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ON THE EXISTENCE AND MONOTONICITY OF CURVED FRONTS IN A PERIODIC SHEAR FLOW*

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Dedicated to Professor Ling Hsiao on the occasion of her 80th birthday

Abstract. In this paper we prove the existence and monotonicity of curved fronts in a periodic shear flow by mixing infinite different traveling fronts coming from different directions with different speeds. This is a generalization of our previous work [4].

Key words. curved fronts, reaction-advection-diffusion equation, existence.

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1. Introduction. In this paper, we consider the following reaction-advection-diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u + q(x)\frac{\partial u}{\partial y} + f(u), \text{ for all } t \in \mathbb{R},\ (x,y) \in \mathbb{R}^2, \tag{1.1}$$

where the advection coefficient $q(x)$ belongs to $C^{0,\delta}(\mathbb{R})$ for some $\delta > 0$, and satisfies

$$\forall x \in \mathbb{R}, \quad q(x+L) = q(x) \quad \text{and} \quad \int_0^L q(x)\,dx = 0 \tag{1.2}$$

for some $L > 0$. The second condition for q is a normalization condition. The nonlinearity f is assumed to satisfy the following conditions

$$\begin{cases} f \text{ is defined on } \mathbb{R}, \text{ Lipschitz continuous, and } f \equiv 0 \text{ in } \mathbb{R} \setminus (0,1), \\ f \text{ is a concave function of class } C^{1,\delta} \text{ in [0,1]}, \\ f'(0) > 0,\ f'(1) < 0, \text{ and } f(s) > 0 \text{ for all } s \in (0,1). \end{cases} \tag{1.3}$$

A typical example of such a function f is the quadratic nonlinearity $f(u) = u(1-u)$, which was initially considered by Fisher [5], Kolmogorov, Petrovsky and Piskunov [14]. The equation (1.1) arises in various biological models, such as population dynamics and gene developments where u stands for the relative concentration of some substance (see [1], [7] for details). In combustion, the equation (1.1) can be used to describe the models of flames in a shear flow, like in simplified Bunsen flames models with a perforated burner, and u stands for the normalized temperature.

In fact, we are interested in the traveling front solutions of (1.1) which have the form

$$u(t,x,y) = \phi(x, y+ct)$$

for all $(t,x,y) \in \mathbb{R} \times \mathbb{R}^2$, and for some positive constant c which denotes the speed of propagation in the vertical direction. Thus, we are led to the following elliptic equation

$$\Delta\phi + (q(x) - c)\partial_y\phi + f(\phi) = 0, \text{ for all } (x,y) \in \mathbb{R}^2, \tag{1.4}$$

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where the notation $\partial_y \phi$ means the partial derivative of the function ϕ with respect to the variable y.

We assume that the solutions ϕ of the equation (1.4) are normalized so that $0 \le \phi \le 1$. We look in this paper for solutions of (1.4) which satisfy the following type of "conical" conditions at infinity

$$\begin{cases} \lim_{l \to -\infty} \sup_{(x,y) \in C^-_{\alpha,\beta,l}} \phi(x,y) = 0, \\ \lim_{l \to +\infty} \inf_{(x,y) \in C^+_{\alpha,\beta,l}} \phi(x,y) = 1, \end{cases} \tag{1.5}$$

where α and β are given in $(0,\pi)$ such that $\alpha + \beta \le \pi$ and the lower and upper cones $C^-_{\alpha,\beta,l}$ and $C^+_{\alpha,\beta,l}$ are defined as follows:

Definition 1.1. *For each real number l, the lower cone $C^-_{\alpha,\beta,l}$ is defined by*

$$C^-_{\alpha,\beta,l} = \{(x,y) \in \mathbb{R}^2, \quad y \le x \cot \alpha + l \quad \text{whenever} \quad x \le 0 \\ \text{and} \quad y \le -x \cot \beta + l \quad \text{whenever} \quad x \ge 0\}$$

and then the upper cone is defined by

$$C^+_{\alpha,\beta,l} = \mathbb{R}^2 \setminus C^-_{\alpha,\beta,l}.$$

(See the figure 1 for a geometrical description).

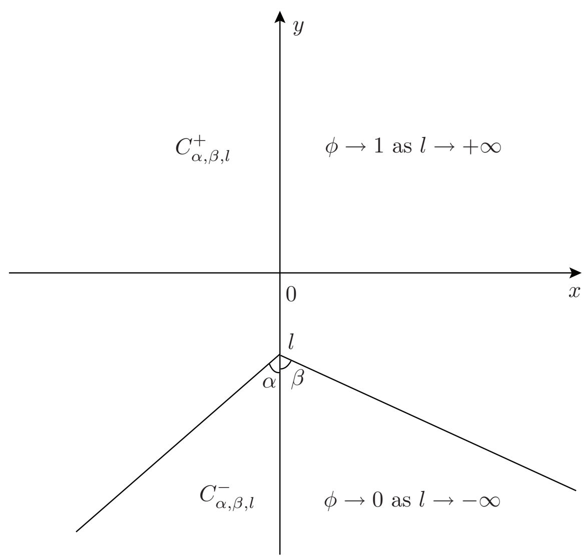

Fig. 1. *the lower and upper cones $C^-_{\alpha,\beta,l}$ and $C^+_{\alpha,\beta,l}$.*

