METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 28, No. 3, pp. 337–354, September 2021

006

STABILITY OF PLANAR RAREFACTION WAVES FOR SCALAR VISCOUS CONSERVATION LAW UNDER PERIODIC PERTURBATIONS*

FEIMIN HUANG† AND QIAN YUAN‡

Dedicated to the 80th Birthday of Ling Hsiao

Abstract. The large time behavior of the solutions to a multi-dimensional viscous conservation law is considered in this paper. It is shown that the solution time-asymptotically tends to the planar rarefaction wave if the initial perturbations are multi-dimensional periodic. The time-decay rate is also obtained. Moreover, a Gagliardo-Nirenberg type inequality is established in the domain $\mathbb{R} \times \mathbb{T}^{n-1} (n \geq 2)$, where $\mathbb{T}^{n-1}$ is the $n-1$-dimensional torus.

Key words. Planar rarefaction wave, periodic perturbation, viscous conservation law.

Mathematics Subject Classification. 35B65, 35L65, 35Q35.

1. introduction. We are concerned with a scalar viscous conservation law, which reads in $\mathbb{R}^n$ as

$$\partial_t u(x,t) + \sum_{i=1}^{n} \partial_i \left(f_i(u(x,t)) \right) = \triangle u(x,t), \quad t > 0, x \in \mathbb{R}^n, \tag{1.1}$$

where $u(x,t) \in \mathbb{R}$, $\partial_i := \partial_{x_i}$ $(i = 1, 2, .., n)$ and $\triangle = \sum\limits_{i=1}^{n} \partial_i^2$; the fluxes $f_i(u)$ $(i = 1, 2, .., n)$ are smooth and $f_1''(u) \geq a_0$ for some constant $a_0 > 0$.

It is well known [11, 21] that the one-dimensional (1-d) conservation law, i.e. the right hand side of (1.1) is zero and $n = 1$, has rich wave phenomena including shocks and rarefaction waves. A centered rarefaction wave $u^R(x_1, t)$ is an entropy solution of the following Riemann problem

$$\begin{cases} \partial_t u^R(x_1,t) + \partial_1 f_1(u^R(x_1,t)) = 0, \\ u^R(x_1, 0) = \begin{cases} \bar{u}_l, \ x_1 < 0, \\ \bar{u}_r, \ x_1 > 0, \end{cases} \end{cases} \tag{1.2}$$

with $\bar{u}_l < \bar{u}_r$, and has an explicit formula as

$$u^R(x_1,t) = \begin{cases} \bar{u}_l, & x_1 \leq f_1'(\bar{u}_l)t, \\ (f_1')^{-1}(\frac{x_1}{t}), & f_1'(\bar{u}_l)t < x_1 \leq f_1'(\bar{u}_r)t, \\ \bar{u}_r, & x_1 > f_1'(\bar{u}_r)t. \end{cases} \tag{1.3}$$

It is shown in [7, 12, 24, 10] that the rarefaction wave given in (1.3) is asymptotically stable for both the inviscid and viscous conservation laws in L^2 framework provided

*Received April 16, 2020; accepted for publication May 28, 2020.

†Academy of Mathematics and Systems Science, CAS, Beijing 100190, China (fhuang@amt.ac.cn); and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China. Feimin Huang is partially supported by NSFC Grant No. 11688101.

‡Corresponding Author. Academy of Mathematics and Systems Science, CAS, Beijing 100190, China (qyuan@amss.ac.cn). Qian Yuan is partially supported by the China Postdoctoral Science Foundation funded project (2019M660831).

that the initial perturbations are L^2 integrable. The analysis for the solutions of conservation laws with periodic initial data is important, cf. [11, 3, 1, 2, 25, 26, 27]. In particular, the large time behavior of the nonlinear waves under periodic perturbations for viscous conservation laws is firstly investigated in [25] with the aid of maximum principle. The 1-d rarefaction wave $u^R(x_1,t)$ becomes the planar rarefaction wave in the multi-dimensional (m-d) case. The main purpose of this paper is to use the energy method to extend the work of [25] to the m-d case, i.e., to show whether the planar rarefaction wave is stable or not under m-d periodic perturbations. Besides, we refer to [5, 7, 8, 9, 16, 17, 4, 13, 14, 6, 15] and the references therein for the other interesting works about the stability of shocks, rarefaction waves and contact discontinuities.

Now we formulate the main result. Since the centered rarefaction wave given in (1.3) is a weak solution and only Lipschitz continuous with respect to x_1, we need to construct a viscous version of rarefaction wave $\tilde{u}^R(x_1,t)$ to replace the original one. Following [17, 24], $\tilde{u}^R(x_1,t)$ can be constructed as a smooth solution of the viscous conservation law (1.1), i.e.,

$$\begin{cases} \partial_t \tilde{u}^R + \partial_1(f_1(\tilde{u}^R)) = \partial_1^2 \tilde{u}^R, \\ \tilde{u}^R(x_1,0) = \frac{\overline{u}_l+\overline{u}_r}{2} + \frac{\overline{u}_r-\overline{u}_l}{2}\frac{e^{x_1}-e^{-x_1}}{e^{x_1}+e^{-x_1}}. \end{cases} \tag{1.4}$$

It is straightforward to check that

$$\lim_{x_1\to-\infty} \tilde{u}^R(x_1,0) = \overline{u}_l < \overline{u}_r = \lim_{x_1\to+\infty} \tilde{u}^R(x_1,0).$$

Consider the scalar equation (1.1) with the following initial data

$$u(x,0) = u_0(x) = \tilde{u}^R(x_1,0) + w_0(x), \qquad x \in \mathbb{R}^n, \tag{1.5}$$

where $w_0(x)$ is a m-d periodic function defined on the n-d torus $\mathbb{T}^n = \prod_{i=1}^n [0,1]$. Without loss of generality (by adding the average constant onto $\overline{u}_l$ and $\overline{u}_r$, respectively), one can assume that

$$\int_{\mathbb{T}^n} w_0(x)dx = 0. \tag{1.6}$$

We are ready to state the main theorem.

THEOREM 1.1. *Assume that the periodic perturbation* $w_0(x) \in H^{[\frac{n}{2}]+2}(\mathbb{T}^n)$ *and satisfies (1.6).*

Then there exists a unique global smooth solution u *of (1.1), (1.5) satisfying*

$$\left\|u(x,t) - \tilde{u}^R(x_1,t)\right\|_{L^\infty(\mathbb{R}^n)} \le C(1+t)^{-\frac{1}{2}}, \quad t > 0, \tag{1.7}$$

where the constant $C > 0$ *is independent of* t.

REMARK 1.2. It is shown in [10] that any viscous rarefaction waves connecting same end states are time-asymptotically equivalent in the $L^\infty(\mathbb{R})$ space with the rate $t^{-\frac{1}{2}}$. We remark that the stability result (1.7) also holds true for more general initial values $\tilde{u}^R(x_1,0)+v_0(x_1)+w_0(x)$ instead of (1.5), where $v_0(x_1)$ is any 1-d function in $L^1(\mathbb{R}) \cap H^1(\mathbb{R})$.

REMARK 1.3. Theorem 1.1 is the first work concerning the m-d periodic perturbations around the nonlinear waves for conservation laws, which shows that the oscillations in all directions around the planar rarefaction wave decay to zero with the rate $t^{-\frac{1}{2}}$, even though the initial perturbations keep oscillating at infinity $|x| \to +\infty$. In other words, the oscillations are eliminated due to the genuine nonlinearity of the equation.

Let us outline the proof of Theorem 1.1. Motivated by [24] and [10] in which the planar rarefaction waves are shown to be stable under m-d perturbations for the scalar equation (1.1), we want to use the energy method to prove Theorem 1.1. However, the initial periodic perturbation $w_0(x)$ is not integrable on $\mathbb{R}^n$, and has no any limit at far fields. Thus, the effective energy method developed in the previous articles can not be applied here directly. To overcome this difficulty, we construct a suitable ansatz $\tilde{u}(x,t)$ which contains the oscillations in the x_1 direction so that the oscillations in the difference between the solution u and the ansatz $\tilde{u}$ is eliminated in the x_1 direction, i.e., $(u-\tilde{u})(x,t) \in L^1_{x_1}(\mathbb{R})$, This is the key point in our proof. Then the stability with time-decay rates (1.7) can be obtained by the L^p energy method developed in [10].

