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ON A CLASS OF DEGENERATE AND SINGULAR MONGE-AMPÈRE EQUATIONS*

HUAIYU JIAN†, YOU LI‡, AND XUSHAN TU§

Dedicated to Professor Ling Hsiao for her 80th birthday

Abstract. In this paper we shall prove the existence, uniqueness and global Hölder continuity for the Dirichlet problem of a class of Monge-Ampère type equations which may be degenerate and singular on the boundary of convex domains. We will establish a relation of the Hölder exponent for the solutions with the convexity for the domains.

Key words. existence, uniqueness, global regularity, degenerate, singular, Monge-Ampère equation.

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1. Introduction. In this paper we study the Monge-Ampère type equation

$$\begin{aligned} \det D^2 u = F(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded convex domain in R^n $(n \geq 2)$, and F satisfies the following (1.2)-(1.3):

$$F(x,t) \in C(\Omega \times (-\infty, 0)) \text{ is non-decreasing in } t \text{ for any } x \in \Omega; \tag{1.2}$$

$$\begin{aligned} &\text{there are constants } \ A > 0,\ \alpha \geq 0,\ \beta \geq n+1 \ \text{ such that} \\ &0 < F(x,t) \leq A d_x^{\beta-n-1} |t|^{-\alpha} \quad \forall (x,t) \in \Omega \times (-\infty, 0), \end{aligned} \tag{1.3}$$

where $d_x = dist(x, \partial\Omega)$. Obviously, this problem is singular and degenerate at the boundary of the domain.

The particular case of problem (1.1) includes a few geometric problems. When $F = |t|^{-(n+2)}$ and u is a solution to problem (1.1), then the Legendre transform of u is a complete affine hyperbolic sphere [3, 4, 6, 11, 16], and $(-u)^{-1} \sum u_{x_i x_j} dx_i dx_j$ gives the Hilbert metric (Poincare metric) in the convex domain Ω [21]. When $F = f(x,u)|t|^{-p}$, problem (1.1) may be obtained from L_p-Minkowski problem [13, 14, 22] and the Minkowski problem in centro-affine geometry[7, 15]. Also see p.440-441 in [17]. Besides, problem (1.1) can be applied to construct non-homogeneous complete Einstein-Kähler metrics on a tubular domain [4, 5].

Cheng and Yau in [4] proved that if Ω is a strictly convex C^2-domain and $F \in C^k$ $(k \geq 3)$ satisfies (1.2)-(1.3), then problem (1.1) admits an unique convex generalized solution $u \in C(\bar{\Omega})$. Moreover, $u \in C^{k+1,\varepsilon}(\Omega) \bigcap C^{\gamma}(\bar{\Omega})$ for any $\varepsilon \in (0,1)$ and some

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†Department of Mathematical Sciences, Tsinghua University, China (hjian@tsinghua.edu.cn).
‡School of Mathematics and statistics, Beijing Technology and Business University, China (thu3141@126.com).
§Department of Mathematical Sciences, Tsinghua University, China (txs17@mails.tsinghua.edu.cn).

$\gamma = C(\beta, \alpha, A, n, \partial\Omega) \in (0, 1)$. We should emphasize that their methods need the strict convexity and the smoothness of Ω, and the differentiability of F.

In this paper we find that the global Hölder regularity for problem (1.1) is independent of the smoothness of Ω and F, and the Hölder exponent depends only on the convexity of the domain. As a result, we can remove the smoothness of Ω as well as the differentiability of F in [4]. Moreover, using the concept of (a, η) type introduced in [11] to describe the convexity of the domain, we obtain a relation of the Hölder exponent for u with the convexity for Ω.

We have noticed that there are many papers on global regularity for equations of Monge-Ampère type. See, for example, [2, 8, 10, 20, 24, 25, 27] and the references therein. But, generally speaking, those results require that the domain Ω should be strictly convex and $\partial\Omega \in C^{1,1}$.

Our first result is stated as the following

THEOREM 1.1. *Supposed that Ω is a bounded convex domain in R^n and $F(x,t)$ satisfies (1.2)-(1.3). Let*

$$\gamma_1 := \begin{cases} \frac{\beta-n+1}{n+\alpha}, & \text{if} \beta < \alpha + 2n - 1, \\ \textit{any number in}(0,1), & \text{if} \beta \geq \alpha + 2n - 1. \end{cases} \tag{1.4}$$

Then problem (1.1) admits an unique convex generalized solution $u \in C^{\gamma_1}(\overline{\Omega})$. Furthermore, $u \in C^{2,\gamma_1}(\Omega)$ if $F(x,t) \in C^{0,1}(\Omega \times (-\infty, 0))$.

Here a generalized solution means the well-known Alexandrov solution. See, for example, [8, 9, 26] for the details.

To improve the regularity for the solution obtained in Theorem 1.1, we use the (a, η) type in [11] to describe the convexity of Ω. From now on, we denote

$$x = (x_1, x_2, \ldots, x_n) = (x', x_n), \quad x' = (x_1, ..., x_{n-1})$$

and

$$|x'| = \sqrt{x_1^2 + ... + x_{n-1}^2}.$$

DEFINITION 1.1. *Supposed that Ω is a bounded convex domain in R^n, and $x_0 \in \partial\Omega$. x_0 is called to be (a, η) type if there are numbers $a \in [1, +\infty)$ and $\eta > 0$, after translation and rotation transforms, we have*

$$x_0 = 0 \quad \textit{and} \quad \Omega \subseteq \{x \in R^n | x_n \geq \eta |x'|^a\}.$$

Ω is called (a, η) type domain if every point of $\partial\Omega$ is (a, η) type.

REMARK 1.1. The convexity requires that the number a should be no less than 1. The less is a, the more convex is the domain. There is no (a, η) type domain for $a \in [1, 2)$, although part of $\partial\Omega$ may be (a, η) type point for $a \in [1, 2)$.

DEFINITION 1.2. *We say that a domain Ω in R^n satisfies exterior (or interior) sphere condition with radius R if for each $x_0 \in \partial\Omega$, there is a $B_R(y_0) \supseteq \Omega$ (or $B_R(y_0) \subseteq \Omega$, respectively) such that $\partial B_R(y_0) \bigcap \partial\Omega \ni x_0$.*

In [11], we have proved that $(2, \eta)$ type domain is equivalent to the domain satisfying exterior sphere condition.

The following two theorems show the relation of the Hölder exponent for u on $\bar{\Omega}$ with the convexity for Ω.

