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OSCILLATORY INTEGRALS INVOLVING THE CARLESON-SJÖLIN CONDITIONS AND SEVERAL APPLICATIONS*

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Dedicated to Professor Ling Hsiao on the occasion of her 80th birthday

Abstract. This short exposition is devoted to a brief overview of several challenging problems in modern harmonic analysis. Some of them have been through a considerably long period of time while the others are developed very recently. The clue that we intend to follow is the development of the restriction theorem in harmonic analysis and the theory of oscillatory integrals satisfying the Carleson-Sjölin conditions. We try to summarize all the recent results on these related problems.

Key words. Fourier integral operators, Kakeya and restriction conjecture, local smoothing, Carleson-Sjölin conditions, oscillatory integral.

Mathematics Subject Classification. 42B25, 42B20.

1. Introduction. Fourier analysis originates from the systematic study of the classical trigonometric series. It grows rapidly to an independent branch in modern mathematics, namely *harmonic analysis*, and is widely applied to almost every analytical problems nowadays. Among all the implements in harmonic analysis, the most frequently used object is Fourier transform, which is a fundamental idea of Fourier analysis, and also the simplest example of the *Fourier integral operator* (FIO).

Nowadays, Fourier integral operators play a considerably important role in the mathematical theory of partial differential equations, differential geometry, ergodicity, dynamical systems, classical and quantum mechanics, etc. There are an enormous amount of interesting questions with strong motivation either from mathematics or from physics, which are well investigated with the aid of FIOs. It turns out from empirical aspect that the approaches based on FIOs usually yield sharp or nearly sharp result. It is in this sense that an in-depth knowledge of as much properties for FIOs as possible is absolutely deserved.

To get a comprehensive survey on all kinds of FIOs arising from practical problems is by no means possible. However, a systematical theory has been developed on a special class of FIOs which is intimately connected to wave equations and we will confine ourselves to various of subjects around these specific integral operators. One more reason for us to focus on this special class is that they find direct applications in several topics such as Riesz means on manifolds, concentration of eigenfunctions along submanifolds (an alternative Fourier restriction phenomena), quantum ergodicity etc. Such operators often satisfy the so called *cinematic curvature* condition, and we would like to call it ccFIOs for brevity. They are not only frequently used in the topics alluded above, but also generate several extremely sophisticated open questions concerning themselves, for instance the *local smoothing conjecture* by Sogge and its applications to the variable coefficient version of maximal functions on manifolds.

This simple note is not intended to give an overall treatment from the didactical purpose. We start with Tomas-Stein's restriction and the Carleson-Sjölin condition in

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Section 2. Some extensions of this theory to the finite type of some oscillatory integral operators used in [53] will be addressed in Section3. In Section 4, we shall recall the Kakeya problem and its relation to the restriction theorem. The 5th section deals with the sharp and non-sharp local smoohing estimates for ccFIOs with symbols in different classes. Section 6 is devoted to the maximal Riesz means on compact manifold where some related problems and conjectures will be discussed. In Section 7, we will turn to a very important breakthrough in the quantitative estimates, which sheds new light on many problems we discused . Finally, we will take a look at several natural questions on the very recent results upon eigenfunctions, where we will encounter the Carleson-Sjölin condition once more.

We hope this brief and concise paper of manuscript can be of some help for those who would like to contribute to this area.

2. The Carleson-Sjölin conditions and Tomas-Stein's restriction theorem. The Bochner-Riesz spherical summation operator plays a fundamental role in the theory of Fourier transforms. These operators are defined on functions on $\mathbb{R}^n$ by the formula

$$\widehat{T_\lambda f}(\xi) = m_\lambda(\xi)\widehat{f}(\xi),$$

where $m_\lambda(\xi) = (1-|\xi|^2)^\lambda$ if $|\xi| \le 1$ and $m_\lambda(\xi) = 0$ elsewhere.

Fefferman [28] pointed out that T_λ is never bounded outside the range

$$p(\lambda) = \frac{2n}{n+1+2\lambda} < p < \frac{2n}{n-1-2\lambda} = \tilde{p}(\lambda),$$

and he also proved for $n \ge 2$ that T_λ is bounded on $L^p(\mathbb{R}^n)$ provided that $p(\lambda) < p < \tilde{p}(\lambda)$ and $\lambda > (n-1)/4$, where he made use of a remarkable observation by E. M. Stein, namely the following *a priori* inequality

$$\Big(\int_{\mathbb{S}^{n-1}} |\widehat{f}(\xi)|^2 d\sigma(\xi)\Big)^{\frac{1}{2}} \le A_p\|f\|_p,\ n>1 \tag{2.1}$$

holds for $1 \le p < 4n/(3n+1)$. This is a primary version of the celebrated restriction theorem

$$\Big(\int_{\mathbb{S}^{n-1}} |\widehat{f}(\xi)|^2 d\sigma(\xi)\Big)^{\frac{1}{2}} \le A_p\|f\|_p,\ 1 \le p \le \tfrac{2(n+1)}{n+3}, \quad n>1. \tag{2.2}$$

It is a very deep conjecture asserting that if the L^2-norm is replaced by L^r-norm in (2.1), i.e.

$$\Big(\int_{\mathbb{S}^{n-1}} |\widehat{f}(\xi)|^r d\sigma(\xi)\Big)^{\frac{1}{r}} \le A_p\|f\|_p,\ n>1$$

then it holds for all $1 \le p < 2n/(n+1)$ and $1 \le r \le \infty$.

The Bochner-Reisz conjecture asserts that the necessary condition $p(\lambda) < p < \tilde{p}(\lambda)$ is also sufficient for T_λ to be bounded on $L^p(\mathbb{R}^n)$ for all $\lambda > 0$ and $n \ge 2$. This conjecture is very deep and still widely open in general cases. However, in the two dimensional case $n=2$, it was completely resolved by Carleson-Sjölin [19], Hörmander [45], Fefferman[29] and Cordoba [22]. Simultaneously, the methods they adopted confirms the two dimensional restriction theorem as well.

As observed by Hörmander, to unify the treatment of the restriction and Bochner-Riesz problem, it is very natural to consider the following oscillatory integral, where we adopt notations from [53]

$$T_\lambda f(z) = \int_{\mathbb{R}^{n-1}} e^{i\lambda\Phi(z,y)} a(z,y) f(y) dy, \tag{2.3}$$

where $a \in C_0^\infty(\mathbb{R}^{2n-1})$ and $\Phi \in C^\infty(\mathbb{R}^{2n-1})$ is real valued, fulfilling the Carleson-Sjölin condition

$$\operatorname{rank}\Big(\frac{\partial^2 \Phi}{\partial z \partial y}\Big) \equiv n-1, \ \ (z,y) \in \operatorname{supp} a(z,y), \tag{2.4}$$

$$\det\Big(\frac{\partial^2}{\partial y_j \partial y_k} \langle \Phi'_z, \theta\rangle\Big) \neq 0, \ \ (z,y) \in \operatorname{supp} a(z,y), \tag{2.5}$$

where $\theta \in \mathbb{R}^n \setminus \{0\}$ is the unique unit vector such that $\nabla_y \langle \Phi'_z, \pm\theta\rangle = 0$.

