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FINITE TIME BLOWUP FOR THE 1-D SEMILINEAR GENERALIZED TRICOMI EQUATION WITH SUBCRITICAL OR CRITICAL EXPONENTS*

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Dedicated to Professor Ling Hsiao's 80th birthday

Abstract. For the 1-D semilinear generalized Tricomi equation with the subcritical or critical exponents

$$\partial_t^2 u - t^m\, \partial_x^2 u = |u|^p, \quad (u(0,x), \partial_t u(0,x)) = (u_0(x), u_1(x)),$$

where $t > 0$, $x \in \mathbb{R}$, $1 < p \le p_m = 1 + \frac{4}{m}$ and $u_i \in C_0^\infty(\mathbb{R})$ $(i = 0, 1)$, we shall prove that the weak solution u generally blows up in finite time. Note that for the 1-D equation $\partial_t^2 u - t^m\, \partial_x^2 u = |u|^p$ with $p > p_m$, the global existence of small value weak solution u has been obtained by us. By this paper and our previous papers, we have given a systematic study on the blowup or global existence of small value solution u to the equation $\partial_t^2 u - t^m \Delta u = |u|^p$ for all space dimensions and all $p > 1$. One main ingredient in this paper is to apply the explicit solution formula of linear generalized Tricomi equation to derive the crucial inequality $G(t) = \int_{\mathbb{R}} u(t,x)dx \ge Ct \ln t$ for large $t > 0$ and $p = p_m$ when $u_0(x) \ge 0, u_1(x) \ge 0$ and $(u_0(x), u_1(x)) \not\equiv 0$.

Key words. Generalized Tricomi equation, subcritical exponent, critical exponent, hypergeometric function, blowup.

Mathematics Subject Classification. 35L70, 35L65, 35L67.

1. Introduction. In our former papers [12, 13, 14], we have given a systematic study on the global existence of the small value solution u versus blowup for the multi-dimensional semilinear generalized Tricomi equation $\partial_t^2 u - t^m\, \Delta u = |u|^p$ with the initial data $(u(0,x), \partial_t u(0,x)) = (u_0(x), u_1(x))$, where $t > 0$, $x \in \mathbb{R}^n$, $m \ge 1$, $n \ge 2$, $p > 1$ and $u_i \in C_0^\infty(\mathbb{R}^n)$ $(i = 0, 1)$. For the 1-D semilinear generalized Tricomi equation

$$\begin{cases} \partial_t^2 u - t^m\, \partial_x^2 u = |u|^p & \text{in } \mathbb{R}_+^{1+1}, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \end{cases} \tag{1.1}$$

the global existence of small data solution u has been obtained for $p > p_m = 1 + \frac{4}{m}$ in [15] through establishing the weighted Strichartz inequalities for the linear equation $\partial_t^2 u - t^m\, \partial_x^2 u = f(t,x)$. In the present paper, we focus on the problem (1.1) with $1 < p \le p = p_m$, where $u_i \in C_0^\infty(\mathbb{R})$ $(i = 0, 1)$ and $\operatorname{supp} u_i \subseteq [-1, 1]$. Concerning the local existence and optimal regularities of the solution u to (1.1) under the weaker regularity assumptions of (u_0, u_1), the reader may consult [23]-[26] and [31]-[32]. Our objective in the paper is to prove that, in general, the weak solution u to (1.1) blows up in finite time.

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THEOREM 1.1 (Blowup). *Let* $1 < p \le p_m$ *in* (1.1), $u_i \ge 0$ *for* $i = 0,1$ *and* $(u_0, u_1) \not\equiv 0$. *Then problem* (1.1) *admits no global weak solution* $u \in C\left([0,\infty), H^1(\mathbb{R})\right) \cap C^1\left([0,\infty), L^2(\mathbb{R})\right)$.