In order to motivate our study, let us first recall some related known results. For the Fisher-KPP equation without advection, Hamel and Nadirashvili [8] proved that there exists an infinite-dimensional manifold of entire solutions (which are defined for all time and for all point $x \in \mathbb{R}^N$). In particular, there are infinite-dimensional manifolds of (planar or nonplanar) traveling fronts. Then, Huang [12] proved the stability of these planar or nonplanar traveling fronts with or without compact supported perturbations. For the reaction-advection-diffusion equations of the type (1.1), a well-known paper about this issue is the one by Berestycki and Hamel [1], where the authors investigated the reaction-diffusion equations with periodic advection in a very general framework, and gave the existence of the pulsating traveling fronts (some of their results will be recalled below).

Haragus and Scheel [10, 11] considered some equations of the type (1.4) with α and β close to $\pi/2$. Later, in our previous paper [4], we considered the reaction-advection-diffusion equation of type (1.1) and its corresponding elliptic equation (1.4) equipped with conical conditions (1.5) for general angles α and β and for general periodic shear flow. We proved the existence, nonexistence and monotonicity for the solutions of the semilinear elliptic equation (1.4) with the non-standard conical conditions at infinity (1.5). For the stability of these curved fronts, we refer to our recent paper [13].

Recalling the existence of the curved fronts, in our previous paper [4], we construct a subsolution and a supersolution for our problem by mixing, in different ways, two pulsating traveling fronts coming from opposite sides (left and right) and having different angles with respect to the vertical axis but having the same vertical speed in some sense. A very natural question is how about mixing different traveling fronts coming from different directions with different speeds more than two? For example, let's say, by mixing infinite different traveling fronts? That's just the main interest of the present paper. Although some generalizations are certainly possible, there are also some difficulty containing the existence for reaction-diffusion models of the type (1.4) in heterogeneous media and with non-almost-planar conical conditions at infinity.

For the monotonicity of the traveling fronts, also in our previous paper [4], we first established a generalized comparison principle in unbounded domains of the form $C^+_{\alpha,\beta,l}$, then together with further estimates on the behavior of any solution ϕ of the problem (1.4)-(1.5) in the lower cone $C^-_{\alpha,\beta,l}$, we proved that the traveling wave solution is increasing in the direction of y-axis. In the present paper, we generalize the above monotonicity in one direction of y-axis to an open cone directed by y-axis. The main method employed here is the so-called "sliding techniques" which are similar to those done by Berestycki and Nirenberg [3].

Before stating the main result of this paper, we first give some notions.

NOTATION 1.1. *Let $\gamma \in (0,\pi/2]$, $q = q(X)$ and $f = f(u)$ be two functions satisfying (1.2) and (1.3) respectively. Let $M = (m_{ij})_{1\le i,j\le 2}$ be a positive definite symmetric matrix, that is*

$$\exists\, c_1 > 0,\ \forall \xi \in \mathbb{R}^2,\ \sum_{1\le i,j\le 2} m_{i,j}\xi_i\xi_j \ge c_1|\xi|^2,$$

where $|\xi|^2 = \xi_1^2 + \xi_2^2$ for any $\xi = (\xi_1,\xi_2) \in \mathbb{R}^2$. $c^*_{M,q\sin\gamma,f} > 0$ denotes the minimal speed of propagation of traveling fronts $0 \le u \le 1$ in the vertical downwards direction $-e = (0,-1)$ for the following reaction-advection-diffusion problem

$$\begin{cases} \frac{\partial u}{\partial t} = \mathrm{div}(M\nabla u) + q(X)\sin\gamma\frac{\partial u}{\partial Y} + f(u),\ t\in\mathbb{R},\ (X,Y)\in\mathbb{R}^2, \\ \forall \tau\in\mathbb{R},\ u(t+\tau, X+L, Y) = u(t+\tau, X, Y) = u(t, X, Y+c\tau),\ \textit{for} \\ \ \textit{all}\ (t,X,Y)\in\mathbb{R}\times\mathbb{R}^2, \\ u(t,X,Y) \underset{Y\to-\infty}{\longrightarrow} 0, \quad u(t,X,Y) \underset{Y\to+\infty}{\longrightarrow} 1, \end{cases} \tag{1.6}$$

where the above limits hold locally in t and uniformly in X. The existence and further qualitative properties of such fronts follow from [1, 6, 9].

