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STUDY OF BOUNDARY LAYERS IN COMPRESSIBLE NON-ISENTROPIC FLOWS*

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Dedicated to Professor Ling Hsiao on the occasion of her 80th birthday

Abstract. In this note, we review our recent study on boundary layer problems in compressible non-isentropic flows with non-slip boundary condition for the velocity, in the small viscosity and heat conductivity limit. By multi-scale analysis, we derive the problems of viscous layer profiles and thermal layer profiles for different scales of viscosity and heat conductivity, from which we obtain the interaction mechanism of viscous layers and thermal layers. Then, when the viscosity goes to zero slower than or at the same rate as the heat conductivity, we give a well-posedness result of the two-dimensional viscous layer problem, which is the Prandtl type equations coupled with a degenerated parabolic equation, in the class of tangential velocity being strictly monotonic in the normal variable. Last, when the viscosity goes to zero faster than the heat conductivity, we study the stability of the thermal layer problem at a shear flow in two or three space variables, which is an inviscid Prandtl type equations coupled with a degenerated parabolic equation.

Key words. compressible non-isentropic flows, small viscosity and heat conductivity limit, viscous layers and thermal layers, well-posedness.

Mathematics Subject Classification. 35M13, 35Q35, 76D10, 76D03, 76N20.

1. Introduction. In this note, we are interested to study the asymptotic behavior of solutions in the small viscosity and heat conductivity limit for the compressible viscous flow with heat conduction. As Prandtl found in his pioneering work [30], away from the boundary the flow is mainly driven by the inertia while the friction force is negligible, and the friction plays a key role in determining the behavior of flow near physical boundaries, where the flow has large vortices. On the other hand, as observed by many mechanicians, cf. [31, 2, 18, 35], the flow with heat conduction also has large gradient of temperature near physical boundaries, thus it is very important to analyse the flow behavior near boundaries.

Consider the following compressible non-isentropic Navier-Stokes equations for an ideal gas model with non-slip boundary condition for the velocity in the domain $\Omega = \{t > 0, x \in \mathbb{R}, y > 0\}$,

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \rho(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}) + \nabla p(\rho, \theta) = \mu \Delta \mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}), \\ c_V \rho(\partial_t \theta + (\mathbf{u} \cdot \nabla)\theta) + p(\rho, \theta)\nabla \cdot \mathbf{u} = \kappa \Delta \theta + \lambda(\nabla \cdot \mathbf{u})^2 + 2\mu D \cdot D, \\ \mathbf{u}|_{y=0} = 0, \quad (\frac{\partial \theta}{\partial y} + \sigma\theta)|_{y=0} = \theta^0, \end{cases} \tag{1.1}$$

where ρ, $\mathbf{u} = (u, v)^T$, θ represent the density, velocity, temperature of gas respectively, and $p(\rho, \theta) = R\rho\theta$ is the pressure, $D = D(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ is deformation tensor,

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$D\cdot D=\sum_{i,j=1}^{2} D_{ij}^2$, and the viscosity and heat conductivity satisfying $\mu>0, \lambda+\mu>0$ and $\kappa>0$. We are interested in the behavior of the flow described by (1.1) in the small viscosity and heat conductivity limit. If letting the viscosity and heat conductivity vanishing in (1.1), formally we deduce the following equations for the compressible non-isentropic inviscid flow in $\Omega=\{t>0, x\in\mathbb{R}, y>0\}$,

$$\begin{cases}\partial_t\rho+\nabla\cdot(\rho\mathbf{u})=0,\\ \rho(\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u})+\nabla p(\rho,\theta)=0,\\ c_V\rho(\partial_t\theta+(\mathbf{u}\cdot\nabla)\theta)+p(\rho,\theta)\nabla\cdot\mathbf{u}=0.\end{cases} \tag{1.2}$$

From the non-slip boundary condition for the viscous heat conducting flow given in (1.1), it is nature to impose the impermeable condition:

$$v|_{y=0}=0 \tag{1.3}$$

for the velocity $\mathbf{u}=(u,v)$ of the inviscid flow (1.2), for which one does not need any constraint for the tangential velocity u and temperature on the boundary $\{y=0\}$ to determine the inviscid heat conducting flow from (1.2) with (1.3) and certain initial data of the velocity $\mathbf{u}$ and temperature θ, as the boundary $\{y=0\}$ is the particle path for both $\mathbf{u}$ and θ under the impermeable condition (1.3). As the Prandtl theory in [30] for the incompressible viscous flow, the inconsistence of the boundary conditions between (1.1) and (1.3), in the small viscosity and heat conductivity limit, there are thin layers near the boundary $\{y=0\}$, in which both the tangential velocity and termperature change rapidly. These layers mainly come from the friction of viscosity and the influence of heat conduction near the boundary, thus these two layers are called the viscous layer and thermal layer respectively, cf. [35, 2].

The first goal of this work is to derive the problems of viscous layer profiles and thermal layer profiles for the problem (1.1) in the small viscosity and heat conductivity limit. As Prandtl's theory [30], the viscous term and heat conduction in (1.1) should be balanced with the convective terms in the momentum equation and energy equation in the viscous layer and thermal layer respectively. Thus, by multi-scale analysis, the viscous layer thickness is $O(\sqrt{\mu})$ as $\lambda=O(\mu)$, while the thermal layer thickness is $O(\sqrt{\kappa})$, moreover the viscous layer and thermal layer behave in quite different ways for different scales of viscosity and heat conductivity. We shall obtain that, when the heat conductivity goes to zero faster than or at the same rate as the viscosity, the viscous layer profiles satisfy the Prandtl type equations coupled with a degenerated parabolic equation. In the case that the viscosity goes to zero faster than the heat conductivity, the thermal layer profiles satisfy an inviscid Prandtl type equations coupled with a degenerated parabolic equation, while the viscous layer profiles satisfy the same boundary layer equations as in the isentropic flow studied in [36, 8]. The second goal of this note is to study the well-posedness of the viscous layer problem, when the heat conductivity goes to zero faster than or at the same rate as the viscosity. The existence and uniqueness of a local classical solutions to this problem shall be obtained by using an energy method in a way similar to that given in [1], in the class that the tangential velocity is strictly monotonic with respect to the normal variable. In the last part of this note, we study that when the viscosity goes to zero faster than the heat conductivity, the stability of the thermal layer problem linearized at a shear flow in two or three space variables, which is an inviscid Prandtl type equations coupled with a degenerated parabolic equation. By using the method of stationary phase, we observed that the large time behavior of the normal velocity depends on the monotonicity of the background shear flow velocity, and on the flow direction as well as in the three space variable case.