REMARK 1.2. *For the 1-D semilinear wave equation* $\partial_t^2 v - \partial_x^2 v = |v|^p$ $(p>1)$, *direct computation shows that the local weak solution* v *will generally blow up in finite time (for example, see [9]). However, for the 1-D semilinear generalized Tricomi equation* $\partial_t^2 u - t^m \partial_x^2 u = |u|^p$, *by Theorem 1.1 and Remark 1.6 in [15] we know that the global small value weak solution* u *exists for* $p > p_m$, *which is established through obtaining the decay property of the solution to the linear Tricomi equation and through deriving some weighted Strichartz inequalities.*

REMARK 1.3. *Galstian in [6] showed that the condition*

$$p > 1 + \frac{4}{m} \tag{1.2}$$

(corresponding to (1.19) in [6] with $\alpha = p-1$*) is necessary for the global existence of the weak solution of problem* (1.1). *It was shown in [6, Theorem 1.3] that under* $1 < p < 1+\frac{4}{m}$, *there exists no global mild solution* $u \in C([0,\infty), L^q(\mathbb{R})) \cap C^1([0,\infty), \mathbf{D}'(\mathbb{R}))$ *to the corresponding integral equation form of problem* (1.1) *for some suitable* $q > 1$ *and special initial data* $(u_0(x), u_1(x))$.

Let us recall some historical backgrounds on the semilinear wave equation

$$\begin{cases} \partial_t^2 v - \Delta v = |v|^p, & (t,x) \in \mathbb{R}_+^{1+n}, \\ v(0,x) = v_0(x), \quad \partial_t v(0,x) = v_1(x), \end{cases} \tag{1.3}$$

where $p > 1$, $n \ge 2$, and $v_i \in C_0^\infty(\mathbb{R}^n)$ $(i = 0,1)$. Let $p_c = p_c(n)$ denote the positive root of the quadratic equation

$$(n-1)\,p^2 - (n+1)\,p - 2 = 0. \tag{1.4}$$

W.Strauss [29] made the following conjecture:

CONJECTURE. *If* $p > p_c(n)$, *then small value solution* v *of problem* (1.3) *exists globally. If* $1 < p < p_c(n)$, *then solution* v *of problem* (1.3) *blows up in finite time.*

So far this conjecture has almost been solved. Namely, F. John in [16] shows that when $n = 3$, the global small value solution v always exists if $p > p_c(3) = 1 + \sqrt{2}$, meanwhile, v will blow up in finite time if $p < p_c(3)$. In addition, this conjecture is shortly verified for $n = 2$ by R. T. Glassey [7]-[8]. Afterwards, F. John's blowup result is then extended by T. Sideris [27], who shows that the solution v blows up for some class of initial data if $p < p_c(n)$ and for all n. Finally, the authors in [17] and [10] prove the global existence of small data solution v for $n \ge 4$ and $p > p_c(n)$ (except some exceptional values of p). On the other hand, for the critical case of $p = p_c(n)$, the solution v generally blows up (for $n = 2, 3$ with $p = p_c(2)$, $p_c(3)$, see [28]; for $n \ge 4$ and $p = p_c(n)$, see [33]).

For the M-D semilinear generalized Tricomi equation

$$\begin{cases} \partial_t^2 u - t^m\, \Delta u = |u|^p, & (t,x) \in \mathbb{R}_+^{1+n}, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \end{cases} \tag{1.5}$$

where $m \geq 1$, $x \in \mathbb{R}^n$ with $n \geq 2$, we have shown in [12, 13, 14] that there exists a critical exponent $p_{\text{crit}}(m,n) > 1$ such that the weak solution u generally blows up when $1 < p \leq p_{\text{crit}}(m,n)$, while there exists a global small value weak solution u when $p > p_{\text{crit}}(m,n)$. Here, $p_{\text{crit}}(m,n)$ is the positive root of the quadratic equation

$$\Big((m+2)\,\frac{n}{2}-1\Big)\,p^2-\Big((m+2)\Big(\frac{n}{2}-1\Big)+3\Big)\,p-(m+2)=0. \tag{1.6}$$

Note that if we formally let $m=n=1$ in (1.6) (in fact, (1.6) only holds for $n \geq 2$), then $p_{\text{crit}}(1,1)=(3+\sqrt{33})/2<5$. In addition, it is easily verified that $p_{\text{crit}}(m,1)$ is strictly less than the real critical exponent $p_m = 1+4/m$ in Theorems 1.1.