For any $\gamma \in (0, \pi)$, we set

$$M = \begin{bmatrix} 1 & \cos\gamma \\ \cos\gamma & 1 \end{bmatrix}$$

in the problem (1.6). Then, there exists a positive constant $c^*_{M,q\sin\gamma,f} = c^*_\gamma$ such that for any $c_\gamma \geq c^*_\gamma$, the problem (1.6) admits a solution $(c_\gamma, \varphi_\gamma)$. For any α and β in $(0, \pi)$ such that $\alpha + \beta < \pi$, we define

$$\Gamma_c = \{\gamma \in (0, \pi); \beta \leq \gamma \leq \pi - \alpha, c\sin\gamma = c_\gamma \geq c^*_\gamma\},$$

where c is the positive constant in the equation (1.4), which denotes the speed of the propagation. Let $\mathcal{M}$ be the set of all non-negative and nonzero Radon measures μ on Γ_c $(0 < \mu(\Gamma_c) < +\infty)$.

The main results of this paper is the following two theorems.

THEOREM 1.1 (Existence). *Let $q(x)$ be a globally $C^{0,\delta}(\mathbb{R})$ function (for some $\delta > 0$) satisfying (1.2). Let f be a nonlinearity satisfying (1.3). Then, for any given α and β in $(0, \pi)$ with $\alpha + \beta < \pi$ and for any $c > 0$ such that $\Gamma_c \neq \emptyset$, the equation (1.1) admits a curved fronts with speed c satisfying the condition (1.5). Namely, there exists a map $\mu \mapsto \phi_\mu$, from $\mathcal{M}$ to the set of curved fronts of (1.4) and (1.5).*

REMARK 1.1. *If we can prove that the map $\mu \mapsto \phi_\mu$ is one-to-one (in other words, for $\mu_1 \neq \mu_2$, the corresponding curved fronts φ_{μ_1} and φ_{μ_2} are different), then the problem (1.1) and (1.5) admits an infinite-dimensional manifold of traveling fronts. But, it is still open and will be our further consideration.*

REMARK 1.2. *If $\frac{c^*_\alpha}{\sin\alpha} \neq \frac{c^*_\beta}{\sin\beta}$, without loss of generality, we can assume $\frac{c^*_\alpha}{\sin\alpha} > \frac{c^*_\beta}{\sin\beta}$, then for any given $c \geq c^* = \max\left(\frac{c^*_\alpha}{\sin\alpha}, \frac{c^*_\beta}{\sin\beta}\right)$, there exists $\varepsilon > 0$ such that $(\beta, \beta + \varepsilon) \subseteq \Gamma_c$ (by the continuity of c^*_γ with respect to γ), which implies that the set Γ_c is not empty. If $\frac{c^*_\alpha}{\sin\alpha} = \frac{c^*_\beta}{\sin\beta}$, then for any given $c > c^*$, there exists $\varepsilon > 0$ such that the set Γ_c contains $(\beta, \beta + \varepsilon) \cup (\pi - \alpha - \varepsilon, \pi - \alpha)$, which is not an empty set too.*

THEOREM 1.2 (Monotomocity). *Under the assumptions of Theorem 1.1, if ϕ is a solution of the problem (1.4)-(1.5), then $\frac{\partial\phi}{\partial\nu}(x, y) > 0$ for all $(x, y) \in \mathbb{R}^2$, where ν is a unit vector in $\mathbb{R}^2$ satisfying*

$$\nu \cdot \nu_0 > \cos\left(\arcsin\left(\frac{2\sqrt{f'(0)} - |q_0|}{c}\right)\right),$$

with $q_0 = \max_{x\in\mathbb{R}} q(x)$, namely, ν belongs to the open cone directed by $\nu_0 = (0, 1)$ with angle $\arcsin\left(\frac{2\sqrt{f'(0)}-|q_0|}{c}\right)$.

REMARK 1.3. *In fact, in our previous paper [4], we proved that the traveling wave is increasing in the direction $\nu_0 = (0, 1)$. In the present work, we generalize the above monotonicity in one direction of ν_0 to an open cone directed by ν_0. Motivated by the work of Berestycki and Nirenberg [3], we will use the method of sliding techniques to prove the monotonicity of the traveling waves.*

2. Proof of the main results. In this section, we prove the existence and monotonicity of curved fronts. Let us first give the following lemma which will be used in the proof of our main theorems.

LEMMA 2.1. *If $f(s)$ is a concave function of class $C^{1,\delta}$ in $[0,1]$ and satisfies the assumptions (1.3), and $f(s)$ is defined by* 0 *for all $s>1$, then*
(i) $f(s)$ is a sub-additive function, that is, $f(s+t)\le f(s)+f(t)$, for all $s,t\in(0,1)$;
(ii) $f(s)\le f'(0)s,\ \forall s\in[0,1]$;
(iii) $f'(s)\le f'(0),\ \forall s\in[0,1]$;
(iv) $f'(s)\le f(s)/s,\ \forall s\in(0,1]$.

Proof. All of the conclusions in this lemma can be proved by the definition of concave functions with $f(0)=0$ directly, and we omit the details here. □

Proof of Theorem 1.1. We employ the method of super- and subsolution to prove that for any given α and β in $(0,\pi)$ with $\alpha+\beta<\pi$ and for any $c>0$ such that $\Gamma_c\neq\emptyset$, the problem (1.1) and (1.5) admits a curved front with speed c.