THEOREM 1.2. *Supposed that Ω is (a, η) type domain in R^n with $a \in (2, +\infty)$, and F satisfies (1.2)-(1.3). Let*

$$\gamma_2 := \begin{cases} \frac{\beta-n+1}{n+\alpha} + \frac{2n-2}{a(n+\alpha)}, & \text{if} \beta < \alpha + 2n - 1 - \frac{2n-2}{a}, \\ \textit{any number in}(0,1), & \text{if} \beta \geq \alpha + 2n - 1 - \frac{2n-2}{a}. \end{cases} \tag{1.5}$$

Then the convex generalized solution to problem (1.1) satisfies

$$u \in C^{\gamma_2}(\overline{\Omega}). \tag{1.6}$$

Furthermore $u \in C^{2,\gamma_2}(\Omega)$ if $F(x,t) \in C^{0,1}(\Omega \times (-\infty, 0))$.

THEOREM 1.3. *Let Ω be a bounded convex domain in R^n and u be a convex generalized solution to problem (1.1).*

(i) Supposed that Ω satisfies exterior sphere condition and F satisfies (1.2)-(1.3). Let

$$\gamma_3 := \begin{cases} \frac{\beta}{n+\alpha}, & \text{if} \beta < \alpha + n, \\ \textit{any number in}(0,1), & \text{if} \alpha + n \leq \beta < \alpha + n + 1, \\ 1, & \text{if} \beta \geq \alpha + n + 1. \end{cases} \tag{1.7}$$

Then

$$u \in C^{\gamma_3}(\overline{\Omega}). \tag{1.8}$$

Furthermore $u \in C^{2,\gamma_3}(\Omega)$ if $F(x,t) \in C^{0,1}(\Omega \times (-\infty, 0))$.

(ii) If Ω satisfies interior sphere condition with radius R and F satisfies (1.2) and

$$A d_x^{\beta-n-1} |t|^{-\alpha} \leq F(x,t), \ \forall (x,t) \in \Omega \times (-\infty, 0) \tag{1.9}$$

for some constants $A > 0$, then

$$|u(y)| \geq C(d_y)^{\gamma_4}, \quad \forall y \in \Omega \tag{1.10}$$

for some constant $C = C(\beta, \alpha, A, n, R) > 0$, where

$$\gamma_4 := \frac{\beta}{n+\alpha} \in (0,1). \tag{1.11}$$

REMARK 1.2. The Hölder regularity result of Theorem 1.1 can be viewed as the limit case of Theorem 1.2 as $a \to \infty$. Theorem 1.3 (i) shows that Theorem 1.2 is true for $a = 2$, since a $(2, \eta)$ type domain is equivalent to that the domain satisfying exterior sphere condition.

In the following Sections 2, 3, and 4, we will prove Theorems 1.1, 1.2, and 1.3, respectively. Our method can be used to study the optimal boundary regularity for minimal graphs in hyperbolic space[12].

2. Proof of Theorem 1.1. We start at a primary result which is useful to prove that a convex function in Ω is Hölder continuous in $\bar{\Omega}$.

LEMMA 2.1. *Let Ω be a bounded convex domain and $u \in C(\overline{\Omega})$ be a convex function in Ω with $u|_{\partial\Omega} = 0$. If there are $\gamma \in (0, 1]$ and $M > 0$ such that*

$$|u(x)| \leq M {d_x}^{\gamma}, \quad \forall x \in \Omega, \tag{2.1}$$

then $u \in C^{\gamma}(\overline{\Omega})$ and

$$|u|_{C^{\gamma}(\overline{\Omega})} \leq M\{1 + [diam(\Omega)]^{\gamma}\}.$$

Proof. This was proved in [11]. Here we copy the arguments for the convenience.

For any two point x_1, $x_2 \in \Omega$, consider the line determined by x_1 and x_2. The line will intersect $\partial\Omega$ at two points y_1 and y_2. Without loss of generality we assume that the four points are y_1, x_1, x_2, y_2 in order. By restricted onto the line, u is one dimension convex function. By the monotonic proposition of convex functions, we have

$$|u(x_2) - u(x_1)| \leq \max\{|u(y_1 + (x_2 - x_1)) - u(y_1)|,\ |u(y_2) - u(y_2 - (x_2 - x_1))|\}.$$

Moreover, since $y_1 \in \partial\Omega$, by the assumption (2.1) we have

$$\begin{aligned} |u(y_1 + (x_2 - x_1)) - u(y_1)| &= |u(y_1 + (x_2 - x_1))| \\ &\leq M\{dist(y_1 + x_2 - x_1, \partial\Omega)\}^{\gamma} \\ &\leq M|x_2 - x_1|^{\gamma}. \end{aligned}$$

Similarly,

$$|u(y_2) - u(y_2 - (x_2 - x_1))| \leq M|x_2 - x_1|^{\gamma}.$$

The above three inequalities, together with (2.1), implies the desired result. □

To prove Theorem 1.1, we need a priori estimate result as follows, which holds without strictly convexity of Ω or any smoothness of Ω and of F.

LEMMA 2.2. *Supposed that Ω is a bounded convex domain in $\mathbb{R}^n$ and $F(x,t)$ satisfies (1.2) and (1.3). If u is a convex generalized solution to problem (1.1), then $u \in C^{\gamma_1}(\overline{\Omega})$ and*

$$|u|_{C^{\gamma_1}(\overline{\Omega})} \leq C(\alpha,\ \beta,\ A,\ diam(\Omega),\ n), \tag{2.2}$$

where γ_1 is given by (1.4).

Proof. First, we may assume

$$\beta < \alpha + 2n - 1. \tag{2.3}$$

Since for the case $\beta \geq \alpha + 2n - 1$, we take a $\hat{\beta} < \alpha + 2n - 1$ such that $\frac{\hat{\beta} - n + 1}{n + \alpha}$ can be any number in $(0, 1)$ (Note $n \geq 2$). Obviously, (1.3) still holds in the case that β is replaced by $\hat{\beta}$. Hence, this case is reduced to the case (2.3).

Next, we assume for the time being that

$$0 \in \overline{\Omega} \subseteq R^n_+. \tag{2.4}$$

Then we are going to construct a sub-solution to problem (1.1).

For brevity, write $l = diam(\Omega)$. Set

$$W = -Mx_n^\gamma \cdot \sqrt{N^2l^2 - r^2}$$

where $r = \sqrt{x_1^2 + ... + x_{n-1}^2}$. We will choose positive constants γ, M, N such that W is an sub-solution to problem (1.1) under the assumptions (2.3) and (2.4).

