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GLOBAL WELL-POSEDNESS OF 3D INCOMPRESSIBLE INHOMOGENEOUS NAVIER-STOKES EQUATIONS*

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Dedicated to the 80th birthday of Professor Ling Hsiao

Abstract. In this paper, we prove the global well-posedness of 3D inhomogeneous incompressible Navier-Stokes equations with initial velocity to be sufficiently small in the critical Besov space, $\dot{B}^{3/p-1}_{p,1}$ for $1 < p < 6$ and with initial density in the critical Besov space and bounded away from vacuum. The key ingredient used in the proof lies in a new estimate to the pressure term. In particular, our result improves the previous ones by Abidi et al. (2013) [3], Zhai and Yin (2017) [32], Burtea (2017) [6] and so on.

Key words. Inhomogeneous Navier-Stokes systems, Littlewood-Paley theory, Well-posedness, Besov spaces.

Mathematics Subject Classification. 35Q35, 35Q30, 76D03.

1. Introduction. We consider the well-posedness of the following 3D incompressible inhomogeneous Navier-Stokes equations with initial data in the critical Besov spaces so that the initial density is bounded away from vacuum and without any size restriction:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u + \nabla \pi = 0 \\ \operatorname{div} u = 0, \\ \rho|_{t=0} = \rho_0, \quad \rho u|_{t=0} = \rho_0 u_0, \end{cases} \tag{1.1}$$

where $\rho = \rho(t,x) \in \mathbb{R}_+$, $u = u(t,x) \in \mathbb{R}^3$ and $\pi = \pi(t,x) \in \mathbb{R}$ stand for the density, velocity field and pressure of the incompressible fluid respectively. The positive constant μ designates the viscous coefficient (we shall take it to be 1 in sequel for simplicity). This system describes a fluid which is obtained by mixing several immiscible fluids that are incompressible and that have different densities. It can also describe a fluid containing a melted substance.

It is obvious that this system (1.1) has the following energy law

$$\frac{1}{2}\int_{\mathbb{R}^3} \rho(t,x)|u(t,x)|^2 \mathrm{d}x + \int_0^t \|\nabla u(t',\cdot)\|^2_{L^2(\mathbb{R}^3)} \mathrm{d}t' = \frac{1}{2}\int_{\mathbb{R}^3} \rho_0(x)|u_0(x)|^2 \mathrm{d}x, \tag{1.2}$$

which can be formally obtained by taking the L^2 inner product of the momentum equation of (1.1) with u and using integration by parts and $\operatorname{div} u = 0$.

Based on the energy law (1.2), Kazhikov [22] first proved the global existence of strong solutions to (1.1) with initial velocity small in $H^1(\mathbb{R}^3)$ and with initial density in $L^\infty(\mathbb{R}^3)$ and away from vacuum. The uniqueness of such solutions was obtained

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only lately in [30]. Simon [31] constructed global weak solutions to (1.1) with finite energy (see also the book by Lions [27] with the variable viscosity).

The other interesting feature of the system (1.1) is the scaling invariance property: if (ρ, u, π) is a solution of (1.1) on $[0,T]\times\mathbb{R}^3$, then the rescaled triplet $(\rho, u, \pi)_\lambda$ defined by

$$(\rho, u, \pi)_\lambda(t,x) \stackrel{\text{def}}{=} \big(\rho(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x), \lambda^2\pi(\lambda^2 t, \lambda x)\big), \quad \lambda \in \mathbb{R} \tag{1.3}$$

is also a solution of (1.1) on $[0, T/\lambda^2] \times \mathbb{R}^3$. This leads to the notion of critical regularity.

To consider the well-posedness of the system (1.1) with initial data in the critical spaces, it is convenient to write (1.1) as

$$\begin{cases} \partial_t a + u\cdot\nabla a = 0 \quad \text{in } \mathbb{R}_+\times\mathbb{R}^3, \\ \partial_t u + u\cdot\nabla u - (1+a)(\Delta u + \nabla\pi) = 0, \\ \operatorname{div} u = 0, \\ (a,u)|_{t=0} = (a_0, u_0), \end{cases} \tag{1.4}$$

where $a \stackrel{\text{def}}{=} \frac{1}{\rho} - 1$ in case the density is away from zero.

Before proceeding, we recall the definitions of dyadic operators and Besov spaces. For $u \in \mathcal{S}'_h$, we set

$$\forall j\in\mathbb{Z}, \quad \dot{\Delta}_j u \stackrel{\text{def}}{=} \varphi(2^{-j}D)u \quad \text{and} \quad \dot{S}_j u = \sum_{j'\leq j-1} \dot{\Delta}_{j'}u = \chi(2^{-j}D)u,$$

where $\chi(\tau)$ and $\varphi(\tau)$ are smooth functions such that

$$\operatorname{Supp}\ \varphi \subset \Big\{\tau\in\mathbb{R}\ /\ \ \frac{3}{4} \leq \tau| \leq \frac{8}{3}\Big\} \quad \text{and} \quad \forall\tau>0\,,\ \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\tau) = 1,$$

$$\operatorname{Supp}\ \chi \subset \Big\{\tau\in\mathbb{R}\ /\ \ |\tau| \leq \frac{4}{3}\Big\} \quad \text{and} \quad \forall\tau\in\mathbb{R}\,,\ \chi(\tau) + \sum_{j\geq 0}\varphi(2^{-j}\tau) = 1.$$

Definition 1.1. Let (p,r) be in $[1,\infty]^2$ and s in $\mathbb{R}$. Let us consider u in $\mathcal{S}'_h(\mathbb{R}^d)$, which means that u is in $\mathcal{S}'(\mathbb{R}^d)$ and satisfies $\lim_{j\to-\infty}\|S_j u\|_{L^\infty} = 0$. We set

$$\|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \big\|\big(2^{js}\|\Delta_j u\|_{L^p}\big)_{j\in\mathbb{Z}}\big\|_{\ell^r(\mathbb{Z})}.$$

- if $s < \frac{d}{p}$ (or $s = \frac{d}{p}$ if $r = 1$), we define $\dot{B}^s_{p,r}(\mathbb{R}^d) \stackrel{\text{def}}{=} \big\{u \in \mathcal{S}'_h(\mathbb{R}^d)\ \big|\ \|u\|_{\dot{B}^s_{p,r}} < \infty\big\}$.
- if $k\in\mathbb{N}$ and if $\frac{d}{p} + k \leq s < \frac{d}{p} + k + 1$ (or $s = \frac{d}{p} + k + 1$ if $r = 1$), then we define $\dot{B}^s_{p,r}(\mathbb{R}^d)$ as the subset of u in $\mathcal{S}'_h(\mathbb{R}^d)$ such that $\partial^\beta u$ belongs to $\dot{B}^{s-k}_{p,r}(\mathbb{R}^d)$ whenever $|\beta| = k$.

We remark that $\dot{B}^s_{2,2}(\mathbb{R}^d)$ coincides with the classical homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^d)$. Similarly, we can also define the inhomogeneous Besov space $B^s_{p,q}(\mathbb{R}^d)$. For simplicity, we henceforth denote $\dot{B}^s_{p,1}(\mathbb{R}^d)$ by $\mathcal{B}^s_p(\mathbb{R}^d)$ and $B^s_{p,1}(\mathbb{R}^d)$ by $B^s_p(\mathbb{R}^d)$.

Given initial data (a_0, u_0) in the critical Besov space $\dot{B}^{\frac{d}{2}}_{2,\infty}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \times \mathcal{B}^{\frac{d}{2}-1}_2(\mathbb{R}^d)$ for $d \geq 2$ and a_0 small, Danchin [9] proved the unique local solvability of the system (1.4). Abidi [1] extended the result in [9, 10] to the case of (1.4) with variable viscosity. More precisely, given initial data $(a_0, u_0) \in \mathcal{B}^{\frac{d}{p}}_p(\mathbb{R}^d) \times \mathcal{B}^{\frac{d}{p}-1}_p(\mathbb{R}^d)$ with $1 < p < 2d$ and

$$\|a_0\|_{\mathcal{B}^{\frac{d}{p}}_p} + \|u_0\|_{\mathcal{B}^{\frac{d}{p}-1}_p} \leq c_0$$

for c_0 sufficiently small, then he proved the global existence of solutions to (1.4) and the uniqueness of such solutions was only proved for $p \in (1, d)$. Abidi and Paicu [4] further generalized the result in [1] with different integral indices for a_0 and u_0. The end point result in this direction is due to Danchin and Mucha [14] with a_0 being small in the multiplier space of $\mathcal{B}^{\frac{d}{p}-1}_p(\mathbb{R}^d)$ for $p \in (1, 2d)$, which in particular solved the uniqueness issue which was left open in [1].

On the other hand, motivated by [28], Paicu and the second author [29] improved the smallness condition in [1, 4] to [1]

$$\Big(\mu\|a_0\|_{\mathcal{B}^{\frac{3}{q}}_q} + \|u_0^h\|_{\mathcal{B}^{-1+\frac{3}{p}}_p}\Big) \exp\Big(C_0\mu^{-2}\|u_0^3\|^2_{\mathcal{B}^{-1+\frac{3}{p}}_p}\Big) \leq c_0\mu$$

for some positive constants c_0, C_0 and $1 < q \leq p < 6$ with $\frac{1}{q} - \frac{1}{p} \leq \frac{1}{3}$.

We observe that all the previous mentioned results require the initial density to be a small perturbation of some positive constant. In general, Ladyzhenskaya and Solonnikov [23] proved the unique solvability of (1.1) in a bounded smooth domain with Dirichlet boundary condition on the velocity field. More precisely, given initial velocity $u_0 \in W^{2-\frac{2}{p},p}(\Omega)$, $p > d$, vanishing on $\partial\Omega$ and $\rho_0 \in C^1(\Omega)$ bounded away from zero, they obtained global well-posedness in dimension $d = 2$ as well as local well-posedness in dimension $d = 3$; if, in addition, u_0 is small in $W^{2-\frac{2}{p},p}(\Omega)$, then global well-posedness result holds.

Concerning initial data in the critical space and without any size restriction for the initial density, Abidi, Gui and the second author [2] first proved the global well-posedness of (1.1) with initial velocity being sufficiently small in $\mathcal{B}^{\frac{1}{2}}_2(\mathbb{R}^3)$ and the initial inhomogeneity in $\mathcal{B}^{\frac{3}{2}}_2(\mathbb{R}^3)$ so that the initial density is bounded away from vacuum. They also investigated similar type of result with initial velocity in the critical Besov space with negative index, viz,

$$(a_0, u_0) \in B^{\frac{3}{q}}_q(\mathbb{R}^3) \times \mathcal{B}^{\frac{3}{p}-1}_p(\mathbb{R}^3) \quad \text{with} \quad q \in [1, 2],\, p \in [3, 4] \tag{1.5}$$

and $\frac{1}{q} + \frac{1}{p} > \frac{5}{6}$ and $\frac{1}{q} - \frac{1}{p} \leq \frac{1}{3}$.

Yet motivated by the study of (1.4) with small density, one may expect that the index p in (1.5) should belong to $(1, 6)$. Toward this direction, Zhai and Yang [32] obtained the local existence result if $1 < q \leq p$, $p \in \big(1, \frac{5+\sqrt{17}}{2}\big)$ and $\frac{1}{2} - \frac{1}{p} < \frac{1}{q} < \frac{1}{3} + \frac{1}{p}$. To prove the uniqueness of such solution, they need more restrictions than that in [3]. When $q = p$, Burtea in [6] extended the result to the range of $p \in (6/5, 4)$. Very recently, Zhai and Yin [33] considered the system (1.4) with variable viscosity

[1]Throughout this paper, all the space norms are defined for functions in $\mathbb{R}^3$ without specific mention.

and extended the range to $1 < q \leq p < 6$ but with the additional assumption: $\|a_0\|_{\mathrm{BMO}} \leq \varepsilon_0$ for some ε_0 sufficiently small.

Let us mention that in all these aforementioned works, the density has to be at least in the Besov space $\dot{B}^{d/p}_{p,\infty}(\mathbb{R}^d)$, which excludes the density function with discontinuities across some hypersurface. We observe that $L^\infty(\mathbb{R}^d)$ is the largest scaling invariant space for the density function. Lately, rapid progresses [15, 17, 16, 18, 21, 24, 25, 26] have been made for the well-posedness of (1.4) with initial density in the bounded function space. Especially in [34], the second author proved the global existence of strong solutions to (1.1) with initial density in $L^\infty(\mathbb{R}^3)$ having a positive lower bound and with initial velocity being sufficiently small in the critical Besov space $\mathcal{B}_2^{\frac{1}{2}}(\mathbb{R}^3)$. This result exactly corresponds to the celebrated well-posedness result of Fujita-Kato in [20] devoted to the classical Navier-Stokes system.

The main contribution of the present paper is to consider the well-posedness of (1.4) with initial data satisfying (1.5) for $1 < p < 6$ and without any smallness assumption on initial density. We also remove the restriction of the relationship $1 < q \leq p$ in (1.5).

The main result of this paper states as follows:

THEOREM 1.2. *Let $1 < p, q < 6$ with $\max\{\frac{1}{2} - \frac{1}{q}, \frac{1}{q} - \frac{1}{3}\} < \frac{1}{p} < \frac{1}{6} + \frac{1}{q}$. Let $a_0 \in B_q^{\frac{3}{q}}(\mathbb{R}^3)$, $u_0 \in \mathcal{B}_p^{\frac{3}{p}-1}(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$, and*

$$1 + a_0 \geq \underline{b} > 0. \tag{1.6}$$

Then there exists a positive time T so that (1.4) has a unique solution $(a, u) \in \mathcal{C}([0,T]; B_q^{\frac{3}{q}}(\mathbb{R}^3)) \times \mathcal{C}([0,T]; \mathcal{B}_p^{\frac{3}{p}-1}(\mathbb{R}^3)) \cap L^1([0,T]; \mathcal{B}_p^{\frac{3}{p}+1}(\mathbb{R}^3))$. Moreover, there exists a small constant c depending on $\|a_0\|_{B_q^{\frac{3}{q}}}$ so that if

$$\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}} \leq c \tag{1.7}$$

and $2 \leq p < 6$, then (1.4) has a unique global solution (a, u) satisfying

$$\begin{aligned} &\|a\|_{\widetilde{L}^\infty_t(B_q^{\frac{3}{q}})} + \|u\|_{\widetilde{L}^\infty_t(\mathcal{B}_p^{\frac{3}{p}-1})} + \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+1})} + \|\nabla\pi\|_{\widetilde{L}^1_t(\mathcal{B}_p^{\frac{3}{p}-1})} \\ &\qquad \leq C\Big(\|a_0\|_{B_q^{\frac{3}{q}}} + \|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\Big) \exp\left(C\sqrt{t}\right). \end{aligned} \tag{1.8}$$

REMARK 1.3. As is well-known in the previous references, the main difficulty to work the well-posedness of (1.4) without smallness assumption on the density function lies in the estimate of the pressure function. For instance in [3], the authors dealt with the estimate of the pressure function by $\|\nabla\pi\|_{L^1_t(L^2)}$, which needs the condition (1.5), and in [6, 32, 33], the authors handled the estimate of pressure function by $\|\nabla\pi\|_{L^1_t(\dot{B}^{3/p-3/2}_{p,1})}$. In this paper, we shall handle the estimate of the pressure function by $\|\nabla\pi\|_{L^1_t(\dot{B}^{3/p-2}_{p,1})}$, which makes us to have the full range of $p \in (1,6)$.

The sketch of the paper is as follows: In Section 2, we shall collect some basic facts on Littlewood-Paley analysis and recall some known estimates; then in Section 3 we apply the Littlewood-Paley theory to study the linearized inhomogeneous Navier-Stokes type equations. With these estimates, we shall prove the local well-posedness part of Theorem 1.2 in Section 4. Finally in the last section, we present the proof to the global existence part of Theorem 1.2.

Let us complete this section with the notations we are going to use in this context.

Notations. Let A, B be two operators, we denote $[A, B] = AB - BA$, the commutator between A and B. For $a \lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a \leq Cb$. We shall denote by $(a|b)$ (or $(a|b)_{L^2}$) the $L^2(\mathbb{R}^3)$ inner product of a and b. For X a Banach space and I an interval of $\mathbb{R}$, we denote by $\mathcal{C}(I; X)$ the set of continuous functions on I with values in X, and by $\mathcal{C}_b(I; X)$ the subset of bounded functions of $\mathcal{C}(I; X)$. For $q \in [1, +\infty]$, the notation $L^q(I; X)$ stands for the set of measurable functions on I with values in X, such that $t \longmapsto \|f(t)\|_X$ belongs to $L^q(I)$. We always denote the Fourier transform of a function u by $\hat{u}$ or $\mathcal{F}(u)$, $\{c_{j,r}\}_{j\in\mathbb{Z}}$ a generic element in the sphere of $\ell^r(\mathbb{Z})$ and $(c_j)_{j\in\mathbb{Z}}$ (resp. $(d_j)_{j\in\mathbb{Z}}$) a generic element in the sphere of $\ell^2(\mathbb{Z})$ (resp. $\ell^1(\mathbb{Z})$).

2. Preliminaries.

2.1. Basic facts on Littlewood-Paley theory. For the convenience of the readers, we recall some basic facts on Littlewood-Paley theory from [5].

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we will use Chemin-Lerner type spaces $\widetilde{L}^q_T(\mathcal{B}^s_p)$.

Definition 2.1. Let $s \leq \frac{3}{p}$ (respectively $s \in \mathbb{R}$), $(q, p) \in [1, +\infty]^2$ and $T \in]0, +\infty]$. We define $\widetilde{L}^q_T(\mathcal{B}^s_p)$ as the completion of $C([0, T]; \mathcal{S}_h(\mathbb{R}^3))$ by the norm

$$\|f\|_{\widetilde{L}^q_T(\mathcal{B}^s_p)} \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} 2^{js} \Big(\int_0^T \|\Delta_j f(t)\|^q_{L^p}\, dt \Big)^{\frac{1}{q}} < \infty,$$

with the usual change if $q = \infty$.