Step 1: Construction of a subsolution. Let $\hat\varphi(x,y)$ be a function defined by

$$\hat\varphi(x,y)=\frac{1}{\mu(\Gamma_c)}\int_{\Gamma_c}\varphi_\gamma(x,x\cos\gamma+y\sin\gamma)d\mu(\gamma).$$

As a consequence of the Lebesgue's dominated convergence theorem, the function $\hat\varphi(x,y)$ is of class $C^2(\mathbb{R}^2)$ and it satisfies

$$\begin{aligned}
&\Delta\hat\varphi(x,y)+(q(x)-c)\partial_y\hat\varphi(x,y)\\
=&\frac{1}{\mu(\Gamma_c)}\int_{\Gamma_c}\operatorname{div}(M\nabla\varphi_\gamma(x,x\cos\gamma+y\sin\gamma))d\mu(\gamma)\\
&+\frac{1}{\mu(\Gamma_c)}\int_{\Gamma_c}(q(x)-c)\sin\gamma\partial_Y\varphi_\gamma(x,x\cos\gamma+y\sin\gamma)d\mu(\gamma)\\
=&-\frac{1}{\mu(\Gamma_c)}\int_{\Gamma_c}f(\varphi_\gamma(x,x\cos\gamma+y\sin\gamma))d\mu(\gamma)\\
\ge&-f\left(\frac{1}{\mu(\Gamma_c)}\int_{\Gamma_c}\varphi_\gamma(x,x\cos\gamma+y\sin\gamma)d\mu(\gamma)\right)\\
=&-f(\hat\varphi),
\end{aligned}$$

since f is concave on $[0,1]$. Thus, $\hat\varphi$ is a subsolution of the equation (1.4). Noticing the fact that both of the functions $\varphi_\alpha(x,-x\cos\alpha+y\sin\alpha)$ and $\varphi_\beta(x,x\cos\beta+y\sin\beta)$ solve the equation (1.4), we are informed that the function $\underline{\phi}_\mu(x,y)$ defined by

$$\underline{\phi}_\mu(x,y)=\max\left(\varphi_\alpha(x,-x\cos\alpha+y\sin\alpha),\varphi_\beta(x,x\cos\beta+y\sin\beta),\hat\varphi(x,y)\right)$$

is a subsolution of the equation (1.4). Moreover, by construction and since $\alpha+\beta<\pi$, we know that $\underline{\phi}_\mu$ satisfies $\lim\limits_{l\to+\infty}\inf\limits_{(x,y)\in C^+_{\alpha,\beta,l}}\underline{\phi}_\mu(x,y)=1$.

Step 2: Construction of a supersolution. For each $\lambda\in\mathbb{R}$ and $\gamma\in\Gamma_c$, call $k(\lambda,\gamma)$ the principal eigenvalue of the operator

$$L_{\lambda,\gamma}\psi:=\psi''+2\lambda\cos\gamma\psi'+(\lambda^2+\lambda q(x)\sin\gamma+f'(0))\psi$$

acting on the set

$$E=\{\psi\in C^2(\mathbb{R});\psi(x+L)=\psi(x),\ \text{for all } x\in\mathbb{R}\}.$$

Then, the number

$$\lambda_\gamma = \min\{\lambda > 0, k(\lambda,\gamma) = c_\gamma \lambda = c\lambda \sin\gamma\}$$

is well-defined (see [6]). Let $\psi_\gamma(x)$ denote the unique positive principal eigenfunction of $L_{\lambda,\gamma}$ with $\lambda = \lambda_\gamma$ such that

$$\|\psi_\gamma\|_{L^\infty(\mathbb{R})} = 1.$$

Let $\tilde{\varphi}(x,y)$ be a function defined by

$$\tilde{\varphi}(x,y) = \frac{1}{\mu(\Gamma_c)} \int_{\Gamma_c} e^{\lambda_\gamma(x\cos\gamma + y\sin\gamma)} \psi_\gamma(x) d\mu(\gamma).$$

Then, the function $\tilde{\varphi}(x,y)$ is of class $C^2(\mathbb{R}^2)$ and satisfies

$$\begin{aligned} &\Delta\tilde{\varphi}(x,y) + (q(x) - c)\partial_y \tilde{\varphi}(x,y) + f(\tilde{\varphi}) \\ &\leq \Delta\tilde{\varphi}(x,y) + (q(x) - c)\partial_y \tilde{\varphi}(x,y) + f'(0)\tilde{\varphi} \\ &= \frac{1}{\mu(\Gamma_c)} \int_{\Gamma_c} e^{\lambda_\gamma(x\cos\gamma + y\sin\gamma)} (L_{\lambda_\gamma,\gamma}\psi_\lambda - c_\gamma \lambda_\gamma \psi_\lambda) d\mu(\gamma) \\ &= 0. \end{aligned}$$