At the end of this introduction, let us review the study of the well-posedness of the boundary layer problem. The first well-posedness theory of the two-dimensional incompressible Prandtl boundary layer equations was developed by Oleinik and her collaborators in [28, 29], in the class of tangential velocity being strictly monotonic with respect to the normal variable by introducing the Crocco transformation. This well-posedness result was re-studied by introducing an energy method in [1] and [27] independently. Under the Oleinik monotonicity assumption and an additional favorite condition of pressure, Xin and Zhang ([40]) obtained a global weak solution to the two-dimensional Prandtl equations. Without the monotonicity assumption of the velocity field, the Prandtl equations are ill-posed in general in the finite order Sobolev spaces, which is related with the separation of boundary layers, and it is studied in [4, 5, 6, 9, 10, 11, 38] and references therein. The well-posedness of the Prandtl equations in the frame of analytic solutions was studied in [17, 32, 32, 26], and the almost global existence of analytic solutions was obtained in [16, 41]. Recently, there are also some interesting results on the existence of solutions in the Gevrey class ([19, 7, 20]). The stability and instability of the three-dimensional Prandtl equations were studied by authors in [22, 23, 24]. All these studies were restricted to the incompressible flow, for which there are only viscous boundary layers near the boundaries. As studied in [35, 2, 13, 18, 34], the compressible flow near boundary is more complicated, there are not only viscous layers but also thermal layers, in which the heat transfers quickly, and there exists interaction between viscous layers and thermal layers. Till now, there are a few works on compressible viscous flows, the linearized compressible Prandtl equations were studied in [39], the well-posedness of the Prandtl equations in two-dimensional isentropic compressible flows in the monotonic class of tangential velocity was studied in [36, 8], the small viscosity limit of the compressible isentropic viscous flow with the Navier-slip condition was obtained in [37]. These results were only limited to the isentropic flow, in which there is no any thermal layer. In [21], we have studied the behavior of viscous layers and thermal layers in nonisentropic compressible circularly symmetric flows with the viscosity and heat conductivity having the same scale.

The remainder of this note is arranged as follows. In §2, we present problems of viscous layer profiles and thermal layer profiles in the small viscosity and heat conductivity limits for the problem (1.1). In §3, the well-posedness of a viscous layer problem is studied by using an energy approach when the tangential velocity is monotonically increasing in the normal variable. Finally in §4, we study the well-posedness and stability of a thermal layer problem, which is an inviscid Prandtl equation coupled with a degenerated parabolic equation.

2. Problems of compressible boundary layers. As pointed out in §1, inspired by Prandtl's theory [30], the inconsistence of the boundary conditions given in (1.1) and (1.3) implies that in the small viscosity and heat conductivity limit in (1.1), there will appear the viscous layer and thermal layer near the boundary $\{y = 0\}$, in which both the tangential velocity and termperature change rapidly.

To study the small viscosity and heat conductivity limit for the problem (1.1), we assume that they depend on a small parameter ϵ in the following way,

$$\mu = \epsilon\mu', \quad \lambda = \epsilon\lambda', \quad \kappa = \epsilon^{\alpha}\kappa' \tag{2.1}$$

with $\alpha > 0$. Inspired by the Prandtl theory [30], the thickness of the viscous layer and thermal layer for the problem (1.1) are $O(\sqrt{\epsilon})$ and $O(\epsilon^{\frac{\alpha}{2}})$ respectively. Next, we derive the problems of the viscous layer and thermal layer for the different scales of α. For convenience, we shall denote by $(\rho^\epsilon, \mathbf{u}^\epsilon, \theta^\epsilon)$ and $(\rho^e, \mathbf{u}^e, \theta^e)$ the solutions to the problems (1.1) with (2.1), and (1.2)-(1.3) respectively in the following discussion.

<u>Case one:</u> $\alpha = 1$, i.e. the viscosity and heat conductivity have the same rate.

In this case, the viscous layer and thermal layer have the same thickness, $O(\sqrt{\epsilon})$, and in the $O(\sqrt{\epsilon})-$neighborhood of the boundary $\{y = 0\}$, the solution of the problem (1.1) with (2.1) can be approximated by

$$\begin{cases} (\rho^\epsilon, u^\epsilon, \theta^\epsilon)(t,x,y) = (\rho^b, u^b, \theta^b)(t,x,\frac{y}{\sqrt{\epsilon}}) + o(1) \\ v^\epsilon(t,x,y) = \sqrt{\epsilon}v^b(t,x,\frac{y}{\sqrt{\epsilon}}) + o(\sqrt{\epsilon}) \end{cases} \tag{2.2}$$

where $\eta = \frac{y}{\sqrt{\epsilon}}$ is the fast variable, and as $\eta = \frac{y}{\sqrt{\epsilon}} \to +\infty$, the boundary layer profiles $(\rho^b, u^b, v^b, \theta^b)$ fastly approach the values of $(\rho^e, u^e, v^e, \theta^e)$ on $\{y = 0\}$ determined by (1.2)-(1.3). Plugging (2.2) into (1.1), it follows that the density profile and temperature profile satisfy the relation

$$\rho^b(t,x,\eta) = R^{-1}\frac{P(t,x)}{\theta^b(t,x,\eta)}, \tag{2.3}$$

with $P(t,x) = R(\rho^e\theta^e)(t,x,0)$ being the pressure of the Euler flow on the boundary $\{y = 0\}$, and the other boundary layer profiles $(u^b, v^b, \theta^b)(t,x,\eta)$ satisfy the following problem for the Prandtl type equations coupled with a degenerate parabolic equation in $\{t > 0, x \in \mathbb{R}, \eta > 0\}$,

$$\begin{cases} \partial_t u^b + (u^b\partial_x + v^b\partial_\eta)u^b + \frac{RP_x}{P}\theta^b = \frac{\mu R}{P}\theta^b\partial_\eta^2 u^b, \\ \partial_x u^b + \partial_\eta v^b = \frac{R}{(R+c_V)P}\big[\kappa\partial_\eta^2\theta^b + \mu(\partial_\eta u^b)^2\big] - \frac{c_V}{(R+c_V)P}\big(P_x \cdot u^b + P_t\big), \\ \partial_t\theta^b + (u^b\partial_x + v^b\partial_\eta)\theta^b = \frac{R}{(R+c_V)P}\theta^b\big[\kappa\partial_\eta^2\theta^b + \mu(\partial_\eta u^b)^2 + P_x \cdot u^b + P_t\big], \\ (u^b, v^b)|_{\eta=0} = 0, \ \partial_\eta\theta^b|_{\eta=0} = 0, \\ \lim\limits_{\eta\to+\infty}(u^b, \theta^b)(t,x,\eta) = (u^e, \theta^e)(t,x,0). \end{cases} \tag{2.4}$$

<u>Case two:</u> $\alpha > 1$, i.e. the heat conductivity vanishes faster than the viscosity.