Lemma 2.2 (Lemma 2.1 of [5]). *Let $\mathcal{B}$ a ball of $\mathbb{R}^3$, and $\mathcal{C}$ a ring of $\mathbb{R}^3$, let $1 \leq p_2 \leq p_1 \leq \infty$. Then there holds:*

$$\begin{aligned} &\textit{if}\ \ \operatorname{Supp}\ \widehat{a} \subset 2^j\mathcal{B} \Rightarrow \|\partial_x^\alpha a\|_{L^{p_1}} \lesssim 2^{j(|\alpha|+3(1/p_2-1/p_1))}\|a\|_{L^{p_2}};\\ &\textit{if}\ \ \operatorname{Supp}\ \widehat{a} \subset 2^j\mathcal{C} \Rightarrow \|a\|_{L^{p_1}} \lesssim 2^{-jN} \sup_{|\alpha|=N} \|\partial_x^\alpha a\|_{L^{p_1}}. \end{aligned}$$

We shall also frequently use Bony's decomposition in homogeneous context:

$$\begin{aligned} uv =& \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v) = \dot{T}_u v + \dot{T}'_v u \quad \text{with} \quad \dot{T}_u v \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v \quad \text{and}\\ &\dot{R}(u, v) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \widetilde{\dot{\Delta}}_j v, \quad \widetilde{\dot{\Delta}}_j u \stackrel{\text{def}}{=} \sum_{|j-j'|\leq 1} \dot{\Delta}_{j'} v, \quad \dot{T}'_v u \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \dot{S}_{j+2} v \dot{\Delta}_j u. \end{aligned} \tag{2.1}$$

We now recall some important laws of product in Besov spaces and some commutator estimates.

Lemma 2.3 (see [5]). *Let $1 \leq p, q \leq \infty$, $-3\min\{\frac{1}{q}, 1-\frac{1}{p}\} < s \leq 1+3\min\{\frac{1}{p}, \frac{1}{q}\}$, $\nabla a \in \mathcal{B}^{\frac{3}{q}}_q$ and $b \in \mathcal{B}^{s-1}_p$. Then there holds*

$$\|[\dot{\Delta}_j, a]b\|_{L^p} \lesssim d_j 2^{-js} \|\nabla a\|_{\mathcal{B}^{\frac{3}{q}}_q} \|b\|_{\mathcal{B}^{s-1}_p}. \tag{2.2}$$

LEMMA 2.4 (see [5]). *For any positive real number s and any $(p,r) \in [1,\infty]^2$, the space $L^\infty \cap B^s_{p,r}$ is an algebra, and a constant C exists such that*

$$\|fg\|_{B^s_{p,r}} \leq \frac{C^{s+1}}{s}\big(\|f\|_{L^\infty}\|g\|_{B^s_{p,r}} + \|f\|_{B^s_{p,r}}\|g\|_{L^\infty}\big), \tag{2.3}$$

(2.3) *holds with $B^s_{p,r}$ being replaced by $\dot{B}^s_{p,r}$.*

LEMMA 2.5 (Lemma 2.5 of [32]). *Let $1 \leq p, q \leq \infty, s_1 \leq \frac{3}{q}, s_2 \leq 3\min\big\{\frac{1}{p}, \frac{1}{q}\big\}$ and $s_1 + s_2 > 3\max\big\{0, \frac{1}{p} + \frac{1}{q} - 1\big\}$. For any $(a,b) \in \mathcal{B}^{s_1}_q \times \mathcal{B}^{s_2}_p$, we have*

$$\|ab\|_{\mathcal{B}_p^{s_1+s_2-\frac{3}{q}}} \lesssim \|a\|_{\mathcal{B}^{s_1}_q}\|b\|_{\mathcal{B}^{s_2}_p}. \tag{2.4}$$

REMARK 2.6. It is easy to observe from Lemma 2.5 that

$$\|ab\|_{\mathcal{B}_p^{\frac{3}{p}-1}} \lesssim \|a\|_{\mathcal{B}_q^{\frac{3}{q}}}\|b\|_{\mathcal{B}_p^{\frac{3}{p}-1}} \quad \text{if} \ \ \frac{1}{3} - \frac{1}{q} \leq \frac{1}{p} \leq \frac{1}{3} + \frac{1}{q}, \tag{2.5}$$

$$\|ab\|_{\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}}} \lesssim \|a\|_{\mathcal{B}_q^{\frac{3}{q}}}\|b\|_{\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}}} \quad \text{if} \ \ \frac{1}{2} - \frac{1}{q} \leq \frac{1}{p} \leq \frac{1}{2} + \frac{1}{q}, \tag{2.6}$$

$$\|ab\|_{\mathcal{B}_p^{\frac{3}{p}-2}} \lesssim \|a\|_{\mathcal{B}_q^{\frac{3}{q}}}\|b\|_{\mathcal{B}_p^{\frac{3}{p}-2}} \quad \text{if} \ \ \frac{2}{3} - \frac{1}{q} \leq \frac{1}{p} \leq \frac{2}{3} + \frac{1}{q}. \tag{2.7}$$

As an application of the above basic facts on Littlewood-Paley theory, we prove the following commutaor's estimates:

LEMMA 2.7. *For $1 < p, q < 6$, one has*

$$\big\|[\dot{\Delta}_j, a]b\big\|_{L^p} \lesssim d_j 2^{-j(\frac{3}{p}-1)}\|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}}\|b\|_{\dot{B}_{p,\infty}^{\frac{3}{p}-\frac{3}{2}}} \quad \textit{if} \ \ \frac{1}{p} < \frac{1}{q} + \frac{1}{2}, \tag{2.8}$$

$$\big\|[\dot{\Delta}_j, a]b\big\|_{L^p} \lesssim d_j 2^{-j(\frac{3}{p}-\frac{3}{2})}\|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}}\|b\|_{\dot{B}_{p,\infty}^{\frac{3}{p}-2}} \quad \textit{if} \ \ \frac{1}{2} - \frac{1}{q} < \frac{1}{p} < \frac{2}{3} + \frac{1}{q}. \tag{2.9}$$

Proof. Let us first consider the case when $q > p$. By applying Bony's decomposition (2.1) and then using a standard commutator's process, we write

$$[\dot{\Delta}_j, a]b = [\dot{\Delta}_j, \dot{T}_a]b + \dot{\Delta}_j(\dot{T}'_b a) - \dot{T}'_{\dot{\Delta}_j b}a. \tag{2.10}$$

Due to the support properties to the Fourier transform of the terms in $\dot{T}_a b$, we deduce from Lemma 2.97 of [5] that

$$\|[\dot{\Delta}_j, \dot{T}_a]b\|_{L^p} \lesssim 2^{-j}\sum_{|j-k|\leq 4}\|\nabla \dot{S}_{k-1}a\|_{L^\infty}\|\dot{\Delta}_j b\|_{L^p}.$$

Yet it is easy to observe that $\|\nabla \dot{S}_{k-1}a\|_{L^\infty} \lesssim d_k 2^{\frac{k}{2}}\|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}}$, so that we obtain

$$\|[\dot{\Delta}_j, \dot{T}_a]b\|_{L^p} \lesssim d_j 2^{-j\left(\frac{1}{2}+s\right)}\|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}}\|b\|_{\dot{B}^s_{p,\infty}} \quad \forall s \in \mathbb{R}. \tag{2.11}$$

For the term $\dot{\Delta}_j(\dot{T}'_b a)$, we split the estimate into two cases. For the first case when $q > p \in (1,2]$, again due to the support properties to the Fourier transform of terms in $\dot{T}'_b a$, we get

$$\|\dot{\Delta}_j(\dot{T}'_b a)\|_{L^p} \lesssim \sum_{j' \geq j-N_0} \|\dot{\Delta}_{j'} a\|_{L^q} \|\dot{S}_{j'+2} b\|_{L^{\frac{pq}{q-p}}}.$$

Observing that for $s < \frac{3}{q}$, we deduce from Lemma 2.2 that $\|\dot{S}_{j'+2} b\|_{L^{\frac{pq}{q-p}}} \lesssim 2^{\left(\frac{3}{q}-s\right)j'} \|b\|_{\dot{B}^s_{p,\infty}}$, which implies that

$$\begin{aligned}\|\dot{\Delta}_j(\dot{T}'_b a)\|_{L^p} &\lesssim \sum_{j' \geq j-N_0} d_{j'} 2^{-j'(s+\frac{1}{2})} \|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}} \|b\|_{\dot{B}^s_{p,\infty}} \\ &\lesssim d_j 2^{-j(s+\frac{1}{2})} \|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}} \|b\|_{\dot{B}^s_{p,\infty}} \quad \text{if} \quad -\frac{1}{2} < s < \frac{3}{q},\end{aligned} \tag{2.12}$$

Along the same line, for the case when $q > p \in (2,6)$, we have

$$\|\dot{\Delta}_j(\dot{T}'_b a)\|_{L^p} \lesssim 2^{\frac{3j}{q}} \sum_{j' \geq j-3} \|\dot{\Delta}_{j'} a\|_{L^q} \|\dot{S}_{j'+2} b\|_{L^p}.$$

Notice that $\|\dot{S}_{j'+2} b\|_{L^p} \lesssim 2^{-j's} \|b\|_{\dot{B}^s_{p,\infty}}$ if $s < 0$, from which, we infer

$$\begin{aligned}\|\dot{\Delta}_j(\dot{T}'_b a)\|_{L^p} &\lesssim 2^{\frac{3}{q}j} \sum_{j' \geq j-N_0} d_{j'} 2^{-\left(s+\frac{3}{q}+\frac{1}{2}\right)j'} \|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}} \|b\|_{\dot{B}^s_{p,\infty}} \\ &\lesssim d_j 2^{-j\left(s+\frac{1}{2}\right)} \|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}} \|b\|_{\dot{B}^s_{p,\infty}} \quad \text{if} \quad -\left(\frac{1}{2}+\frac{3}{q}\right) < s < 0.\end{aligned} \tag{2.13}$$

Similarly, we have

$$\begin{aligned}\|\dot{T}'_{\dot{\Delta}_j b} a\|_{L^p} &\lesssim \sum_{j' \geq j-N_0} \|\dot{\Delta}_{j'} a\|_{L^\infty} \|\dot{S}_{j'+2} \dot{\Delta}_j b\|_{L^p} \\ &\lesssim \|\dot{\Delta}_j b\|_{L^p} \sum_{j' \geq j-N_0} d_{j'} 2^{-\frac{j'}{2}} \|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}} \\ &\lesssim d_j 2^{-j\left(s+\frac{1}{2}\right)} \|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}} \|b\|_{\dot{B}^s_{p,\infty}} \quad \forall s \in \mathbb{R}.\end{aligned} \tag{2.14}$$

Taking $s = \frac{3}{p} - \frac{3}{2}$ in (2.11-2.14) leads to (2.8) for $q > p$. The other case when $q \leq p$ in (2.8) has been proved in Lemma 2.9 of [32].

While by taking $s = \frac{3}{p} - 2$ in (2.11-2.14), we obtain (2.9) for $q > p$. For case when $q \leq p$, all the terms except the second one in (2.10) can be proved as in the previous case. It remains to handle the second term in (2.10). Indeed if $1 < q \leq p < 2$, we have $\|\dot{S}_{j'+2} b\|_{L^{\frac{p}{p-1}}} \lesssim 2^{j'(\frac{3}{p}-1)} \|b\|_{\dot{B}^{\frac{3}{p}-2}_{p,\infty}}$ and $\|\dot{\Delta}_{j'} a\|_{L^p} \lesssim 2^{(\frac{3}{q}-\frac{3}{p})j'} \|\dot{\Delta}_{j'} a\|_{L^q}$, so that

$$\begin{aligned}\|\dot{\Delta}_j(\dot{T}'_b a)\|_{L^p} &\lesssim 2^{3j(1-\frac{1}{p})} \sum_{j' \geq j-N_0} \|\dot{\Delta}_{j'} a\|_{L^p} \|\dot{S}_{j'+2} b\|_{L^{\frac{p}{p-1}}} \\ &\lesssim 2^{3j(1-\frac{1}{p})} \sum_{j' \geq j-N_0} d_{j'} 2^{-\frac{3j'}{2}} \|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}} \|b\|_{\dot{B}^{\frac{3}{p}-2}_{p,\infty}} \\ &\lesssim d_j 2^{-j(\frac{3}{p}-\frac{3}{2})} \|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}} \|b\|_{\dot{B}^{\frac{3}{p}-2}_{p,\infty}}.\end{aligned}$$

For the second case when $1 < q \le p \in [2,6[$, we observe that $\frac{2p}{p-2} > q \Leftrightarrow \frac{1}{p} + \frac{1}{q} > \frac{1}{2}$ so that we can take r between q and $\frac{2p}{p-2}$, which satisfies $\frac{1}{2} < \frac{1}{p} + \frac{1}{r} < 1$. Then it follows from Lemma 2.2 that

$$\begin{aligned}\|\dot{\Delta}_j(\dot{T}'_b a)\|_{L^p} &\lesssim 2^{\frac{3j}{r}} \sum_{j' \ge j - N_0} \|\dot{\Delta}_{j'} a\|_{L^r} \|\dot{S}_{j'+2} b\|_{L^p} \\ &\lesssim 2^{\frac{3j}{r}} \sum_{j' \ge j - N_0} d_{j'} 2^{-3\left(\frac{1}{r}+\frac{1}{p}-\frac{1}{2}\right)} \|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}} \|b\|_{\dot{B}_{p,\infty}^{\frac{3}{p}-2}} \\ &\lesssim 2^{-j\left(\frac{3}{p}-\frac{3}{2}\right)} d_j \|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}} \|b\|_{\dot{B}_{p,\infty}^{\frac{3}{p}-2}}.\end{aligned}$$

This completes the proof of (2.9) for $q \le p$. Therefore, we finish the proof of this Lemma. □

REMARK 2.8. It follows from the proof of Lemma 2.7 that for $1 < p, q < 6$,

$$\big\|\nabla[\dot{\Delta}_j, a]b\big\|_{L^p} \lesssim d_j 2^{-j\left(\frac{3}{p}-2\right)} \|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}} \|b\|_{\dot{B}_{p,\infty}^{\frac{3}{p}-\frac{3}{2}}} \quad \text{if } \frac{1}{p} < \frac{1}{q} + \frac{1}{2}, \tag{2.15}$$

$$\big\|\nabla[\dot{\Delta}_j, a]b\big\|_{L^p} \lesssim d_j 2^{-j\left(\frac{3}{p}-\frac{5}{2}\right)} \|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}} \|b\|_{\dot{B}_{p,\infty}^{\frac{3}{p}-2}} \quad \text{if } \frac{1}{2} - \frac{1}{q} < \frac{1}{p} < \frac{2}{3} + \frac{1}{q}. \tag{2.16}$$

Indeed notice from the support properties to terms in (2.10), we only need to handle the estimate of the term $\nabla \dot{T}'_{\dot{\Delta}_j b} a$. By Lebnitz formula, we have

$$\nabla \dot{T}'_{\dot{\Delta}_j b} a = \dot{T}'_{\nabla \dot{\Delta}_j b} a + \dot{T}'_{\dot{\Delta}_j b} \nabla a.$$

The estimate to the term $\dot{T}'_{\nabla \dot{\Delta}_j b} a$ can be followed along the same line as (2.12)-(2.14). Whereas to handle the remaining term, we distinguish the case when $p \ge q$ and $p < q$. In case $q \le p < 6$, we have

$$\begin{aligned}\|\dot{T}'_{\dot{\Delta}_j b} \nabla a\|_{L^p} &\lesssim \sum_{j' \ge j - N_0} \|\dot{\Delta}_{j'} \nabla a\|_{L^p} \|\dot{S}_{j'+2} \dot{\Delta}_j b\|_{L^\infty} \\ &\lesssim \|\dot{\Delta}_j b\|_{L^\infty} \sum_{j' \ge j - N_0} d_{j'} 2^{-j'\left(\frac{3}{p}-\frac{1}{2}\right)} \|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}} \\ &\lesssim d_j 2^{-j\left(s-\frac{1}{2}\right)} \|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}} \|b\|_{\dot{B}_{p,\infty}^s} \quad \forall s \in \mathbb{R}.\end{aligned}$$

While for $p < q < 6$, we have

$$\begin{aligned}\|\dot{T}'_{\dot{\Delta}_j b} \nabla a\|_{L^p} &\lesssim \sum_{j' \ge j - N_0} \|\dot{\Delta}_{j'} \nabla a\|_{L^q} \|\dot{S}_{j'+2} \dot{\Delta}_j b\|_{L^{\frac{qp}{q-p}}} \\ &\lesssim \|\dot{\Delta}_j b\|_{L^{\frac{qp}{q-p}}} \sum_{j' \ge j - N_0} d_{j'} 2^{-j'\left(\frac{3}{q}-\frac{1}{2}\right)} \|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}} \\ &\lesssim d_j 2^{-j\left(s-\frac{1}{2}\right)} \|a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}} \|b\|_{\dot{B}_{p,\infty}^s} \quad \forall s \in \mathbb{R}.\end{aligned}$$

Taking $s = \frac{3}{p} - \frac{3}{2}$ and $s = \frac{3}{p} - 2$ in the above inequalities gives rise to (2.15) and (2.16).

2.2. Some known estimates. For the reader's convenience, we recall the following two propositions concerning the estimate of the pressure function, the proof of which can be found, for instance, in [12, 6]:

Proposition 2.9 (see [12]). *Consider $a \in L^\infty(\mathbb{R}^3)$ and a constant $\underline{a}$ such that*

$$a \geq \underline{a} > 0.$$

For all vector fields f with coefficients in $L^2(\mathbb{R}^3)$, there exists a tempered distribution π unique up to constant functions such that $\nabla\pi \in L^2(\mathbb{R}^3)$ and equation

$$-\mathrm{div}\,(a\nabla\pi) = \mathrm{div}\, f \tag{2.17}$$

is satisfied. In addition, we have

$$\underline{a}\|\nabla\pi\|_{L^2} \leq \|\mathbb{Q}f\|_{L^2}, \tag{2.18}$$

where $\mathbb{Q} = Id - \mathbb{P}$ and $\mathbb{P}$ is the Leray projector over divergence-free vector fields.

Proposition 2.10 (see [6]). *Let $\frac{6}{5} < p < 6$ with $p \neq 2$ and $1 \leq q < \infty$ satisfy $\frac{1}{2} - \frac{1}{q} < \frac{1}{p} < \frac{1}{2} + \frac{1}{q}$, $a \in \mathcal{B}_q^{\frac{3}{q}}(\mathbb{R}^3)$ with*

$$1 + a \geq \underline{\kappa} > 0.$$

Let $f \in \mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}}(\mathbb{R}^3)$ and $\nabla\pi \in \mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}}(\mathbb{R}^3)$ solve the equation

$$-\mathrm{div}\,((1+a)\nabla\pi) = \mathrm{div}\, f. \tag{2.19}$$

Then we have

$$\|\nabla\pi\|_{\dot{B}_{p,2}^{\frac{3}{p}-\frac{3}{2}}} \lesssim (1 + \|a\|_{\mathcal{B}_q^{\frac{3}{q}}})\|\mathbb{Q}f\|_{\dot{B}_{p,2}^{\frac{3}{p}-\frac{3}{2}}}. \tag{2.20}$$

Remark 2.11. As far as we know, such a estimates for elliptic equation with variable coefficients in Besov space is first given by Zhai and Yin in [32], and then it is improved by Burtea [6]. Indeed it follows from Proposition 2.9 that the estimate (2.20) also holds for $p = 2$.