Thus, $\tilde{\varphi}(x,y)$ is a supersolution of the equation (1.4). As we have done in the proof of the Theorem 1.1 [4], by the sub-additivity of f, we know that the following function

$$\bar{\phi}_\mu(x,y) = \min\left(\varphi_\alpha(x, -x\cos\alpha + y\sin\alpha) + \varphi_\beta(x, x\cos\beta + y\sin\beta) + \tilde{\varphi}(x,y), 1\right)$$

is a supersolution of the equation (1.4). Furthermore, we can prove that the supersolution $\bar{\phi}_\mu(x,y)$ satisfies

$$\lim_{l\to-\infty} \inf_{(x,y)\in C^-_{\alpha,\beta,l}} \bar{\phi}_\mu(x,y) = 0. \tag{2.1}$$

Obviously, it suffices to prove

$$\lim_{l\to-\infty} \inf_{(x,y)\in C^-_{\alpha,\beta,l}} \tilde{\varphi}(x,y) = 0.$$

We assume by contradiction that there exist $\varepsilon > 0$, $(x_n, y_n) \in C^-_{\alpha,\beta,l_n}$, $l_n \leq 0$ and $l_n \to -\infty$ as $n \to +\infty$, such that

$$\lim_{n\to+\infty} \inf_{(x_n,y_n)\in C^-_{\alpha,\beta,l_n}} \tilde{\varphi}(x_n,y_n) = \varepsilon > 0. \tag{2.2}$$

Then, for any given $\beta \leq \gamma \leq \pi - \alpha$, we have

$$0 \leq e^{\lambda_\gamma(x_n\cos\gamma + y_n\sin\gamma)} \psi_\gamma(x_n) \leq \psi_\gamma(x_n) \leq 1$$

and

$$\lim_{n\to+\infty} \inf_{(x_n,y_n)\in C^-_{\alpha,\beta,l_n}} e^{\lambda_\gamma(x_n\cos\gamma + y_n\sin\gamma)} \psi_\gamma(x_n) = 0.$$

Thus, it follows from the Lebesgue's dominated convergence theorem that

$$\begin{aligned}&\lim_{n\to+\infty}\inf_{(x_n,y_n)\in C^-_{\alpha,\beta,l_n}}\tilde{\varphi}(x_n,y_n)\\ =&\frac{1}{\mu(\Gamma_c)}\int_{\Gamma_c}\lim_{n\to+\infty}\inf_{(x_n,y_n)\in C^-_{\alpha,\beta,l_n}}e^{\lambda_\gamma(x_n\cos\gamma+y_n\sin\gamma)}\psi_\gamma(x_n)d\mu(\gamma)=0.\end{aligned}$$

So, the assumption (2.2) is false, and thus, the condition (2.1) is satisfied.

Finally, it follows from the result of [7] that

$$\varphi_\gamma(x,y)\le e^{\lambda_\gamma y}\psi_\lambda(x)$$

holds for all $(x,y)\in\mathbb{R}^2$. So, $\hat{\varphi}(x,y)\le\tilde{\varphi}(x,y)$ for all $(x,y)\in\mathbb{R}^2$. Then, we have $0\le\underline{\phi}_\mu(x,y)\le\bar{\phi}_\mu(x,y)\le 1$ in $\mathbb{R}^2$. Thus, for any $c\ge c^*$ and $\mu\in\mathcal{M}$, the problem (1.4)–(1.5) admits a curved front (c,ϕ_μ) such that $\underline{\phi}_\mu\le\phi_\mu\le\bar{\phi}_\mu$. The proof of this theorem is then complete. ∎

Now we are in the position to prove the monotonicity of the fronts. We first have the following lemma, which is the key lemma to prove the monotonicity of the fronts.

Lemma 2.2. *Let $u(x,y,t)$ be an entire solution of (1.1) such that the fields $\frac{u_t}{u}$ and $\frac{\nabla_{x,y}u}{u}$ are globally bounded. Then, for each vector $\rho=(\rho_1,\rho_2)\in\mathbb{R}^2$ such that $|\rho|<2\sqrt{f'(0)}-|q_0|$, where $q_0=\max_{x\in\mathbb{R}}q(x)$, one has $u_t+\rho\cdot\nabla_{x,y}u>0$ in $\mathbb{R}^2\times\mathbb{R}$.*

Proof. To this end, it is enough to prove that $\partial_t u(x,y,t)+\rho\cdot\nabla_{x,y}u(x,y,t)\ge 0$ for all $(x,y,t)\in\mathbb{R}^2\times\mathbb{R}$. Indeed, suppose the latter is true. By a simple calculation, on can find that the function $U=\partial_t u+\rho\cdot\nabla_{x,y}u$ satisfies a linear parabolic equation. Then, from the parabolic maximum principle, we are informed that $U>0$ for all $(x,y,t)\in\mathbb{R}^2\times\mathbb{R}$, and the conclusion of Lemma 2.2 will follow.

Let us now denote by $v(x,y,t)$ the function

$$v(x,y,t)=\frac{\partial_t u(x,y,t)+\rho\cdot\nabla_{x,y}u(x,y,t)}{u(x,y,t)}.$$

By assumption, this function v is globally bounded and one then only has to prove that $\inf_{\mathbb{R}^2\times\mathbb{R}}v\ge 0$.