About the linear Tricomi equation $\partial_t^2 u - t\partial_x^2 u = 0$, it arises from the transonic gas dynamics (see [3, 22]). There are extensive results for both linear and semilinear Tricomi equations in n space dimensions ($n \in \mathbb{N}$). For instances, the authors of [1, 30, 32] have computed the forward fundamental solution of the linear Tricomi equation $\partial_t^2 u - t\Delta u = 0$ explicitly. The authors of [11, 19, 18, 21, 20] have obtained a series of interesting results on the existence and uniqueness of solutions u to the semilinear Tricomi equation $\partial_t^2 u - t\Delta u = f(t,x,u)$ in the bounded domains, under certain restrictions on the nonlinearity $f(t,x,u)$. The authors of [2, 23, 25, 26] have established the local existence as well as the singularity structure of low regularity solutions to the Cauchy problem of semilinear Tricomi equations in the degenerate hyperbolic region and the elliptic-hyperbolic mixed region, respectively.

We now comment on the proof of Theorem 1.1. To prove Theorem 1.1, we define the function $G(t)=\int_{\mathbb{R}} u(t,x)\,\mathrm{d}x$ as usual. By (1.1), and intend to derive a Riccati-type ordinary differential inequality for $G(t)$. By an analysis similar to [12], and motivated by [33], we can obtain a Riccati-type ordinary differential inequality for $G(t)$ through a delicate analysis of (1.1). From this and Lemma 4 in [27], the blowup result for $1<p<p_m$ in Theorem 1.1 is established under the positivity assumptions of $u_0(x)$ and $u_1(x)$. For the critical case of $p=p_m$, through applying some basic properties of the hypergeometric function and the explicit integral expression of the solution u to (1.1), then a logarithmic improvement of the lower bound of $G(t)$ with $G(t) \geq Ct\ln t$ for large t is achieved. The lower bound $Ct\ln t$ of $G(t)$ together with Lemma 2.1 in [33] yields the blowup result for $p=p_m$ in Theorem 1.1 under the positivity assumptions of u_0 and u_1. Here it is pointed out that in the case of $n \geq 2$, we have applied the method in [33] and used the Radon transform to gain the expected increasing estimate of $G(t)$ for large t in [14]. However, for the critical case of $p=p_m$ in $n=1$, it is difficult to apply the analysis in [14] to get the sufficient increasing estimate of $G(t)$ since the related Radon transform does not exist for 1-D case.

2. Proof of Theorem 1.1 for $1<p<p_m$. In this section, we start to prove Theorem 1.1 for $1<p<p_m$. Motivated by [33], we introduce the function $G(t)=\int_{\mathbb{R}} u(t,x)\,\mathrm{d}x$. By some estimates from [12], we can obtain a Riccati-type differential inequality for $G(t)$ so that the blowup of $G(t)$ will be deduced from the following basic result:

LEMMA 2.1 ([27, Lemma 4]). *Suppose that $G \in C^2([a,b);\mathbb{R})$ and, for $a \leq t < b$,*

$$G(t) \geq C_0(R+t)^{\alpha}, \tag{2.1}$$

$$G''(t) \geq C_1(R+t)^{-q}G(t)^p, \tag{2.2}$$

where C_0, C_1, and R are some positive constants. Suppose further that $p>1$, $\alpha \geq 1$, and $(p-1)\,\alpha \geq q-2$. Then b is finite.

In view of $\operatorname{supp} u_i \subseteq (-1,1)$ $(i = 0,1)$ and the finite propagation speed for solutions of the hyperbolic equations, one has that, for any fixed $t > 0$, the support of $u(t,\cdot)$ with respect to the variable x is contained in the interval $\big(-1-\phi(t), 1+\phi(t)\big)$, where $\phi(t) = \frac{2}{m+2}\, t^{\frac{m+2}{2}}$. Indeed, let $T = \frac{2}{m+2}\, t^{\frac{m+2}{2}}$, then the equation in (1.1) becomes

$$\partial_T^2 u - \partial_x^2 u + \frac{m}{m+2}\frac{\partial_T u}{T} = (\frac{m}{2}+1)^{-\frac{2m}{m+2}} T^{-\frac{2m}{m+2}}\, |u|^p. \tag{2.3}$$

Note that for the semilinear wave equation (2.3), there is the finite propagation speed property for $T > 0$: $-1-T < x < 1+T$. Return to the original equation (1.1), one directly derives the finite propagation speed property for $t > 0$: $-1-\phi(t) < x < 1+\phi(t)$. Therefore, it follows from an integration by parts that

$$G''(t) = \int_{\mathbb{R}} |u(t,x)|^p\, dx \geq \frac{\left|\int_{\mathbb{R}} u(t,x)\, dx\right|^p}{\left(\int_{|x|\leq 1+\phi(t)} dx\right)^{p-1}} \geq C(1+t)^{-\frac{m+2}{2}(p-1)}\, |G(t)|^p,$$

which means that $G(t)$ fulfills inequality (2.2) with $q = \frac{m+2}{2}\,(p-1)$ (once inequality (2.1) has been verified, we then know that G is positive).