In this case, the width of viscous layer is of order $\sqrt{\epsilon}$, while the width of thermal layer is of order $\epsilon^{\alpha/2}$, which is much smaller than $\sqrt{\epsilon}$. As in Prandtl [30], in the $O(\sqrt{\epsilon})-$neighborhood of the boundary $\{y = 0\}$, by taking the following ansatz for the solution of (1.1) with (2.1),

$$\begin{cases} (\rho^\epsilon, u^\epsilon, \theta^\epsilon)(t,x,y) = (\rho^b, u^b, \theta^b)(t,x,\frac{y}{\sqrt{\epsilon}}) + o(1), \\ v^\epsilon(t,x,y) = \sqrt{\epsilon}v^b(t,x,\frac{y}{\sqrt{\epsilon}}) + o(\sqrt{\epsilon}) \end{cases} \tag{2.5}$$

with $\eta = \frac{y}{\sqrt{\epsilon}}$ being the fast variable, from (1.1) we deduce that the relation (2.2) still holds for ρ^b and θ^b, and the viscous layer profiles (u^b, v^b, θ^b) satisfy the following problem for the Prandtl type equations coupled with a transport equation of temperature in $\{t > 0, x \in \mathbb{R}, \eta > 0\}$,

$$\begin{cases} \partial_t u^b + (u^b\partial_x + v^b\partial_\eta)u^b + \frac{RP_x}{P}\theta^b = \frac{\mu R}{P}\theta^b\partial_\eta^2 u^b, \\ \partial_x u^b + \partial_\eta v^b = \frac{\mu R}{(R+c_V)P}(\partial_\eta u^b)^2 - \frac{c_V}{(R+c_V)P}\big(P_x \cdot u^b + P_t\big), \\ \partial_t\theta^b + (u^b\partial_x + v^b\partial_\eta)\theta^b = \frac{R}{(R+c_V)P}\theta^b\big[\mu(\partial_\eta u^b)^2 + P_x \cdot u^b + P_t\big], \\ u^b|_{\eta=0} = v^b|_{\eta=0} = 0, \\ \lim\limits_{\eta\to+\infty}(u^b, \theta^b)(t,x,\eta) = (u^e, \theta^e)(t,x,0). \end{cases} \tag{2.6}$$

On the other hand, in the $O(\epsilon^{\alpha/2})-$neighborhood of the boundary $\{y=0\}$, we take the following expansions for the thermal layer profiles of (1.1),

$$\begin{cases}(\rho^\epsilon,\theta^\epsilon)(t,x,y)=(\rho^B,\theta^B)(t,x,\frac{y}{\epsilon^{\frac{\alpha}{2}}})+o(1),\\ u^\epsilon(t,x,y)=\epsilon^{\frac{\alpha-1}{2}}u^B(t,x,\frac{y}{\epsilon^{\frac{\alpha}{2}}})+o(\epsilon^{\frac{\alpha-1}{2}})\\ v^\epsilon(t,x,y)=\epsilon^{\frac{\alpha}{2}}v^B(t,x,\frac{y}{\epsilon^{\frac{\alpha}{2}}})+o(\epsilon^{\frac{\alpha}{2}})\end{cases} \tag{2.7}$$

with $\zeta=\frac{y}{\epsilon^{\frac{\alpha}{2}}}$ being the fast variable, from (1.1) we deduce that the relation (2.2) still holds for $\rho^B(t,x,\zeta)$ and $\theta^B(t,x,\eta)$, and $u^B(t,x,\zeta)$ is given by

$$u^B(t,x,\zeta)=\zeta(\partial_\eta u^b)(t,x,0), \tag{2.8}$$

and $(v^B(t,x,\zeta),\theta^B(t,x,\zeta))$ satisfy the following problem in $\{t>0,x\in\mathbb{R},\zeta>0\}$,

$$\begin{cases}\partial_t\theta^B+v^B\partial_\zeta\theta^B=\frac{R}{(R+c_V)P}\theta^B\Big(\kappa\partial_\zeta^2\theta^B+\mu\big(\overline{\partial_\eta u^b}\big)^2+P_t\Big),\\ \partial_\zeta\theta^B|_{\zeta=0}=0,\quad \lim\limits_{\zeta\to+\infty}\theta^B(t,x,\zeta)=\theta^b(t,x,0),\end{cases} \tag{2.9}$$

and

$$\partial_\zeta v^B=\frac{R}{(R+c_V)P}\big(\kappa\partial_\zeta^2\theta^B+\mu\big(\overline{\partial_\eta u^b}\big)^2\big)-\frac{c_VP_t}{(R+c_V)P}, \tag{2.10}$$

with $\overline{\partial_\eta u^b}(t,x)=\partial_\eta u^b(t,x,0)$.

REMARK 2.1. *From the equation* (2.10) *and the boundary condition* $v^B(t,x,0)=0$, *one can represent* $v^B(t,x,\zeta)$ *in terms of* θ^B, *which is plugged into the first equation of* (2.9) *to get a scalar equation for* $\theta^B(t,x,\zeta)$, *which is parabolic in* (t,ζ) *with* x *being a parameter.*

Case three: $0<\alpha<1$, i.e. the viscosity vanishes faster than the heat conductivity.

In this case, the width of thermal layer is of order $\epsilon^{\alpha/2}$, much larger than $\sqrt{\epsilon}$, the width of viscous layer. As in Prandtl [30], in the $O(\epsilon^{\alpha/2})-$neighborhood of the boundary $\{y=0\}$, by taking the following ansatz of the thermal layer for (1.1) with (2.1),

$$\begin{cases}(\rho^\epsilon,u^\epsilon,\theta^\epsilon)(t,x,y)=(\rho^B,u^B,\theta^B)(t,x,\frac{y}{\epsilon^{\frac{\alpha}{2}}})+o(1)\\ v^\epsilon(t,x,y)=\epsilon^{\frac{\alpha}{2}}v^B(t,x,\frac{y}{\epsilon^{\frac{\alpha}{2}}})+o(\epsilon^{\frac{\alpha}{2}})\end{cases} \tag{2.11}$$

with $\zeta=\frac{y}{\epsilon^{\frac{\alpha}{2}}}$ being the fast variable, from (1.1) we know that the density profile and temperature profile still satisfy the relation (2.2), and the profiles $(u^B,v^B,\theta^B)(t,x,\zeta)$ satisfy the following problem for the inviscid Prandtl equations coupled with a degenerate parabolic equation in $\{t>0,x\in\mathbb{R},\zeta>0\}$,

$$\begin{cases}\partial_tu^B+(u^B\partial_x+v^B\partial_\zeta)u^B+\frac{RP_x}{P}\theta^B=0,\\ \partial_xu^B+\partial_\zeta v^B=\frac{\kappa R}{(R+c_V)P}\partial_\zeta^2\theta^B-\frac{c_V}{(R+c_V)P}\big(P_x\cdot u^B+P_t\big),\\ \partial_t\theta^B+(u^B\partial_x+v^B\partial_\zeta)\theta^B=\frac{R}{(R+c_V)P}\theta^B\big[\kappa\partial_\zeta^2\theta^B+P_x\cdot u^B+P_t\big],\\ v^B|_{\zeta=0}=\partial_\zeta\theta^B|_{\zeta=0}=0,\\ \lim\limits_{\zeta\to+\infty}(u^B,\theta^B)(t,x,\zeta)=(u^e,\theta^e)(t,x,0).\end{cases} \tag{2.12}$$