For $i, j \in \{1, 2, ..., n-1\}$, write $W_i = \frac{\partial W}{\partial x_i}, W_{ij} = \frac{\partial^2 W}{\partial x_i \partial x_j}$. Then we have

$$\begin{aligned}
W_i &= Mx_n^\gamma \cdot \frac{x_i}{\sqrt{N^2l^2 - r^2}}, \\
W_{ij} &= Mx_n^\gamma \cdot \frac{1}{\sqrt{N^2l^2 - r^2}}(\delta_{ij} + \frac{x_ix_j}{N^2l^2 - r^2}), \\
W_n &= -M\gamma x_n^{\gamma-1} \cdot \sqrt{N^2l^2 - r^2}, \\
W_{in} &= M\gamma x_n^{\gamma-1} \cdot \frac{x_i}{\sqrt{N^2l^2 - r^2}}, \\
W_{nn} &= M\gamma(1-\gamma)x_n^{\gamma-2} \cdot \sqrt{N^2l^2 - r^2}.
\end{aligned}$$

Denote

$$D^2W := \begin{pmatrix} G & \xi \\ \xi^T & W_{nn} \end{pmatrix}$$

where $\xi^T = (W_{n1}, ..., W_{n(n-1)})$, and G is the $(n-1)$-order matrix. Then

$$detD^2W = detG \cdot (W_{nn} - \xi^T G^{-1}\xi).$$

Since all the eigenvalues of G are

$$Mx_n^\gamma \frac{1}{\sqrt{N^2l^2 - r^2}}, ..., \quad Mx_n^\gamma \frac{1}{\sqrt{N^2l^2 - r^2}}, \quad Mx_n^\gamma \frac{N^2l^2}{(N^2l^2 - r^2)\sqrt{N^2l^2 - r^2}},$$

$$detG = M^{n-1}N^2l^2x_n^{(n-1)\gamma} \cdot (\frac{1}{\sqrt{N^2l^2 - r^2}})^{n+1}.$$

It is direct to verify that

$$G\xi = \frac{N^2l^2Mx_n^\gamma}{(N^2l^2 - r^2)^{\frac{3}{2}}}\xi.$$

It follows that

$$\begin{aligned}
\xi^T G^{-1}\xi &= \frac{(N^2l^2 - r^2)^{\frac{3}{2}}}{N^2l^2Mx_n^\gamma}|\xi|^2 \\
&= \frac{M\gamma^2}{N^2l^2}x_n^{\gamma-2}r^2\sqrt{N^2l^2 - r^2}.
\end{aligned}$$

Hence, we obtain that

$$\begin{aligned} detD^2W &= detG(W_{nn} - \xi^T G^{-1}\xi) \\ &= M^{n-1}N^2l^2x_n^{(n-1)\gamma}(\frac{1}{\sqrt{N^2l^2-r^2}})^{n+1}M\gamma x_n^{\gamma-2}\sqrt{N^2l^2-r^2} \\ &\cdot [1-(1+\frac{r^2}{N^2l^2})\gamma] \\ &= M^nN^2l^2\gamma x_n^{n\gamma-2}(\frac{1}{\sqrt{N^2l^2-r^2}})^n[1-(1+\frac{r^2}{N^2l^2})\gamma]. \end{aligned} \tag{2.5}$$

We want to prove

$$detD^2W \geq F(x,W) \quad \text{in} \quad \Omega. \tag{2.6}$$

Since (1.3) and (2.4) imply that

$$F(x,W) \leq Ad_x^{\beta-n-1}|W|^{-\alpha} \leq Ax_n^{\beta-n-1}|W|^{-\alpha},$$

we see that (2.6) can be deduced from

$$detD^2W \geq Ax_n^{\beta-n-1}|W|^{-\alpha} \quad \text{in} \quad \Omega, \tag{2.7}$$

which is equivalent to

$$detD^2W \cdot \frac{1}{A}x_n^{n+1-\beta}|W|^{\alpha} \geq 1 \quad \text{in} \quad \Omega. \tag{2.8}$$

By (2.5), (2.8) is nothing but

$$\frac{1}{A}M^{n+\alpha}N^2l^2\gamma x_n^{(n+\alpha)\gamma-(\beta-n+1)}[1-(1+\frac{r^2}{N^2l^2})\gamma]\cdot(\sqrt{N^2l^2-r^2}\ )^{\alpha-n} \geq 1 \quad \text{in} \quad \Omega. \tag{2.9}$$

Now we choose $\gamma = \frac{\beta-n+1}{n+\alpha}$ such that

$$(n+\alpha)\gamma - (\beta-n+1) = 0.$$

Since $\gamma \in (0,1)$ by (2.3) and $r = |x'| \leq diam(\Omega) = l$ in Ω, we first take $N = C(\gamma)$ large enough such that

$$1-(1+\frac{r^2}{N^2l^2})\gamma > 0.$$

Noting $N^2l^2 - r^2 \in [(N^2-1)l^2,\ N^2l^2]$, we then take $M = C(A,\alpha,\gamma,N,n,l)$ large enough such that

$$\frac{1}{A}M^{n+\alpha}N^2l^2\gamma x_n^{(n+\alpha)\gamma-(\beta-n+1)}[1-(1+\frac{r^2}{N^2l^2})\gamma]\cdot(\sqrt{N^2l^2-r^2}\ )^{\alpha-n} \geq 1,$$

which is (2.9) and thus we have proved (2.6).

Finally, for any point $y \in \Omega$, letting $z \in \partial\Omega$ be the nearest boundary point to y, by some translations and rotations, we assume $z = 0$, $\Omega \subseteq R_+^n$ and the line yz is the $x_n - axis$. This is to say that (2.4) is satisfied. Therefore we have (2.6). Obviously,

$W \leq 0$ on $\overline{\Omega}$. Hence, W is a sub-solution to problem (1.1). By comparison principle for generalized solutions (see [8, 9, 26] for example), we have

$$|u(y)| \leq |W(y)| \leq MNly_n^{\frac{\beta-n+1}{n+\alpha}} = MNld_y^{\frac{\beta-n+1}{n+\alpha}},$$

which, together with Lemma 2.1, implies the desired result (2.2).

Note that we have used the fact that problem (1.1) is invariant under translation and rotation transforms, since $detD^2u$ is invariant and $F(x, u)$ is transformed to the one satisfying the same condition as F. This fact will be again used a few times in the following. □

Proof of Theorem 1.1. We prove the theorem by three steps.

Step 1. Suppose that Ω is bounded convex but $F(x, t) \in C^k(\Omega \times (-\infty, 0))$ $(k \geq 3)$ satisfies (1.2) and (1.3).