To establish (2.1), we introduce the following two functions: The first one is

$$\varphi(x) = e^x. \tag{2.4}$$

The second function is the so-called modified Bessel function

$$K_\nu(t) = \int_0^\infty e^{-t\cosh z} \cosh(\nu z) dz, \quad \nu \in \mathbb{R},$$

which is a solution of the equation

$$\left(t^2 \frac{d^2}{dt^2} + t\frac{d}{dt} - (t^2+\nu^2)\right) K_\nu(t) = 0, \quad t > 0.$$

From [5, page 24], we have

$$K_\nu(t) = \sqrt{\frac{\pi}{2t}}\, e^{-t}\left(1 + O(t^{-1})\right) \quad \text{as } t \to \infty, \tag{2.5}$$

provided that $\operatorname{Re}\nu > -1/2$. Set

$$\lambda(t) = C_m\, t^{\frac{1}{2}} K_{\frac{1}{m+2}}\left(\frac{2}{m+2}\, t^{\frac{m+2}{2}}\right), \quad t > 0, \tag{2.6}$$

where the constant $C_m > 0$ is chosen so that $\lambda(t)$ satisfies

$$\begin{cases} \lambda''(t) - t^m \lambda(t) = 0, & t \geq 0 \\ \lambda(0) = 1, \quad \lambda(\infty) = 0. \end{cases} \tag{2.7}$$

Introduce the test function ψ with

$$\psi(t,x) = \lambda(t)\varphi(x).$$

Let

$$G_1(t) = \int_{\mathbb{R}} u(t,x)\psi(t,x)\,dx.$$

Then

$$G''(t) = \int_{\mathbb{R}} |u(t,x)|^p\,dx \geq \frac{|G_1(t)|^p}{\left(\int_{|x|\leq 1+\phi(t)} \psi(t,x)^{\frac{p}{p-1}}\,dx\right)^{p-1}}. \tag{2.8}$$

Since $\psi(t,x) > 0$ for all $x \in \mathbb{R}$ and $t \geq 0$, one can repeat the proof of Lemma 2.3 in [12] with few modifications to get

LEMMA 2.2. *Under the assumptions of* Theorem 1.1, *there exists a* $t_0 > 0$ *such that*

$$G_1(t) \geq C\,t^{-\frac{m}{2}}, \quad t \geq t_0. \tag{2.9}$$

Relying on Lemma 2.2, we are now able to prove Theorem 1.1 for $1 < p < p_m$. By (2.5) and (2.6), we have that

$$\lambda(t) \sim t^{-\frac{m}{4}} e^{-\phi(t)} \quad \text{as } t \to \infty.$$

Next we estimate the denominator $\left(\int_{|x|\leq 1+\phi(t)} \psi(t,x)^{\frac{p}{p-1}}\,dx\right)^{p-1}$ in (2.8). Note that

$$\left(\int_{|x|\leq 1+\phi(t)} \psi(t,x)^{\frac{p}{p-1}}\,dx\right)^{p-1} = \lambda(t)^p \left(\int_{|x|\leq 1+\phi(t)} \varphi(x)^{\frac{p}{p-1}}\,dx\right)^{p-1}$$

and

$$|\varphi(x)| \leq Ce^{|x|}.$$

Then

$$\int_{|x|\leq 1+\phi(t)} \varphi(x)^{\frac{p}{p-1}}\,dx \leq Ce^{\frac{p}{p-1}(1+\phi(t))}$$

and

$$\begin{aligned}\left(\int_{|x|\leq 1+\phi(t)} \psi(t,x)^{\frac{p}{p-1}}\,dx\right)^{p-1} &\leq Ct^{-\frac{m}{4}p} e^{-p\phi(t)} e^{p(1+\phi(t))} \\ &\leq Ct^{-\frac{m}{4}p}.\end{aligned} \tag{2.10}$$