On the other hand, in the $O(\sqrt{\epsilon})-$neighborhood of the boundary $\{y=0\}$, by taking the following ansatz for the viscous layer of (1.1) with (2.1),

$$\begin{cases}(\rho^\epsilon, u^\epsilon, \theta^\epsilon)(t,x,y) = (\rho^b, u^b, \theta^b)(t,x,\frac{y}{\sqrt{\epsilon}}) + o(1)\\ v^\epsilon(t,x,y) = \sqrt{\epsilon} v^b(t,x,\frac{y}{\sqrt{\epsilon}}) + o(\sqrt{\epsilon})\end{cases} \tag{2.13}$$

with $\eta = \frac{y}{\sqrt{\epsilon}}$ being the fast variable, from (1.1) we deduce that $\rho^b(t,x,\eta) \equiv \rho^B(t,x,0)$, $\theta^b(t,x,\eta) = \theta^B(t,x,0)$, and the viscous layer profiles $(u^b(t,x,\eta), v^b(t,x,\eta))$ satisfy the following problem, which is the same as the isentropic, compressible Prandtl equation studied already in [36, 8],

$$\begin{cases}\partial_t u^b + (u^b\partial_x + v^b\partial_\eta)u^b + \frac{RP_x}{\rho^b} = \frac{\mu}{\rho^b}\partial_\eta^2 u^b,\\ \partial_x u^b + \partial_\eta v^b = -\frac{1}{\rho^b}(\partial_x \rho^b u^b + \partial_t \rho^b),\\ (v^b, v^b)|_{\eta=0} = 0, \quad \lim\limits_{\eta\to+\infty} u^b(t,x,\eta) = u^B(t,x,0).\end{cases} \tag{2.14}$$

The detail derivation of the above boundary layer problems can be found in [25].

3. Well-posedness of viscous boundary layer problems. The aim of this section is to describe the main idea for solving the problems (2.4) and (2.6) of viscous boundary layers given in §2, when the heat conductivity goes to zero faster or at the same rate as that of the viscosity in (1.1). Consider the following problem in $\Omega_T = \{0 \le t < T, x\in\mathbb{R}, y>0\}$, with $\mu>0$ and $\kappa\ge 0$.

$$\begin{cases}\partial_t u + (u\partial_x + v\partial_y)u + \frac{RP_x}{P}\theta = \frac{\mu R}{P}\theta\partial_y^2 u,\\ \partial_t\theta + (u\partial_x + v\partial_y)\theta = \frac{R}{(R+c_V)P}\theta\big[\kappa\partial_y^2\theta + \mu(\partial_y u)^2 + P_x\cdot u + P_t\big],\\ \partial_x u + \partial_y v = \frac{R}{(R+c_V)P}\big[\kappa\partial_y^2\theta + \mu(\partial_y u)^2\big] - \frac{c_V}{(R+c_V)P}\big(P_x\cdot u + P_t\big),\\ \lim\limits_{y\to+\infty}(u,\theta)(t,x,y) = (U,\Theta)(t,x),\\ (u,\theta)|_{t=0} = (u_0,\theta_0)(x,y),\end{cases} \tag{3.1}$$

with one of the boundary conditions

$$\begin{cases}(u,v)|_{y=0} = 0,\ \partial_y\theta|_{y=0} = 0, & \text{when} \quad \mu>0, \kappa>0,\\ (u,v)|_{y=0} = 0, & \text{when} \quad \mu>0, \kappa=0.\end{cases} \tag{3.2}$$

We shall use the energy approach developed in [1] to study this problem, *in the class that the tangential velocity $u(t,x,y)$ is monotonically increasing in the variable $y>0$*. To simplify the calculation, assume that the outflow is an uniform one:

$$(U,P,\Theta)(t,x) \ \equiv\ (1,p,q),$$

for two positive constants p and q. To light the burden of notations, denote $\frac{\mu R}{p}, \frac{\kappa R}{p}, \frac{1}{R+C_V}$ by μ,κ,ν for simplicity. Let $(u^s(t,x,y), v^s(t,x,y), \theta^s(t,x,y))$ be a smooth background state satisfying the boundary conditions (3.2) and $\lim\limits_{y\to+\infty}(u^s,\theta^s) = (1,q)$. Moreover, we assume that

$$\partial_y u^s(t,x,y) > 0, \quad \theta^s(t,x,y) \ge \delta \tag{3.3}$$

for all $(t,x,y)\in\Omega_T$, with $\delta>0$ being a positive constant.

The linearized problem of (3.1)-(3.2) at (u^s, v^s, θ^s) is given as the following one in Ω_T :

$$\begin{cases} \partial_t u + (u^s\partial_x + v^s\partial_y)u + (u\partial_x + v\partial_y)u^s - \mu\partial_y^2 u^s\theta - \mu\theta^s\partial_y^2 u = f_1, \\ \partial_t\theta + (u^s\partial_x + v^s\partial_y)\theta + (u\partial_x + v\partial_y)\theta^s - \nu\big[\kappa\partial_y^2\theta^s + \mu(\partial_y u^s)^2\big]\theta \\ \qquad\qquad\qquad\qquad\qquad -\nu\theta^s\big[\kappa\partial_y^2\theta + 2\mu\partial_y u^s\partial_y u\big] = f_2, \\ \partial_x u + \partial_y v - \nu\big[\kappa\partial_y^2\theta + 2\mu\partial_y u^s\partial_y u\big] = f_3, \\ \lim\limits_{y\to+\infty}(u,\theta)(t,x,y) = 0, \quad (u,\theta)|_{t\le 0} = 0, \end{cases} \tag{3.4}$$

with the boundary conditions

$$\begin{cases} (u,v)|_{y=0} = 0, \ \partial_y\theta|_{y=0} = 0, & \text{when} \quad \mu > 0, \kappa > 0, \\ (u,v)|_{y=0} = 0, & \text{when} \quad \mu > 0, \kappa = 0. \end{cases} \tag{3.5}$$