To prove Theorem 1.1, we establish a Gagliardo-Nirenberg type inequality in the domain $\Omega := \mathbb{R} \times \mathbb{T}^{n-1}$, which is the second novelty in this paper. The Gagliardo-Nirenberg (GN) inequality is very useful in the field of PDEs and usually holds in the whole space $\mathbb{R}^n$ or in the domain with zero Dirichlet boundary condition. Let us recall the GN inequality which reads as, for any $1 \leq p \leq +\infty$ and integer $0 \leq j < m$,

$$\left\|\nabla_x^j u\right\|_{L^p(\mathbb{R}^n)} \leq C \left\|\nabla_x^m u\right\|_{L^r(\mathbb{R}^n)}^{\theta} \left\|u\right\|_{L^q(\mathbb{R}^n)}^{1-\theta}, \tag{1.8}$$

where $\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n}\right)\theta + \frac{1}{q}(1-\theta)$, $\frac{j}{m} \leq \theta \leq 1$ and C is a constant independent of u. However, the n-d GN inequality (1.8) does not hold in the domain Ω generally owing to the following counterexample.

Counterexample. It is noted that any 1-d function $0 \neq f(x_1) \in C_c^\infty(\mathbb{R})$ is periodic in the x_i direction for $i = 2, \cdots, n$ (one cannot exclude the 1-d case since the initial perturbations satisfying (1.6) include the 1-d periodic functions). Then corresponding to the case $j = 0, p = \frac{n}{n-1}, \theta = m = r = 1$ in (1.8), which is exactly the n-d Sobolev inequality, one can let $\zeta_d(x_1) = f(d^{-1}x_1)$, $d > 0$, and a direct computation implies that

$$\left\|\zeta_d\right\|_{L^{\frac{n}{n-1}}(\Omega)} = C_{n,d} \left\|\nabla_x \zeta_d\right\|_{L^1(\Omega)}, \tag{1.9}$$

where $C_{n,d} := d^{\frac{n-1}{n}} \|f\|_{L^{\frac{n}{n-1}}(\mathbb{R})} / \|f'\|_{L^1(\mathbb{R})}$. Since $C_{n,d} \to +\infty$ as $d \to +\infty$, we conclude that in general the n-d GN inequality (1.8) is not true in Ω without additional conditions.

Instead, based on a function-decomposition in (3.3) below, we establish a Gagliardo-Nirenberg type inequality on the domain Ω.

Theorem 1.4 (GN type inequality on $\Omega = \mathbb{R} \times \mathbb{T}^{n-1}$). *Let $u \in L^q(\Omega)$ and $\nabla^m u \in L^r(\Omega)$ where $1 \leq q, r \leq +\infty$ and $m \geq 1$, and u is periodic in the x_i direction for $i = 2, \cdots, n$. Then there exists a decomposition $u(x) = \sum_{k=0}^{n-1} u^{(k)}(x)$ such that each $u^{(k)}$ satisfies the $k+1$-dimensional GN inequality (1.8), i.e.,*

$$\left\|\nabla^j u^{(k)}\right\|_{L^p(\Omega)} \leq C \left\|\nabla^m u\right\|_{L^r(\Omega)}^{\theta_k} \left\|u\right\|_{L^q(\Omega)}^{1-\theta_k}, \tag{1.10}$$

for any $0 \le j < m$ *and* $1 \le p \le +\infty$ *satisfying* $\frac{1}{p} = \frac{j}{k+1} + \left(\frac{1}{r} - \frac{m}{k+1}\right)\theta_k + \frac{1}{q}(1-\theta_k)$ *and* $\frac{j}{m} \le \theta_k \le 1$, *with the following exceptional cases,*

1) *if* $j = 0, rm < k+1$ *and* $q = +\infty$, *additional assumption that* $u \to 0$ *as* $|x_1| \to +\infty$ *is required;*
2) *if* $1 < r < +\infty$ *and* $m - j - \frac{k+1}{r}$ *is a non-negative integer, additional assumption that* $\theta_k < 1$ *is required.*

Hence, it holds that

$$\left\|\nabla^j u\right\|_{L^p(\Omega)} \le C \sum_{k=0}^{n-1} \left\|\nabla^m u\right\|_{L^r(\Omega)}^{\theta_k} \left\|u\right\|_{L^q(\Omega)}^{1-\theta_k}, \tag{1.11}$$

where the constant $C > 0$ *is independent of* u.

Remark 1.5. It is noted that the first term for $k = 0$ on the right-hand side (RHS) of (1.11) is necessary since u can be a 1-d function defined on $\mathbb{R}$. For example, considering the same function $\zeta_d(x_1)$ as in the counterexample, when corresponding to the case $j = 0, p = r = 2$ and $m = q = 1$ in (1.11), a direct computation gives that

$$\|\zeta_d\|_{L^2(\Omega)} = Cd^{\frac{1}{2}(3\theta-1)} \|\nabla\zeta_d\|_{L^2(\Omega)}^{\theta} \|\zeta_d\|_{L^1(\Omega)}^{1-\theta} \quad \text{for any } 0 \le \theta \le 1, \tag{1.12}$$

where $C > 0$ is independent of d. The estimate (1.11) can be true only for $\theta = \theta_0 = \frac{1}{3}$. Otherwise, one can let $d \to 0+$ if $\theta < \frac{1}{3}$ and $d \to +\infty$ if $\theta > \frac{1}{3}$, respectively to get the contradiction.

Remark 1.6. An interesting interpolation inequality involving $\nabla^2 u$ in the domain $\mathbb{R} \times \mathbb{T}^2$ was established in [23].

The rest of the paper is organized as follows. Some preliminaries on the viscous rarefaction waves and the construction of the ansatz are given in Section 2. Section 3 is devoted to the proof of Theorem 1.4 and an interpolation inequality, Corollary 3.5. In Section 4, we show the desired a priori estimates, Theorem 4.2; thereafter, the proof of Theorem 1.1 is completed.

2. Preliminaries and Ansatz. Some properties of the viscous rarefaction wave are listed as follows.

Lemma 2.1. *The viscous rarefaction wave* $\tilde{u}^R(x_1, t)$ *solving (1.4) satisfies*

$$\lim_{t \to +\infty} \left\|\left(\tilde{u}^R - u^R\right)(\cdot, t)\right\|_{L^\infty(\mathbb{R})} = 0. \tag{2.1}$$

Moreover, for any $t > 0$ *it holds that*

$$0 < \partial_1 \tilde{u}^R(x_1, t) \le \min\left\{\frac{C}{t}, \max_{x_1 \in \mathbb{R}} \left(u_0^R\right)'\right\} \qquad \forall x_1 \in \mathbb{R}, \tag{2.2}$$

$$\left\|\left(\tilde{u}^R(\cdot, t) - \overline{u}_l\right)\left(\tilde{u}^R(\cdot, t) - \overline{u}_r\right)\right\|_{L^1(\mathbb{R})} \le C(1+t), \tag{2.3}$$

$$\left\|\partial_1 \tilde{u}^R(\cdot, t)\right\|_{L^p(\mathbb{R})} \le Ct^{-1+\frac{1}{p}}, \qquad p \in [1, +\infty), \tag{2.4}$$

where the constant $C > 0$ *is independent of* t.

Proof. The L^∞ time-asymptotically equivalence (2.1) has been proved in [7, 10]. By the maximum principles for (1.4) and the equation of the derivative $v = \partial_1 \tilde{u}^R$,

$$\partial_t v - \partial_1^2 v + f_1''(\tilde{u}^R) v^2 + f_1'(\tilde{u}^R) \partial_1 v = 0, \tag{2.5}$$

one can get that $\sup_{x_1\in\mathbb{R},t\geq 0}\left|\tilde{u}^R(x_1,t)\right| \leq \left\|u_0^R\right\|_{L^\infty(\mathbb{R})}$ and $0 < \partial_1\tilde{u}^R(x_1,t) \leq \max_{x_1\in\mathbb{R}}\left(u_0^R\right)'(x_1)$. Besides, it was shown in [19] that the solution $\tilde{u}^R$ of the 1-d convex conservation law (1.4) satisfies the well-known Oleĭnik entropy condition, i.e., $\partial_1\tilde{u}^R(x_1,t) \leq \frac{C}{t}$ for all $x_1 \in \mathbb{R}$, where $C > 0$ is independent of t.

Since $\overline{u}_l < \tilde{u}^R < \overline{u}_r$ and both $\tilde{u}^R$ and $\partial_1\tilde{u}^R$ are uniformly bounded, it follows from (1.4) that

$$\frac{d}{dt}\left[\int_{-\infty}^0 \left(\tilde{u}^R(x_1,t)-\overline{u}_l\right)dx_1 + \int_0^{+\infty}\left(\overline{u}_r-\tilde{u}^R(x_1,t)\right)dx_1\right] \leq C, \quad t>0,$$

which implies that

$$\int_{-\infty}^0 \left(\tilde{u}^R(x_1,t)-\overline{u}_l\right)dx_1 + \int_0^{+\infty}\left(\overline{u}_r-\tilde{u}^R(x_1,t)\right)dx_1 \leq C(1+t), \quad t>0,$$

and thus (2.3) holds true. And (2.4) can follow from

$$\left\|\partial_1\tilde{u}^R\right\|_{L^p(\mathbb{R})}^p \leq \left\|\partial_1\tilde{u}^R\right\|_{L^\infty(\mathbb{R})}^{p-1}\int_{\mathbb{R}}\partial_1\tilde{u}^R(x_1,t)dx_1 \leq Ct^{-p+1}.$$

□

Now we construct the ansatz. Set

$$g(x_1,t) := \frac{\tilde{u}^R(x_1,t)-\overline{u}_l}{\overline{u}_r-\overline{u}_l}, \quad x_1\in\mathbb{R},\ t\geq 0, \tag{2.6}$$

which is smooth, belongs to the interval $(0,1)$ and satisfies $\partial_1 g(x_1,t) > 0$ for any $x_1\in\mathbb{R}, t\geq 0$.