Therefore, it follows from (2.8)-(2.10) that, for $t \geq t_0$,

$$G''(t) \geq Ct^{-\frac{m}{4}p}. \tag{2.11}$$

If $-\frac{mp}{4} > -1$, then $p < \frac{4}{m}$. In this case we have that from (2.11)

$$G(t) \geq C\,(1+t)^{2-\frac{mp}{4}}$$

and one can let $\alpha = 2 - \frac{mp}{4}$ in (2.1). The condition $(p-1)\,\alpha \geq q-2$ in Lemma 2.1 can be written as

$$(p-1)\left(2 - \frac{mp}{4}\right) > \frac{m+2}{2}(p-1) - 2,$$

which is equivalent to

$$mp^2 + (m-4)p - (2m+4) < 0. \tag{2.12}$$

Note that the positive root of $mp^2 + (m-4)p - (2m+4) = 0$ is

$$p_1 = p_1(m) = \frac{4 - m + \sqrt{9m^2 + 8m + 16}}{2m}.$$

Then by (2.12) we have $p < p_1$. It is easy to know $p_1 > \frac{4}{m}$, thus by Lemma 2.1 one knows that the local solution u of (1.1) will blow up when $1 < p < \frac{4}{m}$ (in this case, only $m = 1, 2, 3$ are permitted).

If $-\frac{mp}{4} = -1$, i.e., $p = \frac{4}{m} > 1$, we then have

$$G(t) \geq (1+t)\ln(1+t) \geq C(1+t);$$

If $-\frac{mp}{4} < -1$, i.e., $p > \frac{4}{m}$, we then also have

$$G(t) \geq C(1+t).$$

Thus for these two cases, $\alpha = 1$ holds in (2.1), and then the condition $(p-1)\,\alpha \geq q-2$ in Lemma 2.1 is equivalent to

$$(p-1) > \frac{m+2}{2}(p-1) - 2 \Longrightarrow p < 1 + \frac{4}{m}.$$

This, together with Lemma 2.1, yields that the local solution u of (1.1) will blow up when $\frac{4}{m} \leq p < 1 + \frac{4}{m}$.

Combining all these results above, we complete the proof of Theorem 1.1 for $1 < p < p_m$.

3. Proof of Theorem 1.1 for $p = p_m$. Before starting the proof of Theorem 1.1 for $p = p_m$, we cite a blowup lemma from [33].

LEMMA 3.1. *Let $p > 1$, $a \geq 1$, and $(p-1)a = q-2$. Suppose $G \in C^2[0,T)$ satisfies, when $t \geq T_0 > 0$,*

$$G(t) \geq K_0(t+M)^a, \tag{3.1}$$

$$G''(t) \geq K_1(t+M)^{-q}G(t)^p, \tag{3.2}$$

where K_0, K_1, T_0 and M are some positive constants. Fixing K_1, there exists a positive constant c_0 independent of M and T_0 such that if $K_0 \geq c_0$, then $T < \infty$.

With this lemma and $G(t) = \int_{\mathbb{R}^n} u(t,x)\,dx$, our subsequent tasks are to derive (3.1) and (3.2) for the solution u of problem (1.1). In view of $\operatorname{supp} u_i \subseteq (-1,1)$ $(i = 0,1)$ and the finite propagation speed for the solutions of hyperbolic equations, one has that, for any fixed $t > 0$, the support of $u(t,\cdot)$ with respect to the variable x is contained in the interval $(-(1+\phi(t)), 1+\phi(t))$ with $\phi(t) = \frac{2}{m+2}\,t^{\frac{m+2}{2}}$.

It follows from the equation in (1.1) and the integration by parts that

$$
\begin{aligned}
G''(t) = \int_{\mathbb{R}} |u(t,x)|^{p_m}\, dx &\geq \frac{\left|\int_{\mathbb{R}} u(t,x)\, dx\right|^{p_m}}{\left(\int_{|x|\leq 1+\phi(t)} dx\right)^{p_m-1}} \\
&\geq C(1+t)^{-\frac{(m+2)(p_m-1)}{2}} |G(t)|^{p_m},
\end{aligned} \tag{3.3}
$$

which means that $G(t)$ fulfills inequality (3.2) with $p = p_m$ and $q = \frac{2(m+2)}{m}$ (once inequality (3.1) has been verified then the positivity of G is demonstrated).