We also recall the estimate for transport equation:

Proposition 2.12 (see Proposition 3.1 in [4]). *Let $(p, q) \in [1,\infty]^2$, $-1 - \frac{3}{p} < s < 1 + 3\min\{\frac{1}{p}, \frac{1}{q}\}$ if $\frac{1}{p} + \frac{1}{q} \leq 1$, or $-1 - \frac{3}{q'} < s < 1 + 3\min\{\frac{1}{p}, \frac{1}{q}\}$ if $\frac{1}{p} + \frac{1}{q} \geq 1$, and $r \in [1,\infty]$ (resp. $s = 1 + 3\min\{\frac{1}{p}, \frac{1}{q}\}$ with $r = 1$). where q' is the conjugate exponent of q. Let v be a divergence free vector field with $\nabla v \in L^1([0,T]; \dot{B}_{p,r}^{\frac{3}{p}} \cap L^\infty(\mathbb{R}^3))$ (resp. $v \in L^1([0,T]; \mathcal{B}_p^{1+\frac{3}{p}}(\mathbb{R}^3))$). Given $f \in L^1([0,T]; \dot{B}_{q,r}^s(\mathbb{R}^3))$ and let $a_0 \in \dot{B}_{q,r}^s$ and $a \in \mathcal{C}([0,T]; \dot{B}_{q,r}^s(\mathbb{R}^3))$ solves*

$$\begin{cases} \partial_t a + u \cdot \nabla a = f, \\ a|_{t=0} = a_0. \end{cases} \tag{2.21}$$

There holds for $0 < t \leq T$

$$\|a\|_{\widetilde{L}_t^\infty(\dot{B}_{q,r}^s)} \leq \|a_0\|_{\dot{B}_{q,r}^s} + C\left\{\int_0^t \|a(\tau)\|_{\dot{B}_{q,r}^s} U'(\tau)d\tau + \|f\|_{L_t^1(\dot{B}_{q,r}^s)}\right\}, \tag{2.22}$$

where $U(t) \stackrel{def}{=} \int_0^t \|\nabla v\|_{\dot B^{\frac{3}{p}}_{p,r}\cap L^\infty} d\tau$ *(resp.* $U(t) \stackrel{def}{=} \int_0^t \|v\|_{\mathcal{B}_p^{1+\frac{3}{p}}} d\tau$*). Similar inequality holds for the inhomogeneous Besov spaces.*

REMARK 2.13. If $f=0$ and $\frac{1}{q}-\frac{1}{3}\le\frac{1}{p}$ then we have further

$$\|a\|_{\widetilde{L}^\infty_t(\mathcal{B}_q^{\frac{3}{q}})} \lesssim \|a_0\|_{\mathcal{B}_q^{\frac{3}{q}}} \mathrm{e}^{CV(t)}, \tag{2.23}$$

$$\|a-\dot S_m a\|_{\widetilde{L}^\infty_t(\mathcal{B}_q^{\frac{3}{q}})} \lesssim \sum_{j\ge m} 2^{\frac{3j}{q}}\|\dot\Delta_j a_0\|_{L^q} + \|a_0\|_{\mathcal{B}_q^{\frac{3}{q}}}\left(\mathrm{e}^{CV(t)}-1\right), \tag{2.24}$$

where $V(t) \stackrel{def}{=} \int_0^t \|v\|_{\mathcal{B}_p^{1+\frac{3}{p}}} d\tau$.

3. *A priori* estimate to the linearized equation of (1.1)**.** The goal of this section is to investigate the *a priori* estimate to smooth enough solutions of the following linearized equations of (1.1), which will be the key ingredient to prove the local well-posedness part of Theorem 1.2.

$$\begin{cases} \partial_t u + v\cdot\nabla u - (1+a)(\Delta u - \nabla\pi) = f, \\ \mathrm{div} u = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{3.1}$$

The main result states as follows:

PROPOSITION 3.1. *Let* p, q, u_0 *be given by Theorem 1.2. Let* (a,v,f) *be smooth enough functions satisfy* $1+a\ge \underline{b}>0$ *and* $\mathrm{div}\, v=0$*. Then for any smooth enough solution* $(u,\nabla\pi)$ *of the system* (3.1) *on* $[0,T]$*, if there exist some sufficiently small positive constant* c *and some integer* $m\in\mathbb{Z}$ *such that*

$$\left(1+\|a\|_{\mathcal{B}_q^{\frac{3}{q}}}\right)^{12}\|a-\dot S_m a\|_{\widetilde{L}^\infty_T(\mathcal{B}_q^{\frac{3}{q}})} \le c \quad \textit{and} \quad \|a\|_{L^\infty_T(\mathcal{B}_q^{\frac{3}{q}})} \le C, \tag{3.2}$$

there holds

$$\begin{aligned} &\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}_p^{\frac{3}{p}-1})} + \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+1})} + \|\nabla\pi\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-1})} \lesssim \int_0^t \|v\|_{\mathcal{B}_p^{\frac{3}{p}+1}}\|u\|_{\mathcal{B}_p^{\frac{3}{p}}}\,\mathrm{d}\tau \\ &+\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}}} + 2^{4m}(A_{ma}(t)+2)^{12}\int_0^t \|u\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\,\mathrm{d}\tau + 2^m(A_{ma}(t)+1)^3\|f\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})} \\ &+2^{4m}(A_{ma}(t)+1)^6\int_0^t\left(\|v\|^{\frac{2}{3}}_{\mathcal{B}_p^{\frac{3}{p}}}\|v\|^{\frac{2}{3}}_{\mathcal{B}_p^{\frac{3}{p}+1}} + \|v\|^{\frac{3}{2}}_{\mathcal{B}_p^{\frac{3}{p}}}\|v\|^{\frac{1}{2}}_{\mathcal{B}_p^{\frac{3}{p}+1}}\right)\|u\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\,\mathrm{d}\tau \\ &+2^{2m}(A_{ma}(t)+1)^4\int_0^t\left(\|v\|_{\mathcal{B}_p^{\frac{3}{p}}} + \|v\|^2_{\mathcal{B}_p^{\frac{3}{p}-1}}\right)\|u\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\,\mathrm{d}\tau, \quad \forall t\in[0,T]. \end{aligned} \tag{3.3}$$

Here in all that follows, we always denote

$$A_{ma}(t) \stackrel{def}{=} 2^{\frac{3}{q}m}\|\dot S_m a\|_{L^\infty_t(L^q)} \quad \textit{and} \quad \tilde{\mathcal{B}}_p^s \stackrel{def}{=} \begin{cases} \mathcal{B}_p^{s-1}\cap \dot B_{p,2}^{s-\frac{3}{2}}, & \textit{if} \quad \dfrac{3}{2}\le p<6, \\ \mathcal{B}_p^{s-1}\cap\mathcal{B}_p^{s-2}, & \textit{if} \quad 1<p<\dfrac{3}{2}. \end{cases} \tag{3.4}$$

Proof. Writing a as $\dot S_m a + (a - \dot S_m a)$. We observe from (3.2) that

$$1 + \dot S_m a = 1 + a + (\dot S_m a - a) \geq \frac{b}{2}. \tag{3.5}$$

Correspondingly, we write the u equation of (3.1) as

$$\begin{aligned} \partial_t u + v\cdot\nabla u - (1+\dot S_m a)\big(\Delta u - \nabla\pi\big) =& f + E_m \quad \text{with} \\ E_m =& (a - \dot S_m a)\big(\Delta u - \nabla\pi\big). \end{aligned} \tag{3.6}$$

We now decompose the proof of Proposition 3.1 into the following steps:

Step 1. The estimate of $\|u\|_{\widetilde L^\infty_t(\mathcal{B}_p^{\frac3p-1})} + \|u\|_{L^1_t(\mathcal{B}_p^{\frac3p+1})}$.

Let $\mathbb{P} = I + \nabla(-\Delta)^{-1}\operatorname{div}$ be the Leray projection operator. Applying the operator $\mathbb{P}\dot\Delta_j$ to (3.6) gives

$$\partial_t \dot\Delta_j u + \mathbb{P}\dot\Delta_j\big(v\cdot\nabla u - (1+\dot S_m a)(\Delta u - \nabla\pi)\big) = \mathbb{P}\dot\Delta_j(f + E_m).$$

By applying a standard commutator process, we find

$$-\mathbb{P}\dot\Delta_j\big((1+\dot S_m a)\Delta u\big) = -\operatorname{div}\big((1+\dot S_m a)\dot\Delta_j\nabla u\big) + \nabla S_m a\cdot\dot\Delta_j\nabla u - [\mathbb{P}\dot\Delta_j, \dot S_m a]\Delta u.$$

We thus get

$$\begin{aligned} \partial_t\dot\Delta_j u + v\cdot\nabla\dot\Delta_j u - \operatorname{div}\big((1+\dot S_m a)\dot\Delta_j\nabla u\big) = -[\mathbb{P}\dot\Delta_j, v]\nabla u \\ - \mathbb{P}\dot\Delta_j(\dot S_m a\nabla\pi) + \mathbb{P}\dot\Delta_j(f+E_m) + [\mathbb{P}\dot\Delta_j, \dot S_m a]\Delta u - \nabla S_m a\cdot\dot\Delta_j\nabla u. \end{aligned} \tag{3.7}$$

Due to $\operatorname{div} v = 0$, by multiplying (3.7) by $|\dot\Delta_j u|^{p-2}\dot\Delta_j u$ and then integrating the resulting equality over $\mathbb{R}^3$, we obtain

$$\begin{aligned} &\frac1p\frac{\mathrm d}{\mathrm dt}\|\dot\Delta_j u\|_{L^p}^p - \int_{\mathbb{R}^3}\operatorname{div}\big((1+\dot S_m a)\dot\Delta_j\nabla u\big)\cdot|\dot\Delta_j u|^{p-2}\dot\Delta_j u\,\mathrm dx \\ &\quad\leq \|\dot\Delta_j u\|_{L^p}^{p-1}\Big(\big\|[\mathbb{P}\dot\Delta_j, v]\nabla u\big\|_{L^p} + \|\dot\Delta_j f\|_{L^p} + \|\dot\Delta_j E_m\|_{L^p} \\ &\qquad + \big\|\mathbb{P}\dot\Delta_j(\dot S_m a\nabla\pi)\big\|_{L^p} + \big\|[\mathbb{P}\dot\Delta_j, \dot S_m a]\Delta u\big\|_{L^p} + \|\nabla S_m a\cdot\dot\Delta_j\nabla u\|_{L^p}\Big). \end{aligned} \tag{3.8}$$

It follows from (3.5) and (11) of [8] that

$$-\int_{\mathbb{R}^3}\operatorname{div}\big((1+\dot S_m a)\dot\Delta_j\nabla u\big)\cdot|\dot\Delta_j u|^{p-2}\dot\Delta_j u\mathrm dx \geq \frac{bC}{2}\left(\frac{p-1}{p^2}\right)2^{2j}\|\dot\Delta_j u\|_{L^p}^p. \tag{3.9}$$

By applying Lemma 2.3, we find

$$\begin{aligned} \big\|[\mathbb{P}\dot\Delta_j, v]\nabla u\big\|_{L^p} &\lesssim d_j(t)2^{-j(\frac3p-1)}\|\nabla v\|_{\mathcal{B}_p^{\frac3p}}\|u\|_{\mathcal{B}_p^{\frac3p-1}}, \\ \big\|[\mathbb{P}\dot\Delta_j, \dot S_m a]\Delta u\big\|_{L^p} &\lesssim d_j(t)2^{-j(\frac3p-1)}\|\nabla\dot S_m a\|_{\mathcal{B}_q^{\frac3q}}\|\nabla u\|_{\mathcal{B}_p^{\frac3p-1}}. \end{aligned}$$

Whereas observing that $\mathbb{P}(\dot S_m a\nabla\pi) = \mathbb{P}(\nabla(\dot S_m a\pi) - \pi\nabla\dot S_m a)) = -\mathbb{P}(\pi\nabla\dot S_m a)$. We deduce from a similar proof of Lemma 2.5 that

$$\|\mathbb{P}\dot\Delta_j(\dot S_m a\nabla\pi)\|_{L^p} \lesssim d_j(t)2^{-j(\frac3p-1)}\|\nabla\dot S_m a\|_{\mathcal{B}_q^{\frac3q-\frac12}}\|\nabla\pi\|_{\mathcal{B}_{p,2}^{\frac3p-\frac32}} \text{ by } \frac1p < \frac16 + \frac1q,$$

and

$$\|\dot\Delta_j E_m\|_{L^p}\lesssim d_j(t)2^{-j(\frac{3}{p}-1)}\|a-\dot S_m a\|_{\mathcal{B}^{\frac{3}{q}}_q}\big(\|u\|_{\mathcal{B}^{\frac{3}{p}+1}_p}+\|\nabla\pi\|_{\mathcal{B}^{\frac{3}{p}-1}_p}\big). \tag{3.10}$$

Finally it is easy to observe that

$$\begin{aligned}\|\nabla S_m a\cdot\dot\Delta_j\nabla u\|_{L^p}&\lesssim\|\nabla S_m a\|_{L^\infty}\|\dot\Delta_j\nabla u\|_{L^p}\\ &\lesssim d_j(t)2^{-j(\frac{3}{p}-1)}\|\dot S_m a\|_{\mathcal{B}^{\frac{3}{q}+1}_q}\|u\|_{\mathcal{B}^{\frac{3}{p}}_p}.\end{aligned}$$

By substituting above estimates into (3.8) and using the fact that

$$\|\dot S_m a\|_{\mathcal{B}^{\frac{3}{q}+\tau}_q}\lesssim 2^{\left(\tau+\frac{3}{q}\right)m}\|\dot S_m a\|_{L^q}\quad\text{for any }\tau>-\frac{3}{q},$$

we deduce that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot\Delta_j u\|_{L^p}+2^{2j}\|\dot\Delta_j u\|_{L^p}&\lesssim d_j(t)2^{-j(\frac{3}{p}-1)}\Big(\|f\|_{\mathcal{B}^{\frac{3}{p}-1}_p}+\|v\|_{\mathcal{B}^{\frac{3}{p}+1}_p}\|u\|_{\mathcal{B}^{\frac{3}{p}-1}_p}\\ &+\|a-\dot S_m a\|_{\mathcal{B}^{\frac{3}{q}}_q}\big(\|u\|_{\mathcal{B}^{\frac{3}{p}+1}_p}+\|\nabla\pi\|_{\mathcal{B}^{\frac{3}{p}-1}_p}\big)\\ &+2^{\left(\frac{1}{2}+\frac{3}{q}\right)m}\|\dot S_m a\|_{L^q}\big(2^{\frac{m}{2}}\|u\|_{\mathcal{B}^{\frac{3}{p}}_p}+\|\nabla\pi\|_{B^{\frac{3}{p}-\frac{3}{2}}_{p,2}}\big)\Big).\end{aligned}$$

Integrating the above inequality over $[0,t]$ and multiplying it by $2^{j(\frac{3}{p}-1)}$, and then summing up the resulting inequality over $j\in\mathbb{Z}$ and using (3.2), we achieve

$$\begin{aligned}\|u\|_{\widetilde L^\infty_t(\mathcal{B}^{\frac{3}{p}-1}_p)}+\|u\|_{L^1_t(\mathcal{B}^{\frac{3}{p}+1}_p)}&\lesssim\|u_0\|_{\mathcal{B}^{\frac{3}{p}-1}_p}+\|f\|_{L^1_t(\mathcal{B}^{\frac{3}{p}-1}_p)}\\ &+\int_0^t\|v\|_{\mathcal{B}^{\frac{3}{p}+1}_p}\|u\|_{\mathcal{B}^{\frac{3}{p}-1}_p}\,\mathrm{d}\tau+\|a-\dot S_m a\|_{L^\infty_t(\mathcal{B}^{\frac{3}{q}}_q)}\|\nabla\pi\|_{L^1_t(\mathcal{B}^{\frac{3}{p}-1}_p)}\\ &+2^{\left(\frac{1}{2}+\frac{3}{q}\right)m}\|\dot S_m a\|_{L^\infty_t(L^q)}\big(2^{\frac{m}{2}}\|u\|_{L^1_t(\mathcal{B}^{\frac{3}{p}}_p)}+\|\nabla\pi\|_{L^1_t(\mathcal{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2})}\big).\end{aligned} \tag{3.11}$$

Step 2. The estimate of $\|\nabla\pi\|_{L^1_t(B^{\frac{3}{p}-1}_{p,1})}$.

In order to do it, we get, by first applying the space divergence to the equation (3.6) that

$$\begin{aligned}\mathrm{div}\,\{(1+a)\nabla\pi\}=&-\mathrm{div}\,(v\cdot\nabla u)+\mathrm{div}\,(\dot S_m a\Delta u)\\ &+\mathrm{div}\,\{(a-\dot S_m a)\Delta u\}+\mathrm{div}\,f.\end{aligned} \tag{3.12}$$

Applying the dyadic operator $\dot\Delta_j$ to (3.12) gives

$$\begin{aligned}\mathrm{div}\,\big((1+\dot S_m a)\nabla\dot\Delta_j\pi\big)=&-\mathrm{div}\,\dot\Delta_j(v\cdot\nabla u)+\mathrm{div}\,\dot\Delta_j(\dot S_m a\Delta u)+\mathrm{div}\,\dot\Delta_j f\\ &+\mathrm{div}\,\dot\Delta_j\big((\dot S_m a-a)(\nabla\pi-\Delta u)\big)-\mathrm{div}\,[\dot\Delta_j,\dot S_m a]\nabla\pi.\end{aligned}$$

Taking L^2 inner product of the above equation with $|\dot\Delta_j\pi|^{p-2}\dot\Delta_j\pi$ and using (3.9), we find

$$\begin{aligned}2^{2j}\|\dot\Delta_j\pi\|_{L^p}\lesssim&\Big(2^j\|\dot\Delta_j f\|_{L^p}+\|\dot\Delta_j\,\mathrm{div}(v\cdot\nabla u)\|_{L^p}+\|\mathrm{div}\,\dot\Delta_j(\dot S_m a\Delta u)\|_{L^p}\\ &+2^j\big\|\dot\Delta_j((\dot S_m a-a)(\nabla\pi-\Delta u))\big\|_{L^p}+\|\mathrm{div}\,[\dot\Delta_j,\dot S_m a]\nabla\pi\|_{L^p}\Big).\end{aligned} \tag{3.13}$$

We now estimate term by term in (3.13). Due to $\operatorname{div} v = 0$, we have $\operatorname{div}(v \cdot \nabla u) = \operatorname{div}(u \cdot \nabla v)$. So that applying (2.5) yields

$$\|\dot{\Delta}_j \operatorname{div}(v \cdot \nabla u)\|_{L^p} \lesssim d_j(t) 2^{-j(\frac{3}{p}-2)} \|\nabla v\|_{\mathcal{B}_p^{\frac{3}{p}}} \|u\|_{\mathcal{B}_p^{\frac{3}{p}-1}}. \tag{3.14}$$

To deal with the estimate of the term $\operatorname{div}(\dot{S}_m a \Delta u)$, we need the following Lemma, the proof of which will be postponed after the proof of Proposition 3.1.