Let $u_l(x,t)$ and $u_r(x,t)$ denote the two periodic solutions of (1.1) (see [20] for the global existence) with the respective periodic data

$$u_l(x,0) = \overline{u}_l + w_0(x) \quad \text{and} \quad u_r(x,0) = \overline{u}_r + w_0(x), \quad x\in\mathbb{R}^n. \tag{2.7}$$

And define

$$w_i(x,t) := u_i(x,t) - \overline{u}_i, \quad i = l \text{ or } r, \tag{2.8}$$

which is a periodic function with zero average for any $t \geq 0$. Then we have

Lemma 2.2. *If $w_0(x)\in H^{\left[\frac{n}{2}\right]+2}(\mathbb{T}^n)$ satisfies (1.6), then for $i = l$ or r, it holds that*

$$\|w_i(\cdot,t)\|_{W^{1,\infty}(\mathbb{R}^n)} \leq C\|w_0\|_{H^{\left[\frac{n}{2}\right]+2}(\mathbb{T}^n)} e^{-2\alpha t}, \quad t\geq 0, \tag{2.9}$$

where the constants $\alpha > 0$ and $C > 0$ are independent of t.

Proof. The proof of Lemma 2.2 is based on basic L^2 energy estimates and the Poincáre inequality on $\mathbb{T}^n$, which is standard and thus omitted. □

We are ready to construct the ansatz as follows:

$$\tilde{u}(x,t) := u_l(x,t)\left(1-g(x_1,t)\right) + u_r(x,t)g(x_1,t), \tag{2.10}$$

which is periodic in the x_i direction for $i = 2, \cdots, n$. By direct calculations, the source term induced by the ansatz $\tilde{u}(x,t)$ is given as follows.

$$\begin{aligned} h(x,t) := \ & \partial_t \tilde{u} + \sum_{i=1}^{n} \partial_i \left(f_i(\tilde{u}(x,t)) \right) - \triangle \tilde{u}(x,t) \\ = \ & (u_r - u_l)\, g(1-g) \sum_{i=1}^{n} \left[\sigma_i(u_l, \tilde{u}) \partial_i w_l - \sigma_i(u_r, \tilde{u}) \partial_i w_r \right] \\ & + (u_r - u_l)\, \sigma_1(\tilde{u}^R, \tilde{u}) \left(\tilde{u} - \tilde{u}^R \right) \partial_1 g - 2\partial_1 \left(w_r - w_l \right) \partial_1 g, \end{aligned} \tag{2.11}$$

where $\sigma_i(u,v) := \int_0^1 f_i''(v + \theta(u-v))d\theta,\ i = 1, \cdots, n$. Thus, the source term $h(x,t)$ is also periodic in the x_i direction for $i = 2, \cdots, n$.

Lemma 2.3. *Under the assumptions of Lemma 2.2, it holds that for any $p \in [1, +\infty]$,*

$$\|h(\cdot,t)\|_{L^p(\Omega)} \le C \|w_0\|_{H^{[\frac{n}{2}]+2}(\mathbb{T}^n)} e^{-\alpha t}, \quad t \ge 0, \tag{2.12}$$

where $\alpha > 0$ is the constant in Lemma 2.2, and $C > 0$ is independent of t.

Proof. By (2.3), (2.4) and Lemma 2.2, it is straightforward to check that (2.12) holds for $p = 1$ and $+\infty$. And for $p \in (1, +\infty)$, (2.12) follows from the interpolation $\|h\|_{L^p} \le \|h\|_{L^\infty}^{1-\frac{1}{p}} \|h\|_{L^1}^{\frac{1}{p}}$. □

3. Interpolation inequality on $\Omega = \mathbb{R} \times \mathbb{T}^{n-1}$. Although the n-d GN inequality (1.8) does not hold in the domain $\Omega = \mathbb{R} \times \mathbb{T}^{n-1}$ generally, it is true with an additional condition, that is,

Lemma 3.1. *Let $u \in L^q(\Omega)$, and its derivatives of order m, $\nabla^m u \in L^r(\Omega)$, where $m \ge 1$ and $1 \le r, q \le +\infty$. And assume that u is periodic in the x_i direction and satisfies*

$$\int_{\mathbb{T}} u(x) dx_i = 0 \qquad \textit{for all } i = 2, 3, \cdots, n. \tag{3.1}$$

Then the n-d GN inequality (1.8) holds true for u.

Proof. Following [18, Lecture II], it suffices to prove the following two extreme cases.

1) For a.e. $x \in \Omega$, it holds that

$$|u(x)|^n \le \int_{\mathbb{R}} |\partial_1 u|\, dx_1 \prod_{i=2}^{n} \int_{\mathbb{T}} |\partial_i u|\, dx_i. \tag{3.2}$$

2) For any $1 \le i \le n$, and $1 \le q < +\infty, 1 < r < +\infty$, it holds that

$$\int |\partial_i u|^p\, dx_i \le C \left(\int \left|\partial_i^2 u\right|^r dx_i \right)^{\frac{p}{2r}} \left(\int |u|^q\, dx_i \right)^{\frac{p}{2q}}, \tag{3.3}$$

where $\frac{2}{p} = \frac{1}{r} + \frac{1}{q}$, and "$\int$" indicates the integral over $\mathbb{R}$ if $i = 1$ or $\mathbb{T}$ if $i > 1$.

Thanks to (3.1), it is straightforward to show (3.2). And (3.3) for $i = 1$ has been verified in [18]. It remains to show this inequality on $\mathbb{T} = [0, 1]$ for $i > 1$. In fact, for any bounded interval $I \subset \mathbb{R}$, it follows from [18, (2.7)] that

$$\int_I |\partial_i u|^p \, dx_i \le C\, |I|^{1+p-\frac{p}{r}} \left(\int_I \left|\partial_i^2 u\right|^r dx_i \right)^{\frac{p}{r}} + C\, |I|^{-\left(1+p-\frac{p}{r}\right)} \left(\int_I |u|^q \, dx_i \right)^{\frac{p}{q}}, \quad (3.4)$$

where $C > 0$ is independent of I. Since u is periodic in the x_i direction with period 1, the first term on the RHS of (3.4) must be greater than the second one, as long as the length of I is equal to a large integer N. Similar to [18, (2.8)], one can cover $[0, N]$ by finite intervals $I_j \subset [0, 2N]$, at each of which, the first term on the RHS of (3.4) (where $I = I_j$) is either greater than or equal to the second one. The proof is to repeat the one in [18]. However, to make this paper complete, we still give the details here. First let $k > 0$ be large enough (but fixed), and consider $I = (0, \frac{N}{k})$ in (3.4). If the first term on the RHS is greater than the second one, set $I_1 = (0, \frac{N}{k})$, and it holds that

$$\int_{I_1} |\partial_i u|^p \, dx_i \le C \left(\frac{N}{k}\right)^{1+p-\frac{p}{r}} \left(\int_{I_1} \left|\partial_i^2 u\right|^r dx_i \right)^{\frac{p}{r}}, \quad (3.5)$$

where the constant $C > 0$ is independent of either k or N. Otherwise, if the second term on the RHS of (3.4) is greater, then due to the choice of N, one can extend the interval $(0, \frac{N}{k})$ to a larger one, $I_1 \subset [0, 2N]$, until the two terms become equal. Then one has that

$$\int_{I_1} |\partial_i u|^p \, dx_i \le C \left(\int_{I_1} \left|\partial_i^2 u\right|^r dx_i \right)^{\frac{p}{2r}} \left(\int_{I_1} |u|^q \, dx_i \right)^{\frac{p}{2q}}, \quad (3.6)$$

where the constant $C > 0$ is independent of either k or N. Starting at the end point of I_1, one can repeat this process to set $I_2, I_3, \cdots$, until $[0, N]$ is covered. For fixed k, there must be finitely many such intervals I_j, each of which is contained in $[0, 2N]$. Then summing the estimates (3.5) and (3.6) together yields that

$$\begin{aligned} \int_0^N |\partial_i u|^p \, dx_i \le\ & Ck \left(\frac{N}{k}\right)^{1+p-\frac{p}{r}} \left(\int_0^N \left|\partial_i^2 u\right|^r dx_i \right)^{\frac{p}{r}} \\ & + C \left(\int_0^{2N} \left|\partial_i^2 u\right|^r dx_i \right)^{\frac{p}{2r}} \left(\int_0^{2N} |u|^q \, dx_i \right)^{\frac{p}{2q}}. \end{aligned}$$

Letting $k \to +\infty$, one has that

$$\int_0^N |\partial_i u|^p \, dx_i \le C \left(\int_0^{2N} \left|\partial_i^2 u\right|^r dx_i \right)^{\frac{p}{2r}} \left(\int_0^{2N} |u|^q \, dx_i \right)^{\frac{p}{2q}},$$

which yields (3.3) due to the fact that u is periodic in x_i direction and $\frac{p}{2r} + \frac{p}{2q} = 1$. □

REMARK 3.2. Lemma 3.1 means that the GN inequality (1.8) holds under the additional condition (3.1). Nevertheless, it is difficult to verify (3.1) for the perturbation $u(x, t) - \tilde{u}(x, t)$ for any x_i direction ($i = 2, \cdots, n$) even though it holds initially, i.e., $\int_{\mathbb{T}} w_0(x) dx_i = 0 \ \forall i = 2, \cdots, n$, which is much stronger than the original condition (1.6).