We choose a sequence of bounded and strictly convex domains $\{\Omega_i\}$ such that

$$\Omega_i \in C^2 \quad \text{and} \quad \Omega_i \subseteq \Omega_{i+1}, i = 1, 2, \cdots, \quad \bigcup_{i=1}^{\infty} \Omega_i = \Omega. \tag{2.10}$$

Then by Theorem 5 in [4], there exists a convex generalized solution u_i to problem (1.1) in the domain Ω_i for each i. We assume $u_i(x) = 0$ for all $x \in R^n \setminus \Omega_i$. By Lemma 2.2, We have the uniform estimations

$$|u_i|_{C^{\frac{\beta-n+1}{n+\alpha}}(\overline{\Omega})} = |u_i|_{C^{\frac{\beta-n+1}{n+\alpha}}(\overline{\Omega_i})} \leq C(\alpha,\ \beta,\ A,\ diam(\Omega),\ n), \tag{2.11}$$

which implies that there is a subsequence, still denoted by itself, convergent to a u in the space $C(\overline{\Omega})$. Moreover, by (2.11) again, we have

$$|u|_{C^{\frac{\beta-n+1}{n+\alpha}}(\overline{\Omega})} \leq C(\alpha,\ \beta,\ A,\ diam(\Omega),\ n).$$

By the well-known convergence result for convex generalized solutions (see Lemma 1.6.1 in [9] for example), we see that u is a convex generalized solution to problem (1.1).

Step 2. Drop the restriction on the smoothness for F.

Supposed that $F_j \in C^k(\Omega \times (-\infty, 0))$ $(k \geq 3)$ satisfy the same assumption as F in the Step 1 and F_j locally uniform convergence to F in as $j \to \infty$. (For example we can take $F_j = F * \eta_{\varepsilon_j}$, ε_j convergence to 0 as j tend to $+\infty$.) Then by the result of Step 1, for each j, there exists a convex generalized solution $u_j \in C^{\frac{\beta-n+1}{n+\alpha}}(\overline{\Omega})$ to problem (1.1) with F replaced by F_j. Moreover, we have

$$|u_j|_{C^{\frac{\beta-n+1}{n+\alpha}}(\overline{\Omega})} \leq C(\alpha,\ \beta,\ A,\ diam(\Omega),\ n) \tag{2.12}$$

for all j. Using this estimate, Lemma 1.6.1 in [9], and the same argument as in Step 1, we obtain a solution u to problem (1.1), which is the limit of a subsequence of u_j in the space space $C(\overline{\Omega})$. Furthermore, we have $u \in C^{\frac{\beta-n+1}{n+\alpha}}(\overline{\Omega})$ by (2.12). The uniqueness for (1.1) is directly from the comparison principle (see [8, 9, 26] for example).

Step 3. We are going to prove $u \in C^{2,\ \frac{\beta-n+1}{n+\alpha}}(\Omega)$ if $F(x, t) \in C^{0,1}(\Omega \times (-\infty, 0))$. It is enough to prove

$$u \in C^{2,\ \frac{\beta-n+1}{n+\alpha}}(\overline{\Omega_1}) \tag{2.13}$$

for any convex $\Omega_1 \subset\subset \Omega$.

Taking a convex Ω' such that $\Omega_1 \subset\subset \Omega' \subset\subset \Omega$, if there exists $z \in \overline{\Omega'} \subset \Omega$ such that $u(z) = 0$, then $u \equiv 0$ in Ω by convexity and the boundary condition $u|_{\partial\Omega} = 0$. Hence we obtain (2.13). Otherwise, $u(x) < 0$ for all $x \in \overline{\Omega'}$. Then $F(x, u(x)) \in C^{\frac{\beta-n+1}{n+\alpha}}(\overline{\Omega'})$ and is positive on $\overline{\Omega'}$. By the Caffarelli's local $C^{2,\alpha}$ regularity in [1] (also see [18] for another proof), we obtain (2.13), too.

3. Proof of Theorem 1.2. In this section we establish the relation between the Hölder exponent and the convexity of the domain Ω and thus prove Theorem 1.2.

Assume that Ω is a (a, η) type domain with $a \in (2, \infty)$, F satisfies (1.2)-(1.3), and u is the unique solution to problem (1.1) as in Theorem 1.1. To prove Theorem 1.2, it is sufficient to prove (1.6). See the Step 3 in the proof of Theorem 1.1.

As (2.3) we may assume

$$\beta < \alpha + 2n - 1 - \frac{2n-2}{a}. \tag{3.1}$$

Hence, in the following we have

$$\gamma_2 = \frac{\beta - n + 1}{n+\alpha} + \frac{2n-2}{a(n+\alpha)} \in (0, 1).$$

By Lemma 2.1, (1.6) can be deduced from

$$|u(y)| \le C\ d_y{}^{\gamma_2}, \quad \forall y \in \Omega \tag{3.2}$$

for some positive constant$C = C(a, n, \alpha, \eta, A, diam\Omega)$.

We are going to prove (3.2). For any $y \in \Omega$, we can find $z \in \partial\Omega$, such that $|y - z| = d_y$. Since the domain Ω is (a, η) type and the problem (1.1) is invariant under translation and rotation transforms, we may assume $z = 0$, and take the line determined by z and y as the $x_n - axis$ such that

$$\Omega \subseteq \{x \in R^n | x_n \ge \eta |x'|^a\}.$$

We will prove (3.2) by three steps.

Step 1. Let

$$W(x_1, ..., x_n) = W(r, x_n) = -[(\frac{x_n}{\varepsilon})^{\frac{2}{a}} - x_1^2 - ... - x_{n-1}^2]^{\frac{1}{b}},$$

where $r = |x'| = \sqrt{x_1^2 + \cdots, x_{n-1}^2}$, b and ε are positive constants to be determined. We want to find a sufficient condition for which W is a sub-solution to problem (1.1).