As remarked in [53], the Carleson-Sjölin condition is an assumption invariant under change of variables. Geometrically, it says that the projection maps

$$\begin{tikzcd}
 & \mathscr{C}_\Phi \equiv \{(z, \Phi'_z, y, -\Phi'_y)\} \arrow[dl, "\mathit{\Pi}_{T^*Y}"'] \arrow[d, "\mathit{\Pi}_Z"'] \arrow[dr, "\mathit{\Pi}_{T_z^*Z}"] & \\
T^*Y & Z & \mathcal{S}_z \subset T_z^*Z
\end{tikzcd}$$

have the property that $\operatorname{rank} d\mathit{\Pi}_{T^*Y} \equiv 2(n-1)$, and $\mathcal{S}_z := \mathit{\Pi}_{T_z^*Z}(\mathscr{C}_\Phi)$ is an immersed hypersurface with nowhere vanishing Gaussian curvature.

The operators defined via oscillatory integral (2.3) are of great importance since they are linked in a sense to the problems of Bochner-Riesz means and restriction theorems as formulated by Hörmander [45]. In particular, if we set $y \mapsto \Psi(y)$ to be an immersion of $\mathbb{R}^n$ as a surface of total curvature $\neq 0$ and

$$Sf(z) = \int_{\mathbb{R}^{n-1}} e^{i\langle z, \Psi(y)\rangle} a_0(y) f(y) dy,$$

with $a_0(y) \in C_0^\infty(\mathbb{R}^{n-1})$, then one can evaluate Sf in certain L^q space which is bounded in the following way

$$\|Sf\|_q \leq C_{q,r} \|f\|_r \tag{2.6}$$

as a consequence of a dispersive estimate for (2.3)

$$\|T_\lambda f\|_q \leq C_{q,r} \lambda^{-\frac{n}{q}} \|f\|_r. \tag{2.7}$$

A remarkable question is then raised by Hörmander asking whether

$$q > \frac{2n}{n-1}, \quad \frac{n+1}{q} \leq \frac{n-1}{r'}, \tag{2.8}$$

known as a necessary condition for (2.7), is also sufficient.

The answer is affirmative when $n = 1$ as indicated above and this corresponds to the celebrated Carleson-Sjölin-Hörmander-Stein's full adjoint restriction theorem in dimension two, as well as Cordoba's theorem on the Bochner-Riesz multiplier on $L^p(\mathbb{R}^2)$. In higher dimensions, Stein [69] proved based on analytic interpolation that (2.7) is true for $r = 2$ and $q \geq \frac{2(n+1)}{n-1}$, known as the adjoint Tomas-Stein restriction theorem. Previously, Tomas [70] proved the restriction estimate (2.2) holds for $1 \leq p < \frac{2(n+1)}{n+3}$, with the endpoint $p = \frac{2(n+1)}{n+3}$ settled by Stein's interpolation. Since this result deals with the case for $r = 2$, Tomas-Stein's estimates is also called the L^2-restriction theorem and it plays a very important role in the theory of nonlinear dispersive equations, known as various kinds of Strichartz's estimates. Bourgain [6] obtained certain counterexamples showing that Hörmander's conjecture, formulated in such a general manner, does not always hold true. In [14], the progress has been made on Hörmander's problems by the multilinear restriction estimates and in a very recent work [39], Hörmander's problem was settled by using the polynomial partitionning argument.

3. Oscillatory integral operators of finite order type. In applications as well as for the sake of theoretical completeness, it is interesting and useful to study the cases where the curvature assumption (2.5) fails on some subset of points. It turns out that such kind of oscillatory integrals occurs frequently in a wide range of subjects of mathematical analysis, where a representative as a substitute of the curvature assumption is the condition of *finite type* imposed on the underlying surfaces[1].

In the past decades, hypersurfaces of finite type have received extensive investigations in harmonic analysis since it connects with a few different mathematical branches. These estimates play a crucial role in a number of problems, such as pseudoconvex domain of finite type in the theory of several complex variables, Riesz means on a convex domain in $\mathbb{R}^n$ with boundary of finite line type, asymptotic estimates on the discrepancy function of the lattice points $D(\lambda) = \#(\lambda B \cap \mathbb{Z}^n) - \#(\lambda^n B)$ as $\lambda \to \infty$, where B is a convex body whose boundary ∂B has finite type properties. Moreover, there are also applications in problems with strong physical background. In [20], the authors contribute a systematic treatment to a class of dispersive estimates for classical and general wave equations, Klein-Gordon equations, Schrödinger equations under the geometric assumptions of finite type, where the phase is inhomogeneous and allowed to have vanishing Gaussian curvature along curves. The restriction theorem for Fourier transforms on certain hypersurfaces with vanishing curvature, together with some specific geometric assumptions, is found useful in the study of the long-time behavior of the random Schrödinger equation

$$i\partial_t \psi(t,x) = \Big[-\frac{1}{2}\Delta_x + \lambda V(x)\Big]\psi(x), \psi(t,x) \in L^2(\mathbb{R}^3) \tag{3.1}$$

in the quantum diffusion theory.

Considering the importance of various finite type properties, we are interested in studying estimates like (2.7) for oscillatory integrals of the form (2.3) with the modified Carleson-Sjölin condition where the curvature assumption (2.5) can be weakened to include the degenerating second fundamental forms. Certainly, a natural route of exploration is to develop this theory with finite type assumption. It can be regarded in a sense as the variable coefficient version of the objects alluded to the last paragraph, and it might be interesting to find its relation to those things as what Hörmander did in [45].

[1]We refer to [20, 69] for precise definition of finite type.

4. Kakeya conjecture and restriction theorems beyond Tomas-Stein. In higher dimensions $n \geq 3$ the adjoint Fourier restriction problem and Bochner-Riesz means are still widely open, especially the non-L^2 restriction conjecture. Indeed, it is demonstrated by Bourgain [6] that in general, Hörmander's conjecture is *not* true. The pathology involves considerations of Kakeya compression phenomenon. We remark that although Hörmander's conjecture fails in general, it still works for some special phase functions including that involved in the Bochner-Riesz means operator. We refer to [14, 37, 38] for the recent development.

The existence of Kakeya type sets exhibits essential distinctions between higher Euclidean spaces and the two dimension case. Even though Kakeya compression phenomena in higher dimensions prevent Hörmander's approach to Bochner-Riesz and restriction problem, understanding the Kakeya problem can also bring about some progress to the knowledge of the latter problems, especially the restriction theorem beyond Tomas-Stein's L^2-scheme. In fact, when the left side of (2.1) is replaced by its L^1-norm, it is possible to be bounded by $\|f\|_{L^p(\mathbb{R}^n)}$ with some p between $2(n+1)/(n+3)$ and $2n/(n+1)$. This is the first result on the non-L^2 restriction inequality, which is due to Bourgain[5] by using some inequalities for Kakeya maximal functions.