To prove Theorem 1.1, we shall establish a GN type inequality on the domain $\mathbb{R}\times\mathbb{T}^{n-1}$ without any additional condition by applying Lemma 3.1 and the following decomposition lemma.

For any $k\ge 1$, set

$$\begin{aligned}\mathcal{A}_k=\Big\{&u(x)\text{ is measurable on }\mathbb{R}\times\mathbb{T}^k\text{ with }x_1\in\mathbb{R},\text{ and periodic in }x_i\in\mathbb{T},\\&\text{satisfying }\int_{\mathbb{T}}u(x)dx_i\equiv 0\text{ for all }i=2,\cdots,k+1\Big\}.\end{aligned}\tag{3.7}$$

Then for any function $u\in\mathcal{A}_k$, the $k+1$-dimensional GN inequality (1.8) holds true. We are ready to decompose general function u in terms of $\mathcal{A}_k$, i.e.,

LEMMA 3.3 (Decomposition Lemma). *One can decompose $u(x)$ as follows,*

$$u(x)=\sum_{k=0}^{n-1}u^{(k)}(x)\quad a.e.\ x\in\Omega,\tag{3.8}$$

where

$$\begin{aligned}&u^{(0)}(x)=u^{(0)}(x_1)=\int_{\mathbb{T}^{n-1}}u(x_1,x_2,\cdots,x_n)dx_2\cdots dx_n,\\&u^{(k)}(x)=\sum_{2\le i_1<\cdots<i_k\le n}u_{i_1,\cdots,i_k}(x_1,x_{i_1},\cdots,x_{i_k}),\quad k=1,\cdots,n-1,\end{aligned}\tag{3.9}$$

where each $u_{i_1,\cdots,i_k}(x_1,x_{i_1},\cdots,x_{i_k})\in\mathcal{A}_k$. Moreover, for any $m\ge 0$ and $1\le p\le+\infty$, it holds that

$$\left\|\nabla_x^m u^{(0)}\right\|_{L^p(\Omega)}+\sum_{k=1}^{n-1}\sum_{2\le i_1<\cdots<i_k\le n}\|\nabla_x^m u_{i_1,\cdots,i_k}\|_{L^p(\Omega)}\le C\left\|\nabla_x^m u\right\|_{L^p(\Omega)},\tag{3.10}$$

where $C>0$ is independent of u.

REMARK 3.4. To avoid excessive words, we omit the assumptions in Lemma 3.3 that the integrals (3.9) and

$$\int_{\mathbb{T}^{n-1-k}}u(x)\frac{dx_2\cdots dx_n}{dx_{i_1}\cdots dx_{i_k}},\quad 1\le k\le n-1,\quad 2\le i_1<\cdots<i_k\le n,$$

should exist, here and hereafter we use $\int_{\mathbb{T}^{n-1-k}}(\cdot)\frac{dx_2\cdots dx_n}{dx_{i_1}\cdots dx_{i_k}}$ to denote the integral that is integrated with respect to $x_2,\cdots,x_n$ except for $x_{i_1},\cdots,x_{i_k}$.

Proof. Starting with $u^{(0)}$ defined in (3.9), set

$$u_i(x_1,x_i)=\int_{\mathbb{T}^{n-2}}\left(u-u^{(0)}\right)\frac{dx_2\cdots dx_n}{dx_i},\quad i=2,\cdots,n.\tag{3.11}$$

It is straightforward to check that $u_i(x_1,x_i)\in\mathcal{A}_1$. Define $u^{(1)}(x):=\sum_{i=2}^n u_i$. Assume that for all $l=1,2,\cdots,k-1$ with $k\ge 2$ and $2\le j_1<\cdots<j_l\le n$, all the functions

$u_{j_1,\cdots,j_l} = u_{j_1,\cdots,j_l}(x_1, x_{j_1}, \cdots, x_{j_l}) \in \mathcal{A}_l$ and thus $u^{(l)} := \sum\limits_{2\le j_1<\cdots<j_l\le n} u_{j_1,\cdots,j_l}$ have been well-defined. Then for any fixed sequence $2 \le i_1 < \cdots < i_k \le n$, set

$$\begin{aligned}
&u_{i_1,\cdots,i_k}(x_1, x_{i_1}, \cdots, x_{i_k}) \\
:=& \int_{\mathbb{T}^{n-1-k}} \Big(u - \sum_{l=0}^{k-1} u^{(l)}\Big) \frac{dx_2\cdots dx_n}{dx_{i_1}\cdots dx_{i_k}} \\
=& \int_{\mathbb{T}^{n-1-k}} \Big(u - \sum_{l=0}^{k-2} u^{(l)} - \sum_{2\le j_1<\cdots<j_{k-1}\le n} u_{j_1,\cdots,j_{k-1}}\Big) \frac{dx_2\cdots dx_n}{dx_{i_1}\cdots dx_{i_k}}.
\end{aligned} \tag{3.12}$$

It is noted that for $r = 1, \cdots, k$, one has

$$\int_{\mathbb{T}} u_{j_1,\cdots,j_{k-1}} dx_{i_r} = \begin{cases} 0 & \text{if } i_r \in \{j_1, \cdots, j_{k-1}\}, \\ u_{j_1,\cdots,j_{k-1}} & \text{if } i_r \notin \{j_1, \cdots, j_{k-1}\}. \end{cases} \tag{3.13}$$

Then it holds that

$$\begin{aligned}
\int_{\mathbb{T}} u_{i_1,\cdots,i_k} dx_{i_r} =& \int_{\mathbb{T}^{n-k}} \Big(u - \sum_{l=0}^{k-2} u^{(l)}\Big) \frac{dx_2\cdots dx_n}{dx_{i_1}\cdots dx_{i_{r-1}} dx_{i_{r+1}}\cdots dx_{i_k}} \\
&- \sum_{2\le j_1<\cdots<j_{k-1}\le n} \int_{\mathbb{T}^{n-k}} u_{j_1,\cdots,j_{k-1}} \frac{dx_2\cdots dx_n}{dx_{i_1}\cdots dx_{i_{r-1}} dx_{i_{r+1}}\cdots dx_{i_k}} \\
=& \; u_{i_1,\cdots,i_{r-1},i_{r+1},\cdots,i_k} - u_{i_1,\cdots,i_{r-1},i_{r+1},\cdots,i_k} \\
=& \; 0,
\end{aligned}$$

which means $u_{i_1,\cdots,i_k} \in \mathcal{A}_k$ and then the decomposition (3.8) for function u holds. It remains to show (3.10). It follows from (3.9) and Minkowski inequality that

$$\begin{aligned}
\left\|\nabla_x^m u^{(0)}\right\|_{L^p(\mathbb{R})} &\le \left\| \|\nabla_x^m u\|_{L^1(\mathbb{T}^{n-1};dx_2\cdots dx_n)} \right\|_{L^p(\mathbb{R};dx_1)} \\
&\le \left\| \|\nabla_x^m u\|_{L^p(\mathbb{R};dx_1)} \right\|_{L^1(\mathbb{T}^{n-1};dx_2\cdots dx_n)} \\
&\le \|\nabla_x^m u\|_{L^p(\Omega)},
\end{aligned}$$

where the last inequality is derived from the Hölder inequality. Similarly, one can obtain from (3.11) that for any $2 \le i \le n$,

$$\|\nabla_x^m u_i\| \le \left\|\nabla_x^m \left(u - u^{(0)}\right)\right\|_{L^p(\Omega)} \le 2 \|\nabla_x^m u\|_{L^p(\Omega)}.$$