Suppose by contradiction that $\inf_{\mathbb{R}^2\times\mathbb{R}}v=-\epsilon<0$. There exists a sequence $(x_n,y_n,t_n)\in\mathbb{R}^2\times\mathbb{R}$ such that $v(x_n,y_n,t_n)\to-\epsilon$ as $n\to+\infty$. Up to extraction of some subsequence, two and only two cases may occur:

Case 1: $u(x_n,y_n,t_n)\to\alpha\in(0,1]$ as $n\to+\infty$,

Case 2: $u(x_n,y_n,t_n)\to 0$ as $n\to+\infty$.

Let us first deal with Case 1. After a straightforward calculation, it is found that the function v satisfies

$$v_t=\Delta v+2\frac{\nabla_{x,y}u}{u}\cdot\nabla_{x,y}v+q(x)\frac{\partial v}{\partial y}+(f'(u)-\frac{f(u)}{u})v+\frac{\rho_1 q'(x)}{u}\frac{\partial u}{\partial y}.$$

Let us set

$$\begin{aligned}u_n(x,y,t)&=u(x+x_n,y+y_n,t+t_n),\\ v_n(x,y,t)&=v(x+x_n,y+y_n,t+t_n),\\ q_n(x)&=q(x+x_n).\end{aligned}$$

From the standard parabolic estimates, the functions u_n converge in $C^1_{loc}(\mathbb{R}_t)$ and $C^2_{loc}(\mathbb{R}^2_{x,y})$ to a function u_∞(up to extraction of some subsequence). The function u_∞ is such that $0 \le u_\infty \le 1$ and it solves

$$\partial_t u_\infty = \Delta u_\infty + q_\infty \frac{\partial u_\infty}{\partial y} + f(u_\infty) \quad \text{in } \mathbb{R}^2 \times \mathbb{R}.$$

Furthermore, since $u(x_n, y_n, t_n) \to \alpha \in (0,1]$ as $n \to +\infty$, one has $u(0,0,0) = \alpha > 0$. Therefore, the function $u_\infty(x,y,t)$ is positive everywhere (because of the strong parabolic maximum principle) and the globally bounded sequences of functions $\frac{\nabla_{x,y} u_n}{u_n}$, $f'(u_n)$ and $\frac{f(u_n)}{u_n}$ converge to the globally bounded functions $\frac{\nabla_{x,y} u_\infty}{u_\infty}$, $f'(u_\infty)$ and $\frac{f(u_\infty)}{u_\infty}$, respectively.

Similarly, the globally bounded functions v_n converge to a globally bounded function v_∞, which is equal to

$$v_\infty = \frac{\partial_t u_\infty + \rho \cdot \nabla_{x,y} u_\infty}{u_\infty}.$$

The function v_∞ is such that $v_\infty(x,y,t) \ge -\epsilon$ for all $(x,y,t) \in \mathbb{R}^2 \times \mathbb{R}$ and $v_\infty(0,0,0) = -\epsilon$. Furthermore, v_∞ satisfies

$$\partial_t v_\infty = \Delta v_\infty + 2\frac{\nabla_{x,y} u_\infty}{u_\infty} \cdot \nabla_{x,y} v_\infty + q(x)\frac{\partial v_\infty}{\partial y} + (f'(u_\infty) - \frac{f(u_\infty)}{u_\infty})v_\infty + \frac{\rho_1 q'(x)}{u_\infty}\frac{\partial u_\infty}{\partial y}.$$

The point $(0,0,0)$ is a global minimum for the function v_∞ and $v_\infty(0,0,0) = -\epsilon < 0$. On the other hand, $u_\infty(0,0,0) = \alpha \in (0,1]$ and $f'(\alpha) - \frac{f(\alpha)}{\alpha} \le 0$ since the function f is concave on $[0,1]$ and $f(0) = 0$. Then it follows that $v_\infty \equiv -\epsilon$ in $\mathbb{R}^2 \times \mathbb{R}^-$. In other words, $\frac{\partial_t u_\infty + \rho \cdot \nabla_{x,y} u_\infty}{u_\infty} \equiv -\epsilon < 0$ in $\mathbb{R}^2 \times \mathbb{R}^-$. Since u_∞ is positive, one gets

$$\partial_t u_\infty + \rho \cdot \nabla_{x,y} u_\infty < 0 \ \text{ in } \mathbb{R}^2 \times \mathbb{R}^-. \tag{2.3}$$

But, since u_∞ is a solution of $\partial_t u_\infty = \Delta u_\infty + q(x)\frac{\partial u_\infty}{\partial y} + f(u_\infty)$ such that $u_\infty \le 1$, one has either $u_\infty \equiv 1$ or $u_\infty < 1$. The case $u_\infty \equiv 1$ is in contradiction with (2.3). The case $u_\infty < 1$ means that the function u_∞ is a solution of (1.1), such that $0 < u_\infty < 1$. Specially, when $\rho_1 = 0$, $\omega(t) = u(0, \rho_2 t, t) = \phi(0, (\rho_2 + c)t) \to 0$ as $t \to -\infty$. But this positive function ω is decreasing for $t \le 0$ by (2.3). One has then reached a contradiction. As a conclusion, Case 1 is ruled out.