Next, we focus on the derivation of (3.1), which is divided into the following two steps:

3.1. A lower bound of u. By Theorem 3.1 in [32] and Lemma 2.1 in [30], we can write the solution u of (1.1) as

$$
\begin{aligned}
u(t,x) = u^0(t,x) & \\
+ C \int_0^t \int_{x-\phi(t)+\phi(s)}^{x+\phi(t)-\phi(s)} & \big(\phi(t)+\phi(s)+x-y\big)^{-\gamma} \\
\times & \big(\phi(t)+\phi(s)-(x-y)\big)^{-\gamma} \\
\times & H\big(\gamma,\gamma,1,z\big)|u(s,y)|^{p_m}\, \mathrm{d}y\mathrm{d}s,
\end{aligned} \tag{3.4}
$$

where $z = \frac{(-x+y+\phi(t)-\phi(s))(-x+y-\phi(t)+\phi(s))}{(-x+y+\phi(t)+\phi(s))(-x+y-\phi(t)-\phi(s))}$, $H\big(\gamma,\gamma,1,z\big)$ is the hypergeometric function with $\gamma = \frac{m}{2(m+2)}$, and u^0 is the solution of the following linear generalized Tricomi equation:

$$
\begin{cases}
\partial_t^2 u - t^m\, \partial_x^2 u = 0 & \text{in } \mathbb{R}_+^{1+1}, \\
u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x).
\end{cases}
$$

By (2.3) in [32], we know that

$$
\begin{aligned}
u^0(t,x) = & 2^{\frac{m+4}{m+2}} \frac{\Gamma(\frac{m}{m+2})}{\Gamma^2(\frac{m}{2(m+2)})} \int_0^1 v_{u_0}\big(\phi(t)s,x\big)(1-s^2)^{-\frac{m+4}{2(m+2)}}\, \mathrm{d}s \\
& + t2^{\frac{m}{m+2}} \frac{\Gamma(\frac{m+4}{m+2})}{\Gamma^2(\frac{m+4}{2(m+2)})} \int_0^1 v_{u_1}\big(\phi(t)s,x\big)(1-s^2)^{-\frac{m}{2(m+2)}}\, \mathrm{d}s,
\end{aligned} \tag{3.5}
$$

where $v_\varphi(t,x)$ is a solution to the Cauchy problem for the 1-D linear wave equation

$$
\begin{cases}
\partial_t^2 v - \partial_x^2 v = 0 & \text{in } \mathbb{R}_+^{1+1}, \\
v(0,x) = \varphi(x), \quad \partial_t v(0,x) = 0.
\end{cases}
$$

It follows from D'Alembert formula that

$$
v_\varphi(t,x) = \frac{1}{2}[\varphi(t+x)+\varphi(t-x)]. \tag{3.6}
$$

Thus by u_0, $u_1 \geq 0$, we know that $u^0 \geq 0$ and

$$
\begin{aligned}
u(t,x) \geq C \int_0^t \int_{x-\phi(t)+\phi(s)}^{x+\phi(t)-\phi(s)} & \big(\phi(t)+\phi(s)+x-y\big)^{-\frac{m}{2(m+2)}} \\
& \times \big(\phi(t)+\phi(s)-(x-y)\big)^{-\frac{m}{2(m+2)}} \\
& \times H\big(\gamma,\gamma,1,z\big)|u(s,y)|^{p_m}\, \mathrm{d}y\mathrm{d}s.
\end{aligned} \tag{3.7}
$$

Notice that $z \in [0,1]$ for $|x-y| \le \phi(t)-\phi(s)$. Therefore, by [4, page 59],

$$\begin{aligned} H(\gamma,\gamma,1,z) &= \frac{1}{\Gamma(\gamma)\Gamma(1-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{-\gamma}(1-zt)^{-\gamma}\,\mathrm{d}t \\ &\ge \frac{1}{\Gamma(\gamma)\Gamma(1-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{-\gamma}\,\mathrm{d}t = \frac{B(\gamma,1-\gamma)}{\Gamma(\gamma)\Gamma(1-\gamma)} = 1. \end{aligned}$$