For $i, j \in \{1, 2, ..., n-1\}$, by direct computation we have

$$\begin{aligned} W_i =& W_r \frac{x_i}{r}, \\ Wij =& \frac{W_r}{r}\delta_{ij} + (W_{rr} - \frac{W_r}{r})\frac{x_i}{r}\frac{x_j}{r}, \\ W_{in} =& W_{rn}\frac{x_i}{r}. \end{aligned} \tag{3.3}$$

Let

$$D^2W := \begin{pmatrix} G & \xi \\ \xi^T & W_{nn} \end{pmatrix}$$

where $\xi^T = (W_{n1}, ..., W_{n(n-1)})$, and G is the matrix of $n-1$ order all of which eigenvalues are

$$\frac{W_r}{r}, ..., \frac{W_r}{r}, W_{rr},$$

and one of which eigenvector with respect to the eigenvalue W_{rr} is ξ. As obtaining (2.5), we have

$$detD^2W = (\frac{W_r}{r})^{n-2}W_{rr}(W_{nn} - \frac{|W_{rn}|^2}{W_{rr}}).$$

Obviously, $W \le 0$ on $\partial\Omega$. Therefore we conclude that W *is a sub-solution to problem (1.1) if and only if*

$$H[W] := (\frac{W_r}{r})^{n-2}(W_{rr}W_{nn} - |W_{rn}|^2)[F(x,W)]^{-1} \ge 1 \ \ in \ \ \Omega. \tag{3.4}$$

We use the expression of W to compute

$$\begin{aligned}
W_r &= \frac{2}{b}((\frac{x_n}{\varepsilon})^{\frac{2}{a}} - r^2)^{\frac{1}{b}-1} \cdot r,\\
W_n &= -\frac{2}{ab}((\frac{x_n}{\varepsilon})^{\frac{2}{a}} - r^2)^{\frac{1}{b}-1} \cdot (\frac{x_n}{\varepsilon})^{\frac{2}{a}-1} \cdot \frac{1}{\varepsilon},\\
W_{rr} &= \frac{4}{b}(1-\frac{1}{b})((\frac{x_n}{\varepsilon})^{\frac{2}{a}} - r^2)^{\frac{1}{b}-2} \cdot r^2 + \frac{2}{b}((\frac{x_n}{\varepsilon})^{\frac{2}{a}} - r^2)^{\frac{1}{b}-1},\\
W_{nn} &= \frac{4(b-1)}{a^2b^2}((\frac{x_n}{\varepsilon})^{\frac{2}{a}} - r^2)^{\frac{1}{b}-2} \cdot (\frac{x_n}{\varepsilon})^{\frac{4}{a}-2} \cdot (\frac{1}{\varepsilon})^2\\
&\quad + \frac{2(a-2)}{a^2b}((\frac{x_n}{\varepsilon})^{\frac{2}{a}} - r^2)^{\frac{1}{b}-1} \cdot (\frac{x_n}{\varepsilon})^{\frac{2}{a}-2} \cdot (\frac{1}{\varepsilon})^2,\\
W_{rn} &= \frac{4(1-b)}{ab^2}((\frac{x_n}{\varepsilon})^{\frac{2}{a}} - r^2)^{\frac{1}{b}-2} \cdot (\frac{x_n}{\varepsilon})^{\frac{2}{a}-1} \cdot r \cdot \frac{1}{\varepsilon}.
\end{aligned}$$

Using the expression of W again we have

$$W_r = \frac{2}{b}|W|^{1-b} \cdot r, \tag{3.5}$$

$$\begin{aligned}
W_n &= -\frac{2}{ab}|W|^{1-b} \cdot (\frac{x_n}{\varepsilon})^{\frac{2}{a}-1} \cdot \frac{1}{\varepsilon},\\
W_{rr} &= \frac{4(b-1)}{b^2}|W|^{1-2b} \cdot r^2 + \frac{2}{b}|W|^{1-b},\\
W_{nn} &= \frac{4(b-1)}{a^2b^2}|W|^{1-2b} \cdot (\frac{x_n}{\varepsilon})^{\frac{4}{a}-2} \cdot \frac{1}{\varepsilon^2} + \frac{2(a-2)}{a^2b}|W|^{1-b} \cdot (\frac{x_n}{\varepsilon})^{\frac{2}{a}-2} \cdot \frac{1}{\varepsilon^2},\\
W_{rn} &= \frac{4(1-b)}{ab^2}|W|^{1-2b}(\frac{x_n}{\varepsilon})^{\frac{2}{a}-1} \cdot r \cdot \frac{1}{\varepsilon}.
\end{aligned}$$

Hence,

$$\begin{aligned}
W_{rr} \cdot W_{nn} - (W_{rn})^2 &= \frac{8(a-2)(b-1)}{a^2b^3}|W|^{2-3b} \cdot (\frac{x_n}{\varepsilon})^{\frac{2}{a}-2} \cdot r^2 \cdot (\frac{1}{\varepsilon})^2\\
&\quad + \frac{8(b-1)}{a^2b^3}|W|^{2-3b} \cdot (\frac{x_n}{\varepsilon})^{\frac{4}{a}-2} \cdot (\frac{1}{\varepsilon})^2\\
&\quad + \frac{4(a-2)}{a^2b^2}|W|^{2-2b} \cdot (\frac{x_n}{\varepsilon})^{\frac{2}{a}-2} \cdot (\frac{1}{\varepsilon})^2\\
&:= I_1 + I_2 + I_3.
\end{aligned} \tag{3.6}$$

To estimate I_1, I_2 and I_3, we will choose a small $\delta = C(a,\alpha,\beta,n) > 0$. Now for this δ, we choose a small $\varepsilon = C(\delta, a, \eta) > 0$ such that

$$\varepsilon(\frac{1}{\delta})^{\frac{a}{2}} \leq \eta. \tag{3.7}$$

Then we have

$$\Omega \subseteq \{x \in R^n | x_n \geq \eta |x'|^a\} \subseteq \{x \in R^n | \delta(\frac{x_n}{\varepsilon})^{\frac{2}{a}} \geq r^2\}. \tag{3.8}$$

By (3.8) we have

$$|W|^b = (\frac{x_n}{\varepsilon})^{\frac{2}{a}} - r^2 \in [(1-\delta)(\frac{x_n}{\varepsilon})^{\frac{2}{a}},\ (\frac{x_n}{\varepsilon})^{\frac{2}{a}}]. \tag{3.9}$$

Since $a > 2$, we have two case: $a \geq \frac{2\alpha+2}{\beta-n+1}$ and $a < \frac{2\alpha+2}{\beta-n+1}$ if $\frac{2\alpha+2}{\beta-n+1} > 2$.

Step 2. Assume that $\frac{2\alpha+2}{\beta-n+1} > 2$ and $2 < a < \frac{2\alpha+2}{\beta-n+1}$. We want to find $b > 1$ and $\varepsilon > 0$ such that (3.4) is satisfied, by which we will prove (3.2).