Lemma 3.2. *Let s, p, q satisfy $s + \frac{3}{p} - 2 > 0$ if $\frac{1}{p} + \frac{1}{q} \leq 1$, or $1 + s - \frac{3}{q} > 0$ if $\frac{1}{p} + \frac{1}{q} > 1$. We have*

$$\|\operatorname{div}(\dot{S}_m a \Delta u)\|_{\mathcal{B}_p^{s-3+\frac{3}{p}-\frac{3}{q}}} \lesssim \|\dot{S}_m a\|_{\mathcal{B}_q^s} \|u\|_{\mathcal{B}_p^{\frac{3}{p}}} \quad \textit{for} \quad s \leq 1 + \frac{3}{q} \tag{3.15}$$

Applying (3.15) for $s = 1 + \frac{3}{q}$ yields

$$\|\operatorname{div} \dot{\Delta}_j (\dot{S}_m a \Delta u)\|_{L^p} \lesssim d_j(t) 2^{-\left(\frac{3}{p}-2\right)j} \|\nabla \dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}}} \|u\|_{\mathcal{B}_p^{\frac{3}{p}}}. \tag{3.16}$$

For the last term in (3.13), we get, by applying (2.15), that

$$\|\operatorname{div} [\dot{\Delta}_j, \dot{S}_m a] \nabla \pi\|_{L^p} \lesssim d_j(t) 2^{-(\frac{3}{p}-2)j} \|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}} \|\nabla \pi\|_{\dot{B}_{p,2}^{\frac{3}{p}-\frac{3}{2}}}. \tag{3.17}$$

Thanks to (3.2), by inserting the estimates, (3.10), (3.14), (3.16) and (3.17) into (3.13), we deduce that

$$\begin{aligned} \|\nabla \pi\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}-1})} &\lesssim 2^{\left(\frac{3}{q}+\frac{1}{2}\right)m} \|\dot{S}_m a\|_{L_t^\infty(L^q)} \Big(2^{\frac{m}{2}} \|u\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}})} + \|\nabla \pi\|_{L_t^1(\dot{B}_{p,2}^{\frac{3}{p}-\frac{3}{2}})} \Big) \\ &+ \|f\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}-1})} + \|a - \dot{S}_m a\|_{L_t^\infty(\mathcal{B}_q^{\frac{3}{q}})} \|u\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}+1})} + \int_0^t \|v\|_{\mathcal{B}_p^{\frac{3}{p}+1}} \|u\|_{\mathcal{B}_p^{\frac{3}{p}-1}} \, d\tau. \end{aligned} \tag{3.18}$$

By summing (3.11) and (3.18) and then using (3.2), we obtain

$$\begin{aligned} &\|u\|_{\widetilde{L}_t^\infty(\mathcal{B}_p^{\frac{3}{p}-1})} + \|u\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}+1})} + \|\nabla \pi\|_{\widetilde{L}_t^1(\mathcal{B}_p^{\frac{3}{p}-1})} \lesssim \|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}} + \|f\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}-1})} \\ &\quad + 2^{\left(\frac{3}{q}+\frac{1}{2}\right)m} \|\dot{S}_m a\|_{L_t^\infty(L^q)} \Big(2^{\frac{m}{2}} \|u\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}})} + \|\nabla \pi\|_{L_t^1(\dot{B}_{p,2}^{\frac{3}{p}-\frac{3}{2}})} \Big) + \int_0^t \|v\|_{\mathcal{B}_p^{\frac{3}{p}+1}} \|u\|_{\mathcal{B}_p^{\frac{3}{p}-1}} \, d\tau. \end{aligned}$$

Note that $\|u\|_{\mathcal{B}_p^{\frac{3}{p}}} \lesssim \|u\|_{\mathcal{B}_p^{\frac{3}{p}-1}}^{\frac{1}{2}} \|u\|_{\mathcal{B}_p^{\frac{3}{p}+1}}^{\frac{1}{2}}$, we get, by applying Young's inequality, that

$$\begin{aligned} &\|u\|_{\widetilde{L}_t^\infty(\mathcal{B}_p^{\frac{3}{p}-1})} + \|u\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}+1})} + \|\nabla \pi\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}-1})} \\ &\lesssim \|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}} + \|f\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}-1})} + 2^{2\left(1+\frac{3}{q}\right)m} \|\dot{S}_m a\|_{L_t^\infty(L^q)}^2 \int_0^t \|u\|_{\mathcal{B}_p^{\frac{3}{p}-1}} \, d\tau \\ &\quad + \int_0^t \|v\|_{\mathcal{B}_p^{\frac{3}{p}+1}} \|u\|_{\mathcal{B}_p^{\frac{3}{p}-1}} \, d\tau + 2^{\left(\frac{3}{q}+\frac{1}{2}\right)m} \|\dot{S}_m a\|_{L_t^\infty(L^q)} \|\nabla \pi\|_{L_t^1(\dot{B}_{p,2}^{\frac{3}{p}-\frac{3}{2}})}. \end{aligned} \tag{3.19}$$

Step 3. The estimate of (3.3) for $p\in[3/2,6)$.

In this step, we shall focus on the case when $\frac{3}{2}\leq p<6$. We first deduce from (3.12) and Proposition 2.10 that

$$\|\nabla\pi\|_{\dot B^{\frac{3}{p}-\frac{3}{2}}_{p,2}}\lesssim\big(1+\|a\|_{B^{\frac{3}{q}}_q}\big)\Big(\|v\cdot\nabla u\|_{\dot B^{\frac{3}{p}-\frac{3}{2}}_{p,2}}+\|\dot S_m a\Delta u\|_{\dot B^{\frac{3}{p}-\frac{3}{2}}_{p,2}}+\|(a-\dot S_m a)\Delta u\|_{\dot B^{\frac{3}{p}-\frac{3}{2}}_{p,2}}+\|f\|_{\dot B^{\frac{3}{p}-\frac{3}{2}}_{p,2}}\Big).\tag{3.20}$$

Due to $\operatorname{div}v=0$, $v\cdot\nabla u=\operatorname{div}(v\otimes u)$, we get, by applying Lemma 2.4, that

$$\|v\cdot\nabla u\|_{\mathcal{B}^{\frac{3}{p}-\frac{3}{2}}_p}\lesssim\|v\|_{\mathcal{B}^{\frac{3}{p}}_p}\|u\|_{\mathcal{B}^{\frac{3}{p}-\frac{1}{2}}_p}+\|v\|_{\dot{\mathcal{B}}^{\frac{3}{p}-\frac{1}{2}}_p}\|u\|_{\mathcal{B}^{\frac{3}{p}}_p}.\tag{3.21}$$

It follows from (2.6) that

$$\begin{aligned}\|\dot\Delta_j((a-\dot S_m a)\Delta u)\|_{\mathcal{B}^{\frac{3}{p}-\frac{3}{2}}_p}&\lesssim\|a-\dot S_m a\|_{\mathcal{B}^{\frac{3}{q}}_q}\|u\|_{\mathcal{B}^{\frac{3}{p}+\frac{1}{2}}_p},\\ \|\dot\Delta_j(\dot S_m a\Delta u)\|_{\mathcal{B}^{\frac{3}{p}-\frac{3}{2}}_p}&\lesssim\|\dot S_m a\|_{\mathcal{B}^{\frac{3}{q}}_q}\|u\|_{\mathcal{B}^{\frac{3}{p}+\frac{1}{2}}_p}.\end{aligned}\tag{3.22}$$

Inserting the above estimates into (3.20) and using (3.2) gives rise to

$$\begin{aligned}\|\nabla\pi\|_{L^1_t(\dot B^{\frac{3}{p}-\frac{3}{2}}_{p,2})}&\lesssim 2^{\frac{3}{q}m}\|\dot S_m a\|_{L^\infty_t(L^q)}\|u\|_{L^1_t(\mathcal{B}^{\frac{3}{p}+\frac{1}{2}}_p)}+\|f\|_{L^1_t(\mathcal{B}^{\frac{3}{p}-\frac{3}{2}}_p)}\\&\quad+\|a-\dot S_m a\|_{L^\infty_t(\mathcal{B}^{\frac{3}{q}}_q)}\int_0^t\|u\|^{\frac14}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|u\|^{\frac34}_{\mathcal{B}^{\frac{3}{p}+1}_p}\,\mathrm{d}\tau\\&\quad+\int_0^t\Big(\|v\|^{\frac12}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|v\|^{\frac12}_{\mathcal{B}^{\frac{3}{p}+1}_p}\|u\|^{\frac34}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|u\|^{\frac14}_{\mathcal{B}^{\frac{3}{p}+1}_p}\\&\quad+\|u\|^{\frac12}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|u\|^{\frac12}_{\mathcal{B}^{\frac{3}{p}+1}_p}\|v\|^{\frac34}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|v\|^{\frac14}_{\mathcal{B}^{\frac{3}{p}+1}_p}\Big)\,\mathrm{d}\tau.\end{aligned}\tag{3.23}$$

Then by substituting (3.23) into (3.19) and using Young's inequality, we obtain

$$\begin{aligned}&\|u\|_{\widetilde L^\infty_t(\mathcal{B}^{\frac{3}{p}-1}_p)}+\|u\|_{L^1_t(\mathcal{B}^{\frac{3}{p}+1}_p)}+\|\nabla\pi\|_{L^1_t(\mathcal{B}^{\frac{3}{p}-1}_p)}\\&\lesssim\|u_0\|_{\mathcal{B}^{\frac{3}{p}-1}_p}+2^{2m}A^2_{ma}(t)\big(1+A_{ma}(t)\big)^6\int_0^t\|u\|_{\mathcal{B}^{\frac{3}{p}-1}_p}\,\mathrm{d}\tau\\&\quad+2^m\big(1+A_{ma}(t)\big)^2\int_0^t\Big(\|v\|^{\frac23}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|v\|^{\frac23}_{\mathcal{B}^{\frac{3}{p}+1}_p}+\|v\|^{\frac32}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|v\|^{\frac12}_{\mathcal{B}^{\frac{3}{p}+1}_p}\Big)\|u\|_{\mathcal{B}^{\frac{3}{p}-1}_p}\,\mathrm{d}\tau\\&\quad+\int_0^t\|v\|_{\mathcal{B}^{\frac{3}{p}+1}_p}\|u\|_{\mathcal{B}^{\frac{3}{p}-1}_p}\,\mathrm{d}\tau+\big(2^{\frac{m}{2}}A_{ma}(t)+1\big)\|f\|_{L^1_t(\tilde{\mathcal{B}}^{\frac{3}{p}}_p)},\end{aligned}\tag{3.24}$$

where $A_{ma}(t)$ is given by (3.4). This proves (3.3) for $p\in[3/2,6)$.

Step 4. The estimate of $\|\nabla\pi\|_{L^1_t(\mathcal{B}^{\frac{3}{p}-\frac{3}{2}}_p)}$.

As in the previous step, we shall handle term by term in (3.13). The estimate of $\|\dot\Delta_j\operatorname{div}(v\cdot\nabla u)\|_{L^p}$ and $\|\dot\Delta_j((a-\dot S_m a)\Delta u\|_{L^p}$ has been given by (3.21) and (3.22) respectively. For the remaining terms, we first get, by applying (2.6), that

$$\big\|\dot\Delta_j((a-\dot S_m a)\nabla\pi\big\|_{L^p}\lesssim d_j(t)2^{-j\left(\frac{3}{p}-\frac{3}{2}\right)}\|a-\dot S_m a\|_{\mathcal{B}^{\frac{3}{q}}_q}\|\nabla\pi\|_{\mathcal{B}^{\frac{3}{p}-\frac{3}{2}}_p}.$$

While applying (3.15) for $s = \frac{1}{2} + \frac{3}{q}$ yields

$$\|\mathrm{div}\, \dot{\Delta}_j(\dot{S}_m a \Delta u)\|_{L^p} \lesssim d_j(t) 2^{-\left(\frac{3}{p}-\frac{5}{2}\right)j} \|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}+\frac{1}{2}}} \|u\|_{\mathcal{B}_p^{\frac{3}{p}}}.$$

Applying (2.16) gives

$$\|\mathrm{div}\, [\dot{\Delta}_j, \dot{S}_m a]\nabla\pi\|_{L^p} \lesssim d_j 2^{-(\frac{3}{p}-\frac{5}{2})j} \|\dot{S}_m a\|_{\mathcal{B}_{q,1}^{\frac{3}{q}+\frac{1}{2}}} \|\nabla\pi\|_{\mathcal{B}_p^{\frac{3}{p}-2}}.$$

By inserting the above estimates into (3.13) and using (3.2), we deduce that

$$\begin{aligned}
\|\nabla\pi\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}})} &\lesssim 2^{(\frac{3}{q}+\frac{1}{2})m} \|\dot{S}_m a\|_{L_t^\infty(L^q)} \big(\|u\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}})} + \|\nabla\pi\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}-2})}\big) \\
&+ \|f\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}})} + \|a - \dot{S}_m a\|_{L_t^\infty(\mathcal{B}_q^{\frac{3}{q}})} \int_0^t \|u\|^{\frac{1}{4}}_{\mathcal{B}_p^{\frac{3}{p}-1}} \|u\|^{\frac{3}{4}}_{\mathcal{B}_p^{\frac{3}{p}+1}}\,\mathrm{d}\tau \\
&+ \int_0^t \Big(\|v\|^{\frac{1}{2}}_{\mathcal{B}_p^{\frac{3}{p}-1}} \|v\|^{\frac{1}{2}}_{\mathcal{B}_p^{\frac{3}{p}+1}} \|u\|^{\frac{3}{4}}_{\mathcal{B}_p^{\frac{3}{p}-1}} \|u\|^{\frac{1}{4}}_{\mathcal{B}_p^{\frac{3}{p}+1}} \\
&+ \|u\|^{\frac{1}{2}}_{\mathcal{B}_p^{\frac{3}{p}-1}} \|u\|^{\frac{1}{2}}_{\mathcal{B}_p^{\frac{3}{p}+1}} \|v\|^{\frac{3}{4}}_{\mathcal{B}_p^{\frac{3}{p}-1}} \|v\|^{\frac{1}{4}}_{\mathcal{B}_p^{\frac{3}{p}+1}} \Big)\,\mathrm{d}\tau.
\end{aligned} \tag{3.25}$$

Step 5. The estimate of $\|\nabla\pi\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}-2})}$ with $1 < p < \frac{3}{2}$.

Once again, we shall start with (3.13). Firstly due to $\mathrm{div}\, v = 0$ and $p < 6$, $v \cdot \nabla u = \mathrm{div}(v \otimes u)$, so that

$$\|\dot{\Delta}_j(v \cdot \nabla u)\|_{L^p} \lesssim d_j(t) 2^{-j(\frac{3}{p}-2)} \big(\|v\|_{\mathcal{B}_p^{\frac{3}{p}}} \|u\|_{\mathcal{B}_p^{\frac{3}{p}-1}} + \|v\|_{\mathcal{B}_p^{\frac{3}{p}-1}} \|u\|_{\mathcal{B}_p^{\frac{3}{p}}}\big).$$

It follows from (2.7) that

$$\big\|\dot{\Delta}_j\big((a - \dot{S}_m a)(\Delta u - \nabla\pi)\big)\big\|_{L^p} \lesssim d_j(t) 2^{-j\left(\frac{3}{p}-2\right)} \|a - \dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}}} \big(\|u\|_{\mathcal{B}_p^{\frac{3}{p}}} + \|\nabla\pi\|_{\mathcal{B}_p^{\frac{3}{p}-2}}\big).$$

Whereas due to $\frac{1}{p} + \frac{1}{q} > \frac{2}{3}$, we get, by applying (3.15) with $s = \frac{3}{q}$, that

$$\|\dot{\Delta}_j\, \mathrm{div}(\dot{S}_m a \Delta u)\|_{L^p} \lesssim d_j(t) 2^{-(\frac{3}{p}-3)j} \|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}}} \|u\|_{\mathcal{B}_p^{\frac{3}{p}}}.$$

To handle the commutator term in (3.13), we need the following Lemma, the proof of which will be postponed after the proof of Proposition 3.1.

LEMMA 3.3. *For $1 < p, q < 6$, one has*

$$\|[\dot{\Delta}_j, \dot{S}_m a]\nabla\pi\|_{L^p} \lesssim d_j(t) 2^{-\left(\frac{3}{p}-2\right)j} \|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}-\frac{1}{2}}} \|\nabla\pi\|_{L^2}, \tag{3.26}$$

$$\|\,\mathrm{div}[\dot{\Delta}_j, \dot{S}_m a]\nabla\pi\|_{L^p} \lesssim d_j(t) 2^{-\left(\frac{3}{p}-3\right)j} \|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}-\frac{1}{2}}} \|\nabla\pi\|_{L^2} \tag{3.27}$$

$$\textit{with} \quad 1 < p < \frac{3}{2} \quad \textit{and} \quad \frac{1}{p} < \frac{1}{6} + \frac{1}{q}.$$

By using above inequalities, we find

$$\begin{aligned}
\|\nabla\pi\|_{L^1_t(B^{\frac{3}{p}-2}_{p,1})} &\lesssim 2^{\frac{3}{q}m}\|\dot S_m a\|_{L^\infty_t(L^q)}\|u\|_{L^1_t(\mathcal{B}^{\frac{3}{p}}_p)} + \|f\|_{L^1_t(\mathcal{B}^{\frac{3}{p}-2}_p)}\\
&+ 2^{(\frac{3}{q}-\frac{1}{2})m}\|\dot S_m a\|_{L^\infty_t(L^q)}\|\nabla\pi\|_{L^1_t(L^2)}\\
&+ \|a-\dot S_m a\|_{L^\infty_t(\mathcal{B}^{\frac{3}{q}}_q)}\int_0^t \|u\|^{\frac{1}{2}}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|u\|^{\frac{1}{2}}_{\mathcal{B}^{\frac{3}{p}+1}_p}\,\mathrm{d}\tau\\
&+ \int_0^t\Big(\|v\|_{\mathcal{B}^{\frac{3}{p}}_p}\|u\|_{\mathcal{B}^{\frac{3}{p}-1}_p} + \|v\|_{\mathcal{B}^{\frac{3}{p}-1}_p}\|u\|_{\mathcal{B}^{\frac{3}{p}}_p}\Big)\,\mathrm{d}\tau.
\end{aligned} \tag{3.28}$$

Step 6. The estimate of (3.3) for $p \in (1, 3/2)$.