Together with (3.7), this yields

$$\begin{aligned} u(t,x) \ge C \int_0^t \int_{x-\phi(t)+\phi(s)}^{x+\phi(t)-\phi(s)} & \big(\phi(t)+\phi(s)+x-y\big)^{-\frac{m}{2(m+2)}} \\ & \times \big(\phi(t)+\phi(s)-(x-y)\big)^{-\frac{m}{2(m+2)}} \\ & \times |u(s,y)|^{p_m}\,\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{3.8}$$

On the other hand, it follows from (3.6) that

$$\int_0^1 v_{u_1}\big(\phi(t)s,x\big)(1-s^2)^{-\frac{m}{2(m+2)}}\,\mathrm{d}s = t^{-1}\int_{x-\phi(t)}^{x+\phi(t)} \frac{u_1(y)}{\big(\phi^2(t)-|x-y|^2\big)^{\frac{m}{2(m+2)}}}\,\mathrm{d}y.$$

Hence

$$\begin{aligned} u(t,x) > u^0(t,x) &\ge C\int_{x-\phi(t)}^{x+\phi(t)} \frac{u_1(y)}{\big(\phi^2(t)-|x-y|^2\big)^{\frac{m}{2(m+2)}}}\,\mathrm{d}y \\ &\ge \frac{C}{\big(\phi^2(t)-\big||x|-1\big|^2\big)^{\frac{m}{2(m+2)}}} \int_{x-\phi(t)}^{x+\phi(t)} u_1(y)\,\mathrm{d}y. \end{aligned}$$

Denote $H(t) = \{x : \phi(t)-1 < |x| < \phi(t)\}$, then for $\phi(t) \ge 2$ and $x \in H(t)$, we have

$$u(t,x) > \frac{C}{\big(\phi^2(t)-\big||x|-1\big|^2\big)^{\frac{m}{2(m+2)}}} \int_{x-\phi(t)}^{x+\phi(t)} u_1(y)\,\mathrm{d}y \ge C\big(\phi(t)\big)^{-\frac{m}{2(m+2)}}. \tag{3.9}$$

Here we comment that the lower bound in (3.9) is essentially optimal. Indeed, if we compute the other term in (3.5), we see that

$$\int_0^1 v_{u_0}\big(\phi(t)s,x\big)(1-s^2)^{-\frac{m+4}{2(m+2)}}\,\mathrm{d}s = t\int_{x-\phi(t)}^{x+\phi(t)} \frac{u_0(y)}{\big(\phi^2(t)-|x-y|^2\big)^{\frac{m+4}{2(m+2)}}}\,\mathrm{d}y.$$

This derives the following lower bound for the first term in (3.5) when $x \in H(t)$

$$\begin{aligned} t\int_{x-\phi(t)}^{x+\phi(t)} \frac{u_0(y)}{\big(\phi^2(t)-|x-y|^2\big)^{\frac{m+4}{2(m+2)}}}\,\mathrm{d}y &\ge \frac{t}{\big(\phi^2(t)-\big||x|-1\big|^2\big)^{\frac{m+4}{2(m+2)}}} \int_{x-\phi(t)}^{x+\phi(t)} u_0(y)\,\mathrm{d}y \\ &\ge C\big(\phi(t)\big)^{-\frac{m}{2(m+2)}}. \end{aligned}$$

Therefore we have

$$u(t,x) \ge C\big(\phi(t)\big)^{-\frac{m}{2(m+2)}}. \tag{3.10}$$

3.2. Derivation of (3.1)**.** Recall that in (3.8), the range of y is

$$x-\phi(t)+\phi(s)\le y\le x+\phi(t)-\phi(s).$$

If $0<\phi(s)\le\frac{\phi(t)-|x|-1}{2}$, then

$$|x-y|\le|x|+|y|\le|x|+\phi(s)+1\le|x|+\frac{\phi(t)-|x|-1}{2}+1\le\phi(t)-\phi(s).$$