To study energy estimates of solutions to the problem (3.4), as in the classical linearized Prandtl equation studied in [1], the key point is to study the terms $v\partial_y u^s$ and $v\partial_y\theta^s$ appeared in the first and the second equations of (3.4) respectively. To do this, as in [1], we introduce

$$w(t,x,y) = \partial_y\Big(\frac{u}{\partial_y u^s}\Big)(t,x,y), \quad \tilde\theta(t,x,y) = \Big(\theta - \frac{\partial_y\theta^s}{\partial_y u^s}\cdot u\Big)(t,x,y), \tag{3.6}$$

then, from (3.4)-(3.5) it follows that $(w,\tilde\theta)$ satisfy the following degenerated parabolic integro-differential equations:

$$\begin{cases} \partial_t w + \partial_x(u^s\cdot w) + \partial_y(\eta\cdot w) - \mu\partial_y\big(\frac{\partial_y^2 u^s}{\partial_y u^s}\cdot\tilde\theta\big) + \partial_y(\zeta\cdot\int_0^y w dy) \\ \qquad -2\mu\partial_y u^s\partial_y^2 u^s\cdot\int_0^y w dy - \mu\partial_y(\theta^s\cdot\partial_y w) + \nu\kappa\partial_y^2\tilde\theta = \partial_y f_1 - f_3, \\ \partial_t\tilde\theta + (u^s\partial_x + v^s\partial_y)\tilde\theta + (\mu-\nu\kappa)\theta^s\partial_y\theta^s\partial_y w + a\tilde\theta + bw \\ \qquad\qquad\qquad + c\int_0^y w dy - \nu\kappa\theta^s\partial_y^2\tilde\theta = f_2, \end{cases} \tag{3.7}$$

with the initial and boundary conditions:

$$\begin{cases} \big[\mu\theta^s\partial_y w + \mu\frac{\partial_y^2 u^s}{\partial_y u^s}(2\theta^s w + \tilde\theta) + f_1\big]\big|_{y=0} = 0, \quad \partial_y\tilde\theta|_{y=0} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{as } \mu > 0, \kappa > 0, \\ \big[\mu\theta^s\partial_y w + \mu\frac{\partial_y^2 u^s}{\partial_y u^s}(2\theta^s w + \tilde\theta) + f_1\big]\big|_{y=0} = 0, \quad \text{as } \mu > 0, \kappa = 0 \\ (w,\tilde\theta)|_{t\le 0} \equiv 0, \end{cases} \tag{3.8}$$

where the coefficient functions η, ζ, a, b depend on the background state (u^s, v^s, θ^s) and its derivatives up to order three, one can see the corresponding functions given in [1].

In a way similar to that given in [1], we can obtain energy estimates of solutions to the problem (3.7)-(3.8) with loss of regularity of the background state (u^s, v^s, θ^s) and the source terms (f_1, f_2, f_3), which implies the energy estimates of solutions to the original linearized problem (3.4)-(3.5). by using the transformation (3.6).

As in [1], from energy estimates for the linearized problem (3.4)-(3.5), we can conclude the local existence of solutions to the nonlinear problem (3.1)-(3.2) in the monotonic class $\partial_y u(t,x,y) > 0$, when $\mu > 0, \kappa \ge 0$ by using the Nash-Moser iteration technique ([1, 15]).

REMARK 3.1. *Noting that the equations in* (3.1) *are of the Prandtl type for the velocity field coupled with a degenerated parabolic or transport equation for the temperature as* $\kappa > 0$ *or* $\kappa = 0$, *as in Oleinik [28] and Oleinik, Samokhin [29], under the assumptions*

$$U(t,x) > 0, \quad u_0(x,y) > 0, \quad \partial_y u_0(x,y) > 0, \quad \text{for all } y > 0, \tag{3.9}$$

in the monotonic class $\partial_y u(t,x,y) > 0$, *by introducing the Crocco transformation,*

$$\tau = t, \quad \xi = x, \quad \eta = \frac{u(t,x,y)}{U(t,x)}, \tag{3.10}$$

the new unknowns

$$w(\tau,\xi,\eta) = \frac{\partial_y u(t,x,y)}{U(t,x)}, \quad \tilde{\theta}(\tau,\xi,\eta) = \frac{\theta(t,x,y)}{\Theta(t,x)}, \tag{3.11}$$

satisfy the following initial-boundary value problem for degenerated parabolic equations in the domain $Q_T = \{(\tau,\xi,\eta) | 0 < \tau < T, \ \xi \in \mathbb{R}, \ 0 < \eta < 1\}$,

$$\begin{cases} \partial_\tau w + \eta U \partial_\xi w + (A - \frac{RP_x\Theta}{PU}\theta)\partial_\eta w + \frac{RP_x\Theta}{PU} w\partial_\eta\theta + B_1 w + Cw^3 \\ \qquad\qquad + \frac{\kappa R\Theta}{(R+c_V)P} w^2 \partial_\eta(w\partial_\eta\theta) - \frac{\mu R\Theta}{P} w^2\partial_\eta(\theta\partial_\eta w) = 0, \\ \partial_\tau\theta + \eta U\partial_\xi\theta + (A - \frac{RP_x\Theta}{PU}\theta)\partial_\eta\theta + B_2\theta - C\theta w^2 \\ \qquad\qquad + \frac{\mu R\Theta}{P}\theta w\partial_\eta w\partial_\eta\theta - \frac{\kappa R\Theta}{(R+c_V)P}\theta w\partial_\eta(w\partial_\eta\theta) = 0, \\ (w\partial_\eta w)|_{\eta=0} = \frac{P_x}{\mu U}, \quad \partial_\eta\theta|_{\eta=0} = 0, \quad (w,\theta)|_{\eta=1} = (0,1), \\ (w,\theta)|_{\tau=0} = (w_0, \tilde{\theta}_0)(\xi,\eta) \ \triangleq \ (\frac{\partial_y u_0}{U}, \ \frac{\theta_0}{\Theta})|_{t=0} \end{cases} \tag{3.12}$$

in which the tilde of θ *are dropped for simplicity. The problem* (3.12) *can be solved by using the iteration scheme:*