Let us now deal with Case 2. Up to extraction of some subsequence, one has

$$u(x_n, y_n, t_n) \to 0, \qquad \text{as } n \to +\infty.$$

Let us set

$$\omega_n(x,y,t) = \frac{u(x + \rho_1 t + x_n, y + \rho_2 t + y_n, t + t_n)}{u(x_n, y_n, t_n)} e^{\frac{1}{2}(\rho_1 x + \rho_2 y)}, \ (x,y,t) \in \mathbb{R}^2 \times \mathbb{R}.$$

Since the fields $\frac{u_t}{u}$ and $\frac{\nabla_{x,y} u}{u}$ are globally bounded, there exists a constant C such that $\omega_n(x,,y,t) \le e^{C(|x|+|y|+|t|)}$ for all $(x,y,t) \in \mathbb{R}^2 \times \mathbb{R}$ and all n. In particular, the sequence $\{\omega_n\}$ is locally bounded and the functions $(x,y,t) \mapsto u(x + x_n, y + y_n, t + t_n)$ approach 0 locally in $\mathbb{R}^2 \times \mathbb{R}$. On the other hand, each function ω_n satisfies

$$\partial_t \omega_n = \Delta \omega_n + q_n \frac{\partial \omega_n}{\partial y} + \left(\frac{f(u)}{u} - \frac{1}{4}|\rho|^2 - \frac{1}{2}\rho_2 q_n \right) \omega_n, \ (x,y,t) \in \mathbb{R}^2 \times \mathbb{R}.$$

From standard parabolic estimates, the function ω_∞ solves

$$\partial_t\omega_\infty = \Delta\omega_\infty + q_\infty\frac{\partial\omega_\infty}{\partial y} + \left(\frac{f(u)}{u} - \frac{1}{4}|\rho|^2 - \frac{1}{2}\rho_2 q_\infty\right)\omega_\infty,\ (x,y,t)\in\mathbb{R}^2\times\mathbb{R}. \quad (2.4)$$

and it satisfies

$$\forall t\in\mathbb{R},\ \forall(x,y)\in\mathbb{R}^2,\ \omega_\infty(x,y,t)\le e^{C(|x|+|y|+|t|)}. \quad (2.5)$$

Due to the definition of ω_n and to the choice of (x_n,y_n,t_n), one has

$$\partial_t\omega_n(0,0,0) = \frac{\partial_t u(x_n,y_n,t_n) + \rho\cdot\nabla_{x,y}u(x_n,y_n,t_n)}{u(x_n,y_n,t_n)} = v(x_n,y_n,t_n)\to-\epsilon,$$

as $n\to+\infty$. Hence,

$$\partial_t\omega_\infty(0,0,0) = -\epsilon. \quad (2.6)$$

For each point $x_0\in\mathbb{R}$, we can first freeze $q_\infty(x)$ at x_0. So, the equation (2.4) can be written as

$$\partial_t\omega_\infty = \Delta\omega_\infty + q_\infty(x_0)\frac{\partial\omega_\infty}{\partial y} + \left(\frac{f(u)}{u} - \frac{1}{4}|\rho|^2 - \frac{1}{2}\rho_2 q_\infty(x_0)\right)\omega_\infty. \quad (2.7)$$

For any $\tau>0$ and $(\lambda,\mu)\in\mathbb{R}^2$, we define a function as follows

$$p(\lambda,\mu,\tau) = \frac{1}{4\pi\tau}e^{(f'(0)-\frac{1}{4}|\rho|^2-\frac{1}{2}\rho_2 q_\infty(\lambda))\tau - \frac{\lambda^2+|\mu+q_\infty(\lambda)\tau|^2}{4\tau}}. \quad (2.8)$$

Because of (2.5), (2.7) and (2.8), $\omega_\infty(x,y,t)$ can be written as

$$\omega_\infty(x,y,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\omega_\infty(\xi,\eta,-k)p(x_0,y-\eta,t+k)d\xi d\eta,$$

for all $k>|t|$. Then let $q_\infty(x)$ melt, that is, let x change in $\mathbb{R}$, we obtain

$$\omega_\infty(x,y,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\omega_\infty(\xi,\eta,-k)p(x-\xi,y-\eta,t+k)d\xi d\eta,$$

for all $k>|t|$. As a consequence,

$$\partial_t\omega_\infty(x,y,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\omega_\infty(\xi,\eta,-k)\partial_t p(x-\xi,y-\eta,t+k)d\xi d\eta.$$

