \tag{4.1}$$

where

$$\begin{aligned}G[\eta](r) := &\frac{\tilde{p}'(v_+)\eta(r)\eta(1)}{\mu u_-}r^{n-1} + \frac{u_-v_+}{2\mu(v_+ + \eta(1))}\frac{1}{r^{n-1}} \\ &+ \frac{u_-\eta(r)}{\mu(v_+ + \eta(1))r^{n-1}} - \frac{(n-1)u_-r^{n-1}}{\mu}\int_r^\infty \frac{\eta(s)}{(v_+ + \eta(1))s^{2n-1}}\,ds \\ &+ \frac{(v_+ + \eta(1))r^{n-1}}{\mu u_-}N(\eta(r)).\end{aligned}$$

Noting $u_- < 0$, and solving the equation (4.1) in terms of η with the inhomogeneous term G and the far field condition $\eta(\infty) = 0$, we obtain

$$\eta(r) = -\int_r^\infty e^{-\frac{\kappa v_+}{|u_-|}(s^n - r^n)}G[\eta](s)\,ds, \quad r \ge 1, \tag{4.2}$$

where we recall $\kappa = -\tilde{p}'(v_+)/(\mu n) > 0$. This time, to prove the existence of the solution of (4.2) with the decay rate estimate (2.10), we look for a solution of (4.2) in the Banach space Y, with its norm $\|\cdot\|_Y$, defined by

$$Y = \{\eta \in C([1,\infty));\ \sup_{r\ge1}|r^{2(n-1)}\eta(r)| < \infty\}, \quad \|\eta\|_Y = \sup_{r\ge1}|r^{2(n-1)}\eta(r)|.$$

By the same way as in the last section, we construct the approximate sequence $\{\eta^{(m)}\}_{m\ge0}$ by

$$\begin{cases}\eta^{(0)}(r) = -\displaystyle\int_r^\infty e^{-\frac{\kappa v_+}{|u_-|}(s^n - r^n)}G[0](s)\,ds, \quad r \ge 1, \\ \eta^{(m+1)}(r) = -\displaystyle\int_r^\infty e^{-\frac{\bar{\kappa} v_+}{|u_-|}(s^n - r^n)}G[\eta^{(m)}](s)\,ds \quad (m \ge 0), \quad r \ge 1.\end{cases} \tag{4.3}$$

To show $\{\eta^{(m)}\}_{m\ge0}$ is a Cauchy sequence in Y for suitably small $|u_-|$, we prepare the following lemma.

LEMMA 4.1. *For $g \in X$, it holds that*

$$r^{2(n-1)}\Big|\int_r^\infty e^{-\frac{\kappa v_+}{|u_-|}(s^n-r^n)}g(s)\,ds\Big| \le \frac{|u_-|}{n\kappa v_+}\sup_{s\ge r}|s^{n-1}g(s)|, \quad r\ge 1.$$

Proof. It holds that

$$\begin{aligned}\Big|\int_r^\infty e^{-\frac{\kappa v_+}{|u_-|}(s^n-r^n)}g(s)\,ds\Big| &\le \int_r^\infty e^{-\frac{n\kappa v_+}{|u_-|}r^{n-1}(s-r)}s^{-(n-1)}s^{(n-1)}|g(s)|\,ds \\ &\le r^{-(n-1)}\int_r^\infty e^{-\frac{n\kappa v_+}{|u_-|}r^{n-1}(s-r)}\,ds\cdot\sup_{s\ge r}|s^{n-1}g(s)| \\ &\le r^{-2(n-1)}\frac{|u_-|}{n\kappa v_+}\sup_{s\ge r}|s^{n-1}g(s)|, \quad r\ge 1.\end{aligned}$$

Thus, the proof of Lemma 4.1 is completed.