$$\begin{cases} \big(\partial_\tau + \eta U\partial_\xi + (A - \frac{RP_x\Theta}{PU}\theta^{n-1})\partial_\eta + B_1 + C(w^{n-1})^2\big) w^n + \frac{RP_x\Theta}{PU} w^{n-1}\partial_\eta\theta^n \\ \qquad + \frac{\kappa R\Theta}{(R+c_V)P}(w^{n-1})^2\partial_\eta(w^{n-1}\partial_\eta\theta^n) - \frac{\mu R\Theta}{P}(w^{n-1})^2\partial_\eta(\theta^{n-1}\partial_\eta w^n) = 0, \\ \partial_\tau\theta^n + \eta U\partial_\xi\theta^n + (A - \frac{RP_x\Theta}{PU}\theta^{n-1})\partial_\eta\theta^n + [B_2 - C(w^{n-1})^2]\theta^n \\ \qquad + \frac{\mu R\Theta}{P}\theta^{n-1}w^{n-1}\partial_\eta w^{n-1}\partial_\eta\theta^n - \frac{\kappa R\Theta}{(R+c_V)P}\theta^{n-1}w^{n-1}\partial_\eta(w^{n-1}\partial_\eta\theta^n) = 0, \\ (w^{n-1}\partial_\eta w^n)|_{\eta=0} = \frac{P_x}{\mu U}, \ \partial_\eta\theta^n|_{\eta=0} = 0, \\ (w^n,\theta^n)|_{\tau=0} = (w_0(\xi,\eta), , \tilde{\theta}_0(\xi,\eta)) \end{cases} \tag{3.13}$$

Note that in (3.13), *we don't need to impose any boundary condition for* (w^n,θ^n) *at* $\{\eta = 1\}$, *as* $(w^n,\theta^n)|_{\eta=1} = (0,1)$ *holds always when it is true for* (w^{n-1},θ^{n-1}), *by noting* $(A - \frac{RP_x\Theta}{PU})|_{\eta=1} = 0$. *One can refer to [25] for the detail of this approach.*

4. Study of thermal layer problems. From §2, we know that when the viscosity goes to zero faster than the heat conductivity, i.e. $0 < \alpha < 1$ in (2.1), the thermal layer profiles satisfy the problem (2.12) of inviscid Prandtl equations coupled with a degenerated parabolic equation. The goal of this section is to present the main idea for studying the existence and stability of solutions to this problem in two and three space variables, developed in [25].

4.1. Existence of a local classical solution. First, we assume that the outer flow pressure

$$P = R(\rho^e\theta^e)|_{y=0} \text{ is only a function of t.} \tag{4.1}$$

Consider the following initial-boundary value problem in $\{(t,x',y):\ t>0, x'\in\mathbb{R}^{d-1}, y>0\}$ with $d=2,3$:

$$\begin{cases} \partial_t \mathbf{u}_h + (\mathbf{u}_h\cdot\nabla_h + u_d\partial_y)\mathbf{u}_h = 0, \\ \partial_t\theta + (\mathbf{u}_h\cdot\nabla_h + u_d\partial_y)\theta = \frac{\kappa}{P}\theta\partial_y^2\theta + \frac{\kappa P_t}{P}\theta, \\ \nabla_h\cdot\mathbf{u}_h + \partial_y u_d = \frac{\kappa}{P}\partial_y^2\theta - \frac{(1-\kappa)P_t}{P}, \\ (u_d,\theta)|_{y=0} = \big(0,\theta^0(t,x')\big), \qquad \lim\limits_{y\to+\infty}\theta(t,x,y) = \Theta(t,x'), \\ (\mathbf{u}_h,\theta)|_{t=0} = (\mathbf{u}_{h0},\theta_0)(x',y), \end{cases} \tag{4.2}$$

where $\mathbf{u}_h = (u_1,\ldots,u_{d-1})^T$ is the tangential velocity, $\nabla_h\cdot\mathbf{u}_h = \sum_{k=1}^{d-1}\partial_{x_k}u_k$, $\Theta(t,x')$ is a positive known function, and $\kappa>0$ is a constant.

THEOREM 4.1. *Suppose that the data given in* (4.2), $\mathbf{u}_{h0}\in C^2, \theta_0\in C^2, \theta^0\in C^1, P\in C^1$ *and* $\Theta\in C^1$ *satisfy the compatibility conditions of* (4.2) *up to order one, and*

$$t^* := \sup\left\{t:\ \inf_{(x',y)\in\mathbb{R}^d_+} det\big(I_{d-1}+s\nabla_h\mathbf{u}_{h0}(x',y)\big)>0,\ \forall s\in[0,t]\right\}>0. \tag{4.3}$$

Then, there is a unique classical solution to (4.2) *in* $[0,t_0)\times\mathbb{R}^d_+$, *for some* $t_0\in(0,t^*]$.

Sketch of the proof: As in (4.2), the tangential velocity $\mathbf{u}_h$ satisfies a transport equation, we shall develope the idea of Hong and Hunter in [14] to study this problem.

Step 1: Introducing the characteristic coordinates:

$$t=\tau,\ x'=x'(\tau,\xi,\eta),\ y=y(\tau,\xi,\eta)$$

with

$$\begin{cases} \frac{\partial}{\partial\tau}x'(\tau,\xi,\eta) \ = \ \mathbf{u}_h\big(\tau,x'(\tau,\xi,\eta),y(\tau,\xi,\eta)\big), \\ \frac{\partial}{\partial\tau}y(\tau,\xi,\eta) \ = \ u_d\big(\tau,x'(\tau,\xi,\eta),y(\tau,\xi,\eta)\big), \\ x'(0,\xi,\eta) \ =\xi,\ y(0,\xi,\eta) \ = \ \eta, \end{cases} \tag{4.4}$$

with $\xi=(\xi_1,\cdots,\xi_{d-1})^T\in\mathbb{R}^{d-1}$, from $(4.2)_1$ we know that on characteristics,

$$\mathbf{u}_h(\tau,x'(\tau,\xi,\eta),y(\tau,\xi,\eta)) \equiv \mathbf{u}_{h0}(\xi,\eta),$$

which implies

$$x' \ = \ \xi+\tau\mathbf{u}_{h0}(\xi,\eta), \tag{4.5}$$

determining uniquely $\xi=\xi(\tau,x',\eta)$ when $0\le\tau\le t^*$.