The remaining functions $u_{i_1,\cdots,i_k}$ defined in (3.13) can be proved in the same way, and then the proof of Lemma 3.3 is completed. □

Proof of Theorem 1.4. We first decompose $u(x) = \sum\limits_{k=0}^{n-1} u^{(k)}(x)$ as in Lemma 3.3. Then it follows from Lemma 3.1 and Lemma 3.3 that $u^{(0)} \in L^q(\mathbb{R}), \nabla^m u^{(0)} \in L^r(\mathbb{R})$, and it satisfies the 1-d GN inequality (1.8); each $u_{i_1,\cdots,i_k} \in L^q(\Omega), \nabla^m u_{i_1,\cdots,i_k} \in L^r(\Omega)$

and it satisfies the $(k+1)$-d GN inequality (1.8). Hence,

$$\begin{aligned}\left\|\nabla^j u^{(0)}\right\|_{L^p(\mathbb{R})} &\leq C\left\|\nabla^m u^{(0)}\right\|_{L^r(\mathbb{R})}^{\theta_0}\left\|u^{(0)}\right\|_{L^q(\mathbb{R})}^{1-\theta_0} \leq C\left\|\nabla^m u\right\|_{L^r(\Omega)}^{\theta_0}\|u\|_{L^q(\Omega)}^{1-\theta_0};\\ \left\|\nabla^j u^{(k)}\right\|_{L^p(\Omega)} &\leq \sum_{2\leq i_1<\cdots<i_k\leq n}\left\|\nabla^j u_{i_1,\cdots,i_k}\right\|_{L^p(\Omega)}\\ &\leq C\sum_{2\leq i_1<\cdots<i_k\leq n}\left\|\nabla^m u_{i_1,\cdots,i_k}\right\|_{L^r(\Omega)}^{\theta_k}\left\|u_{i_1,\cdots,i_k}\right\|_{L^q(\Omega)}^{1-\theta_k}\\ &\leq C\left\|\nabla^m u\right\|_{L^r(\Omega)}^{\theta_k}\|u\|_{L^q(\Omega)}^{1-\theta_k}, \qquad 1\leq k\leq n-1,\end{aligned}$$

where the indices $\theta_0, \cdots, \theta_{n-1}$ are that introduced in Theorem 1.4. The proof is finished. □

In order to use the L^p energy method developed in [10], we shall establish the following interpolation inequality by Theorem 1.4.

Corollary 3.5 (Interpolation Inequality in Ω). *For any $2 \leq p < \infty$ and $1 \leq q \leq p$, it holds that*

$$\|u\|_{L^p(\Omega)} \leq C\sum_{k=0}^{n-1}\left\|\nabla(|u|^{\frac{p}{2}})\right\|_{L^2(\Omega)}^{\frac{2\gamma_k}{1+\gamma_k p}}\|u\|_{L^q(\Omega)}^{\frac{1}{1+\gamma_k p}}, \tag{3.14}$$

where $\gamma_k = \frac{k+1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)$ and the constant $C = C(p,q,n) > 0$ is independent of u.

Proof. For the function $v(x) = |u(x)|^{\frac{p}{2}}$, it follows from Theorem 1.4 that

$$\|v\|_{L^2(\Omega)} \leq C\sum_{k=0}^{n-1}\|\nabla v\|_{L^2(\Omega)}^{\theta_k}\|v\|_{L^{2r_k}(\Omega)}^{1-\theta_k},$$

where $\frac{1}{2} = \left(\frac{1}{2}-\frac{1}{k+1}\right)\theta_k + \frac{1}{2r_k}(1-\theta_k)$ with $\frac{1}{2} \leq r_k \leq 1$ and $0 \leq \theta_k \leq 1$ for $k = 0, 1, \cdots, n-1$. This yields that

$$\|u\|_{L^p(\Omega)} \leq C\sum_{k=0}^{n-1}\left\|\nabla|u|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{\frac{2}{p}\theta_k}\|u\|_{L^{pr_k}(\Omega)}^{1-\theta_k}. \tag{3.15}$$

If $2 \leq p \leq 2q$, choosing $r_k = \frac{q}{p} \geq \frac{1}{2}$ for $k = 0, \cdots, n-1$ can finish the proof. If $p > 2q$, let $\frac{1}{2} < r_k < 1$, which will be determined later. Since $q < pr_k < p$, it follows from interpolation that

$$\|u\|_{L^{pr_k}(\Omega)} \leq \|u\|_{L^q(\Omega)}^{1-\rho_k}\|u\|_{L^p(\Omega)}^{\rho_k}, \tag{3.16}$$

where $\rho_k = \frac{pr_k - q}{(p-q)r_k} \in (0,1)$. Plugging (3.16) into (3.15) yields that

$$\|u\|_{L^p(\Omega)} \leq C\sum_{k=0}^{n-1}\left\|\nabla|u|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{\frac{2}{p}\theta_k}\|u\|_{L^q(\Omega)}^{(1-\rho_k)(1-\theta_k)}\|u\|_{L^p(\Omega)}^{\rho_k(1-\theta_k)}. \tag{3.17}$$

For $k = 0, 1, \cdots, n-1$ with $n \geq 2$ and $r_k \in \left(\frac{1}{2}, 1\right)$, one can obtain by simple calculations that

$$\rho_k(1-\theta_k) = \frac{p}{p-q}\times\frac{r_k - \frac{q}{p}}{\frac{k+1}{2}-\frac{k-1}{2}r_k} \quad \in \left(\frac{2(p-2q)}{(k+3)(p-q)}, 1\right). \tag{3.18}$$

Thus, for any $\delta \in \left(\frac{2(p-2q)}{3(p-q)}, 1\right)$, there exist $r_0, r_1, \cdots, r_{n-1} \in \left(\frac{1}{2}, 1\right)$ such that $\rho_k(1-\theta_k) = \delta$ for any $k = 0, 1, \cdots, n-1$. Then (3.17) yields that

$$\|u\|_{L^p(\Omega)} \leq C \sum_{k=0}^{n-1} \left\| \nabla |u|^{\frac{p}{2}} \right\|_{L^2(\Omega)}^{\frac{2\theta_k}{p(1-\delta)}} \|u\|_{L^q(\Omega)}^{\frac{(1-\rho_k)(1-\theta_k)}{1-\delta}}.$$

A direct calculation implies that $\frac{2\theta_k}{p(1-\delta)} = \frac{2\gamma_k}{1+\gamma_k p}$ and $\frac{(1-\rho_k)(1-\theta_k)}{1-\delta} = \frac{1}{1+\gamma_k p}$ for all $k = 0, \cdots, n-1$, where $\gamma_k = \frac{k+1}{2}\left(\frac{1}{q} - \frac{1}{p}\right)$, and the proof is completed. □

LEMMA 3.6 ([10] Lemma 2.2). *For any $2 \leq p < \infty$, it holds that*

$$\|\partial_i u\|_{L^p(\Omega)} \leq C \left\| \partial_i \left(|\partial_i u|^{\frac{p}{2}} \right) \right\|_{L^2(\Omega)}^{\frac{2}{p+2}} \|u\|_{L^p(\Omega)}^{\frac{2}{p+2}}, \quad i = 1, \cdots, n, \tag{3.19}$$

where the constant $C > 0$ is independent of u.

Proof. The inequality (3.19) has been established in [10] for the whole space $\mathbb{R}^n$, and it is still true in the domain Ω with the aid of integration by parts and the Hölder inequality as in [10]. □

4. A priori estimates and proof. Denote the perturbation by $\phi(x,t) := u(x,t) - \tilde{u}(x,t)$, which is periodic in the x_i direction for $i = 2, \cdots, n$, and satisfies

$$\partial_t \phi + \sum_{i=1}^{n} \partial_i \left[f_i(\tilde{u} + \phi) - f_i(\tilde{u}) \right] = \triangle \phi - h, \tag{4.1}$$

$$\phi(x, 0) = 0. \tag{4.2}$$

REMARK 4.1. In fact, given the initial data $u_0(x) = \tilde{u}^R(x_1, 0) + v_0(x_1) + w_0(x)$ stated in Remark 1.2, the initial condition (4.2) turns to $\phi(x,0) = v_0(x_1) \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$ instead, which makes no difference in the following proof.

We shall prove the global existence and large time behavior of solution $\phi(x,t)$ to the Cauchy problem (4.1) and (4.2). The global existence can be established by obtaining the a priori estimates (4.3) and (4.4) below, since the local existence of the solution ϕ to (4.1) with the initial data in $L^1(\Omega) \cap H^1(\Omega)$ is standard.