The Kakeya maximal functions were introduced by Bourgain [5] to evaluate the Hausdorff dimension of Kakeya sets in $\mathbb{R}^n$. The Nikodym type maximal function was employed by Cordoba to resolve the two dimensional Bochner-Riesz summation problem. In [5], Bourgain formulated them in a standard form that now people use

$$f_\delta^*(\xi) = \delta^{-(n-1)} \sup_{a \in \mathbb{R}^n} \int_{T_a^\delta(\xi)} |f(x)| dx \tag{4.1}$$

where $\xi \in \mathbb{S}^{n-1}$ and $T_a^\delta(\xi)$ is a tube of dimension $1 \times \underbrace{\delta \times \cdots \times \delta}_{n-1}$ centered at a. The Nikodym maximal function is dual to the Kakeya maximal function as follows

$$f_\delta^{**}(x) = \delta^{-(n-1)} \sup_{\xi \in \mathbb{S}^{n-1}} \int_{T_x^\delta(\xi)} |f(y)| dy. \tag{4.2}$$

The Kakeya maximal conjecture asserts that for $1 \leq p \leq n$

$$\|f_\delta^*\|_{L^p(\mathbb{S}^{n-1})} \leq C_\varepsilon \Big(\frac{1}{\delta}\Big)^{\frac{n}{p}-1+\varepsilon} \|f\|_{L^p(\mathbb{R}^n)}, \forall\, \varepsilon > 0. \tag{4.3}$$

Bourgain in [5] made a crucial observation that although the restriction conjecture implies the Kakeya maximal conjecture, it is possible to derive some progress on the restriction theorem conversely.

The Kakeya conjecture was settled by Cordoba [22] in dimension two. In higher dimensions, Drury [24] proved the case when $p = (n+1)/2$. Bourgain [5] improved this result for each $n \geq 3$ to $p = \varepsilon_n + (n+1)/2$ for some $0 < \varepsilon_n < 1/2$ by using the bush argument. Four years later, Wolff [78] refined Bourgain's bush argument and obtained the classical $(n+2)/2-$estimate by using his induction on scale argument. Based on some observations from the arithmetic combinatorics, Bourgain [8] was able to improve Wolff's result further for $n \geq 8$, where some part of his records are refreshed later by Katz-Tao [46].

Wolff's $L^{\frac{n+2}{2}}-$estimate for $n \geq 3$ also coincides with the two dimensional case exactly well. By inputting some new ingredients of observation, induction on scales argument with the one involving the auxiliary maximal functions, Sogge [60] was able to generalize Wolff's estimate to the Nikodym type estimate in the case where the ambient space is three dimensional non-Euclidean space. This strategy indeed can be used to give a new proof of Wolff's $L^{\frac{n+2}{2}}$-estimates for $n \geq 3$, and we refer to the work [50] and [81] for more details. The key assumption on the manifold is the *totally geodesic* property, which is fulfilled by manifolds of constant curvatures. Without this assumption, there are counterexamples to show the easily deduced estimate $p = (n+1)/2$ can not be improved to more general index of p [59].

There are some other improvements related to the Kakeya problem, but we just focus on the most impressive ones. The interested readers may consult [41, 44, 47, 48, 74, 83] for further investigations. We will return to Kakeya problem and its applications to some very recent results of problems in mathematical analysis.

5. The deepest conjecture: sharp and non-sharp local smoothing for ccFIOs. Let Y and Z be C^∞ paracompact manifolds of dimension n and $n+1$ and $I^\sigma(Z, Y; \mathscr{C})$ denote a class of $\sigma-$order Fourier integral operators, which is determined by the properties of the canonical relation of $\mathscr{C}$.

We assume $\mathscr{C}$ is a canonical relation from $T^*Y\backslash 0$ to $T^*Z\backslash 0$ which is homogeneous, Lagrangian with respect to the symplectic form $d\zeta \wedge dz - d\eta \wedge dy$ and closed in $T^*Z \setminus 0 \times T^*Y \setminus 0$. Thus $\mathscr{C} \subset T^*Z \setminus 0 \times T^*Y \setminus 0$ is a conic submanifold of dimension $2n+1$.

Consider

$$\begin{tikzcd} & \mathscr{C} \arrow[dl, "\Pi_{T^*Y}"'] \arrow[d, "\Pi_Z"'] \arrow[dr, "\Pi_{T^*_z Z}"] & \\ T^*Y \setminus 0 & Z & T^*_z Z \setminus 0 \end{tikzcd}$$

We assume the following nondegeneracy condition

$$\operatorname{rank} d\Pi_{T^*Y} \equiv 2n, \quad \operatorname{rank} d\Pi_Z \equiv n+1.$$

To describe the second assumption, we denote by

$$\Gamma_{z_0} = \Pi_{T^*_{z_0} Z}(\mathscr{C}).$$

Then Γ_{z_0} is a smooth immersed hypersurface in $T^*_{z_0} Z \setminus 0$. We assume that for every $\zeta \in \Gamma_{z_0}$, $n-1$ principal curvatures do not vanish. These two assumptions constitute the so called *cinematic curvature* condition formulated in [57]. In following context, the Fourier integral operators which satisfy the *cinematic curvature* condition are called ccFIOs.

The main object in [53] is to establish some L^p-local smoothing estimate for ccFIOs belonging to $I^{\mu-1/4}(Z, Y; \mathscr{C})$, where μ satisfies a condition like

$$\mu < -(n-1)\Big(\frac{1}{2} - \frac{1}{p}\Big) + \varepsilon(p,n),$$

for some $\varepsilon(p,n) > 0$. Here, we consider a ccFIO $\mathscr{F} \in I^{\mu-1/4}$ which can be written microlocally as the form

$$\mathscr{F} f(z) = \int_{\mathbb{R}^n} e^{i\varphi(z,\eta)} a(z,\eta) \widehat{f}(\eta) d\eta, \tag{5.1}$$

where φ satisfies the conditions described in Section 2 of [53], and $a(z,\eta) \in C^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^n)$ is compactly supported in z having the property that

$$|\partial_z^\gamma \partial_\eta^\alpha a(z,\eta)| \leq C_{\alpha,\gamma}(1+|\eta|)^{\mu-\rho|\alpha|+\delta|\gamma|}, \rho = 1, \delta = 0.$$

This is an abstract geometrically invariant generalization of the local smoothing conjecture for wave equations, formulated by Sogge [57], and it is the deepest one among a series of analytical conjectures. There are many sophisticated papers dedicated to improving the previously known result, however, we chose to avoid them in this paper since otherwise it would consume a considerable content which is less related to our theme here. The interested readers may consult with [35]. In two recent papers[32, 33], the authors, collaborated with C. Gao, obtained certain improvement at the level of abstact frame work. In the constant coefficient case, the problem is settled by Guth, Wang and Zhang [40] in 2D, where the variable coefficient case is solved by Gao, Liu, Miao and Xi in [34].