By using Lemma 4.1, we show that for any fixed $v_+ > 0$, there exist positive constants ϵ_0 and C such that if $|u_-| \le \epsilon_0$, then there exists a positive constant M satisfying

$$\|\eta^{(m)}\|_Y \le M \le C|u_-|^2, \quad m\ge 0. \tag{4.4}$$

Let us show (4.4) by mathematical induction:

Case $m=0$. Using Lemma 4.1, we have

$$\begin{aligned}|r^{2(n-1)}\eta^{(0)}(r)| &\le \frac{|u_-|}{n\kappa v_+}\sup_{r\ge 1}|r^{n-1}G[0](r)| \\ &\le C|u_-|\sup_{r\ge 1}|r^{n-1}(\frac{u_-}{2\mu}\frac{1}{r^{n-1}})| \le C|u_-|^2.\end{aligned}\tag{4.5}$$

Hence, we first ask the constant M to satisfy

$$C|u_-|^2 \le M, \tag{4.6}$$

so that (4.5) gives $\|\eta^{(0)}\|_Y \le M$.

Case $m = k+1$ $(k\ge 0)$. Suppose $\|\eta^{(k)}\|_Y \le M$. Here we ask the constant M satisfy another assumption

$$M \le \frac{v_+}{2}, \tag{4.7}$$

which, in particular, implies

$$|\eta^{(k)}(1)| \le \|\eta^{(k)}\|_Y \le M \le \frac{v_+}{2}. \tag{4.8}$$

Then, by using Lemma 4.1, (4.7),(4.8), and also the Taylor's theorem, we estimate

$\eta^{(k+1)}$ defined by (4.3) in the same way as in the last section :

$$\begin{aligned}
\sup_{r\geq 1}|r^{2(n-1)}\eta^{(k+1)}(r)| &\leq \frac{|u_-|}{n\kappa v_+}\sup_{r\geq 1}\left|r^{n-1}\left(\frac{n\kappa v_+\eta^{(k)}(r)\eta^{(k)}(1)}{u_-}r^{n-1}\right)\right| \\
&+\frac{|u_-|}{n\kappa v_+}\sup_{r\geq 1}\left|r^{n-1}\left(\frac{u_-v_+}{2\mu(v_++\eta^{(k)}(1))}\frac{1}{r^{n-1}}\right)\right| \\
&+\frac{|u_-|}{n\kappa v_+}\sup_{r\geq 1}\left|r^{n-1}\left(\frac{u_-\eta^{(k)}(r)}{\mu(v_++\eta^{(k)}(1))r^{n-1}}\right)\right| \\
&+\frac{|u_-|}{n\kappa v_+}\sup_{r\geq 1}\left|r^{n-1}\left(\frac{(n-1)u_-r^{n-1}}{\mu}\int_r^\infty \frac{\eta^{(k)}(s)}{(v_++\eta^{(k)}(1))s^{2n-1}}\,ds\right)\right| \\
&+\frac{|u_-|}{n\kappa v_+}\sup_{r\geq 1}\left|r^{n-1}\left(\frac{(v_++\eta^{(k)}(1))r^{n-1}}{\mu u_-}N(\eta^{(k)}(r))\right)\right| \\
&\leq CM^2+C|u_-|^2+C|u_-|^2M+C|u_-|^2M+CM^2 \\
&\leq C|u_-|^2+CM^2.
\end{aligned} \tag{4.9}$$

Therefore, we further assume

$$C|u_-|^2\leq \frac{M}{2},\qquad CM\leq\frac{1}{2}, \tag{4.10}$$

so that (4.9) gives the desired estimate $\|\eta^{(k+1)}\|_Y\leq M$. It is easy to see there exists a positive constant ϵ_0 such that if $|u_-|\leq\epsilon_0$, all the assumptions (4.6),(4.7), and (4.10) hold, and in particular, M can be chosen by

$$M=2C|u_-|^2,$$

which proves the uniform boundedness of $\eta^{(m)}$ $(m\geq 0)$ in Y with the estimate (4.4). Once the uniform estimate (4.4) is obtained, the remaining arguments on the convergence to the limit η, and the regularity and uniqueness of the limit are very standard as in the last section, so we omit the details. Thus, the proof for the result (II) in Theorem 2.1 is completed.

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