THEOREM 4.2 (A priori estimates). *Assume that $\phi(x,t)$ is the unique smooth solution to* (4.1),(4.2) *for any $t \in [0,T]$, then it holds that*

$$\|\phi(t)\|_{L^p(\Omega)} \leq C_p (1+t)^{-\frac{1}{2}+\frac{1}{2p}} \qquad \forall p \in [1, +\infty), \tag{4.3}$$

$$\|\nabla \phi(t)\|_{L^p(\Omega)} \leq C_p (1+t)^{-1+\frac{1}{2p}}, \qquad \forall p \in [2, +\infty), \tag{4.4}$$

where the constants $C > 0$ and $C_p > 0$ are independent of t.

Proof. The uniform bound of $\|\phi(t)\|_{L^\infty(\Omega)}$ follows from the maximum principle easily. And following the L^p energy method as in [10], we first prove (4.3).

Step 1. We first show the L^1 estimates. Given $\delta > 0$, let $S_\delta(\eta)$ be a C^2 convex approximation to the function $|\eta|$, e.g.

$$S_\delta(\eta) = \begin{cases} -\eta, & \eta \leq -\delta; \\ -\frac{\eta^4}{8\delta^3} + \frac{3\eta^2}{4\delta} + \frac{3\delta}{8}, & -\delta < \eta \leq \delta; \\ \eta, & \eta > \delta. \end{cases}$$

Multiplying $S'_\delta(\phi)$ on both sides of (4.1) yields that

$$\begin{aligned}
&\partial_t S_\delta(\phi) + S''_\delta(\phi)\,|\nabla\phi|^2 + \int_0^\phi S''_\delta(\eta)\,(f'_1(\tilde u+\eta) - f'_1(\tilde u))\,d\eta\,\partial_1\tilde u^R \\
= &\sum_{i=1}^n \partial_i\{\cdots\} - \int_0^\phi S''_\delta(\eta)\,(f'_1(\tilde u+\eta) - f'_1(\tilde u))\,d\eta\,\partial_1\left(\tilde u - \tilde u^R\right) \\
&-\sum_{i=2}^n \int_0^\phi S''_\delta(\eta)\,(f'_i(\tilde u+\eta) - f'_i(\tilde u))\,d\eta\,\partial_i\tilde u - S'_\delta(\phi)h,
\end{aligned} \tag{4.5}$$

where

$$\{\cdots\} = S'_\delta(\phi)\partial_i\phi - S'_\delta(\phi)\,(f_i(\tilde u+\phi) - f_i(\tilde u)) + \int_0^\phi S''_\delta(\eta)\,(f_i(\tilde u+\eta) - f_i(\tilde u))\,d\eta.$$

Since $S''_\delta \geq 0, f''_1 > 0, \partial_1\tilde u^R > 0$ and $|\phi| \leq S_\delta(\phi)$, integrating (4.5) over Ω, together with Lemmas 2.2 and 2.3, gives that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega S_\delta(\phi)dx \leq\ & Ce^{-\alpha t}\int_\Omega \sum_{i=1}^n \left|\int_0^\phi S''_\delta(\eta)\,(f'_i(\tilde u+\eta) - f'_i(\tilde u))\,d\eta\right| dx + C\,\|h\|_{L^1(\Omega)} \\
\leq\ & Ce^{-\alpha t}\int_\Omega \left|\int_0^\phi S''_\delta(\eta)\,|\eta|\,d\eta\right| dx + Ce^{-\alpha t} \\
\leq\ & Ce^{-\alpha t}\,\|\phi\|_{L^1(\Omega)} + Ce^{-\alpha t} \\
\leq\ & Ce^{-\alpha t}\int_\Omega S_\delta(\phi)dx + Ce^{-\alpha t},
\end{aligned}$$

where $C > 0$ is independent of δ. Then by the Gronwall inequality and letting $\delta \to 0+$, one has that

$$\|\phi(t)\|_{L^1(\Omega)} \leq C. \tag{4.6}$$

For $p \in [2, +\infty)$, multiplying (4.1) by $|\phi|^{p-2}\phi$ yields that

$$\begin{aligned}
&\frac{1}{p}\partial_t|\phi|^p + \frac{4(p-1)}{p^2}\left|\nabla|\phi|^{\frac{p}{2}}\right|^2 + (p-1)\int_0^\phi (f'_1(\tilde u+\eta) - f'_1(\tilde u))\,|\eta|^{p-2}\mathrm{d}\eta\,\partial_1\tilde u^R \\
= &\sum_{i=1}^n \partial_i\{\cdots\} - (p-1)\int_0^\phi (f'_1(\tilde u+\eta) - f'_1(\tilde u))|\eta|^{p-2}\mathrm{d}\eta\,\partial_1\left(\tilde u - \tilde u^R\right) \\
&-(p-1)\sum_{i=2}^n \int_0^\phi (f'_i(\tilde u+\eta) - f'_i(\tilde u))|\eta|^{p-2}\mathrm{d}\eta\,\partial_i\tilde u - |\phi|^{p-2}\phi h,
\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
\{\cdots\} =\ & |\phi|^{p-2}\phi\partial_i\phi - (f_i(\tilde u+\phi) - f_i(\tilde u))\,|\phi|^{p-2}\phi \\
&+ (p-1)\int_0^\phi (f_i(\tilde u+\eta) - f_i(\tilde u))\,|\eta|^{p-2}\mathrm{d}\eta.
\end{aligned}$$

Since

$$(p-1)\int_0^\phi (f'_1(\tilde u+\eta) - f'_1(\tilde u))|\eta|^{p-2}\mathrm{d}\eta\,\partial_1\tilde u^R \geq C\partial_1\tilde u^R\,|\phi|^p \geq 0,$$

then integrating (4.7) over Ω, together with Lemma 2.3, gives that

$$\begin{aligned}
&\frac{d}{dt}\|\phi\|_{L^p(\Omega)}^p+\left\|\nabla|\phi|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2+\left\|\left(\partial_1\tilde{u}^R\right)^{\frac{1}{p}}\phi\right\|_{L^p(\Omega)}^p\\
\le\;& Ce^{-\alpha t}\|\phi\|_{L^p(\Omega)}^p+C\left(\|\phi\|_{L^p(\Omega)}^p\right)^{1-\frac{1}{p}}\|h\|_{L^p(\Omega)}\\
\le\;& Ce^{-\alpha t}\|\phi\|_{L^p(\Omega)}^p+Ce^{-\alpha t}.
\end{aligned}\tag{4.8}$$

Multiplying (4.8) by $(1+t)^\beta$, where $\beta>\frac{n}{2}(p-1)$ is a constant, then integrating the resulting equation over $(0,T)$ yields that

$$\begin{aligned}
&(1+T)^\beta\|\phi(T)\|_{L^p(\Omega)}^p+\int_0^T(1+t)^\beta\left\|\nabla|\phi|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2dt\\
\le\;& C\int_0^T e^{-\alpha t}(1+t)^\beta\|\phi\|_{L^p(\Omega)}^p\,dt+C+C\int_0^T(1+t)^{\beta-1}\|\phi\|_{L^p(\Omega)}^p\,dt\\
\le\;& C+C\int_0^T(1+t)^{\beta-1}\|\phi\|_{L^p(\Omega)}^p\,dt.
\end{aligned}\tag{4.9}$$

It follows from Corollary 3.5 and (4.6) that

$$\begin{aligned}
&C\int_0^T(1+t)^{\beta-1}\|\phi\|_{L^p(\Omega)}^p\,dt\\
\le\;& C\sum_{k=0}^{n-1}\int_0^T(1+t)^{\beta-1}\left\|\nabla|\phi|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{\frac{2\gamma_k p}{1+\gamma_k p}}\|\phi\|_{L^1(\Omega)}^{\frac{p}{1+\gamma_k p}}dt\\
\le\;& \frac{1}{2}\int_0^T(1+t)^\beta\left\|\nabla|\phi|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2dt+C(1+T)^{\beta-\gamma_0 p},
\end{aligned}\tag{4.10}$$

where $\gamma_k=\frac{k+1}{2}\left(1-\frac{1}{p}\right)$. This, together with (4.9), yields that

$$(1+T)^\beta\|\phi(T)\|_{L^p(\Omega)}^p+\int_0^T(1+t)^\beta\left\|\nabla|\phi|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2dt\le C+C(1+T)^{\beta-\frac{p-1}{2}}.$$

Thus (4.3) holds for $p\in[2,+\infty)$. The case for $p\in(1,2)$ follows from the interpolation.