In this step, we use (3.12) and Proposition 2.9 to get

$$\|\nabla\pi\|_{L^2} \lesssim \|v\cdot\nabla u\|_{L^2} + \|\dot S_m a\Delta u\|_{L^2} + \Big\|(a-\dot S_m a)\Delta u\Big\|_{L^2} + \|f\|_{L^2} \tag{3.29}$$

For case of $1 < p < \frac{3}{2}$, we have $\mathcal{B}^{\frac{3}{p}-\frac{3}{2}}_p(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$, it is enough for estimate every term on the right hand side of (3.29) in the space of $\mathcal{B}^{\frac{3}{p}-\frac{3}{2}}_p(\mathbb{R}^3)$. However, these estimates are already obtained by **Step 3.** As a result, it comes out

$$\begin{aligned}
\|\nabla\pi\|_{L^1_t(L^2)} &\lesssim 2^{\frac{3}{q}m}\|\dot S_m a\|_{L^\infty_t(L^q)}\|u\|_{L^1_t(\mathcal{B}^{\frac{3}{p}+\frac{1}{2}}_p)} + \|f\|_{L^1_t(\mathcal{B}^{\frac{3}{p}-\frac{3}{2}}_p)}\\
&+ \|a-\dot S_m a\|_{L^\infty_t(\mathcal{B}^{\frac{3}{q}}_q)}\int_0^t \|u\|^{\frac{1}{4}}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|u\|^{\frac{3}{4}}_{\mathcal{B}^{\frac{3}{p}+1}_p}\,\mathrm{d}\tau\\
&+ \int_0^t\Big(\|v\|^{\frac{1}{2}}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|v\|^{\frac{1}{2}}_{\mathcal{B}^{\frac{3}{p}+1}_p}\|u\|^{\frac{3}{4}}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|u\|^{\frac{1}{4}}_{\mathcal{B}^{\frac{3}{p}+1}_p}\\
&\qquad + \|u\|^{\frac{1}{2}}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|u\|^{\frac{1}{2}}_{\mathcal{B}^{\frac{3}{p}+1}_p}\|v\|^{\frac{3}{4}}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|v\|^{\frac{1}{4}}_{\mathcal{B}^{\frac{3}{p}+1}_p}\Big)\,\mathrm{d}\tau.
\end{aligned} \tag{3.30}$$

We remark that the advantage of (3.30) than (3.25) is that the pressure term does not appear on the right-hand side of (3.30).

By inserting (3.30) into (3.28), we get for $1 < p < \frac{3}{2}$

$$\begin{aligned}
\|\nabla\pi\|_{L^1_t(\mathcal{B}^{\frac{3}{p}-2}_p)} &\lesssim A_{ma}(t)\|u\|_{L^1_t(\mathcal{B}^{\frac{3}{p}}_p)} + 2^{-\frac{m}{2}}A^2_{ma}(t)\|u\|_{L^1_t(\mathcal{B}^{\frac{3}{p}+\frac{1}{2}}_p)}\\
&+ (2^{-\frac{m}{2}}A_{ma}(t)+1)\|a-\dot S_m a\|_{L^\infty_t(\mathcal{B}^{\frac{3}{q}}_q)}\int_0^t\Big(\|u\|^{\frac{1}{4}}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|u\|^{\frac{3}{4}}_{\mathcal{B}^{\frac{3}{p}+1}_p} + \|u\|^{\frac{1}{2}}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|u\|^{\frac{1}{2}}_{\mathcal{B}^{\frac{3}{p}+1}_p}\Big)\,\mathrm{d}\tau\\
&+ 2^{-\frac{m}{2}}A_{ma}(t)\int_0^t \|v\|^{\frac{1}{2}}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|v\|^{\frac{1}{2}}_{\mathcal{B}^{\frac{3}{p}+1}_p}\|u\|^{\frac{3}{4}}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|u\|^{\frac{1}{4}}_{\mathcal{B}^{\frac{3}{p}+1}_p}\,\mathrm{d}\tau\\
&+ 2^{-\frac{m}{2}}A_{ma}(t)\int_0^t \|u\|^{\frac{1}{2}}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|u\|^{\frac{1}{2}}_{\mathcal{B}^{\frac{3}{p}+1}_p}\|v\|^{\frac{3}{4}}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|v\|^{\frac{1}{4}}_{\mathcal{B}^{\frac{3}{p}+1}_p}\,\mathrm{d}\tau + \|f\|_{L^1_t(\mathcal{B}^{\frac{3}{p}-2}_p)}\\
&+ \int_0^t\big(\|v\|_{\mathcal{B}^{\frac{3}{p}}_p}\|u\|_{\mathcal{B}^{\frac{3}{p}-1}_p} + \|v\|_{\mathcal{B}^{\frac{3}{p}-1}_p}\|u\|_{\mathcal{B}^{\frac{3}{p}}_p}\big)\,\mathrm{d}\tau + 2^{-\frac{m}{2}}A_{ma}(t)\|f\|_{L^1_t(\mathcal{B}^{\frac{3}{p}-\frac{3}{2}}_p)}.
\end{aligned} \tag{3.31}$$

As $m \in \mathbb{Z}^+$, by inserting (3.31) into (3.25), we arrive at

$$
\begin{aligned}
\|\nabla\pi\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}})} &\lesssim 2^{\frac{m}{2}} A_{ma}(t)\big(1+A_{ma}(t)\big)\|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}})} + A^3_{ma}(t)\|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+\frac{1}{2}})}\\
&+ \big(A_{ma}(t)+1\big)^2\Big(\|a-\dot{S}_m a\|_{L^\infty_t(\mathcal{B}_q^{\frac{3}{q}})}\int_0^t \big(\|u\|^{\frac14}_{\mathcal{B}_p^{\frac{3}{p}-1}}\|u\|^{\frac34}_{\mathcal{B}_p^{\frac{3}{p}+1}} + \|u\|^{\frac12}_{\mathcal{B}_p^{\frac{3}{p}-1}}\|u\|^{\frac12}_{\mathcal{B}_p^{\frac{3}{p}+1}}\big)\,\mathrm{d}\tau\\
&+ \int_0^t \Big(\|v\|^{\frac12}_{\mathcal{B}_p^{\frac{3}{p}-1}}\|v\|^{\frac12}_{\mathcal{B}_p^{\frac{3}{p}+1}}\|u\|^{\frac34}_{\mathcal{B}_p^{\frac{3}{p}-1}}\|u\|^{\frac14}_{\mathcal{B}_p^{\frac{3}{p}+1}} + \|u\|^{\frac12}_{\mathcal{B}_p^{\frac{3}{p}-1}}\|u\|^{\frac12}_{\mathcal{B}_p^{\frac{3}{p}+1}}\|v\|^{\frac34}_{\mathcal{B}_p^{\frac{3}{p}-1}}\|v\|^{\frac14}_{\mathcal{B}_p^{\frac{3}{p}+1}}\Big)\,\mathrm{d}\tau\Big)\\
&+ 2^{\frac{m}{2}} A_{ma}(t)\int_0^t \big(\|v\|_{\mathcal{B}_p^{\frac{3}{p}}}\|u\|_{\mathcal{B}_p^{\frac{3}{p}-1}} + \|v\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\|u\|_{\mathcal{B}_p^{\frac{3}{p}}}\big)\,\mathrm{d}\tau + 2^{\frac{m}{2}} A_{ma}(t)\|f\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-2})}\\
&+ \big(A_{ma}(t)+1\big)^2\|f\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}})}.
\end{aligned}
\tag{3.32}
$$

Finally, by summing up the estimates (3.19) and (3.32) and then applying Young's inequality to resulting one, we obtain (3.3). This finishes the proof of Proposition 3.1. □

REMARK 3.4. In the particular case when $v = u$ in Proposition 3.1, we have

$$
\begin{aligned}
&\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}_p^{\frac{3}{p}-1})} + \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+1})} + \|\nabla\pi\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-1})} \le C\exp\big(Ct2^{4m}(A_{ma}(t)+2)^{12}\big)\\
&\quad\times\Big(\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}} + 2^m(A_{ma}(t)+1)^3\|f\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})}\\
&\qquad + 2^{16m}(A_{ma}(t)+1)^{24}\int_0^t \big(\|u\|^3_{\mathcal{B}_p^{\frac{3}{p}-1}} + \|u\|^5_{\mathcal{B}_p^{\frac{3}{p}-1}} + \|u\|_{\mathcal{B}_p^{\frac{3}{p}+1}}\|u\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\big)\,\mathrm{d}\tau\Big).
\end{aligned}
\tag{3.33}
$$

In fact, we only need to replace the estimates of $\|\dot{\Delta}_j(v\cdot\nabla u)\|_{L^p}$ by the following ones when $v = u$:

$$
\begin{aligned}
&\|u\cdot\nabla u\|_{\mathcal{B}_p^{\frac{3}{p}-1}} \lesssim \|u\otimes u\|_{\mathcal{B}_p^{\frac{3}{p}}} \lesssim \|u\|^2_{\mathcal{B}_p^{\frac{3}{p}}} \lesssim \|u\|_{\mathcal{B}_p^{\frac{3}{p}+1}}\|u\|_{\mathcal{B}_p^{\frac{3}{p}-1}},\\
&\|u\cdot\nabla u\|_{\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}}} \lesssim \|u\otimes u\|_{\mathcal{B}_p^{\frac{3}{p}-\frac{1}{2}}} \lesssim \|u\|_{\mathcal{B}_p^{\frac{3}{p}}}\|u\|_{\mathcal{B}_p^{\frac{3}{p}-\frac{1}{2}}} \lesssim \|u\|^{\frac54}_{\mathcal{B}_p^{\frac{3}{p}-1}}\|u\|^{\frac34}_{\mathcal{B}_p^{\frac{3}{p}+1}},\\
&\|u\cdot\nabla u\|_{\mathcal{B}_p^{\frac{3}{p}-2}} \lesssim \|u\otimes u\|_{\mathcal{B}_p^{\frac{3}{p}-1}} \lesssim \|u\|_{\mathcal{B}_p^{\frac{3}{p}}}\|u\|_{\mathcal{B}_p^{\frac{3}{p}-1}} \lesssim \|u\|^{\frac32}_{\mathcal{B}_p^{\frac{3}{p}-1}}\|u\|^{\frac12}_{\mathcal{B}_p^{\frac{3}{p}+1}}.
\end{aligned}
$$

With some bookkeeping, we find

$$
\begin{aligned}
&\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}_p^{\frac{3}{p}-1})} + \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+1})} + \|\nabla\pi\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-1})}\\
&\quad\lesssim \|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}} + 2^{4m}(A_{ma}(t)+2)^{12}\int_0^t \|u\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\,\mathrm{d}\tau\\
&\qquad + 2^{16m}(A_{ma}(t)+1)^{24}\int_0^t \big(\|u\|^3_{\mathcal{B}_p^{\frac{3}{p}-1}} + \|u\|^5_{\mathcal{B}_p^{\frac{3}{p}-1}}\big)\,\mathrm{d}\tau\\
&\qquad + \int_0^t \|u\|_{\mathcal{B}_p^{\frac{3}{p}+1}}\|u\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\,\mathrm{d}\tau + 2^m(A_{ma}(t)+1)^3\|f\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})}.
\end{aligned}
$$

Applying the Gronwall's inequality leads to (3.33).

Let us now present the proof of Lemmas 3.2 and 3.3.

Proof of Lemma 3.2. Indeed due to $\operatorname{div} u = 0$, by applying Bony's decomposition, we write

$$\operatorname{div}(\dot S_m a \Delta u) = \dot T_{\nabla \dot S_m a}\Delta u + \operatorname{div}\dot T_{\Delta u}\dot S_m a + \operatorname{div}\dot R(\dot S_m a, \Delta u). \tag{3.34}$$

We first observe that due to $s \leq 1 + \frac{3}{q}$, there holds $\|\dot S_{k-1}\nabla \dot S_m a\|_{L^\infty} \lesssim 2^{\left(\frac{3}{q}+1-s\right)k}\|\dot S_m a\|_{\mathcal{B}^s_q}$ and thus

$$\begin{aligned}\|\dot\Delta_j(\dot T_{\nabla \dot S_m a}\Delta u)\|_{L^p} &\lesssim \sum_{|j-k|\leq 4}\|\dot S_{k-1}\nabla \dot S_m a\|_{L^\infty}\|\dot\Delta_k \Delta u\|_{L^p}\\ &\lesssim d_j(t)2^{-\left(s-3+\frac{3}{p}-\frac{3}{q}\right)j}\|\dot S_m a\|_{\mathcal{B}^s_q}\|u\|_{\mathcal{B}^{\frac{3}{p}}_p}.\end{aligned}$$

For the second term in (3.34), we first handle the case when $q \leq p$

$$\begin{aligned}\|\operatorname{div}\dot\Delta_j(\dot T_{\Delta u}\dot S_m a)\|_{L^p} &\lesssim 2^j\sum_{|j-k|\leq 4}\|\dot S_{k-1}\Delta u\|_{L^\infty}\|\dot\Delta_k \dot S_m a\|_{L^p}\\ &\lesssim 2^j\sum_{|j-k|\leq 4}2^{\left(\frac{3}{q}-\frac{3}{p}+2\right)k}\|u\|_{L^\infty}\|\dot\Delta_k \dot S_m a\|_{L^q}\\ &\lesssim d_j(t)2^{-\left(s-3+\frac{3}{p}-\frac{3}{q}\right)j}\|\dot S_m a\|_{\mathcal{B}^s_q}\|u\|_{\mathcal{B}^{\frac{3}{p}}_p},\end{aligned}$$

while for case of $p < q$, we have $\|\dot S_{k-1}\Delta u\|_{L^{\frac{pq}{q-p}}} \lesssim d_k(t)2^{\left(\frac{3}{q}-\frac{3}{p}+2\right)k}\|u\|_{\mathcal{B}^{\frac{3}{p}}_p}$ and

$$\begin{aligned}\|\operatorname{div}\dot\Delta_j(\dot T_{\Delta u}\dot S_m a)\|_{L^p} &\lesssim 2^j\sum_{|j-k|\leq 4}\|\dot S_{k-1}\Delta u\|_{L^{\frac{pq}{q-p}}}\|\dot\Delta_k \dot S_m a\|_{L^q}\\ &\lesssim d_j 2^{-\left(s-3+\frac{3}{p}-\frac{3}{q}\right)j}\|\dot S_m a\|_{\mathcal{B}^s_q}\|u\|_{\mathcal{B}^{\frac{3}{p}}_p}.\end{aligned}$$

For the last term of (3.34), we shall distinguish the case when $\frac{1}{p}+\frac{1}{q} \leq 1$ and $\frac{1}{p}+\frac{1}{q} > 1$. In case $\frac{1}{p}+\frac{1}{q} \leq 1$, there is $r \geq 1$ such that $\frac{1}{r} = \frac{1}{p}+\frac{1}{q}$ and

$$\begin{aligned}\|\operatorname{div}\dot\Delta_j(\dot R(\dot S_m a, \Delta u))\|_{L^p} &\lesssim 2^j 2^{3\left(\frac{1}{r}-\frac{1}{p}\right)j}\sum_{k\geq j-3}\|\dot\Delta_k \dot S_m a\|_{L^q}\|\widetilde{\dot\Delta}_k\Delta u\|_{L^p}\\ &\lesssim 2^{\left(\frac{3}{q}+1\right)j}\sum_{k\geq j-3}2^{-k\left(s-2+\frac{3}{p}\right)}d_k(t)\|\dot S_m a\|_{\mathcal{B}^s_q}\|u\|_{\mathcal{B}^{\frac{3}{p}}_p}\\ &\lesssim d_j(t)2^{-\left(s-3+\frac{3}{p}-\frac{3}{q}\right)j}\|\dot S_m a\|_{\mathcal{B}^s_q}\|u\|_{\mathcal{B}^{\frac{3}{p}}_p},\end{aligned}$$

where in the last step, we used the fact that $s-2+\frac{3}{p} > 0$.

Whereas for the case when $\frac{1}{p}+\frac{1}{q} > 1$, we have $p < q' = \frac{q}{q-1}$ and then

$$\begin{aligned}\|\operatorname{div}\dot\Delta_j(\dot R(\dot S_m a, \Delta u))\|_{L^p} &\lesssim 2^{\left(4-\frac{3}{p}\right)j}\sum_{k\geq j-3}\|\dot\Delta_k \dot S_m a\|_{L^q}\|\widetilde{\dot\Delta}_k\Delta u\|_{L^{q'}}\\ &\lesssim 2^{(4-\frac{3}{p})j}\sum_{k\geq j-3}2^{-k\left(1+s-\frac{3}{q}\right)k}d_k(t)\|\dot S_m a\|_{\mathcal{B}^s_q}\|u\|_{\mathcal{B}^{\frac{3}{p}}_p}\\ &\lesssim d_j(t)2^{-\left(s-3+\frac{3}{p}-\frac{3}{q}\right)j}\|\dot S_m a\|_{\mathcal{B}^s_q}\|u\|_{\mathcal{B}^{\frac{3}{p}}_p} \quad \text{if} \quad 1+s-\frac{3}{q} > 0.\end{aligned}$$

As a result, it comes out (3.15). □

Proof of Lemma 3.3. By applying Bony's decomposition and a standard commutator's process, we write

$$[\dot{\Delta}_j, \dot{S}_m a]\nabla\pi = [\dot{\Delta}_j, \dot{T}_{\dot{S}_m a}]\nabla\pi - \dot{T}'_{\dot{\Delta}_j \nabla\pi}\dot{S}_m a + \dot{\Delta}_j \dot{T}_{\nabla\pi}\dot{S}_m a + \dot{\Delta}_j \dot{R}(\dot{S}_m a, \nabla\pi). \quad (3.35)$$

We henceforth handle term by term in (3.35).