For ccFIOs associated to this class of symbols $a(z,\eta)$, the local smoothing estimates obtained in [53] is far from being optimal. However, if one considers the same local smoothing properties for ccFIOs with symbols as above with $\rho = \delta = 1/2$, then it is possible to obtain the sharp result which can be found in Theorem 4.8 of [53].

This motivates to study the intermediate cases between the two extremes where we may take $\rho+\delta = 1, 0 \leq \delta \leq \rho \leq 1$ and classify for what (ρ,δ) it is possible to get the sharp local smoothing estimates. Such operators are also useful in practice. For example, they naturally appear in the microlocal wave packet decompositions of the argument of [2] and play a crucial role in the microlocal Kakeya-Nikodym averages of eigenfunctions to which we will return.

6. The maximal Bochner-Riesz means on compact manifolds. To exhibit the importance of the local smoothing conjecture, we examine its relation to the weaker conjecture which is also stronger than the rest series of conjectures such as restriction, Kakeya etc [71]. This conjecture is about the maximal version of Bochner-Riesz means.

Let f be a Schwartz function on $\mathbb{R}^2$ and define its Bochner-Riesz means of order $\delta > 0$ to be

$$B_R^\delta(f)(x) = \int_{\mathbb{R}^2} (1-|\xi/R|^2)_+^\delta \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi,\ R > 0.$$

It is proved by Carleson-Sjölin, Hörmander and Cordoba that if $0 < \delta \leq 1/2$ and $\frac{4}{3+2\delta} < p < \frac{4}{1-2\delta}$, for all $f \in L^p(\mathbb{R}^2)$, one has $B_R^\delta(f) \to f$ in $L^p(\mathbb{R}^2)$. Since $\delta > 1/2$ implies the integrability of the kernel of the Bochner-Riesz multiplier, the result for $1 \leq p \leq \infty$ follows from Young's inequality.

A more sophisticated problem is about the almost everywhere convergence of the limit $\lim\limits_{R\to\infty} B_R^\delta(f)(x)$. For $p > 2$, this question was completely settled by Carbery [17]. In particular, we have $B_R^\delta(f)(x) \to f(x), a.e.$ as $R \to \infty$ for arbitrary $L^p(\mathbb{R}^2)$ functions f with $p > 2$ if and only if $\delta > \max(0, \frac{1}{2} - \frac{2}{p})$. An alternative approach can be found in Chapter 2 of [59], on account of the square-function method and Kakeya type maximal inequalities which is reminiscent of the argument in [53] in dealing with local smoothing effect of ccFIOs. Indeed, the local smoothing conjecture is strong enough to imply the correct estimates on maximal Bochner-Riesz operator. Let us take a digression to this fact.

Local smoothing conjecture implies maximal Bochner-Riesz. It suffices to consider $p = 4$ for the same reason in Chapter 2 in [59]. Take $\eta \in C_0^\infty(\mathbb{R}^2)$ adapted to the interval $[1/2 \le |\xi| \le 2]$, and define

$$\widehat{T_R^\delta f}(\xi) = \eta\Big(\frac{\xi}{R}\Big)\Big(1 - \Big|\frac{\xi}{R}\Big|\Big)_+^\delta \widehat{f}(\xi).$$

Let

$$\mathcal{E}_R^\delta f(x) \triangleq B_R^\delta f(x) - T_R^\delta f(x),$$

then

$$\sup_{R>0} \mathcal{E}_R^\delta(f)(x) \le c\, Mf(x),$$

since $(1-\eta(\xi/R))(1-|\xi/R|)_+^\delta \in C_0^\infty(\mathbb{R}^n)$, where $Mf(x)$ is the Hardy-Littlewood maximal function. Noting that

$$T_*^\delta f(x) \triangleq \sup_{R>0} |T_R^\delta f(x)|^p \le \sum_{k\in\mathbb{Z}} \sup_{2^k \le R \le 2^{k+1}} \big|T_R^\delta\big(\sum_{|j-k|\le 2} \dot{\Delta}_j f(x)\big)\big|^p,$$

we have by square-function estimates and Young's inequality

$$\big\|T_*^\delta f\big\|_p^p \le \sum_k \Big\|\big(\sum_{|j-k|\le 2} |\dot{\Delta}_j f|^2\big)^{\frac{1}{2}}\Big\|_p^p \le \|f\|_p^p, \tag{6.1}$$

provided

$$\big\|\sup_{2^k \le R \le 2^{k+1}} |T_R^\delta f|\,\big\|_p \le c\,\|f\|_p, \quad \operatorname{supp} \widehat{f} \subset [2^{k-3} \le |\xi| \le 2^{k+3}]. \tag{6.2}$$

After a re-scaling, we have

$$\sup_{2^k \le R \le 2^{k+1}} |T_R^\delta f(x)| \le \sup_{1\le R\le 2} |B_R^\delta f_k|(2^k x),$$

where $f_k(x) = f(2^{-k}x)$. We may thus reduce (6.2) to

$$\big\|\sup_{1\le R\le 2} |B_R^\delta f|\,\big\|_p \le c_\delta\,\|f\|_p, \quad \operatorname{supp} f \subset [2^{-3} \le |\xi| \le 2^3]. \tag{6.3}$$

Using that t_+^δ is the inverse Fourier transform of $(t+i0)^{-\delta-1}$, we rewrite

$$B_R^\delta f(x) = B_{R,0}^\delta f(x) + \sum_{k\ge 1} B_{R,k}^\delta f(x),$$

where

$$B_{R,k}^\delta f(x) := R^{-\delta} \int e^{-iRt} \eta_k(t)(t+i0)^{-\delta-1} e^{it\sqrt{-\Delta}} f(x)\,dt,$$

with $\eta_k(t) = \eta(2^{-k}t)$ for $k \ge 1$ and $\eta_0(t) = 1 - \sum_{k\ge 1}\eta_k(t)$. Invoking the elementary inequality

$$\sup_s |G(s)|^2 \le |G(0)|^2 + 2\|G\|_2\|G'\|_2$$

and introducing a smooth function $\rho(R)$ supported in $[\frac{1}{2}, 4]$, we have

$$\sup_{1\le R\le 2} |B^{\delta}_{R,\,k} f|^2(x) \le c\, \|\rho \widehat{F}_k\|_{L^2_R}\, \|(\rho \widehat{F}_k)'\|_{L^2_R},$$

where $F_k(t) := \eta_k(t)(t+i0)^{-\delta-1} e^{it\sqrt{-\Delta}} f$. By Plancherel, this is estimated further by

$$\Big[\int_{2^k\le t\le 2^{k+1}} |e^{it\sqrt{-\Delta}} f|^2 \frac{dt}{t^{-2(1+\delta)}}\Big]^{\frac{1}{2}} \Big[\int_{2^k\le t\le 2^{k+1}} |e^{it\sqrt{-\Delta}} f|^2 \frac{dt}{t^{-2\delta}}\Big]^{\frac{1}{2}}. \tag{6.4}$$