Since $a > 2$ and $b > 1$, I_1, I_2 and I_3 in (3.6) are all positive.

$$W_{rr} \cdot W_{nn} - (W_{rn})^2 \geq I_2 = \frac{8(b-1)}{a^2 b^3} |W|^{2-3b} \cdot (\frac{x_n}{\varepsilon})^{\frac{4}{a}-2} \cdot (\frac{1}{\varepsilon})^2.$$

Observe that $d_x \leq x_n$ in Ω. Hence, by (1.3), (3.4) and (3.5) we obtain

$$\begin{aligned} H[W] &= (\frac{W_r}{r})^{n-2}(W_{rr}W_{nn} - |W_{rn}|^2)[F(x,W)]^{-1} \\ &\geq (\frac{2}{b})^{n-2} \cdot |W|^{(1-b)(n-2)} \cdot \frac{8(b-1)}{a^2b^3}|W|^{2-3b} \cdot (\frac{x_n}{\varepsilon})^{\frac{4}{a}-2} \cdot (\frac{1}{\varepsilon})^2 \cdot \frac{1}{A} d_x^{n+1-\beta} |W|^\alpha \\ &\geq (\frac{2}{b})^{n-2} \cdot |W|^{(1-b)(n-2)} \cdot \frac{8(b-1)}{a^2b^3}|W|^{2-3b} \cdot (\frac{x_n}{\varepsilon})^{\frac{4}{a}-2} \cdot (\frac{1}{\varepsilon})^2 \cdot \frac{1}{A} x_n^{n+1-\beta} |W|^\alpha. \end{aligned}$$

It follows from (3.9) that

$$\begin{aligned} x_n &\leq \varepsilon(\frac{1}{1-\delta})^{\frac{a}{2}} |W|^{\frac{ab}{2}} \quad \text{in } \Omega, \\ (\frac{x_n}{\varepsilon})^{\frac{4}{a}-2} &\geq [\ (\frac{1}{1-\delta})^{\frac{a}{2}} |W|^{\frac{ab}{2}}\ ]^{\frac{4}{a}-2}, \\ x_n^{n+1-\beta} &\geq [\ \varepsilon(\frac{1}{1-\delta})^{\frac{a}{2}} |W|^{\frac{ab}{2}}\ ]^{n+1-\beta}. \end{aligned}$$

Therefore, we arrive at

$$\begin{aligned} H[W] &\geq (\frac{2}{b})^{n-2} \cdot |W|^{(1-b)(n-2)} \cdot \frac{8(b-1)}{a^2b^3}|W|^{2-3b} \cdot (\frac{1}{1-\delta})^{2-a} |W|^{\frac{ab}{2}(\frac{4}{a}-2)} \\ &\quad \cdot (\frac{1}{\varepsilon})^2 \cdot \frac{1}{A} \varepsilon^{n+1-\beta} (\frac{1}{1-\delta})^{\frac{a}{2}(n+1-\beta)} |W|^{\frac{ab}{2}(n+1-\beta)} |W|^\alpha \\ &= (\frac{1}{\varepsilon})^{\beta-n+1} \frac{1}{A} (\frac{2}{b})^{n-2} \cdot \frac{8(b-1)}{a^2b^3} \cdot (\frac{1}{1-\delta})^{2-a+\frac{a}{2}(n+1-\beta)} \\ &\quad \cdot |W|^{(1-b)(n-2)+2-3b+\frac{ab}{2}(\frac{4}{a}-2)+\frac{ab}{2}(n+1-\beta)+\alpha}. \end{aligned}$$

Now, we set

$$(1-b)(n-2) + 2 - 3b + \frac{ab}{2}(\frac{4}{a} - 2) + \frac{ab}{2}(n+1-\beta) + \alpha = 0,$$

which is equivalent to

$$b = \frac{2(n+\alpha)}{a(\beta-n+1)+2n-2}.$$

Since $a \in (2, \frac{2\alpha+2}{\beta-n+1})$, we see that $b > 1$. Observing that $\beta - n + 1 > 0$, we can choose $\varepsilon = C(a, \eta, A, \alpha, \beta, n) > 0$ small enough again, such that $H[W] \geq 1$. This proves (3.4), which is to say that W is a sub-solution to problem (1.1). By comparison principle, we have

$$|u(x)| \leq |W(x)|, \quad \forall x \in \Omega.$$

Restricting this inequality onto the x_n axis, we obtain

$$|u(y)| \leq (\frac{y_n}{\varepsilon})^{\frac{2}{ab}} = (\frac{d_y}{\varepsilon})^{\frac{\beta-n+1}{n+\alpha}+\frac{2n-2}{a(n+\alpha)}},$$

which is (3.2) exactly.

Step 3. Assume that $a \geq \frac{2\alpha+2}{\beta-n+1}$. Note that $a > 2$ by the assumption of the theorem. We will find $b \in (0,1)$ and $\varepsilon > 0$ such that the function W is a sub-solution to problem (1.1), and thus prove (3.2).

By (3.9) we have

$$I_1 \geq \frac{8(a-2)(b-1)}{a^2b^3}|W|^{2-3b} \cdot (\frac{x_n}{\varepsilon})^{\frac{2}{a}-2} \cdot \delta(\frac{x_n}{\varepsilon})^{\frac{2}{a}} \cdot (\frac{1}{\varepsilon})^2 = \delta(a-2)I_2.$$

Since $a > 2$, $b \in (0,1)$ and (3.9) yields

$$(\frac{x_n}{\varepsilon})^{\frac{4}{a}-2} \leq |W|^{b(2-a)},$$

we obtain

$$\begin{aligned}
I_1 + I_2 &\geq (1+\delta(a-2))I_2 \\
&\geq (1+\delta(a-2))\frac{8(b-1)}{a^2b^3}|W|^{2-3b} \cdot |W|^{b(2-a)} \cdot (\frac{1}{\varepsilon})^2 \\
&= (1+\delta(a-2))\frac{8(b-1)}{a^2b^3}|W|^{2-b-ab} \cdot (\frac{1}{\varepsilon})^2.
\end{aligned}$$

Again by (3.9), we have

$$(\frac{x_n}{\varepsilon})^{\frac{2}{a}-2} \geq (\frac{1}{1-\delta})^{1-a}|W|^{b(1-a)}.$$

Hence, we have

$$\begin{aligned}
I_3 &\geq \frac{4(a-2)}{a^2b^2}|W|^{2-2b} \cdot (\frac{1}{1-\delta})^{1-a}|W|^{b(1-a)} \cdot (\frac{1}{\varepsilon})^2 \\
&= \frac{4(a-2)}{a^2b^2}(\frac{1}{1-\delta})^{1-a} \cdot |W|^{2-b-ab} \cdot (\frac{1}{\varepsilon})^2.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&W_{rr} \cdot W_{nn} - (W_{rn})^2 \\
=\ & I_1 + I_2 + I_3 \\
\geq\ & [(1+\delta(a-2))\frac{8(b-1)}{a^2b^3} + \frac{4(a-2)}{a^2b^2}(\frac{1}{1-\delta})^{1-a}]|W|^{2-b-ab} \cdot (\frac{1}{\varepsilon})^2 \\
:=\ & \sigma(a,b,\delta)|W|^{2-b-ab} \cdot (\frac{1}{\varepsilon})^2,
\end{aligned}$$

where

$$\sigma(a,b,\delta) = (1+\delta(a-2))\frac{8(b-1)}{a^2b^3} + \frac{4(a-2)}{a^2b^2}(\frac{1}{1-\delta})^{1-a}.$$