For the first term on the right-hand side of (3.35), we first deal with the case when $p < 2$ so that there is r satisfying $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$, which together with the fact: $\frac{1}{p} < \frac{1}{2} + \frac{1}{q}$, implies that $r > q$, from which, we infer $\|\dot{S}_{k-1}\nabla\dot{S}_m a\|_{L^r} \lesssim d_k(t)2^{3k\left(1-\frac{1}{p}\right)}\|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}-\frac{1}{2}}}$, and

$$\begin{aligned}\|[\dot{\Delta}_j, \dot{T}_{\dot{S}_m a}]\nabla\pi\|_{L^p} &\lesssim 2^{-j}\sum_{|j-k|\le 4}\|\dot{S}_{k-1}\nabla\dot{S}_m a\|_{L^r}\|\dot{\Delta}_k\nabla\pi\|_{L^2}\\ &\lesssim d_j(t)2^{-(\frac{3}{p}-2)j}\|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}-\frac{1}{2}}}\|\nabla\pi\|_{L^2},\end{aligned}$$

while for the case when $p \ge 2$, we have $\|\dot{S}_{k-1}\nabla\dot{S}_m a\|_{L^\infty} \lesssim d_k(t)2^{\frac{3k}{2}}\|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}-\frac{1}{2}}}$ and thus

$$\begin{aligned}\|[\dot{\Delta}_j, \dot{T}_{\dot{S}_m a}]\nabla\pi\|_{L^p} &\lesssim 2^{-j}\sum_{|j-k|\le 4}2^{\left(\frac{3}{2}-\frac{3}{p}\right)k}\|\dot{S}_{k-1}\nabla\dot{S}_m a\|_{L^\infty}\|\dot{\Delta}_k\nabla\pi\|_{L^2}\\ &\lesssim d_j 2^{-(\frac{3}{p}-2)j}\|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}-\frac{1}{2}}}\|\nabla\pi\|_{L^2}.\end{aligned}$$

For the term $\dot{T}'_{\dot{\Delta}_j\nabla\pi}\dot{S}_m a$, we decompose the estimate into two cases. When $1 < q \le p$, we deduce from Lemma 2.2 that

$$\begin{aligned}\|\dot{T}'_{\dot{\Delta}_j\nabla\pi}\dot{S}_m a\|_{L^p} &\lesssim \sum_{k\ge j-N_0}\|\dot{\Delta}_k\dot{S}_m a\|_{L^p}\|\dot{S}_{k+2}\dot{\Delta}_j\nabla\pi\|_{L^\infty}\\ &\lesssim 2^{\frac{3j}{2}}\|\dot{\Delta}_j\nabla\pi\|_{L^2}\sum_{k\ge j-N_0}d_k(t)2^{-k\left(\frac{3}{p}-\frac{1}{2}\right)}\|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}-\frac{1}{2}}} \qquad (3.36)\\ &\lesssim d_j(t)2^{-\left(\frac{3}{p}-2\right)j}\|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}-\frac{1}{2}}}\|\nabla\pi\|_{L^2}.\end{aligned}$$

For second case when $q > p$, due to $\frac{1}{p} < \frac{1}{q} + \frac{1}{3}$, $\frac{pq}{q-p} > 2$, then we deduce from Lemma 2.2 that

$$\begin{aligned}\|\dot{T}'_{\dot{\Delta}_j\nabla\pi}\dot{S}_m a\|_{L^p} &\lesssim \sum_{k\ge j-N_0}\|\dot{\Delta}_k\dot{S}_m a\|_{L^q}\|\dot{S}_{k+2}\dot{\Delta}_j\nabla\pi\|_{L^{\frac{pq}{q-p}}}\\ &\lesssim 2^{3j\left(\frac{1}{2}+\frac{1}{q}-\frac{1}{p}\right)}\|\nabla\pi\|_{L^2}\sum_{k\ge j-N_0}d_k(t)2^{-\left(\frac{3}{q}-\frac{1}{2}\right)k}\|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}-\frac{1}{2}}} \qquad (3.37)\\ &\lesssim d_j 2^{-(\frac{3}{p}-2)j}\|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}-\frac{1}{2}}}\|\nabla\pi\|_{L^2}.\end{aligned}$$

For the term $\dot{\Delta}_j\dot{T}_{\nabla\pi}\dot{S}_m a$, when $q \le p$, we have $\|\dot{S}_{k-1}\nabla\pi\|_{L^\infty} \le 2^{\frac{3k}{2}}\|\nabla\pi\|_{L^2}$, and thus

$$\begin{aligned}\|\dot{\Delta}_j\dot{T}_{\nabla\pi}\dot{S}_m a\|_{L^p} &\lesssim \sum_{|j-k|\le 4}\|\dot{\Delta}_k\dot{S}_m a\|_{L^p}\|\dot{S}_{k-1}\nabla\pi\|_{L^\infty}\\ &\lesssim d_j 2^{-\left(\frac{3}{p}-2\right)j}\|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}-\frac{1}{2}}}\|\nabla\pi\|_{L^2},\end{aligned}$$

and when $q > p$, it follows from Lemma 2.2 that $\|\dot{S}_{k-1}\nabla\pi\|_{L^{\frac{pq}{q-p}}} \lesssim 2^{3\left(\frac{1}{2}+\frac{1}{q}-\frac{1}{p}\right)k}\|\nabla\pi\|_{L^2}$, and thus

$$\begin{aligned}\|\dot{\Delta}_j\dot{T}_{\nabla\pi}\dot{S}_m a\|_{L^p} &\lesssim \sum_{|j-k|\leq 4}\|\dot{\Delta}_k\dot{S}_m a\|_{L^q}\|\dot{S}_{k-1}\nabla\pi\|_{L^{\frac{pq}{q-p}}}\\ &\lesssim d_j(t)2^{-\left(\frac{3}{p}-2\right)j}\|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}-\frac{1}{2}}}\|\nabla\pi\|_{L^2}.\end{aligned}$$

Finally let us turn to the last term, $\dot{\Delta}_j\dot{R}(\dot{S}_m a,\nabla\pi)$, in (3.35). When $1<q<2$, we have $q'>2$, we thus deduce from Lemma 2.2 that

$$\begin{aligned}\|\dot{\Delta}_j\dot{R}(\dot{S}_m a,\nabla\pi)\|_{L^p} &\lesssim 2^{3\left(1-\frac{1}{p}\right)j}\sum_{k\geq j-3}\|\dot{\Delta}_k\dot{S}_m a\|_{L^q}\|\widetilde{\dot{\Delta}}_k\nabla\pi\|_{L^{q'}}\\ &\lesssim 2^{3\left(1-\frac{1}{p}\right)j}\sum_{k\geq j-3}d_k(t)2^{-k}\|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}-\frac{1}{2}}}\|\nabla\pi\|_{L^2}\\ &\lesssim d_j 2^{-\left(\frac{3}{p}-2\right)j}\|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}-\frac{1}{2}}}\|\nabla\pi\|_{L^2}.\end{aligned}$$

Similarly when $q\geq 2$, we get

$$\begin{aligned}\|\dot{\Delta}_j\dot{R}(\dot{S}_m a,\nabla\pi)\|_{L^p} &\lesssim 2^{3j\left(\frac{1}{2}+\frac{1}{q}-\frac{1}{p}\right)}\sum_{k\geq j-3}\|\dot{\Delta}_k\dot{S}_m a\|_{L^q}\|\widetilde{\dot{\Delta}}_k\nabla\pi\|_{L^2}\\ &\lesssim 2^{3j\left(\frac{1}{2}+\frac{1}{q}-\frac{1}{p}\right)}\sum_{k\geq j-3}d_k(t)2^{-\left(\frac{3}{q}-\frac{1}{2}\right)}\|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}-\frac{1}{2}}}\|\nabla\pi\|_{L^2}\\ &\lesssim d_j 2^{-\left(\frac{3}{p}-2\right)j}\|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}-\frac{1}{2}}}\|\nabla\pi\|_{L^2}.\end{aligned}$$

By summing up the above estimate, we conclude the proof of (3.26).

On the other hand, due to the support properties to the Fourier transform of the terms in (3.35), in order to prove (3.27), it remains to check the estimate of $\operatorname{div}\big(\dot{T}'_{\dot{\Delta}_j\nabla\pi}\dot{S}_m a\big)$. Indeed applying Lebnitz formula, we get

$$\operatorname{div}\big(\dot{T}'_{\dot{\Delta}_j\nabla\pi}\dot{S}_m a\big)=\dot{T}'_{\dot{\Delta}_j\Delta\pi}\dot{S}_m a+\dot{T}'_{\dot{\Delta}_j\nabla\pi}\nabla\dot{S}_m a. \tag{3.38}$$

It follows from the derivation of (3.36) and (3.37) that

$$\|\dot{T}'_{\dot{\Delta}_j\Delta\pi}\dot{S}_m a\|_{L^p}\lesssim d_j(t)2^{-\left(\frac{3}{p}-3\right)j}\|\dot{S}_m a\|_{\mathcal{B}_q^{\frac{3}{q}-\frac{1}{2}}}\|\nabla\pi\|_{L^2}.$$

We now turn to the estimate of the second term in (3.38). Corresponding to (3.36), when $1<p<\frac{3}{2}$ and $\frac{1}{p}<\frac{1}{6}+\frac{1}{q}$, we deduce from Lemma 2.2 that

$$\begin{aligned}\|\dot{T}'_{\dot{\Delta}_j\nabla\pi}\nabla\dot{S}_m a\|_{L^p} &\lesssim \sum_{k\geq j-N_0}\|\dot{\Delta}_k\nabla\dot{S}_m a\|_{L^{\frac{6p}{6-p}}}\|\dot{S}_{k+2}\dot{\Delta}_j\nabla\pi\|_{L^6}\\ &\lesssim 2^j\|\nabla\pi\|_{L^2}\sum_{k\geq j-N_0}d_k(t)2^{-3k\left(\frac{1}{p}-\frac{2}{3}\right)}\left\|\dot{S}_m a\right\|_{\mathcal{B}_q^{\frac{3}{q}-\frac{1}{2}}}\\ &\lesssim d_j 2^{-\left(\frac{3}{p}-3\right)j}\|\dot{S}_m a\|_{\dot{B}_{q,1}^{\frac{3}{q}-\frac{1}{2}}}\|\nabla\pi\|_{L^2},\end{aligned}$$

where we use the fact that $1<p<\frac{3}{2}$ and $\frac{1}{p}<\frac{1}{6}+\frac{1}{q}\Rightarrow\frac{6p}{6-p}>q$. This completes the proof of (3.27). □

4. Proof of Theorem 1.2: Existence. In this section, we shall apply the *a priori* estimate obtained in the previous section to prove the local well-posedness part of Theorem 1.2.

THEOREM 4.1. *Under the assumption of Theorem 1.2, there is a positive time* T *such that* (1.4) *has a unique local solution* $(a,u)\in\mathcal{C}_b([0,T];B_q^{\frac{3}{q}})\times\mathcal{C}_b([0,T];\mathcal{B}_p^{\frac{3}{p}-1}\cap L^1([0,T];\mathcal{B}_p^{\frac{3}{p}+1}))$. *Furthermore, if* c*, which depends on* $\|a_0\|_{B_q^{\frac{3}{q}}}$*, is small enough in* (1.7)*, then* $T\geq 1$*, and there holds*

$$\|u\|_{\widetilde{L}^\infty([0,T];\mathcal{B}_p^{\frac{3}{p}-1})}+\|u\|_{L^1([0,T];\mathcal{B}_p^{\frac{3}{p}+1})}+\|(\partial_t u,\nabla\pi)\|_{L^1([0,T];\tilde{\mathcal{B}}_p^{\frac{3}{p}})}\lesssim\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}. \tag{4.1}$$

Moreover, for any $t_0>0$*, there holds*

$$\begin{aligned}&\|u\|_{\widetilde{L}^\infty([t_0,T];\mathcal{B}_p^{\frac{3}{p}})}+\|u\|_{L^1([t_0,T];\mathcal{B}_p^{\frac{3}{p}+2})}+\|\nabla\pi\|_{L^1([t_0,T];\mathcal{B}_p^{\frac{3}{p}})}\\&\qquad\lesssim\|a_0\|_{B_q^{\frac{3}{q}}}\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\big(1+{t_0}^{-\frac{1}{2}}\big)\exp\left(C\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\right).\end{aligned}\tag{4.2}$$

4.1. Existence part of Theorem 4.1. As usual, we shall first construct approximate solutions of (1.4), and then perform the uniform estimates for such approximate solutions, finally the local existence of solutions to (1.4) follows by a compactness argument.

Step 1. Construction of approximate solutions.

For $n\in\mathbb{N}$, let

$$a_0^n\stackrel{\text{def}}{=}\dot{S}_n a_0-\dot{S}_{-n}a_0\quad\text{and}\quad u_0^n\stackrel{\text{def}}{=}\dot{S}_n u_0-\dot{S}_{-n}u_0.$$

Then it follows from Theorem 0.2 of [10] that the system (1.4) has a unique local-in-time smooth solution $(a^n,u^n,\nabla\pi^n)$ on $[0,T^n]$ with the initial data (a_0^n,u_0^n).

Step 2. Uniform estimates of the approximate solutions.

We shall search a suitable small $T<\inf_{n\in\mathbb{N}}T^n$ such that $(a^n,u^n,\nabla\pi^n)$ is uniformly bounded in the space

$$E_T\stackrel{\text{def}}{=}\widetilde{L}_T^\infty(B_q^{\frac{3}{q}})\times\big((\widetilde{L}_T^\infty(\mathcal{B}_p^{\frac{3}{p}-1})\cap L_T^1(\mathcal{B}_p^{\frac{3}{p}+1})\big)\times L_T^1(\mathcal{B}_p^{\frac{3}{p}-1}),$$

where p,q are given in Theorem 1.2.

In order to do so, we denote $u_L^n(t)\stackrel{\text{def}}{=}\mathrm{e}^{t\Delta}u_0^n$ and $\bar{u}^n\stackrel{\text{def}}{=}u^n-u_L^n$. Then it is easy to observe that

$$\begin{aligned}&\|u_L^n\|_{\widetilde{L}_T^\infty(\mathcal{B}_p^{\frac{3}{p}-1})}\lesssim\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\quad\text{and}\\&\|u_L^n\|_{\widetilde{L}_T^1(\mathcal{B}_p^{\frac{3}{p}+1})}\lesssim\sum_{j\in\mathbb{Z}}2^{j\left(\frac{3}{p}-1\right)}\big(1-\mathrm{e}^{-ct2^{2j}}\big)\|\dot{\Delta}_j u_0\|_{L^p},\end{aligned}\tag{4.3}$$

and $(a^n,\bar{u}^n,\nabla\pi^n)$ solves

$$\begin{cases}\partial_t a^n+(u_L^n+\bar{u}^n)\cdot\nabla a^n=0,\\\partial_t\bar{u}^n+u_L^n\cdot\nabla\bar{u}^n-(1+a^n)(\Delta\bar{u}^n-\nabla\pi^n)=H_n,\\\operatorname{div}\bar{u}^n=0,\\(a^n,\bar{u}^n)\mid_{t=0}=(a_0^n,0),\end{cases}\tag{4.4}$$

where

$$H_n \overset{\text{def}}{=} -u_L^n\cdot\nabla u_L^n - \bar u^n\cdot\nabla u_L^n - \bar u^n\cdot\nabla\bar u^n + a^n\Delta u_L^n.$$

For $a_0\in\mathcal{B}_q^{\frac{3}{q}}$, we fix $m\in\mathbb{N}$ by

$$m \overset{\text{def}}{=} \inf\Big\{k\in\mathbb{N}\ \mid \sum_{\ell\geq k} 2^{\frac{3\ell}{q}}\|\dot\Delta_\ell a_0\|_{L^q}\leq c_0\underline{b}\big(1+\|a_0\|_{\mathcal{B}_q^{\frac{3}{q}}}\big)^{-12}\Big\} \tag{4.5}$$

for some sufficiently small positive constant c_0, which will be chosen later on.

Notice that $\operatorname{div}(u_L^n+\bar u^n)=0$, we get, by applying Proposition 2.12 to a^n that

$$\begin{aligned}
&\|a^n\|_{L_t^\infty(\mathcal{B}_q^{\frac{3}{q}})}\lesssim \|a_0\|_{\mathcal{B}_q^{\frac{3}{q}}}\exp\Big(C\big(\|\nabla u_L^n\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}})}+\|\nabla\bar u^n\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}})}\big)\Big),\\
&\|a^n\|_{L_t^\infty(L^q)}\leq\|a_0^n\|_{L^q}\leq\|a_0\|_{L^q}\quad\text{and}\quad \|a^n\|_{L_t^\infty(L^\infty)}\leq\|a_0^n\|_{L^\infty}\leq\|a_0\|_{\mathcal{B}_q^{\frac{3}{q}}}.
\end{aligned}\tag{4.6}$$

On the other hand, under assumption that

$$\big(1+\|a^n\|_{\mathcal{B}_q^{\frac{3}{q}}}\big)^{12}\|a^n-\dot S_m a^n\|_{\widetilde L_T^\infty(\mathcal{B}_q^{\frac{3}{q}})}\leq 2c_0\underline{b}, \tag{4.7}$$

we deduce from Proposition 3.1 and (4.4) that

$$\begin{aligned}
Z_n(t)\overset{\text{def}}{=}&\|\bar u^n\|_{\widetilde L_t^\infty(\mathcal{B}_p^{\frac{3}{p}-1})}+\|\bar u^n\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}+1})}+\|\nabla\pi^n\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}-1})}\\
\lesssim& B_{ma^n}(t)\Big\{\int_0^t\|\bar u^n\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\,\mathrm{d}\tau+\|H_n\|_{L_t^1(\tilde{\mathcal{B}}_p^{\frac{3}{p}})}\\
&+\int_0^t\big(\|u_L^n\|^{\frac{2}{3}}_{\mathcal{B}_p^{\frac{3}{p}-1}}\|u_L^n\|^{\frac{2}{3}}_{\mathcal{B}_p^{\frac{3}{p}+1}}+\|u_L^n\|^{\frac{3}{2}}_{\mathcal{B}_p^{\frac{3}{p}-1}}\|u_L^n\|^{\frac{1}{2}}_{\mathcal{B}_p^{\frac{3}{p}+1}}\big)\|\bar u^n\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\,\mathrm{d}\tau\\
&+\int_0^t\big(\|u_L^n\|_{\mathcal{B}_p^{\frac{3}{p}+1}}+\|u_L^n\|_{\mathcal{B}_p^{\frac{3}{p}}}+\|u_L^n\|^2_{\mathcal{B}_p^{\frac{3}{p}-1}}\big)\|\bar u^n\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\,\mathrm{d}\tau\Big\},
\end{aligned}\tag{4.8}$$

where $B_{ma^n}(t)\overset{\text{def}}{=}2^{4m}(A_{ma^n}(t)+2)^{12}$.