Thus by Cauchy-Schwarz

$$\begin{aligned}\|\sup_{1<R<2} |B^{\delta}_{R,k} f|\,\|^2_p, \le & \|e^{it\sqrt{-\Delta}} f\|_{L^p_x L^2_{t\sim 2^k}(t^{-2(1+\delta)}dt)} \|e^{it\sqrt{-\Delta}} f\|_{L^p_x L^2_{t\sim 2^k}(t^{-2\delta}dt)} \\ \le & 2^{-k(1+\delta-\frac{1}{2}-\frac{\epsilon}{2}+\delta-\frac{1}{2}-\frac{\epsilon}{2})} \|e^{it\sqrt{-\Delta}} f\|^2_{L^p_x L^2_{t\sim 2^k}(t^{-(1+\epsilon)}dt)}.\end{aligned}$$

We only need to show

$$\|e^{it\sqrt{-\Delta}} f\|^2_{L^p_x L^2_{t\sim 2^k}(t^{-(1+\epsilon)}dt)} \le c_\epsilon\, \|f\|_p. \tag{6.5}$$

Changing variables $t \to 2^k t$ and setting $\widehat{f}^k(\xi) = 2^{-kd} \widehat{f}(2^{-k}\xi)$, one verifies by simple scaling that (6.5) is equivalent to the following weaker version of local smoothing estimate

$$\|e^{it\sqrt{\Delta}} f\|_{L^p_x L^2_t[1,\,2]} \le c\, \|f\|_{\epsilon,\,p}, \quad \operatorname{supp} \widehat{f} \subset \{\xi,\ 2^{k-3} \le |\xi| < 2^{k+3}\},$$

where $\|\cdot\|_{\epsilon,\,p}$ denote the norm of Sobolev space $W^{\epsilon,\,p}$. □

Now that we have understood in a partial sense the local smoothing for ccFIOs on manifolds, it is natural to ask the question how to use that strategy to obtain the maximal estimate for Reisz means on some Riemann manifolds (M, g). The authors in [53] proved the maximal inequality

$$\begin{cases} \|S^{\delta}_* f\|_{L^p(M)} \le C\|f\|_p, \ \ 2 \le p < \infty, \ \ S^{\delta}_* f(x) \triangleq \sup\limits_{R>0} \big|S^{\delta}_R f(x)\big|, \\ S^{\delta}_R f(x) \triangleq \sum\limits_{k\le R} \Big(1 - \dfrac{k}{R}\Big)^{\delta} E_k f(x), \ \ E_k \text{ projection operator} \end{cases} \tag{6.6}$$

under the same condition as Carbery's for

$$2 \le p < \infty, \ \ \delta > \max\Big\{2\Big|\frac{1}{2} - \frac{1}{p}\Big| - \frac{1}{2},\ 0\Big\},$$

where the two dimensional manifold (M, g) is assumed to be Zoll in the sense that all the geodesic flow on M is periodic with a minimal period. We are interested in general manifolds (M, g) with other different type of geometric assumptions.

Further more, we also would like to investigate the case for $p < 2$. This becomes much more delicate and difficult to deal with as observed in [73], where it is shown that this maximal inequality (6.6) fails if $\delta < \frac{3}{2p} - 1$. Tao also obtained some positive result on the almost everywhere convergence under the condition that $1 < p < 2$ and

$$\delta > \max\Big(\frac{3}{4p} - \frac{3}{8},\ \frac{7}{6p} - \frac{2}{3}\Big), \tag{6.7}$$

which to the authors' knowledge is the best result so far. Our second problem in this section cares for the possibility of extending Tao's result alluded above to the Zoll surfaces. This would be added to the progress to the result of [53] for $p < 2$. However, it seems not easy to carry Tao's argument directly to handle the maximal Bochner-Riesz operator on compact Zoll surfaces, for instance $\mathbb{S}^2$.

One may also consider the higher dimensional counterparts of the results for Bochner-Riesz in [53]. To attack this, the bilinear or multilinear estimates for oscillatory integrals will necessarily be involved. We will not pursue these matters here.

At the end of this section, we include the chain of implications of the series of conjectures as collected in [71]:

$$\begin{array}{rcl} \text{local smoothing} \Rightarrow \text{maximal Bochner-Riesz} & & \\ \Downarrow & & \\ \text{Bochner-Riesz} \Rightarrow \text{restriction} \Rightarrow \text{Kakeya.} & & \end{array}$$

Here we do not consider the parabolic case.

7. Multilinear Kakeya/restriction theorems opens up a new approach. From this section on, we shall turn to the very recent progress on harmonic analysis that we promised in the previous contexts and the objects we are ready to consider are the main subjects of the aim of this article.

Before we go further, we have to confess that all of the above problems and conjectures had arrived at a extremely high level, where the techniques are developed almost to their limits. However, in very recent years, a completely new method arises. This method is so effective that even crude manipulations based on its idea would bring about certain progress.

Roughly speaking, this method is composed of two aspects. The first is the use of the *multilinear restriction theorem* due to Bennett-Carbery-Tao [1], while the second one is the polynomial method motivated by Dvir's proof of the Kakeya conjecture in finite fields [27] and developed by Guth [36] to prove the endpoint multilinear Kakeya conjecture of Bennett-Carbery-Tao [1]. The polynomial method can also be applied to some other problems such as the Erdös distinct distance problem, however we will not include it in this paper.

It has turned out that the multilinear restriction theorem and multilinear Kakeya theorem are much deeper observations in approaching to the linear ones. In fact, the curvature assumption that the linear restriction conjecture relies on is replaced by the transversality condition in the multilinear case, which can still be verified even though the curvature condition no longer holds. The multilinear restriction and Kakeya conjecture was formulated by Bennett-Carbery-Tao [1], where the authors proved the non-endpoint case and the endpoint case of the multilinear Kakeya conjecture was resolved by Guth [36]. The above multilinear restrictin theorem with full multiplicity is a special case of a more general theory concerning the varying multiplicities with respect to the dimension [23]. In particular, the classical bilinear Fourier restriction theorem is initiated from the work of Bourgain [7] and Klainerman-Machedon [31], which is developed further by Tao-Vargas-Vega [76], and Tao-Vargas [77] with applications to PDEs. The first sharp result on the cone was obtained by Wolff [80] by using the celebrated induction on scale argument, apart from the endpoint case which is settled by Tao [72]. The sharp result on the paraboloid is established by Tao [75] based on the same idea of Wolff without proving the endpoint case. By using the enhanced induction argument in [72] and the method of descent, this question is recently settled by the second author [82]. Interestingly, one finds applications of the bilinear restriction estimates in the Calderón's problem for conductivities [42, 43, 30] and it is pointed out in [HKL] that further improvements would be available provided the endpoint bilinear estimates were true.