We will need

$$\sigma(a,b,\delta) = (1+\delta(a-2))\frac{8(b-1)}{a^2b^3} + \frac{4(a-2)}{a^2b^2}(\frac{1}{1-\delta})^{1-a} > 0. \tag{3.10}$$

Using above estimates, together with (1.3) and (3.9) we have

$$\begin{aligned}
&H[W]\\
&= (\frac{W_r}{r})^{n-2}(W_{rr}W_{nn} - |W_{rn}|^2)(F(x,W))^{-1}\\
&\geq (\frac{2}{b})^{n-2}\cdot |W|^{(1-b)(n-2)}\cdot \sigma(a,b,\delta)|W|^{2-b-ab}\cdot(\frac{1}{\varepsilon})^2\cdot (F(x,W))^{-1}\\
&\geq (\frac{2}{b})^{n-2}\cdot |W|^{(1-b)(n-2)}\cdot \sigma(a,b,\delta)|W|^{2-b-ab}\cdot(\frac{1}{\varepsilon})^2\cdot \frac{1}{A}d_x^{n+1-\beta}|W|^{\alpha}\\
&\geq (\frac{2}{b})^{n-2}\cdot |W|^{(1-b)(n-2)}\cdot \sigma(a,b,\delta)|W|^{2-b-ab}\cdot(\frac{1}{\varepsilon})^2\cdot \frac{1}{A}x_n^{n+1-\beta}|W|^{\alpha}\\
&= (\frac{1}{\varepsilon})^{\beta-n+1}(\frac{2}{b})^{n-2}\frac{1}{A}\cdot\sigma(a,b,\delta)|W|^{2-b-ab}\cdot|W|^{(1-b)(n-2)}\cdot(\frac{x_n}{\varepsilon})^{n+1-\beta}|W|^{\alpha}\\
&\geq (\frac{1}{\varepsilon})^{\beta-n+1}(\frac{2}{b})^{n-2}\frac{1}{A}\cdot\sigma(a,b,\delta)|W|^{2-b-ab}\cdot|W|^{(1-b)(n-2)}\\
&\quad\cdot(\frac{1}{1-\delta})^{\frac{a(n+1-\beta)}{2}}|W|^{\frac{ab(n+1-\beta)}{2}}\cdot|W|^{\alpha}\\
&= (\frac{1}{\varepsilon})^{\beta-n+1}(\frac{2}{b})^{n-2}\frac{1}{A}\cdot(\frac{1}{1-\delta})^{\frac{a(n+1-\beta)}{2}}\sigma(a,b,\delta)|W|^{2-b-ab+(1-b)(n-2)+\frac{ab(n+1-\beta)}{2}+\alpha}.
\end{aligned}$$

Now, we set

$$2-b-ab+(1-b)(n-2)+\frac{ab(n+1-\beta)}{2}+\alpha = 0, \tag{3.11}$$

which ie equivalent to

$$b = \frac{2(n+\alpha)}{a(\beta-n+1)+2n-2}.$$

Since $a \geq \frac{2\alpha+2}{\beta-n+1}$, we see that $b\in(0,1]$. notice that (3.10) is equivalent to

$$(a-2)(1-\delta)^{a-1} > (1+\delta(a-2))(\frac{2(1-b)}{b}). \tag{3.12}$$

Since $\gamma_2 = \frac{\beta-n+1}{n+\alpha} + \frac{2n-2}{a(n+\alpha)} \in (0,1)$ by (3.1), we see that

$$a-2 > \frac{a(\beta-n+1)+2n-2}{n+\alpha} - 2 = (\frac{2(1-b)}{b}).$$

Using this and taking $\delta = C(a,\alpha,\beta,n) > 0$ small enough, we obtain (3.12) and thus (3.10).

Finally, choosing a positive

$$\varepsilon = C(a,\eta,A,\alpha,\beta,b(a,\alpha,\beta,n),\delta(a,\alpha,\beta,n)) = C(a,\eta,A,\alpha,\beta,n)$$

smaller if necessary, by (3.10) and (3.11) we obtain that $H[W]\geq 1$ in Ω, which implies W is an sub-solution to problem (1.1) by (3.4). As in the end of Step 2, we have proved (3.2).

4. Proof of Theorem 1.3. As the proof of Theorem 1.2, the proof of (i) of Theorem 1.3 follows directly from

$$|u(y)| \leq C\ {d_y}^{\gamma_3}, \quad \forall y \in \Omega \tag{4.1}$$

for some positive constant $C = C(a, n, \alpha, \eta, A, diam\Omega)$.

For any $y \in \Omega$, we can find $z \in \partial\Omega$, such that $|y - z| = d_y$. Since the domain Ω satisfies exterior sphere condition with radius R and the problem (1.1) is invariant under translation and rotation transforms, we may assume

$$z = \mathbf{0} \in \partial\Omega \bigcap \partial B_R(y_0), \quad \Omega \subseteq B_R(y_0). \tag{4.2}$$

Since $z = \mathbf{0}$ satisfies $|y - z| = d_y$, the tangent plane of Ω at $z = \mathbf{0}$ is unique. And it is easy to check y is on the line dertermined by $\mathbf{0}$ and y_0. Hence $d_y = |y| = |y_0| - |y_0 - y| = R - |y_0 - y|$.

Consider the function

$$W(x) = -M(R^2 - |x - y_0|^2)^b = -M(R^2 - r^2)^b, \tag{4.3}$$

where $r = |x - y_0|$, M and b are positive constants to be determined later. As (3.3), we obtain that

$$detD^2W = (\frac{W_r}{r})^{n-1}W_{rr}.$$

But

$$W_r = 2Mbr(R^2 - r^2)^{b-1},$$

$$W_{rr} = 2Mb(R^2 - r^2)^{b-2}[R^2 - (2b - 1)r^2].$$

Hence

$$detD^2W = (2Mb)^n(R^2 - r^2)^{n(b-1)-1}[R^2 - (2b - 1)r^2]. \tag{4.4}$$

Observing that $W \leq 0$ on $\partial\Omega$, we see that W is a sub-solution to problem (1.1) if and only if

$$H[W] := (2Mb)^n(R^2 - r^2)^{n(b-1)-1}[R^2 - (2b - 1)r^2][F(x, W)]^{-1} \geq 1 \tag{4.5}$$

for all $x \in \Omega$ and $r = |x - y_0|$.