This together with (3.8) yields

$$\begin{aligned}u(t,x)\ge{}&C\int_0^t\int_{x-\phi(t)+\phi(s)}^{x+\phi(t)-\phi(s)}\big(\phi(t)+\phi(s)+x-y\big)^{-\frac{m}{2(m+2)}}\\&\times\big(\phi(t)+\phi(s)-(x-y)\big)^{-\frac{m}{2(m+2)}}|u(s,y)|^{p_m}\,\mathrm{d}y\mathrm{d}s\\\ge{}&C\int_0^{\phi^{-1}(\frac{\phi(t)-|x|-1}{2})}\int_{\mathbb{R}}\big(\phi(t)+\phi(s)+x-y\big)^{-\frac{m}{2(m+2)}}\\&\times\big(\phi(t)+\phi(s)-(x-y)\big)^{-\frac{m}{2(m+2)}}|u(s,y)|^{p_m}\,\mathrm{d}y\mathrm{d}s\end{aligned}\tag{3.11}$$

Note that if $x>0$, then

$$\phi(t)+\phi(s)-(x-y)=\phi(t)-|x|+\phi(s)+y\le\phi(t)-|x|+2\phi(s)+1\le2\big(\phi(t)-|x|\big),$$

$$\phi(t)+\phi(s)+(x-y)=\phi(t)+|x|+\phi(s)-y\le8\phi(t);$$

if $x\le0$, then

$$\phi(t)+\phi(s)+(x-y)=\phi(t)-|x|+\phi(s)-y\le\phi(t)-|x|+2\phi(s)+1\le2\big(\phi(t)-|x|\big),$$

$$\phi(t)+\phi(s)-(x-y)=\phi(t)+|x|+\phi(s)+y\le8\phi(t).$$

Consequently, by (3.11) the solution u admits the following lower bound

$$\begin{aligned}u(t,x)\ge{}&C\int_0^{\phi^{-1}(\frac{\phi(t)-|x|-1}{2})}\int_{H(s)}\big(\phi(t)\big)^{-\frac{m}{2(m+2)}}\big(\phi(t)-|x|\big)^{-\frac{m}{2(m+2)}}|u(s,y)|^{p_m}\,\mathrm{d}y\mathrm{d}s\\\ge{}&\big(\phi(t)\big)^{-\frac{m}{2(m+2)}}\big(\phi(t)-|x|\big)^{-\frac{m}{2(m+2)}}\int_2^{\frac{\phi(t)-|x|-1}{2}}\\&\times\int_{H(s)}\big(\phi(s)\big)^{-\frac{mp_m}{2(m+2)}}\,\mathrm{d}y\big(\phi(s)\big)^{-\frac{m}{m+2}}\mathrm{d}\phi(s)\\={}&\big(\phi(t)\big)^{-\frac{m}{2(m+2)}}\big(\phi(t)-|x|\big)^{-\frac{m}{2(m+2)}}\int_2^{\frac{\phi(t)-|x|-1}{2}}\big(\phi(s)\big)^{-\frac{3m+4}{2(m+2)}}\mathrm{d}\phi(s)\\\ge{}&C\big(\phi(t)\big)^{-\frac{m}{2(m+2)}}\big(\phi(t)-|x|\big)^{-\frac{m}{2(m+2)}}.\end{aligned}$$

Thus, we have

$$\begin{aligned}G''(t)=\int_{\mathbb{R}}|u(t,x)|^{p_m}\,\mathrm{d}x\ge{}&C\int_0^{\phi(t)-1}\big(\phi(t)\big)^{-\frac{mp_m}{2(m+2)}}\big(\phi(t)-|x|\big)^{-\frac{mp_m}{2(m+2)}}\,\mathrm{d}|x|\\\ge{}&C\big(\phi(t)\big)^{-\frac{mp_m}{2(m+2)}}\big(\phi(t)\big)^{1-\frac{mp_m}{2(m+2)}}=C\big(\phi(t)\big)^{-\frac{2}{m+2}}=Ct^{-1},\end{aligned}$$

which derives

$$G(t) \geq Ct \ln t \tag{3.12}$$

for large t. Namely, (3.1) is shown.

We next finish the proof of Theorem 1.1 for $p = p_m$.

Let $a = 1$ in (3.1). Due to $p = p_m$ and $q = \frac{2(m+2)}{m}$, then the basic condition $(p-1)a = q-2$ in Lemma 3.1 holds. Based on Lemma 3.1 and Subsections 3.1-3.2, we know that the solution u to problem (1.1) for $p = p_m$ blows up in finite time. Therefore the proof of Theorem 1.1 for $p = p_m$ is completed.

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