It was realized immediately in the work [14] that the result in [1] and the polynomial method will become a fundamental step in penetrating almost all the classical conjectures we have discussed before this section and indeed, the authors in [14] obtained new results on the restriction conjecture by showing that

$$\|\widehat{fd\sigma}\|_{L^p(\mathbb{R}^n)} \leq C_p\|f\|_{L^\infty(S,d\sigma)}$$

is valid for S a surface with nowhere vanishing Gauss curvature and $p > p_n$, where

$$p_n = \begin{cases} 2(4n+3)/(4n-3), & \text{if } n \equiv 0\,(\mathrm{mod} 3), \\ (2n+1)/(n-1), & \text{if } n \equiv 1\,(\mathrm{mod} 3), \\ 4(n+1)/(2n-1), & \text{if } n \equiv 2\,(\mathrm{mod} 3). \end{cases} \tag{7.1}$$

If S enjoys some symmetries, the standard factorization theorem along with the above result would yield the following restriction theorem

$$\|\widehat{f}\|_{L^p(S,d\sigma)} \leq C_p\|f\|_{L^p(\mathbb{R}^n)}$$

for $1 \leq p < p_n'$. Moreover, the three dimension case is refined in [14] to $p > 3.3$ by establishing an induction-on-scales inequality, which we would like to call the Bourgain-Guth inequality. The improvements in higher dimensions is obtained by Guth [38]. It is very essential in getting improvement of the results on Carleson's problem about the maximal Schrödinger operator

$$\mathscr{S}(f)(x) = \sup_{0<t<1}\left|e^{it\Delta}f(x)\right|, x \in \mathbb{R}^n.$$

In fact, by using the induction-on-scales inequality alluded above, Bourgain[9] obtained

$$\left\|\mathscr{S}f\right\|_{L^2(B(0,1))} \leq C\|f\|_{H^s(\mathbb{R}^n)},\ s > \frac{1}{2} - \frac{1}{4n}. \tag{7.2}$$

The case $n = 1$ was completely settled by Carleson [18]. Previous to [9], one only knows (7.2) for $s > 3/8$ in the case $n = 2$ [49] and $s > 1/2$ in the higher dimensional case $n \geq 3$ [54]. We refer two recent works concerning this classical problems[25, 26].

We need to indicate that in [54], Sjölin proved the more general result concerning the fractional order operator

$$\Big\|\sup_{0<t<1}|e^{it(-\Delta)^{a/2}}f|\Big\|_{L^2(B(0,1))} \leq C\|f\|_{H^s(\mathbb{R}^n)},\ s > 1/2, \tag{7.3}$$

for all $a > 1$ and in particular, one can refine the method to get $s \geq 1/2$ if $n = 2$. Dealing with the general case $a > 1$ is more delicate than $a = 2$ and it is proved in [51] that if $n = 2$, then (7.3) is valid for all $s > 3/8$, and if $n \geq 3$ one is able to obtain (7.3) for some $s < 1/2$. We mainly adopted the argument of [9] however rebuild the Bourgain-Guth inequality and handled some other intricate points where the argument in [9] breaks down. It is interesting to ask whether (7.3) holds in the higher dimensions for all $s > (2n-1)/4n$ and $a > 1$.

The method based on multilinear restriction estimates developed by Bourgain and Guth [14] has further applications. In the rest part of this section, we will introduce some of the most important results.

7.1. Moment inequalities for trigonometric polynomials. Let

$$f(x) = \sum_{z\in\mathcal{E}} a_z e^{2\pi i x\cdot z}$$

where $\mathcal{E}$ stands for the set of $\mathbb{Z}^n$-points on some dilate λS of a fixed compact surface S in $\mathbb{R}^n$ with positive definite second fundamental form. Assume S is the standard unit sphere in $\mathbb{R}^n$. Then f is the eigenfunction for the Laplacian on $\mathbb{T}^n$ and

$$-\Delta f = \lambda^2 f.$$

It is conjectured that

$$\|f\|_{L^q(\mathbb{T}^n)} \leq C_q \|f\|_{L^2(\mathbb{T}^n)}, \quad q < \frac{2n}{n-2}, \tag{7.4}$$

and

$$\|f\|_{L^q(\mathbb{T}^n)} \leq C_q \lambda^{n(\frac{1}{2}-\frac{1}{q})-1} \|f\|_{L^2(\mathbb{T}^n)}, \quad q > \frac{2n}{n-2}. \tag{7.5}$$

Using the idea in [14], Bourgain [10] proved

$$\|f\|_{L^p(\mathbb{T}^n)} \leq C_\varepsilon \lambda^\varepsilon \|f\|_{L^2(\mathbb{T}^n)},\ p = \frac{2n}{n-1}. \tag{7.6}$$

The argument in [10] also yields some further improvement for the scaling invariant Strichartz estimates for periodic Schrödinger equations, namely for $q > \frac{2(n+3)}{n}$ and $n \geq 4$

$$\|e^{it\Delta} P_{\leq N}\phi\|_{L^q(\mathbb{T}^{n+1})} \leq C_q N^{\frac{n}{2}-\frac{n+2}{q}} \|\phi\|_2. \tag{7.7}$$

In a successive work, Bourgain and Demeter [12] proved the following nearly sharp Strichartz estimate in a very general framework

$$\|e^{it\Delta} P_{\leq N}\phi\|_{L^p(\mathbb{T}^n_\theta\times I)} \leq C_\varepsilon N^{\frac{n}{2}-\frac{n+2}{p}+\varepsilon} |I|^{1/p} \|\phi\|_2,\ p \geq \frac{2(n+1)}{n-1}, \tag{7.8}$$

where $\mathbb{T}^n_\theta = \prod_{j=1}^n (\mathbb{R}/\theta_j\mathbb{Z})$ with $1/2 < \theta_j < 2$.

The proof relies on a very deep result called the ℓ^2-decoupling theorem. To describe this theorem, we denote by

$$P^{n-1} := \{(\xi_1, \cdots, \xi_{n-1}, \xi_1^2 + \cdots + \xi_{n-1}^2) \in \mathbb{R}^n : |\xi_j| \leq 1/2\}$$

and $\mathcal{N}_\delta$ be the $\delta-$neighborhood of P^{n-1} and let $\mathcal{P}_\delta$ be a finitely overlapping cover of $\mathcal{N}_\delta$ with curved regions θ of the form

$$\theta = \big\{(\xi_1, \cdots, \xi_{n-1}, \eta + \xi_1^2 + \cdots + \xi_{n-1}^2) : (\xi_1, \cdots, \xi_{n-1}) \in C_\theta, |\eta| \leq 2\delta\big\}$$

where C_θ runs over all cubes

$$c + \big[-\delta^{1/2}/2, \delta^{1/2}/2\big]^{n-1}, \quad c \in \frac{\delta^{1/2}}{2}\mathbb{Z}^{n-1} \cap \big[-1/2, 1/2\big]^{n-1}.$$

We denote by f_θ the Fourier restriction of f to θ. The parabola in P^{n-1} can be replaced by any surface and the decomposition as above is the same.