While due to $\operatorname{div}u_L^n=\operatorname{div}\bar u^n=0$, we get, by applying the law of product, (2.5) and (2.7), that

$$\begin{aligned}
&\|H_n\|_{L_t^1(\tilde{\mathcal{B}}_p^{\frac{3}{p}})}\\
&\lesssim\int_0^t\Big(\|u_L^n\|_{\mathcal{B}_p^{\frac{3}{p}+1}}\big(\|u_L^n\|_{\mathcal{B}_p^{\frac{3}{p}-1}}+\|\bar u^n\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\big)\mathrm{d}\tau+\|u_L^n\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\big(\|u_L^n\|_{\mathcal{B}_p^{\frac{3}{p}}}+\|\bar u^n\|_{\mathcal{B}_p^{\frac{3}{p}}}\big)\\
&\quad+\|\bar u^n\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\big(\|\bar u^n\|_{\mathcal{B}_p^{\frac{3}{p}}}+\|\bar u^n\|_{\mathcal{B}_p^{\frac{3}{p}+1}}\big)+\|a^n\|_{\mathcal{B}_q^{\frac{3}{q}}}\big(\|u_L^n\|_{\mathcal{B}_p^{\frac{3}{p}+1}}+\|u_L^n\|_{\mathcal{B}_p^{\frac{3}{p}}}\big)\Big)\mathrm{d}\tau.
\end{aligned}$$

By applying Young's inequality, we get

$$\begin{aligned}
B_{ma^n}(t)\|H_n\|_{L_t^1(\tilde{\mathcal{B}}_p^{\frac{3}{p}})}\leq&\|u_L^n\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}+1})}+\eta\|\bar u^n\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}+1})}\\
&+CB^2_{ma^n}(t)\int_0^t\Big(\|\bar u^n\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\|\bar u^n\|_{\mathcal{B}_p^{\frac{3}{p}+1}}+\|a^n\|_{\mathcal{B}_q^{\frac{3}{q}}}\|u_L^n\|_{\mathcal{B}_p^{\frac{3}{p}+1}}\\
&+\|u_L^n\|_{\mathcal{B}_p^{\frac{3}{p}+1}}\big(\|u_L^n\|_{\mathcal{B}_p^{\frac{3}{p}-1}}+\|\bar u^n\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\big)+\|u_L^n\|^3_{\mathcal{B}_p^{\frac{3}{p}-1}}\\
&+\|\bar u^n\|^3_{\mathcal{B}_p^{\frac{3}{p}-1}}+\|\bar u^n\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\|u_L^n\|^2_{\mathcal{B}_p^{\frac{3}{p}-1}}+\|a^n\|^2_{\mathcal{B}_q^{\frac{3}{q}}}\|u_L^n\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\Big)\mathrm{d}\tau,
\end{aligned}\tag{4.9}$$

Yet thanks to (4.6) and the definitions of $A_{ma^n}(t), B_{ma^n}(t)$, we have

$$\begin{aligned}
A_{ma^n}(t) &\stackrel{\text{def}}{=} 2^{\frac{3}{q}m}\|\dot{S}_m a^n\|_{L^\infty_t(L^q)} \lesssim 2^{\frac{3}{q}m}\|a_0\|_{L^q}\\
B_{ma^n}(t) &\stackrel{\text{def}}{=} 2^{4m}(A_{ma^n}(t)+2)^{12} \lesssim 2^{4m}\big(2^{\frac{3}{q}m}\|a_0\|_{L^q}+2\big)^{12}.
\end{aligned}$$

For simplicity, we henceforth denote C_m to be a constant, which depends only on m and $\|a_0\|_{L^q\cap\mathcal{B}^{\frac{3}{q}}_q}$. Then by inserting (4.9) into (4.8) and then choosing η sufficiently small, we arrive at

$$\begin{aligned}
Z_n(t) \le& C_m\Big\{t\big(\|u^n_L\|^3_{L^\infty_t(\mathcal{B}^{\frac{3}{p}-1}_p)} + \|a_0\|^2_{B^{\frac{3}{q}}_{q,1}}\|u^n_L\|_{L^\infty_t(\mathcal{B}^{\frac{3}{p}-1}_p)}\big)\\
&+\big(\|u^n_L\|_{L^\infty_t(\mathcal{B}^{\frac{3}{p}-1}_p)} + \|a_0\|_{\mathcal{B}^{\frac{3}{q}}_q}\big)\|u^n_L\|_{L^1_t(\mathcal{B}^{\frac{3}{p}+1}_p)}\\
&+\int_0^t\Big(\|\bar{u}^n\|^2_{\mathcal{B}^{\frac{3}{p}-1}_p} + \|\bar{u}^n\|_{\mathcal{B}^{\frac{3}{p}+1}_p} + D(\tau)\Big)\|\bar{u}^n\|_{\mathcal{B}^{\frac{3}{p}-1}_p}\,\mathrm{d}\tau\Big\} \qquad (4.10)\\
\text{with}\quad D(t) \stackrel{\text{def}}{=}& 1 + \|u^n_L\|_{\mathcal{B}^{\frac{3}{p}+1}_p} + \|u^n_L\|^2_{\mathcal{B}^{\frac{3}{p}-1}_p} + \|u^n_L\|^{\frac{2}{3}}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|u^n_L\|^{\frac{2}{3}}_{\mathcal{B}^{\frac{3}{p}+1}_p}\\
&+\|u^n_L\|^{\frac{3}{2}}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|u^n_L\|^{\frac{1}{2}}_{\mathcal{B}^{\frac{3}{p}+1}_p} + \|u^n_L\|^{\frac{1}{2}}_{\mathcal{B}^{\frac{3}{p}-1}_p}\|u^n_L\|^{\frac{1}{2}}_{\mathcal{B}^{\frac{3}{p}+1}_p}.
\end{aligned}$$

Applying Gronwall's inequality to (4.10) leads to

$$\begin{aligned}
Z_n(t) \le C_m\exp\Big(C\int_0^t D(\tau)\mathrm{d}\tau\Big)\Big\{&t\big(\|u^n_L\|^3_{L^\infty_t(\mathcal{B}^{\frac{3}{p}-1}_p)} + \|a_0\|^2_{\mathcal{B}^{\frac{3}{q}}_q}\|u^n_L\|_{L^\infty_t(\mathcal{B}^{\frac{3}{p}-1}_p)}\big)\\
&+Z^2_n(t) + tZ^3_n(t) + \big(\|u^n_L\|_{L^\infty_t(\mathcal{B}^{\frac{3}{p}-1}_p)} + \|a_0\|_{\mathcal{B}^{\frac{3}{q}}_q}\big)\|u^n_L\|_{L^1_t(\mathcal{B}^{\frac{3}{p}+1}_p)}\Big\}.
\end{aligned}$$

Yet it follows from (4.3) that

$$\begin{aligned}
\int_0^t D(\tau)\mathrm{d}\tau \lesssim& \|u_0\|_{\mathcal{B}^{\frac{3}{p}-1}_p} + \Big(t + t\|u_0\|^2_{\mathcal{B}^{\frac{3}{p}-1}_p} + t^{\frac{1}{3}}\|u_0\|^{\frac{4}{3}}_{\mathcal{B}^{\frac{3}{p}-1}_p} + t^{\frac{1}{2}}\|u_0\|^2_{\mathcal{B}^{\frac{3}{p}-1}_p} + t^{\frac{1}{2}}\|u_0\|_{\mathcal{B}^{\frac{3}{p}-1}_p}\Big)\\
\stackrel{\text{def}}{=}& W(t).
\end{aligned}$$

Thus, under the assumptions of (1.7) and (4.7), we deduce from (4.3) for $t \le 1$,

$$Z_n(t) \le C_m\Big(Z^2_n(t) + Z^3_n(t) + ct + \sum_{j\in\mathbb{Z}} 2^{j(\frac{3}{p}-1)}\Big(1-\mathrm{e}^{-ct2^{2j}}\Big)\|\dot{\Delta}_j u_0\|_{L^p}\Big). \qquad (4.11)$$

For any sufficiently small $\varepsilon > 0$, which will be chosen later on, there exists $N_0 \in \mathbb{N}$ so that

$$C_m\sum_{j\ge N_0} 2^{j(\frac{3}{p}-1)}\Big(1-\mathrm{e}^{-ct2^{2j}}\Big)\|\dot{\Delta}_j u_0\|_{L^p} \le \frac{\varepsilon}{2}.$$

While we observe that

$$\sum_{j\le N_0} 2^{j(\frac{3}{p}-1)}\Big(1-\mathrm{e}^{-ct2^{2j}}\Big)\|\dot{\Delta}_j u_0\|_{L^p} \le Ct\sum_{j\le N_0} 2^{j(\frac{3}{p}+1)}\|\dot{\Delta}_j u_0\|_{L^p} \le Ct2^{2N_0}\|u_0\|_{\mathcal{B}^{\frac{3}{p}-1}_p}.$$

Let us take

$$T^n_1 \stackrel{\text{def}}{=} \min(T^n, T_1) \quad\text{and}\quad T_1 \stackrel{\text{def}}{=} \min\Big(1, \big(2C_m c(1+2^{2N_0})\big)^{-1}\varepsilon\Big) \qquad (4.12)$$

Then we deduce from (4.11) that

$$Z_n(t) \leq C_m\big(Z_n^2(t) + Z_n^3(t)\big) + \varepsilon \quad \text{for} \quad t < T_1^n. \tag{4.13}$$

We define

$$T_1^\star \stackrel{\text{def}}{=} \sup\{\, t \leq T_1,\ Z_n(t) \leq 2\varepsilon \,\}. \tag{4.14}$$

We claim that $T_1^\star = T_1$ as long as ε is sufficiently small. Otherwise if $T_1^\star < T_1$, we take $\varepsilon \leq \varepsilon_1$ with ε_1 being determined by $C_m(\varepsilon_1 + \varepsilon_1^2) \leq \frac{1}{4}$. Then we deduce from (4.13) that

$$Z_n(t) \leq \frac{4}{3}\varepsilon \quad \text{for } t \leq T_1^\star,$$

which contradict with the definition of $T_1^\star$ defined by (4.14). This in turn shows that $T_1^\star = T_1$.

By virtue of (4.14), we get, by applying (4.3) and (4.6) that for $t \leq T_1$,

$$\|a^n\|_{\widetilde{L}^\infty_{T_1}(\mathcal{B}^{\frac{3}{q}}_q)} \leq C\|a_0\|_{\mathcal{B}^{\frac{3}{q}}_q}.$$

So that $t \leq T_1$, we get, by applying (2.24) (see also Proposition 3.2 of [3]) that

$$\begin{aligned}\big(1 + \|a^n\|_{\widetilde{L}^\infty_{T_1}(\mathcal{B}^{\frac{3}{q}}_q)}\big)^{12}\|a^n - \dot{S}_m a^n\|_{\widetilde{L}^\infty_{T_1}(\mathcal{B}^{\frac{3}{q}}_q)} &\leq C\big(1 + \|a_0\|_{\mathcal{B}^{\frac{3}{q}}_q}\big)^{12}\\ \times \Big\{\sum_{j\geq m} 2^{\frac{3j}{q}}\|\dot{\Delta}_j a_0\|_{L^q} + \|a_0\|_{\mathcal{B}^{\frac{3}{q}}_q}\Big(\exp\big[C\big(\|\bar{u}^n\|_{L^1_{T_1}(\mathcal{B}^{\frac{3}{p}+1}_p)} + \|u^n_L\|_{L^1_{T_1}(\mathcal{B}^{\frac{3}{p}+1}_p)}\big)\big] - 1\Big)\Big\},\end{aligned}$$

which together with $e^x - 1 \leq xe^x$ for $x \geq 0$, (1.7), (4.3), (4.5) and (4.14) ensures that

$$\big(1 + \|a^n\|_{\widetilde{L}^\infty_{T_1}(\mathcal{B}^{\frac{3}{q}}_q)}\big)^{12}\|a^n - \dot{S}_m a^n\|_{\widetilde{L}^\infty_{T_1}(\mathcal{B}^{\frac{3}{q}}_q)} \leq c_0\underline{b} + C\big(1 + \|a_0\|_{\mathcal{B}^{\frac{3}{q}}_q}\big)^{13}(c + 2\varepsilon). \tag{4.15}$$

We take c in (1.7) and ε to be so small that

$$C\big(1 + \|a_0\|_{\mathcal{B}^{\frac{3}{q}}_q}\big)^{13}(c + 2\varepsilon) \leq \frac{1}{2}c_0\underline{b},$$

then (4.15) ensures that

$$\big(1 + \|a^n\|_{\widetilde{L}^\infty_{T_1}(\mathcal{B}^{\frac{3}{q}}_q)}\big)^{12}\|a^n - \dot{S}_m a^n\|_{\widetilde{L}^\infty_{T_1}(\mathcal{B}^{\frac{3}{q}}_q)} \leq \frac{3}{2}c_0\underline{b}.$$

Then a continuous argument shows that $T_1^n < T^n$ and thus $T_1^n = T_1$ defined in (4.14).

This together with (4.3) and (4.6) ensures

$$\{a^n, u^n, \nabla\pi^n\}_{n\in\mathbb{N}} \quad \text{is uniformly bounded in } E_{T_1}. \tag{4.16}$$

Step 3. Convergence of the approximation solutions

Thanks to(4.16), we can repeat the same argument as that in Step 3 to the proof of Theorem 5.1 in [9] (see also [3]) to show that there exists a subsequence of $\big\{a^n, u^n, \nabla\pi^n\big\}_{n\in\mathbb{N}}$ which converges to a solution $\{a, u, \nabla\pi\}$, which belongs to $\mathcal{C}_b([0,T_2];B_q^{\frac{3}{q}}(\mathbb{R}^3))\times\mathcal{C}_b([0,T_2];\mathcal{B}_p^{\frac{3}{p}-1}(\mathbb{R}^3))\cap L^1([0,T_2];\mathcal{B}_p^{\frac{3}{p}+1}(\mathbb{R}^3))\times L^1([0,T_2];\mathcal{B}_p^{\frac{3}{p}-1}(\mathbb{R}^3))$, of (1.4). Moreover, there exists some integer m so that

$$\big(1+\|a\|_{\widetilde{L}^\infty_T(\mathcal{B}_q^{\frac{3}{q}})}\big)^{12}\|a-\dot{S}_m a\|_{\widetilde{L}^\infty_T(\mathcal{B}_q^{\frac{3}{q}})}\le 2c_0\underline{b}. \tag{4.17}$$

Step 4. Large time well-posedness of (1.4) for small initial velocity.

Let $(a^n,u^n,\nabla\pi^n)$ be the unique solution of (1.4) with initial data (a_0^n,u_0^n). We denote

$$X_n(t)\overset{\text{def}}{=}\|u^n\|_{\widetilde{L}^\infty_t(\mathcal{B}_p^{\frac{3}{p}-1})}+\|u^n\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+1})}+\|\nabla\pi^n\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-1})}$$

Then under the assumption of (4.7), we get, by applying Remark 3.4 and (4.6), that

$$\begin{aligned}X_n(t)\le& C_m\exp\left(C_m t\right)\Big(\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}+t\left(X_n^5(t)+X_n^3(t)\right)+X_n^2(t)\Big)\\ \le& C_m\exp\left(C_m t\right)\Big(\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}+X_n^5(t)+X_n^2(t)\Big).\end{aligned} \tag{4.18}$$

While thanks to (4.5), we get, by applying (2.24) to a^n equation of (1.4), that

$$\begin{aligned}\big(1+\|a^n\|_{\widetilde{L}^\infty_T(\mathcal{B}_q^{\frac{3}{q}})}\big)^{12}\|a^n-\dot{S}_m a^n\|_{\widetilde{L}^\infty_T(\mathcal{B}_q^{\frac{3}{q}})}&\le c_0\underline{b}+\|a_0\|_{\mathcal{B}_q^{\frac{3}{q}}}\left(\exp(C_0X_n(t))-1\right)\\ &\le c_0\underline{b}+C_0X_n(t)\|a_0\|_{\mathcal{B}_q^{\frac{3}{q}}}\exp(C_0X_n(t)).\end{aligned} \tag{4.19}$$

So that (4.18) holds provided that

$$C_0X_n(t)\|a_0\|_{\mathcal{B}_q^{\frac{3}{q}}}\exp(C_0X_n(t))\le c_0\underline{b}. \tag{4.20}$$

We henceforth take $\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}$ to be so small that

$$16\lambda^2\big(1+\|a_0\|_{\mathcal{B}_q^{\frac{3}{q}}}\big)\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\exp\left(4\lambda\right)\le\min\{1,c_0\underline{b}\}\quad\text{with}\quad\lambda=\max\{C_0,C_m\}. \tag{4.21}$$

Then we claim that there exists a positive time $T>1$ so that

$$X_n(T)\le 2C_m\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\exp\left(2C_m\right).$$

Indeed, let

$$T^*\overset{\text{def}}{=}\sup\big\{T>0: X_n(T)\le 2C_m\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\exp\left(2C_m\right)\ \big\}. \tag{4.22}$$

Then for $T<T*$, it is easy to observe that (4.20) holds by (4.21). If $T^*\le\frac{3}{2}$, then we infer from (4.21) that $T<T^*$

$$C_m\exp\left(C_mT\right)\left(X_n^4(t)+X_n(t)\right)\le\frac{1}{4},$$

which together with (4.18) implies

$$X_n(T)\le C_m\exp\left(2C_4\right)\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}+\frac{1}{4}X_n(T),$$

and thus

$$X_n(T) \leq \frac{4}{3} C_m \exp\left(2C_m\right) \|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}. \tag{4.23}$$

This contradicts (4.22), and thus $T^* \geq \frac{3}{2}$.

Hence we deduce from Proposition 2.12 and (4.22) that

$$\|a^n\|_{\widetilde{L}_t^\infty(\mathcal{B}_q^{\frac{3}{q}})} \leq C\|a_0\|_{\mathcal{B}_q^{\frac{3}{q}}} \mathrm{e}^{CX_n(t)} \leq C_m \|a_0\|_{\mathcal{B}_q^{\frac{3}{q}}}. \tag{4.24}$$

Therefore, $(a^n, u^n, \nabla\pi^n)$ is uniformly bounded in E_{T^*} and (4.1) holds for u^n. With such estimates of $(a^n, u^n, \nabla\pi^n)$, we can repeat the argument in **Step 3** to conclude that the lifespan to the solution $(a, u, \nabla\pi)$ obtained there is greater than 1. Furthermore there holds (4.1) except the term of $\|\nabla\pi\|_{L^1([0,T];\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}})}$ and $\|\partial_t u\|_{L^1([0,T];\tilde{\mathcal{B}}_p^{\frac{3}{p}})}$. Indeed the estimate of $\|\nabla\pi\|_{L_t^1(B_{p,1}^{\frac{3}{p}-2})}$ for $1 < p < \frac{3}{2}$ can be deduced from (3.28) and (3.30). Whereas the estimate of $\|\nabla\pi\|_{L_t^1(\dot{B}_{p,2}^{\frac{3}{p}-\frac{3}{2}})}$ for $\frac{3}{2} \leq p < 6$ can be obtained by a direct application of (3.23). And then the estimate of $\|\partial_t u\|_{L^1([0,T];\tilde{\mathcal{B}}_p^{\frac{3}{p}})}$ follows from the equation (1.4). This completes the existence part of Theorem 4.1.