Step 2. We now prove (4.4). Let $\psi_i:=\partial_i\phi$. Taking the derivative on (4.1) with respect to x_i yields that

$$\partial_t\psi_i+\sum_{j=1}^n\partial_j\left(f_j'(\tilde{u})\psi_i\right)+\sum_{j=1}^n\partial_j\left[\left(f_j'(\tilde{u}+\phi)-f_j'(\tilde{u})\right)(\partial_i\tilde{u}+\psi_i)\right]=\triangle\psi_i-\partial_i h.$$

Multiplying the result by $|\psi_i|^{p-2}\psi_i$, we arrive at

$$\begin{aligned}
&\frac{1}{p}\partial_t|\psi_i|^p+\frac{4(p-1)}{p^2}\left|\nabla|\psi_i|^{\frac{p}{2}}\right|^2+\frac{p-1}{p}f_1''(\tilde{u})\partial_1\tilde{u}^R|\psi_i|^p\\
=\;& \sum_{j=1}^n\partial_j\{\cdots\}-\frac{p-1}{p}f_1''(\tilde{u})\partial_1\left(\tilde{u}-\tilde{u}^R\right)|\psi_i|^p-\frac{p-1}{p}\sum_{j=2}^n f_j''(\tilde{u})\partial_j\tilde{u}|\psi_i|^p\\
&+(p-1)\sum_{j=1}^n\left(f_j'(\tilde{u}+\phi)-f_j'(\tilde{u})\right)(\partial_i\tilde{u}+\psi_i)|\psi_i|^{p-2}\partial_j\psi_i\\
&-\partial_i\left(|\psi_i|^{p-2}\psi_i h\right)+(p-1)|\psi_i|^{p-2}\partial_i\psi_i h,
\end{aligned}\tag{4.11}$$

where

$$\{\cdots\} = |\psi_i|^{p-2}\psi_i\partial_j\psi_i - \frac{1}{p}f_j'(\tilde{u})|\psi_i|^p - \left(f_j'(\tilde{u}+\phi) - f_j'(\tilde{u})\right)(\partial_i\tilde{u}+\psi_i)|\psi_i|^{p-2}\psi_i.$$

Then integrating ((4.11)) over Ω and using Lemma 2.3 yield that

$$\begin{aligned} &\frac{d}{dt}\|\psi_i\|_{L^p(\Omega)}^p + \left\|\nabla|\psi_i|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 + \left\|\left(\partial_1\tilde{u}^R\right)^{\frac{1}{p}}\psi_i\right\|_{L^p(\Omega)}^p \\ \le\ & Ce^{-\alpha t}\|\psi_i\|_{L^p(\Omega)}^p + \underbrace{C\int_\Omega |\partial_i\tilde{u}||\phi||\psi_i|^{p-2}|\nabla\psi_i|\,dx}_{I_1} \\ & + \underbrace{C\int_\Omega |\phi||\psi_i|^{p-1}|\nabla\psi_i|\,dx}_{I_2} + \underbrace{C\int_\Omega |\psi_i|^{p-2}|\nabla\psi_i||h|\,dx}_{I_3}. \end{aligned} \tag{4.12}$$

First, it follows from Lemma 2.3 that

$$\begin{aligned} I_3 &= C\int_\Omega \left|\nabla|\psi_i|^{\frac{p}{2}}\right|\left(e^{-\alpha t}|\psi_i|^p\right)^{\frac{p-2}{2p}} e^{\frac{p-2}{2p}\alpha t}|h|\,dx \\ &\le \frac{1}{8}\left\|\nabla|\psi_i|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2 + Ce^{-\alpha t}\|\psi_i\|_{L^p(\Omega)}^p + Ce^{-\alpha\left(\frac{p}{2}+1\right)t}, \end{aligned} \tag{4.13}$$

where the Hölder inequality for $\frac{1}{2}+\frac{p-2}{2p}+\frac{1}{p}=1$ is used. Following [10], one can claim that

$$\int_\Omega |\phi||\psi_i|^{\frac{p}{2}}\left|\nabla|\psi_i|^{\frac{p}{2}}\right|dx \le C\sum_{k=0}^{n-1}\|\phi\|_{L^{2(k+1)}(\Omega)}\|\psi_i\|_{L^p(\Omega)}^{\frac{p}{4}}\left\|\nabla|\psi_i|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{\frac{3}{2}}. \tag{4.14}$$

In fact, we first decompose $v := |\psi_i|^{\frac{p}{2}} = \sum_{k=0}^{n-1} v^{(k)}$ as in Theorem 1.4, then

$$\begin{aligned} &\int_\Omega |\phi||\psi_i|^{\frac{p}{2}}\left|\nabla|\psi_i|^{\frac{p}{2}}\right|dx \le \sum_{k=0}^{n-1}\int_\Omega |\phi|\left|v^{(k)}\right||\nabla v|\,dx \\ \le\ & \|\phi\|_{L^2(\Omega)}\left\|v^{(0)}\right\|_{L^\infty(\mathbb{R})}\|\nabla v\|_{L^2(\Omega)} + \|\phi\|_{L^4(\Omega)}\left\|v^{(1)}\right\|_{L^4(\Omega)}\|\nabla v\|_{L^2(\Omega)} \\ & + \sum_{k=2}^{n-1}\|\phi\|_{L^{2(k+1)}(\Omega)}\left\|\left|v^{(k)}\right|^{\frac{1}{2}}\right\|_{L^4(\Omega)}\left\|\left|v^{(k)}\right|^{\frac{1}{2}}\right\|_{L^{\frac{4(k+1)}{k-1}}(\Omega)}\|\nabla v\|_{L^2(\Omega)} \\ \le\ & \sum_{k=0}^{n-1}\|\phi\|_{L^{2(k+1)}(\Omega)}\|v\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla v\|_{L^2(\Omega)}^{\frac{3}{2}}, \end{aligned}$$

which yields (4.14). Then combining (4.3) and (4.14), the term I_2 in (4.12) satisfies that

$$\begin{aligned} I_2 &\le C(1+t)^{-\frac{1}{4}}\|\psi_i\|_{L^p(\Omega)}^{\frac{p}{4}}\left\|\nabla|\psi_i|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{\frac{3}{2}} \\ &\le C(1+t)^{-1}\|\psi_i\|_{L^p(\Omega)}^p + \frac{1}{8}\left\|\nabla|\psi_i|^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2. \end{aligned} \tag{4.15}$$

For I_1, if $i \neq 1$, it follows from the Hölder inequality and (4.3) that

$$\begin{aligned} I_1 &\le Ce^{-\alpha t}\int_\Omega |\phi|\,|\psi_i|^{\frac{p-2}{2}}\,|\psi_i|^{\frac{p-2}{2}}\,|\nabla\psi_i|\,dx \\ &\le Ce^{-\alpha t}\,\|\phi\|_{L^p(\Omega)}\,\|\psi_i\|_{L^p(\Omega)}\left\|\nabla|\psi_i|^{\frac{p}{2}}\right\|_{L^2(\Omega)} \\ &\le Ce^{-\alpha t}+Ce^{-\alpha t}\,\|\psi_i\|^p_{L^p(\Omega)}+\frac{1}{8}\left\|\nabla|\psi_i|^{\frac{p}{2}}\right\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.16}$$

If $i = 1$, then

$$I_1 \le\ C\int_\Omega \partial_1\tilde{u}^R\,|\phi|\,|\psi_1|^{p-2}\,|\nabla\psi_1|\,dx + Ce^{-\alpha t}\int_\Omega |\phi|\,|\psi_1|^{p-2}\,|\nabla\psi_1|\,dx, \tag{4.17}$$

where the second term on the RHS of (4.17) can be estimated in the same way as in (4.16). For the first term, one can use Lemma 3.6 to obtain that

$$\begin{aligned} \int_\Omega \partial_1\tilde{u}^R\,|\phi|\,|\psi_1|^{p-2}\,|\nabla\psi_1|\,dx &\le C\left\|\partial_1\tilde{u}^R\right\|_{L^\infty(\Omega)}\|\phi\|^{\frac{2p}{p+2}}_{L^p(\Omega)}\left\|\nabla|\psi_1|^{\frac{p}{2}}\right\|^{\frac{2p}{p+2}}_{L^2(\Omega)} \\ &\le C\left\|\partial_1\tilde{u}^R\right\|^{\frac{p+2}{2}}_{L^\infty(\Omega)}\|\phi\|^p_{L^p(\Omega)}+\frac{1}{8}\left\|\nabla|\psi_1|^{\frac{p}{2}}\right\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.18}$$

Thus, collecting equations (4.13) and (4.15) to (4.18) and applying (2.2), one has that

$$\begin{aligned} &\frac{d}{dt}\|\psi_i\|^p_{L^p(\Omega)}+\left\|\nabla|\psi_i|^{\frac{p}{2}}\right\|^2_{L^2(\Omega)} \\ &\le Ce^{-\alpha t}+C(1+t)^{-1}\|\psi_i\|^p_{L^p(\Omega)}+C(1+t)^{-\frac{p+2}{2}}\|\phi\|^p_{L^p(\Omega)}. \end{aligned} \tag{4.19}$$