First, we consider the case

$$\beta < n + \alpha + 1. \tag{4.6}$$

As (2.3), we need only to consider the case $\beta < n + \alpha$. We take

$$b = \frac{\beta}{n + \alpha} = \gamma_3. \tag{4.7}$$

Then in this case $b = \gamma_3 \in (0, 1)$ and $|2b - 1| < 1$. Hence,

$$R^2 - (2b - 1)r^2 \geq (1 - |2b - 1|)R^2. \tag{4.8}$$

It follows from (4.2) that

$$d_x \le R - |x - y_0| = R - r, \quad \forall x \in \Omega. \tag{4.9}$$

Therefore, by (1.3), (4.5), (4.8) and (4.9) that

$$\begin{aligned} &H[W] \\ &\ge (1-|2b-1|)R^2(2Mb)^n(R^2-r^2)^{n(b-1)-1}\frac{1}{A}(d_x)^{n+1-\beta}|W|^\alpha \\ &\ge (1-|2b-1|)R^2\frac{1}{A}(2Mb)^n(R^2-r^2)^{n(b-1)-1}(R-r)^{n+1-\beta}|W|^\alpha \\ &= (1-|2b-1|)R^2\frac{1}{A}M^\alpha(2Mb)^n(R^2-r^2)^{n(b-1)+b\alpha-1}(R-r)^{n+1-\beta} \\ &= (1-|2b-1|)R^2\frac{1}{A}M^{n+\alpha}(2b)^n(R+r)^{n(b-1)+b\alpha-1}(R-r)^{n(b-1)+b\alpha+n-\beta}. \end{aligned} \tag{4.10}$$

Note that

$$n(b-1) + b\alpha + n - \beta = 0 \tag{4.11}$$

by (4.7). Hence, by (4.10) and (4.11) we can choose a large $M = C(A, b, R, \alpha, n, \beta)$ such that

$$H[W] \ge 1 \quad in \quad \Omega. \tag{4.12}$$

Next, we consider the case

$$\beta \ge n + \alpha + 1.$$

In this case, we take

$$b = 1 = \gamma_3.$$

Then, by (1.3) and (4.4) we have

$$\begin{aligned} H[W] &= (2M)^n[F(x,W)]^{-1} \\ &\ge \frac{1}{A}2^nM^{n+\alpha}(R+r)^\alpha(R-r)^{\alpha+n+1-\beta} \\ &= \frac{1}{A}2^nM^{n+\alpha}(R+r)^\alpha. \end{aligned}$$

Therefore, (4.12) still holds true.

To sum up, we have obtained (4.5). By comparison principle, we see that

$$W(x) \le u(x) \le 0. \tag{4.13}$$

In particular, we obtain that

$$|u(y)| \le |W(y)| = M(R + |y - y_0|)^{\gamma_3}(R - |y - y_0|)^{\gamma_3} \le M(2R)^{\gamma_3}(d_y)^{\gamma_3}.$$

This is the desired (4.1) and hence we have proved the (i) of Theorem 1.3.

To prove (ii) of Theorem 1.3, we notice that $u \in C(\bar{\Omega})$ and $u < 0$ in Ω and $u = 0$ on $\partial\Omega$. By comparing the graph of the convex function u with the cone whose vortex is $(x_0, u(x_0))$ and whose upper bottom is $\bar{\Omega}$, where $u(x_0) = \min_{\bar{\Omega}} u$, we see easily that

(1.10) is true for $\gamma_4 \geq 1$. Hence, we need only to consider that case $\gamma_4 < 1$ in the following, which implies that $\beta < n+1$.

Since (1.10) holds naturally for all $y \in \{x \in \Omega : d_x \geq \frac{R}{2}\}$, where R is the radius of the interior sphere for the Ω. Hence, it is sufficient to prove

$$|u(y)| \geq (d_y)^{\gamma_4}, \quad \forall y \in \{x \in \Omega : d_x < \frac{R}{2}\}. \tag{4.14}$$

Take such a y. We can find $z \in \partial\Omega$, such that $|y - z| = d_y$. we may assume

$$z = \mathbf{0} \in \partial\Omega \bigcap \partial B_R(y_0), \quad B_R(y_0) \subseteq \Omega. \tag{4.15}$$

Since the tangent plane of Ω at $z = \mathbf{0}$ is unique. And it is easy to check that y is on the line determined by $\mathbf{0}$ and y_0. Hence $d_y = |y| = |y_0| - |y_0 - y| = R - |y_0 - y|$.

Observing that in this case, instead of (4.9) we have

$$d_x \geq R - |x - y_0| = R - r, \quad \forall x \in B_R(y_0). \tag{4.16}$$

First, we require $b \in (0,1)$, which implies $2b - 1 \in (-1,1)$. Similar to the arguments of (i), by (4.16) we find that the function W, given by (4.3), satisfies

$$\begin{aligned} H[W] &\leq \frac{1}{A}(2Mb)^n(R^2 - r^2)^{n(b-1)-1}[R^2 - (2b-1)r^2]d_x^{n+1-\beta}|W|^\alpha \\ &\leq \frac{1}{A}M^\alpha(2Mb)^n 2R^2[R^2 - r^2)]^{n(b-1)-1+b\alpha}(R-r)^{n+1-\beta} \\ &= \frac{1}{A}M^{\alpha+n}(2b)^n 2R^2(R+r)^{n(b-1)-1+b\alpha}(R-r)^{n(b-1)+b\alpha+n-\beta}. \end{aligned} \tag{4.17}$$

Taking $b = \frac{\beta}{n+\alpha} = \gamma_4 \in (0,1)$, we have

$$n(b-1) + b\alpha + n - \beta = 0. \tag{4.18}$$

Using (4.17)-(4.18), we see that W is a super-solution to problem (1.1) in the domain $B_R(y_0)$ for sufficiently small $M = C(A, b, R, \alpha, n, \beta) > 0$. Since u is a solution on Ω and $u|_{\partial B_R(y_0)} \leq 0$, thus u is a sub-solution on $B_R(y_0)$. Therefore, we have

$$\begin{aligned} |u(y)| &\geq |W(y)| \\ &= M(R + |y - y_0|)^{\gamma_4}(R - |y - y_0|)^{\gamma_4} \\ &\geq MR^{\gamma_4}(d_y)^{\gamma_4}, \end{aligned}$$

which is the desired (4.14) exactly. In this way, the proof of Theorem 1.3 has been completed.

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