Step 2: Determine $\eta=\eta(t,x',y)$.
Set $J(\tau,\xi,\eta)=\frac{\partial(x',y)}{\partial(\xi,\eta)}$. By a direct computation, we have

$$\partial_\tau J \ = J\cdot(\nabla_h\cdot\mathbf{u}_h+\partial_y u_d) = J\cdot\left[\frac{\partial_\tau\bar\theta}{\bar\theta}-\frac{P_\tau(\tau)}{P(\tau)}\right]$$

where $\bar{\theta}(\tau,\xi,\eta)=\theta(\tau,x'(\tau,\xi,\eta),y(\tau,\xi,\eta))$, which implies

$$J(\tau,\xi,\eta) \;=\; \frac{P(0)}{P(\tau)\theta_0(\xi,\eta)}\bar{\theta}(\tau,\xi,\eta). \tag{4.6}$$

Thus, from (4.6) and the definition of J, we get

$$\frac{\partial}{\partial\eta}y(\tau,\xi(\tau,x',\eta),\eta) \;=\; \frac{\tilde{\theta}(\tau,x',\eta)}{a(\tau,x',\eta)}, \tag{4.7}$$

with $\tilde{\theta}(\tau,x',\eta)=\bar{\theta}(\tau,\xi(\tau,x',\eta),\eta)$, and

$$a(\tau,x',\eta)=\frac{P(\tau)}{P(0)}\theta_0\big(\xi(\tau,x',\eta),\eta\big)\cdot det(I_{d-1}+\tau\nabla_h\mathbf{u}_{h0})\big(\xi(\tau,x',\eta),\eta\big).$$

By noting $y=0$ when $\eta=0$ from $u_d|_{y=0}=0$, from (4.7) we immediately obtain

$$y=\int_0^\eta \frac{\tilde{\theta}(\tau,x',z)}{a(\tau,x',z)}dz, \tag{4.8}$$

which gives the transformation $\eta=\eta(t,x',y)$, being known once $\theta(t,x,y)$ is given.

Step 3: Determine u_d.

Set $\tilde{y}(\tau,x',\eta)=\int_0^\eta \frac{\tilde{\theta}(\tau,x',z)}{a(\tau,x',z)}dz$. Since $y(\tau,\xi,\eta)=\tilde{y}(\tau,\xi+\tau\mathbf{u}_{h0}(\xi,\eta),\eta)$, and $y_\tau(\tau,\xi,\eta)=\mathbf{u}_d(\tau,x'(\tau,\xi,\eta),y(\tau,\xi,\eta))$, we get

$$\begin{aligned} &u_d(t,x',y) \\ &=\int_0^{\eta(t,x',y)} \partial_t\Big(\frac{\tilde{\theta}}{a}\Big)(t,x',z)dz+\int_0^{\eta(t,x',y)}\Big[b\big(t,x',\eta(t,x',y)\big)\cdot\nabla_h\Big(\frac{\tilde{\theta}}{a}\Big)(t,x',z)\Big]dz. \end{aligned} \tag{4.9}$$

being known when $\theta(t,x,y)$ is given, where $b(t,x',z)=\mathbf{u}_{h0}\big(\xi(t,x',z),z\big)$.

Step 4: Determine θ.

Finally, from the problem of $\bar{\theta}(\tau,\xi,\eta)=\theta(\tau,x'(\tau,\xi,\eta),y(\tau,\xi,\eta))$ given in (4.2), we get that $\tilde{\theta}(t,x',z)=\bar{\theta}(t,\xi(t,x',z),z)$ can be determined by solving the following initial-boundary value problem for a degenerated parabolic equation:

$$\begin{cases} \partial_t\tilde{\theta}+b\cdot\nabla_h\tilde{\theta}-\frac{\kappa P_t}{P}\tilde{\theta}-\frac{\kappa a}{P}\partial_z\big(\frac{a}{\tilde{\theta}}\partial_z\tilde{\theta}\big)=0, \\ \tilde{\theta}|_{z=0}=\theta^0(t,x'), \qquad \lim\limits_{z\to+\infty}\tilde{\theta}=\Theta(t,x'), \\ \tilde{\theta}|_{t=0}=\theta_0(x',z); \end{cases} \tag{4.10}$$

REMARK 4.1. *If the constraint* (4.1) *is not true, one can study the well-posedness of the thermal layer problem* (2.12) *in the monotonic class* $\partial_y u(t,x,y)>0$, *in a way similar to the way presented in §3. One can find the detail in [25].*

4.2. Linear stability of 3-d thermal layers at a shear flow. For a given shear flow

$$(\mathbf{u}_h,u_3,\theta)(t,x',y) \;=\; \Big(\mathbf{U}_h(y),0,1\Big),$$

of the thermal layer problem (4.2) in three space variables, with $\mathbf{u}_h = (u_1, u_2)$ and $x' = (x_1, x_2)$, the linearized problem of (4.2) at this shear flow can be written as the following one in $\{t > 0, x' \in \mathbb{R}^2, y > 0\}$:

$$\begin{cases} \partial_t \mathbf{u}_h + \mathbf{U}_h(y) \cdot \nabla_{x'} \mathbf{u}_h + \mathbf{U}_h'(y) u_3 = 0, \\ \partial_t \theta + \mathbf{U}_h(y) \cdot \nabla_{x'} \theta = \partial_y^2 \theta, \\ \nabla_{x'} \cdot \mathbf{u}_h + \partial_y u_3 = \partial_y^2 \theta, \\ u_3|_{y=0} = 0, \quad \mathbf{u}_h|_{t=0} = \mathbf{u}_{h0}(x', y), \\ \theta|_{y=0} = 0, \quad \theta|_{t=0} = \theta_0(x', y). \end{cases} \tag{4.11}$$

From $(4.11)_2 - (4.11)_5$, we can determine the temperature $\theta(t, x', y)$ immediately. Next, from the problem of $(\mathbf{u}_h, u_3)$ given in (4.11), one can easily deduce

$$(\mathbf{u}_h, u_3)(t, x', y) = (\tilde{\mathbf{u}}_h, \tilde{u}_3)(t, x', y) + (\bar{\mathbf{u}}_h, \bar{u}_3)(t, x', y), \tag{4.12}$$

with

$$\begin{cases} \tilde{\mathbf{u}}_h(t, x', y) = & \mathbf{u}_{h0}\big(x' - t\mathbf{U}_h(y), y\big) + t\mathbf{U}_h'(y) \int_0^y (\nabla_h \cdot \mathbf{u}_{h0})\big(x' - t\mathbf{U}_h(z), z\big) dz, \\ \tilde{u}_3(t, x', y) = & -\int_0^y \Big\{ (\nabla_{x'} \cdot \mathbf{u}_{h0})\big(x' - t\mathbf{U}_h(z), z\big) dz \\ & -t \int_0^y \big[\mathbf{U}_h(y) - \mathbf{U}_h(z)\big] \cdot \nabla_{x'}(\nabla_{x'} \cdot \mathbf{u}_{h0})\big(x' - t\mathbf{U}_h(z), z\big) \Big\} dz, \end{cases} \tag{4.13}$$

and

$$\begin{cases} \bar{\mathbf{u}}_h(t, x', y) = & \mathbf{U}_h'(y) \int_0^y \theta_0\big(x' - t\mathbf{U}_h(z), z\big) dz - \mathbf{U}_h'(y) \int_0^y \theta\big(t, x', z\big) dz, \\ \bar{u}_3(t, x', y) = & \theta_y(t, x', y) - \theta_y(t, x', 0) \\ & - \int_0^y \Big\{ \big[\mathbf{U}_h(y) - \mathbf{U}_h(z)\big] \cdot \nabla_{x'} \theta_0\big(x' - t\mathbf{U}_h(z), z\big) \Big\} dz \\ & + \int_0^y \Big\{ \big[\mathbf{U}_h(y) - \mathbf{U}_h(z)\big] \cdot \nabla_{x'} \theta\big(t, x', z\big) \Big\} dz. \end{cases} \tag{4.14}$$