Multiplying (4.19) by $(1+t)^\beta$ with $\beta > p+1$ and then integrating the result over $[0, T]$, one has that

$$\begin{aligned} &(1+t)^\beta\,\|\psi_i(t)\|^p_{L^p(\Omega)}+\int_0^T(1+t)^\beta\left\|\nabla|\psi_i|^{\frac{p}{2}}\right\|^2_{L^2(\Omega)}dt \\ \le\ & C+C\int_0^T(1+t)^{\beta-1}\|\psi_i(t)\|^p_{L^p(\Omega)}\,dt+C\int_0^T(1+t)^{\beta-\frac{p+2}{2}}\|\phi\|^p_{L^p(\Omega)}\,dt. \end{aligned} \tag{4.20}$$

It follows from Lemma 3.6 that

$$\begin{aligned} &\int_0^T(1+t)^{\beta-1}\|\psi_i(t)\|^p_{L^p(\Omega)}\,dt \\ \le\ & C\int_0^T(1+t)^{\beta-1}\left\|\nabla|\psi_i|^{\frac{p}{2}}\right\|^{\frac{2p}{p+2}}_{L^2(\Omega)}\|\phi\|^{\frac{2p}{p+2}}_{L^p(\Omega)}\,dt \\ \le\ & \frac{1}{2}\int_0^T(1+t)^\beta\left\|\nabla|\psi_i|^{\frac{p}{2}}\right\|^2_{L^2(\Omega)}dt+C\int_0^T(1+t)^{\beta-\frac{p+2}{2}}\|\phi\|^p_{L^p(\Omega)}\,dt. \end{aligned}$$

From (4.3), one can get that

$$\begin{aligned} \int_0^T(1+t)^{\beta-\frac{p+2}{2}}\|\phi(t)\|^p_{L^p(\Omega)}\,dt &\le C\int_0^T(1+t)^{\beta-\frac{p+2}{2}}(1+t)^{-\frac{p-1}{2}}\,dt \\ &\le C(1+T)^{\beta-p+\frac{1}{2}}. \end{aligned}$$

The proof of (4.2) is finished. □

Proof of Theorem 1.1. It follows from Theorems Theorem 1.4 and (4.2) that

$$\begin{aligned}\|\phi\|_{L^\infty(\Omega)} &\leq C\sum_{k=0}^{n-1}\|\nabla\phi\|_{L^{p_k}(\Omega)}^{\theta_k}\|\phi\|_{L^{q_k}(\Omega)}^{1-\theta_k}\\ &\leq C\sum_{k=0}^{n-1}(1+t)^{-\left[\left(1-\frac{1}{2p_k}\right)\theta_k+\left(\frac{1}{2}-\frac{1}{2q_k}\right)(1-\theta_k)\right]},\end{aligned}$$

where $0=\left(\frac{1}{p_k}-\frac{1}{k+1}\right)\theta_k+\frac{1}{q_k}(1-\theta_k)$, $\max\{k+1,2\}<p_k<+\infty$ and $1\leq q_k<+\infty$ for $k=0,1,\cdots,n-1$. A direct calculation yields that

$$\left(1-\frac{1}{2p_k}\right)\theta_k+\left(\frac{1}{2}-\frac{1}{2q_k}\right)(1-\theta_k)=\frac{1}{2}+\frac{k\theta_k}{2(k+1)}\geq\frac{1}{2},$$

which implies that

$$\|\phi\|_{L^\infty(\mathbb{R}^n)}=\|\phi\|_{L^\infty(\Omega)}\leq C(1+t)^{-\frac{1}{2}}.$$

Thus,

$$\left\|u-\tilde{u}^R\right\|_{L^\infty(\mathbb{R}^n)}\leq\|\phi\|_{L^\infty(\mathbb{R}^n)}+\left\|\tilde{u}-\tilde{u}^R\right\|_{L^\infty(\mathbb{R}^n)}\leq C(1+t)^{-\frac{1}{2}}.$$

□

REFERENCES

[1] C. M. Dafermos, *Large time behavior of periodic solutions of hyperbolic systems of conservation laws*, Journal of Differential Equations, 121:1 (1995), pp. 183–202.

[2] C. M. Dafermos, *Long time behavior of periodic solutions to scalar conservation laws in several space dimensions*, SIAM J. Math. Anal., 45:4 (2013), pp. 2064–2070.

[3] J. Glimm and P. Lax, *Decay of solutions of systems of nonlinear hyperbolic conservation laws*, Memoirs of the American Mathematical Society, No. 101, American Mathematical Society, Providence, R.I., 1970.

[4] J. Goodman, *Nonlinear asymptotic stability of viscous shock profiles for conservation laws*, Arch. Ration. Mech. Anal., 95:4 (1986), pp. 325–344.

[5] L. He and F. Huang, *Nonlinear stability of large amplitude viscous shock wave for general viscous gas*, J. Differential Equations, 269:2 (2020), pp. 1226–1242. .

[6] F. Huang, Z. Xin and T. Yang, *Contact discontinuities with general perturbation for gas motion*, Adv. Math., 219 (2008), pp. 1246–1297.

[7] A. M. Ilin and O. A. Oleinik, *Asymptotic behavior of solutions of the cauchy problem for some quasi-linear equations for large values of the time*, Mat. Sb. (N.S.), 51:93 (1960), pp. 91–216.

[8] S. Kawashima and A. Matsumura, *Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion*, Comm. Math. Phys., 101:1 (1985), pp. 97–127.

[9] S. Kawashima, A. Matsumura and K. Nishihara, *Asymptotic behavior of solutions for the equations of a viscous heat-conductive gas*, Proc. Japan Acad. Ser. A Math. Sci., 62:7 (1986), pp. 249–252.

[10] S. Kawashima, S. Nishibata and M. Nishikawa, *L^p energy method for multi- dimensional viscous conservation laws and application to the stability of planar waves*, J. Hyperbolic Differ. Equations, 1:3 (2004), pp. 581–603.

[11] P. Lax, *Hyperbolic systems of conservation laws II*, Communications on Pure and Applied Mathematics, 10:4 (1957), pp. 537–566.

[12] T. Liu, *Invariants and asymptotic behavior of solutions of a conservation law*, Proceedings of the American Mathematical Society, 71:2 (1978), 227.

[13] T. Liu and Z. Xin, *Nonlinear Stability of Rarefaction Waves for Compressible Navier Stokes Equations*, Communications in Mathematical Physics, 118 (1988), pp. 451–465.

[14] T. Liu and Z. Xin, *Pointwise decay to contact discontinuities for systems of viscous conservation laws*, Asian J. Math., 1 (1997), pp. 34–84.

[15] T. Liu and Y. Zeng, *Time-asymptotic behavior of wave propagation around a viscous shock profile*, Comm. Math. Phys., 290:1 (2009), pp. 23–82.

[16] A. Matsumura and K. Nishihara, *On the stability of traveling wave solutions of a one-dimensional model system for compressible viscous gas*, Japan J. Appl. Math., 2 (1985), pp. 17–25.

[17] A. Matsumura and K. Nishihara, *Asymptotics toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas*, Japan Journal of Applied Mathematics, 3:1 (1986), pp. 1–13.

[18] L. Nirenberg, *On elliptic partial differential equations*, Ann. della Sc. Norm. Super. di Pisa, Cl. di Sci., 2:13 (1959), pp. 115–162.

[19] O. A. Oleinik, *Discontinuous solutions of non-linear differential equations*, Uspehi Mat. Nauk (N.S.), 12:3(75) (1957), pp. 3–73.

[20] D. Serre, *L^1-stability of nonlinear waves in scalar conservation laws*, Handbook of Differential Equations: Evolutionary Equations, (2002), pp. 473–553.

[21] J. Smoller, *Shock Waves and Reaction Diffusion Equations*, vol. 258, Springer-Verlag, New York-Berlin, 1994.

[22] A. Szepessy and Z. Xin, *Nonlinear stability of viscous shock waves*, Arch. Ration. Mech. Anal., 122 (1993), pp. 53–103.

[23] T. Wang and Y. Wang, *Nonlinear stability of planar rarefaction wave to the three-dimensional Boltzmann equation*, Kinetic and Related Models, 12:3 (2019), pp. 637–679.

[24] Z. Xin, *Asymptotic stability of planar rarefaction waves for viscous conservation laws in several dimensions*, Trans. Amer. Math. Soc., 319:2 (1990), pp. 805–820.

[25] Z. Xin, Q. Yuan and Y. Yuan, *Asymptotic stability of shock profiles and rarefaction waves under periodic perturbations for 1-D convex scalar viscous conservation laws*, arxiv:1902.09772 (2019), pp. 1–42.

[26] Z. Xin, Q. Yuan and Y. Yuan, *Asymptotic stability of shock waves and rarefaction waves under periodic perturbations for 1-d convex scalar conservation laws*, SIAM J. Math. Anal., 51:4 (2019), pp. 2971–2994.

[27] Q. Yuan and Y. Yuan, *On Riemann solutions under different initial periodic perturbations at two infinities for 1-d scalar convex conservation laws*, J. Differ. Equ., (2019), pp. 1–15.