The ℓ^2-decoupling theorem tells us that for any compact C^2 hypersurface S in $\mathbb{R}^n$ with positive definite second fundamental form and $\mathrm{supp}\widehat{f} \subset \mathcal{N}_\delta$, we have

$$\|f\|_p \leq C_\epsilon \delta^{-\frac{n-1}{4}+\frac{n+1}{2p}-\epsilon}\Big(\sum_{\theta\in\mathcal{P}_\delta}\|f_\theta\|_p^2\Big)^{1/2}, \tag{7.9}$$

for all $p \geq 2(n+1)/(n-1)$ and any $\epsilon > 0$.

Apart from (7.8), the above decoupling theorem gives the best result so far for (7.5) when $n \geq 4$

$$\Big\|\sum_{\substack{z\in\mathbb{Z}^n\\ z_1^2+\cdots+z_n^2=N^2}} a_z e^{2\pi i z\cdot x}\Big\|_{L^p(\mathbb{T}^n)} \leq C_\epsilon N^{n(\frac{1}{2}-\frac{1}{p})-1+\epsilon}\Big(\sum_z |a_z|^2\Big)^{1/2}, \tag{7.10}$$

holds for $p \geq \frac{2(n-1)}{n-3}$ and any $\epsilon > 0$.

There are many other applications of this result to various number theoretic problems, such as Diophantine inequalities, additive energies and incidence geometry etc. We refer to [12] for details. One should also notice that the method pioneered in [14] work as well in [12] for the decoupling estimate associated to the cone

$$C^{n-1} = \{(\xi_1,\cdots,\xi_{n-1},\sqrt{\xi_1^2+\cdots+\xi_{n-1}^2}); 1 \leq \sqrt{\xi_1^2+\cdots+\xi_{n-1}^2} \leq 2\}.$$

Such inequalities as observed by Wolff [79]would improve the known results on the conjecture of "local smoothing" of wave equations that we considered in the previous sections.

7.2. Further improvements on the restriction theorem. Motivated by the work [27], [14] and [79], Guth [37, 38] applied the polynomial partitioning to the restriction theorem and got a small improvement of the result in [14]. In particular, it is shown that Stein's restriction theorem for the unit sphere in $\mathbb{R}^3$ is valid for $1 \leq p < \frac{13}{9}$. As the author said in [37], this method is the first application of polynomial partitioning to the restriction theorem. We believe it has great potential to be explored more.

7.3. Decoupling and applications to classical analytic number theory. Let $e(z) = e^{2\pi i z}$, $z \in \mathbb{R}$, and define

$$J_{s,n}(N) = \int_{[0,1]^n}\Big|\sum_{j=1}^N e(x_1 j + x_2 j^2 + \cdots + x_n j^n)\Big|^{2s} dx_1\cdots dx_n.$$

Bourgain-Demeter-Guth [13] proved the main conjecture in Vinogradov's Mean Value Theorem up to an $\epsilon-$loss

$$J_{s,n}(N) \leq C_\epsilon(N^{s+\epsilon} + N^{2s-\frac{n(n+1)}{2}+\epsilon}). \tag{7.11}$$

The method is pure harmonic analysis and the relevant machinery is called *decouplings*. The decoupling theory has since proved to be a very successful tool for a wide variety of problems in number theory that involve exponential sums. We refer to [13] for details and references therein.

8. Concentration of eigenfunction on compact Riemman manifolds. In this section, we turn to another topic, that is the concentration of eigenfunctions on compact Riemann manifolds, which is a very fascinating study field and developing actively because of its connection with mathemaical theory of quantum mechanics.

Let (M, g) be a compact, n-dimensional smooth Riemannian manifold without boundary. We denote by $\mathbf{\Delta}$ the associated positive Laplace-Beltrami operator, and by $\gamma : [a, b] \to M$ a smooth curve parametrized by arc length. Let $(e_\lambda)_\lambda$, $\lambda \geq 0$ be the eigenfunctions of $\mathbf{\Delta}$ such that $\mathbf{\Delta} e_\lambda = \lambda^2 e_\lambda$, $\|e_\lambda\|_2 = 1$ and Π_λ be the projection operator on $L^2(M)$ defined by

$$\Pi_\lambda f := \sum_{\lambda_j \in [\lambda, \lambda+1]} (f|e_{\lambda_j}) e_{\lambda_j},$$

where $(\cdot|\cdot)$ is the usual L^2 inner product with respect to the Riemannian volume form.

A classical estimate on the spectral projector is due to Sogge [58], namely

$$\|\Pi_\lambda f\|_{L^p(M)} \leq C\lambda^{\delta(n,p)} \|f\|_{L^2(M)}, \tag{8.1}$$

where

$$\delta(n,p) = \begin{cases} n\Big(\dfrac{1}{2} - \dfrac{1}{p}\Big) - \dfrac{1}{2}, & p \geq \dfrac{2(n+1)}{n-1}, \\ \dfrac{(n-1)}{2}\Big(\dfrac{1}{2} - \dfrac{1}{p}\Big), & 2 \leq p \leq \dfrac{2(n+1)}{n-1}. \end{cases}$$

To prove (8.1), one first observes that $\|\Pi_\lambda\|_{L^2 \to L^2} \leq C$ and

$$\|\Pi_\lambda f\|_{L^\infty(M)} \leq C\lambda^{\frac{n-1}{2}} \|f\|_{L^2(M)},$$

which appeared in Hörmander's proof of the sharp Weyl's asymptotic formula. Thus by interpolation, it suffices to show

$$\|\Pi_\lambda f\|_{L^{2\frac{n+1}{n-1}}(M)} \leq C\lambda^{\frac{n-1}{2(n+1)}} \|f\|_{L^2(M)}. \tag{8.2}$$

The crucial point is that by using Fourier transform, one can rewrite the spectral projector Π_λ by means of half-wave operator $e^{it\sqrt{\mathbf{\Delta}}}$, which from Hörmander's parametrix construction can be written up to a negligible error term as an oscillatory integral $\lambda^{\frac{n-1}{2}} \mathcal{T}_\lambda f(x)$

$$\mathcal{T}_\lambda f(x) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x,y)} a_\lambda(x,y) f(y) dy,$$

where $\phi(x, y) = d_g(x, y)$ is the geodesic distance between x and y. The key point is that ϕ satisfies the so-called $n \times n$ Carleson-Sjölin condition in [59]. Now Stein's argument for the endpoint estimate of Hörmander oscillatory integral as in the first section demonstrates that

$$\|\mathcal{T}_\lambda f\|_{L^{2\frac{n+1}{n-1}}(\mathbb{R}^n)} \leq C\lambda^{-\frac{n(n-1)}{2(n+1)}} \|f\|_{L^2},$$

and (8.2) follows.