Notice that

$$\begin{aligned}\partial_\tau p(\lambda,\mu,\tau) &= -\frac{1}{\tau}p + \left(f'(0) - \frac{1}{4}|\rho|^2 - \frac{1}{2}\rho_2 q_\infty(\lambda) + \frac{\lambda^2+\mu^2-q_\infty^2(\lambda)\tau^2}{4\tau^2}\right)p\\ &\ge -\frac{1}{\tau}p + \left(f'(0) - \frac{1}{4}|\rho|^2 - \frac{1}{2}\rho_2 q_\infty(\lambda) - \frac{q_\infty^2(\lambda)}{4}\right)p\end{aligned}$$

holds for all $\tau>0$ and $(\lambda,\mu)\in\mathbb{R}^2$. Since ω_∞ is nonnegative, it follows that

$$\begin{aligned}\partial_t\omega_\infty(x,y,t) \ge \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}&\omega_\infty(\xi,\eta,-k)\cdot\\ &\left(-\frac{1}{(t+k)} + f'(0) - \frac{1}{4}|\rho|^2 - \frac{1}{2}\rho_2 q_\infty(x-\xi) - \frac{q_\infty^2(x-\xi)}{4}\right)pd\xi d\eta.\end{aligned}$$

Passing to the limit $k \to +\infty$ in the above formula leads to

$$\partial_t \omega_\infty(x,y,t) \geq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \omega_\infty(\xi,\eta,-k)\cdot \left(f'(0) - \frac{1}{4}|\rho|^2 - \frac{1}{2}\rho_2 q_\infty(x-\xi) - \frac{q_\infty^2(x-\xi)}{4}\right) p d\xi d\eta.$$

Now we need to prove that

$$f'(0) - \frac{1}{4}|\rho|^2 - \frac{1}{2}\rho_2 q_\infty(\lambda) - \frac{q_\infty^2(\lambda)}{4} \geq 0. \tag{2.9}$$

In fact, since $|\rho| \leq 2\sqrt{f'(0)} - |q_0|$, we have $(|\rho| + |q_\infty(\lambda)|)^2 \leq 4f'(0)$, that is,

$$4f'(0) - q_\infty^2(\lambda) \geq |\rho|^2 + 2|\rho|\cdot|q_\infty(\lambda)|.$$

Then, we have

$$f'(0) - \frac{q_\infty^2(\lambda)}{4} \geq \frac{1}{4}|\rho|^2 + \frac{1}{2}\rho_2 q_\infty(\lambda),$$

which implies that the inequality (2.9) holds. Since $\omega_\infty \geq 0$, one gets $\partial_t\omega_\infty(x,y,t) \geq 0$ for all $(x,y,t) \in \mathbb{R}^2 \times \mathbb{R}$. That is in contradiction with (2.6). Therefore, Case 2 is ruled out too and the proof of Lemma 2.2 is complete. □

Let us now come back to the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\nu \in S$ be such that

$$\nu\cdot\nu_0 > \cos\arcsin\left(\frac{2\sqrt{f'(0)} - |q_0|}{c}\right), \quad \text{where } \nu_0 = (0,1).$$

Let ρ be the vector defined by

$$\rho = (c(\nu_0\cdot\nu)\nu - c\nu_0) = (c\sin\gamma\cos\gamma, -c\cos^2\gamma).$$

One has

$$\begin{aligned}
|\rho|^2 &= (c^2\sin^2\gamma\cos^2\gamma + c^2\cos^4\gamma)\\
&= c^2\cos^2\gamma\\
&< c^2\left(1 - \cos^2\left(\arcsin\left(\frac{2\sqrt{f'(0)} - |q_0|}{c}\right)\right)\right)\\
&= (2\sqrt{f'(0)} - |q_0|)^2.
\end{aligned}$$

Let us now check that the function u satisfies the assumptions of Lemma 2.2, that is to say that $\frac{u_t}{u}$ and $\frac{\nabla_{x,y}u}{u}$ are globally bounded. Indeed, since u is written as $u(x,y,t) = \phi(x, y+ct)$, one has $\frac{u_t}{u} = c\frac{\partial_y\phi}{\phi}$ and $\frac{\partial_x u}{u} = \frac{\partial_x\phi}{\phi}$, $\frac{\partial_y u}{u} = \frac{\partial_y\phi}{\phi}$. Therefore, one only has to check that $\frac{\partial_x u}{u} = \frac{\partial_x\phi}{\phi}$, $\frac{\partial_y u}{u} = \frac{\partial_y\phi}{\phi}$ is bounded. Since ϕ is a positive solution of $\Delta\phi + (q(x) - c)\frac{\partial\phi}{\partial y} + f(\phi) = 0$ in $\mathbb{R}^2$, Schauder interior estimates and Harnack-type inequalities imply that there exists $K > 0$, such that $|\partial_x\phi(x,y)| \leq K\phi(x,y)$, $|\partial_y\phi(x,y)| \leq K\phi(x,y)$, which was the desired result. As a consequence, Lemma 2.2 can be applied and yields $\partial_t u + \rho\cdot\nabla_{x,y}u > 0$ in $\mathbb{R}^2\times\mathbb{R}$. Due to the definition of v, it follows that $c\nu_0\cdot\nabla v + \rho\cdot\nabla v > 0$ in $\mathbb{R}^2$, i.e. $c(\nu_0\cdot\nu)\nu\cdot\nabla v > 0$. Since $\nu_0\cdot\nu > 0$ and $c > 0$, one gets $\nu\cdot\nabla v > 0$ in $\mathbb{R}^2$. The proof of Theorem 1.2 is complete. □

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