From the above representation, we have:

THEOREM 4.2. *For the problem* (4.11) *in three space variables, we have the following results:*

(1) *If there is $k \in \mathbb{R}$ such that $U_2(y) = kU_1(y)$ holds for all $y \geq 0$, and $U_1(y)$ has no critical point in $y \geq 0$, then $\|u_3\|(t, y)$ is bounded uniformly in t, with $\|\cdot\|$ denoting the L^2-norm in $x' \in \mathbb{R}^2$.*
(2) *If there is $k \in \mathbb{R}$ such that $U_2(y) = kU_1(y)$ holds for all $y \geq 0$, and $U_1(y)$ has a single, non-degenerate critical point at $y = y_0 > 0$, then when $y > y_0$, for sufficiently large t,*

$$\|u_3\|(t, y) \ \geq \ C\sqrt{t} \, \frac{|U_1(y) - U_1(y_0)|}{\sqrt{|U_1''(y_0)|}}.$$

(3) *If for any given $k \in \mathbb{R}$, $U_2(y) = kU_1(y)$ does not hold for all $y \geq 0$, then there is a point $y_0 > 0$ such that, when $y > y_0$ we have that for sufficiently large t,*

$$\|u_3\|(t, y) \ \geq \ C\sqrt{t}.$$

<u>Idea of proof:</u> As $\theta(t, x', y)$ is a solution of a linear degenerated parabolic equation $(4.11)_2$, from (4.14) it is easy to estimate $(\bar{\mathbf{u}}_h, \bar{u}_3)$, so we only need to estimate $(\tilde{\mathbf{u}}_h, \tilde{u}_3)(t, x', y)$ from (4.13).

Denote by $\widehat{u}(\xi)$ with $\xi=(\xi_1,\xi_2)$, the Fourier transform of u in the $x'=(x_1,x_2)$ variables. From (4.13), it is easy to have

$$\begin{aligned}&\widehat{\widetilde{u}_3}(t,\xi,y) \qquad (4.15)\\ &=-\int_0^y\Big\{i\xi\cdot\widehat{\mathbf{u}_{h0}}(\xi,y)-t\big[\xi\cdot\big(\mathbf{U}_h(y)-\mathbf{U}_h(z)\big)\big]\cdot\big[\xi\cdot\widehat{\mathbf{u}_{h0}}(\xi,z)\big]\Big\}e^{-it\xi\cdot\mathbf{U}_h(z)}dz.\end{aligned}$$

(1) If $U_2(y)=kU_1(y)$ for some constant $k\in\mathbb{R}$, then from (4.15) we deduce

$$\widehat{\widetilde{u}_3}(t,\xi,y)=-\int_0^y\big[1-t(\xi_1+k\xi_2)\big(U_1(y)-U_1(z)\big)\big]\big[\xi\cdot\widehat{\mathbf{u}_{h0}}(\xi,z)\big]e^{-it(\xi_1+k\xi_2)U_1(z)}dz. \tag{4.16}$$

When $U_1(y)$ has no any critical point for all $y\geq 0$, by integration by parts and using Parseval's identity, one can get from (4.16) that $\|u_3(t,\cdot,y)\|_{L^2(\mathbb{R}^2)}$ is bounded uniformly in t. On the other hand, when $U_1(y)$ has a non-degenerate critical point at $y=y_0$, then by using the method of stationary phase in (4.16) we can have the estimate given in Theorem 4.2(2).

(2) Under the assumption given in Theorem 4.2(3), by a contradiction argument, one can show that there is a point $y_0>0$ such that

$$U_1'(y_0)U_2''(y_0)\ \neq\ U_2'(y_0)U_1''(y_0). \tag{4.17}$$

Without loss of generality, we may assume that $U_1'(y_0)>0$ and $U_1'(y_0)U_2''(y_0)-U_2'(y_0)U_1''(y_0)>0$, which implies there is an interval $S_\delta\subseteq(y_0-\delta,y_0+\delta)$ such that

$$U_1'(y)>0,\qquad U_1'(y)U_2''(y)-U_2'(y)U_1''(y)>0,\qquad \forall\, y\in S_\delta, \tag{4.18}$$

so the function $\frac{U_2'(y)}{U_1'(y)}$ is monotonically increasing in S_δ.

Denote by $I_\delta^R:=\Big\{\xi=(\xi_1,\xi_2)\in\mathbb{R}^2\backslash\{0\}:\ |\xi|\leq R,\text{ and }\exists y\in S_\delta,\ s.t.\ \xi\cdot\mathbf{U}_h'(y)=0\Big\}$. From the monotonicity of $\frac{U_2'(y)}{U_1'(y)}$ in S_δ, we know that for fixed $\xi\in I_\delta^R$, there is only one point $y\in S_\delta$ satisfying $\xi\cdot\mathbf{U}_h'(y)=0$. Thus, when $y>y_0$, $\delta\leq y-y_0$, we have $S_\delta\subseteq(0,y)$, for any $\xi\in I_\delta^R$ there exists a unique $y_\xi\in S_\delta$ such that $\xi\cdot\mathbf{U}_h'(y_\xi)=0$ and $\xi\cdot\mathbf{U}_h''(y_\xi)\neq 0$. At such (ξ,y), by applying the method of stationary phase in (4.15) one can finally conclude the estimate given in Theorem 4.2(3).

The detail calculation for these estimates can be found in [20].

Remark 4.2. *It is easy to know that the results of Theorem 4.2 hold as well when the temperature θ is a constant, i.e. for the corresponding problem of the three-dimensional linearized inviscid Prandtl equation. Moreover, from these results, one sees that the long time behavior of the flow to the three-dimensional linearized inviscid Prandtl equation coupld with or without the degenerated parabolic equation of the temperature depends on the flow pattern of the three-dimensional background shear flow $(\mathbf{U}_h(y),0)$, which is similar to the phenomenon which we observed in [23] for the instability of a shear flow in the three-dimensional Prandtl equation.*

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