The parametrix we used above is very fundamental in the study of eigenfunctions and we refer to [59] for details. From this, one can see the oscillatory integral that we started with from the very beginning of this paper plays the key role in the analysis dealing with eigenfunctions.

The estimate (8.1) is saturated by letting $M = \mathbb{S}^n$ and choosing f as the highest weight spherical harmonics for $2 \leq p \leq 2\frac{n+1}{n-1}$, and as the zonal functions for $p \geq 2\frac{n+1}{n-1}$.

Although the relation (8.1) is optimal in general, one can gain some improvement under certain geometric assumptions for $p > 2\frac{n+1}{n-1}$. In the $p = \infty$ case, Sogge and Zelditch [63] proved that if the set of geodesic loops has measure zero, *i.e.* $|\mathcal{L}_x| = 0$ for every $x \in M$, where

$$\mathcal{L}_x = \left\{\xi \in S_x^*M : \Phi_t(x,\xi) = (x,\eta), \text{for some } |t| > 0, \eta \in S_x^*M\right\},$$

then

$$\frac{\|e_\lambda\|_{L^p(M)}}{\|e_\lambda\|_2} = o(\lambda^{\delta(n,p)}),\ p > \frac{2(n+1)}{n-1}.$$

See also [65] for an exposé in a straightforward way. This result was somehow improved by Sogge-Toth-Zelditch [62]. We refer to [66] for the precise statement, where the results on the real analytic Riemannian manifolds and the manifolds of nonpositive sectional curvatures.

If $2 < p < 2\frac{n+1}{n-1}$, the picture is completely different and the attempt to improve the estimation on eigenfunctions becomes subtle. Given a compact Riemannian manifold (M, g) and denote by $\varPi$ the space of unit length geodesics, one shall use the Kakeya-Nikodym norms

$$|\!|\!| e_\lambda |\!|\!|_{KN} = \sup_{\gamma\in\varPi} \|e_\lambda\|_{L^2(T_{\lambda^{-1/2}}(\gamma))},$$

which is introduced by Sogge [64] as a way to refine the L^p-L^2 estimates in two dimensions. In particular, it is shown in [64] that the following three statements are equivalent

$$\lambda_{j_k}^{-\delta(2,p)}\|e_{\lambda_{j_k}}\|_{L^p(M)} \to 0,\ \forall\ 2 < p < 6 \tag{8.3}$$

$$|\!|\!| e_{\lambda_{j_k}} |\!|\!|_{KN} \to 0 \tag{8.4}$$

and

$$\lambda_{j_k}^{-1/4} \sup_{\gamma\in\varPi}\Big(\int_\gamma |e_{\lambda_{j_k}}|^2 ds\Big)^{1/2} \to 0. \tag{8.5}$$

Previous to [64], Burq et al [16] showed that in two-dimensionas, one has[2]

$$\sup_{\gamma\in\varPi}\Big(\int_\gamma |e_\lambda|^2 ds\Big)^{1/2} \leq C\lambda^{1/4}\|e_\lambda\|_{L^2(M)}. \tag{8.6}$$

A general result was obtained by Bourgain

$$\sup_{\gamma\in\varPi}\Big(\int_\gamma |e_\lambda|^2 ds\Big)^{1/2} \leq C\lambda^{1/2p}\|e_\lambda\|_{L^p(M)},\ 2 \leq p \leq \infty. \tag{8.7}$$

[2] See also [21] for some sharp restriction theorem for eigenfunctions in dimension three.

Both of these two results, as were used in one of the steps to show the equivalency of (8.3)-(8.5), are based on the Hörmander parametrix construction which is used in the proof of Sogge's classical estimate on eigenfunctions that we recalled at the beginning of this section.

The higher dimensional analogue of the main result in [64] was found by Blair and Sogge [3], where the key point in their argument is exploring the bilinear oscillatory integral estimates for the oscillatory integrals satisfying Carleson-Sjölin conditions. In [2] and [4], the above results were refined microlocally. It is interesting to find the applications of these refined estimates to the theory of nonlinear partial differential equations.

The final result we want to mention is the possible refinement for the critical index $p = p_c$. Notice that this is the transition point where the behavior of the highest weight spherical harmonics concentrating along curves and the zonal functions accumulating around a pole confront each other at this level. Thus it becomes very hard to find an appropriate norm to formulate a unified form of refinement. However, Sogge in [67] improved critical eigenfunction estimates in the case of nonpositive curvature. It involved many ideas including Blair-Sogge's improvement for $2 < p < p_c$, classical improved supnorm estimates of Bérard, Bourgain's proof of weak-type estimates for Stein-Tomas Fourier restriction theorem as well as Bak-Seeger's improved estimates in the Lorentz space.

THEOREM 8.1 (Sogge, [67]). *Assume that (M, g) is of nonpositive curvature. Then there is a constant $C = C(M, g)$ so that for $\lambda \gg 1$*

$$\|e_\lambda\|_{L^{\frac{2(n+1)}{n-1}}} \leq C\lambda^{\frac{n-1}{2(n+1)}} (\log\log\lambda)^{-\frac{2}{(n+1)^2}}. \tag{8.8}$$

We refer to Sogge's survey article [66] for more informations.

We end up this paper by mentioning a bilinear version of the Kakeya-Nicodym maximal average initiated in [52], as a further refinement of [15]. Let (M, g) be a two-dimensional compact boundaryless Riemannian manifold and we adopt the above notations. Then we have for $0 < \lambda \leq \mu$ and e_λ, e_μ being two eigenfunctions of $\sqrt{-\Delta}$ associated to the frequecies λ and μ. Then for every $\varepsilon > 0$ small, we have $C_\varepsilon > 0$, such that

$$\|e_\lambda e_\mu\|_{L^2(M)} \leq C_\varepsilon \lambda^{\varepsilon/2} \|e_\mu\|_{L^2(M)} |||e_\lambda|||_{KN(\lambda,\varepsilon)} \tag{8.9}$$

and

$$\|e_\lambda e_\mu\|_{L^2(M)} \leq C_\varepsilon \lambda^{\varepsilon/2} \|e_\lambda\|_{L^2(M)} |||e_\mu|||_{KN(\lambda,\varepsilon)} \tag{8.10}$$

where

$$|||f|||_{KN(\lambda,\varepsilon)} = \left(\sup_{\gamma\in\Pi} \lambda^{1/2-\varepsilon} \int_{T_{\lambda^{-1/2+\varepsilon}}} |f(x)|^2 dx\right)^{1/2}.$$

Notice that from the quantum unique ergodicity conjecture, it is natural to expect that the $\varepsilon-$loss in these estimate should be removed. This result is obtained by combining Sogge's Kakeya-Nikodym maximal averages [64] and the bilinear Carleson-Sjölin conditions used by [15].

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