4.2. Uniqueness part of Theorem 4.1. In order to do so, we first consider the *a priori* estimates for following linearized system:

$$\begin{cases} \partial_t u - (1+a)\Delta u + (1+a)\nabla\pi = f \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \\ \operatorname{div} u = g, \\ \partial_t g = \operatorname{div} R, \\ u|_{t=0} = 0. \end{cases} \tag{4.25}$$

Proposition 4.2. *Let p, q be given Theorem 1.2 and a satisfy* (3.2). *Let $(u, \nabla\pi)$ be a smooth enough solution of* (4.25). *Then there holds*

$$\begin{aligned} &\|u\|_{\widetilde{L}_t^\infty(\tilde{\mathcal{B}}_p^{\frac{3}{p}})} + \|u\|_{L_t^1(\tilde{\mathcal{B}}_p^{\frac{3}{p}+2})} + \|(\partial_t u, \nabla\pi)\|_{L_t^1(\tilde{\mathcal{B}}_p^{\frac{3}{p}})} \\ &\quad \lesssim \int_0^t \|u\|_{\tilde{\mathcal{B}}_p^{\frac{3}{p}}} \,\mathrm{d}\tau + \|(f, \nabla g, R)\|_{L_t^1(\tilde{\mathcal{B}}_p^{\frac{3}{p}})}. \end{aligned} \tag{4.26}$$

Proof. Along the same line to the proof of Proposition 3.1, we first get, by a similar derivation of (3.11)), that

$$\begin{aligned} &\|u\|_{\widetilde{L}_t^\infty(\mathcal{B}_p^{\frac{3}{p}-1})} + \|u\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}+1})} \\ &\quad \lesssim 2^{(1+\frac{3}{q})m} \|\dot{S}_m a\|_{L_t^\infty(L^q)} \big(\|u\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}})} + \|\nabla\pi\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}})}\big) \\ &\qquad + \|f\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}-1})} + \|a - \dot{S}_m a\|_{L_t^\infty(\mathcal{B}_q^{\frac{3}{q}})} \|\nabla\pi\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}-1})}, \end{aligned} \tag{4.27}$$

and

$$\begin{aligned} &\|u\|_{\widetilde{L}_t^\infty(\dot{B}_{p,2}^{\frac{3}{p}-\frac{3}{2}})} + \|u\|_{L_t^1(\dot{B}_{p,2}^{\frac{3}{p}-\frac{3}{2}})} \\ &\quad \lesssim 2^{(1+\frac{3}{q})m} \|\dot{S}_m a\|_{L_t^\infty(L^q)} \big(\|u\|_{L_t^1(\dot{B}_{p,2}^{\frac{3}{p}-\frac{1}{2}})} + \|\nabla\pi\|_{L_t^1(\dot{B}_{p,2}^{\frac{3}{p}-\frac{3}{2}})}\big) \\ &\qquad + \|f\|_{L_t^1(\dot{B}_{p,2}^{\frac{3}{p}-1})} + \|a - \dot{S}_m a\|_{L_t^\infty(\mathcal{B}_q^{\frac{3}{q}})} \|\nabla\pi\|_{L_t^1(\dot{B}_{p,2}^{\frac{3}{p}-\frac{3}{2}})}, \end{aligned} \tag{4.28}$$

and for $1 < p < \frac{3}{2}$

$$\begin{aligned} &\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}_p^{\frac{3}{p}-2})} + \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}})} \\ &\lesssim 2^{\frac{3m}{q}} \|\dot{S}_m a\|_{L^\infty_t(L^q)} \big(2^m \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-1})} + \|\nabla\pi\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-2})}\big) \\ &\quad + \|f\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-2})} + \|a - \dot{S}_m a\|_{L^\infty_t(\mathcal{B}_q^{\frac{3}{q}})} \|\nabla\pi\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-2})}. \end{aligned} \tag{4.29}$$

In order to handle the estimate of the pressure function, we apply div to the u equation (4.25) to get

$$\begin{aligned} \operatorname{div}\big((1 + \dot{S}_m a)\nabla\pi\big) =& \operatorname{div}(\nabla g - R) + \operatorname{div}(\dot{S}_m a \Delta u) \\ &+ \operatorname{div}\big((\dot{S}_m a - a)(\nabla\pi - \Delta u)\big) + \operatorname{div} f. \end{aligned} \tag{4.30}$$

Thanks to (4.30), we get, by a similar derivation of (3.18), that

$$\begin{aligned} \|\nabla\pi\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-1})} \lesssim & 2^{(\frac{1}{2}+\frac{3}{q})m} \|\dot{S}_m a\|_{L^\infty_t(L^q)} \big(2^{\frac{m}{2}} \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}})} + \|\nabla\pi\|_{L^1_t(\dot{B}_{p,2}^{\frac{3}{p}-\frac{3}{2}})}\big) \\ &+ \big\|(f, \nabla g, R)\big\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-1})} + \|a - \dot{S}_m a\|_{L^\infty_t(\mathcal{B}_q^{\frac{3}{q}})} \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+1})}. \end{aligned} \tag{4.31}$$

By summing up (4.27) and (4.31) and then using (3.2), we obtain

$$\begin{aligned} &\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}_p^{\frac{3}{p}-1})} + \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+1})} + \|\nabla\pi\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-1})} \\ &\lesssim 2^{(\frac{1}{2}+\frac{3}{q})m} \|\dot{S}_m a\|_{L^\infty_t(L^q)} \big(2^{\frac{m}{2}} \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}})} + \|\nabla\pi\|_{L^1_t(\dot{B}_{p,2}^{\frac{3}{p}-\frac{3}{2}})}\big) \\ &\quad + \|(f, \nabla g, R)\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-1})}. \end{aligned} \tag{4.32}$$

While thanks to (4.30), we get, by a similar derivation of (3.23), that for $\frac{3}{2} \leq p < 6$

$$\begin{aligned} \|\nabla\pi\|_{\widetilde{L}^1_t(\dot{B}_{p,2}^{\frac{3}{p}-\frac{3}{2}})} \lesssim & \|a - \dot{S}_m a\|_{L^\infty_t(\mathcal{B}_q^{\frac{3}{q}})} \|u\|_{L^1_t(\dot{B}_{p,2}^{\frac{3}{p}+\frac{1}{2}})} \\ &+ 2^{\frac{3}{q}m} \|\dot{S}_m a\|_{L^\infty_t(L^q)} \|u\|_{L^1_t(\dot{B}_{p,2}^{\frac{3}{p}+\frac{1}{2}})} + \|(f, \nabla g, R)\|_{L^1_t(\dot{B}_{p,2}^{\frac{3}{p}-\frac{3}{2}})}. \end{aligned} \tag{4.33}$$

Then by inserting (4.33) into (4.32) and then applying Young's inequality, we find

$$\begin{aligned} &\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}_p^{\frac{3}{p}-1})} + \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+1})} + \|\nabla\pi\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-1})} \\ &\lesssim 2^m (A_{ma}(t) + 1) \|(f, \nabla g, R)\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})} \\ &\quad + 2^{2m} (A_{ma}(t) + 1)^2 \|\dot{S}_m a\|_{L^\infty_t(L^q)} \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}})} \\ &\quad + 2^{4m} (A_{ma}(t) + 1)^4 \int_0^t \|u\|_{\mathcal{B}_p^{\frac{3}{p}-1}} \, d\tau. \end{aligned} \tag{4.34}$$

Similarly, we can derive the estimate of $\|u\|_{\widetilde{L}^\infty_t(\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}})}$ from (4.28) and (4.33). We thus conclude the proof of (4.26) for $\frac{3}{2} \leq p < 6$.

For the case of $1 < p < \frac{3}{2}$, we have

$$\begin{aligned} \|\nabla \pi\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}})} \lesssim & 2^{\left(\frac{3}{q}+\frac{1}{2}\right)m} \|\dot{S}_m a\|_{L^\infty_t(L^q)} \big(\|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}})} + \|\nabla \pi\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-2})}\big) \\ & + \big\|(f, \nabla g, R)\big\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}})} + \|a - \dot{S}_m a\|_{L^\infty_t(\mathcal{B}_q^{\frac{3}{q}})} \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+\frac{1}{2}})} \end{aligned} \tag{4.35}$$

Whereas we get, by a similar derivation of (3.28) that for $1 < p < \frac{3}{2}$

$$\begin{aligned} \|\nabla \pi\|_{L^1_t(B_{p,1}^{\frac{3}{p}-2})} \lesssim & 2^{\left(\frac{3}{q}-\frac{1}{2}\right)m} \|\dot{S}_m a\|_{L^\infty_t(L^q)} \big(2^{\frac{m}{2}} \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}})} + \|\nabla \pi\|_{L^1_t(L^2)}\big) \\ & + \big\|(f, \nabla g, R)\big\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-2})} + \|a - \dot{S}_m a\|_{L^\infty_t(\mathcal{B}_q^{\frac{3}{q}})} \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}})}. \end{aligned} \tag{4.36}$$

To handle the estimate of $\|\nabla \pi\|_{L^1_t(L^2)}$, by taking divergence to the u equation of (4.25), we have

$$\operatorname{div}\big((1+a)\nabla \pi\big) = \operatorname{div}(a\Delta u) + \operatorname{div}(f + \nabla g - R)$$

Applying Proposition 2.9 yields

$$\|\nabla \pi\|_{L^2} \lesssim \|a\Delta u\|_{L^2} + \|(f, \nabla g, R)\|_{L^2}.$$

Notice that for $1 < p \le 2$, $\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}}(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$, so that we infer

$$\|\nabla \pi\|_{L^1_t(L^2)} \lesssim \|a\|_{L^\infty_t(\mathcal{B}_q^{\frac{3}{q}})} \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+\frac{1}{2}})} + \|(f, \nabla g, R)\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}})}. \tag{4.37}$$

Since $a \in L^\infty_T(\mathcal{B}_q^{\frac{3}{q}})$, by combining (4.32) and (4.35)-(4.37), we find

$$\begin{aligned} & \|u\|_{\widetilde{L}^\infty_t(\mathcal{B}_p^{\frac{3}{p}-1})} + \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+1})} + \|\nabla \pi\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-1})} \\ & \lesssim 2^m (A_{ma}(t)+1)^2 \|(f, \nabla g, R)\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})} \\ & \quad + 2^{2m} (A_{ma}(t)+1)^3 \|\dot{S}_m a\|_{L^\infty_t(L^q)} \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}})} \\ & \quad + 2^{12m} (A_{ma}(t)+1)^{12} \int_0^t \|u\|_{\mathcal{B}_p^{\frac{3}{p}-1}} \, d\tau. \end{aligned} \tag{4.38}$$

By the same way, we apply (4.29), (4.36) and (4.37) to get

$$\begin{aligned} & \|u\|_{\widetilde{L}^\infty_t(\mathcal{B}_p^{\frac{3}{p}-2})} + \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}})} + \|\nabla \pi\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-2})} \\ & \lesssim 2^m (A_{ma}(t)+1) \|\dot{S}_m a\|_{L^\infty_t(L^q)} \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-1})} + (A_{ma}(t)+1)^2 \\ & \quad \times \Big(\big\|(f, \nabla g, R)\big\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}-2} \cap \mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}})} + \|\dot{S}_m a\|_{L^\infty_t(L^q)} \big(\|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}})} + \|u\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+\frac{1}{2}})}\big)\Big). \end{aligned} \tag{4.39}$$

Thus, combining (4.38) and (4.39) and then using Young's inequality, we get the inequality (4.26) for $1 < p < \frac{3}{2}$. Finally, by using the first equation of (4.25), we easy to get the estimate of $\|\partial_t u\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})}$. The proof of this proposition is finished. □

As in [14, 15, 19, 21], we shall prove the uniqueness part of Theorem 4.1 by using the Lagrangian formulation of (1.4). Toward this, let us recall some basic facts concerning Lagrangian coordinates and trajectories of particles from [14]. We first observe from the existence result of Theorem 4.1 that by choosing T to be sufficiently small, there holds

$$\int_0^T \|\nabla u\|_{\mathcal{B}_p^{\frac{3}{p}}}\, \mathrm{d}t \leq \frac{1}{3}. \tag{4.40}$$

So that for any $y \in \mathbb{R}^3$, the following ordinary differential equation has a unique solution on $[0, T]$:

$$\frac{dX(t,y)}{dt} = u(t, X(t,y)) =: v(t,y), \qquad X(t,y)|_{t=0} = y. \tag{4.41}$$

An integration in time leads to the following relation between the Eulerian coordinates x and the Lagrangian coordinates y:

$$x = X(t,y) = y + \int_0^t v(\tau, y)\, \mathrm{d}\tau. \tag{4.42}$$

Let $Y(t,\cdot)$ be the inverse mapping of $X(t,\cdot)$. Then $\nabla_x Y(t,x) = (\nabla_y X(t,y))^{-1}$ for $x = X(t,y)$. If $\|\nabla_y X - \mathrm{Id}\|_{L^\infty}$ is small enough, we have

$$\nabla_x Y = (\mathrm{Id} + (\nabla_y X - \mathrm{Id}))^{-1} = \sum_{k=0}^{\infty} (-1)^k \left(\int_0^t \nabla_y v(\tau, y)\, \mathrm{d}\tau \right)^k. \tag{4.43}$$

Let $A(t,y) := (\nabla_y X(t,y))^{-1} = \nabla_x Y(t,x)$ and $A^\top$ the transpose matrix of A. Then one has

$$\nabla_x u(t,x) = A(t,x)^\top \nabla_y v(t,y), \quad \mathrm{div}_x\, u(t,x) = \mathrm{div}_y(A(t,y)v(t,y)).$$

While by applying the chain rule, one also has

$$\mathrm{div}_y(A\cdot) = A^\top : \nabla_y$$

where " : " denotes the Frobenius inner product for matrices. As in [14, 15], we denote

$$\begin{gathered}\nabla_u \stackrel{\text{def}}{=} A^\top \cdot \nabla_y, \quad \mathrm{div}_u \stackrel{\text{def}}{=} \mathrm{div}_y(A\cdot), \text{ and } \Delta_u \stackrel{\text{def}}{=} \mathrm{div}_u \nabla_u, \\ b(t,y) \stackrel{\text{def}}{=} a(t, X(t,y)), \quad v(t,y) \stackrel{\text{def}}{=} u(t, X(t,y)), \text{ and } P(t,y) \stackrel{\text{def}}{=} \pi(t, X(t,y)).\end{gathered} \tag{4.44}$$

Notice that for any $t > 0$, the solution of (1.4) obtained in Theorem 4.1 satisfies the smoothness assumption of [15, Proposition 2], so that the triple $(b, v, \nabla P)$ defined by (4.44) fulfills

$$\begin{cases} \partial_t b = 0, \\ \partial_t v - (1+b)(\Delta_u v - \nabla_u P) = 0, \\ \mathrm{div}_u v = 0, \\ b|_{t=0} = a_0, \ v|_{t=0} = u_0. \end{cases} \tag{4.45}$$

Obviously, $b = b(t,y) \equiv a_0(y)$ for all $t > 0$ and is independent of the solution u. Moreover, it follows from Proposition 3.7 in [13] that $(v, \nabla P)$ shares the same regularities as $(u, \nabla\pi)$.

Let $(a_i, u_i, \pi_i), i = 1, 2$, be two solutions of (1.4) which satisfy (4.1). Let $X_i, (v_i, P_i), A_i$, $i = 1, 2$ be given by (4.42) and (4.44). Then the pair $(\delta v, \delta P)$, where

$$\delta v \stackrel{\text{def}}{=} v_2 - v_1, \quad \delta P \stackrel{\text{def}}{=} P_2 - P_1,$$

solves the system

$$\begin{cases} \partial_t \delta v - (1 + a_0)(\Delta \delta v - \nabla \delta P) = \delta f_1 + \delta f_2, \\ \operatorname{div} \delta v = \delta g, \\ \partial_t \delta g = \operatorname{div} \delta R, \\ \delta v|_{t=0} = 0, \end{cases} \tag{4.46}$$

with

$$\begin{aligned} \delta f_1 &\stackrel{\text{def}}{=} (1 + a_0)[(\mathrm{Id} - A_2^\top)\nabla \delta P - \delta A^\top \nabla P_1], \\ \delta f_2 &\stackrel{\text{def}}{=} (1 + a_0)\operatorname{div}[(A_2 A_2^\top - \mathrm{Id})\nabla \delta v + (A_2 A_2^\top - A_1 A_1^\top)\nabla v_1], \\ \delta g &\stackrel{\text{def}}{=} (\mathrm{Id} - A_2) : D\delta v - \delta A : D v_1, \\ \delta R &\stackrel{\text{def}}{=} \partial_t[(\mathrm{Id} - A_2)\delta v] - \partial_t[\delta A v_1]. \end{aligned}$$

Let us first handle the estimate of $\delta A \stackrel{\text{def}}{=} A_2 - A_1$, which will be frequently used in the sequel. Notice that for the matrices $C_i(t,y) \stackrel{\text{def}}{=} \int_0^t \nabla_y v_i(\tau, y)\,\mathrm{d}\tau$, $i = 1, 2$, we observe from (4.43) *cf.* [15, 21] that

$$\delta A(t,y) = \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} C_1^j \left(\int_0^t \nabla_y \delta v \,\mathrm{d}\tau \right) C_2^{k-1-j}, \tag{4.47}$$

The term $C_i(t,y)$ can be estimated in view of (4.43) (with $\nabla_x Y$ replaced by $\nabla_y X$) and (4.40) by

$$|C_i(t,y)| \le \int_0^t |\nabla_x u_i(\tau, X(\tau,y)|\,\mathrm{d}\tau \cdot \sup_{\tau,y} |\nabla_y X(\tau,y| \le \frac{1}{3} \sum_{k=0}^{\infty} \big(\frac{1}{3}\big)^k = \frac{1}{2}, \quad i = 1, 2.$$

As a result, it comes out

$$|\delta A(t,y| \le c_1 \int_0^t |\nabla \delta v(\tau, y)|\,\mathrm{d}\tau, \quad c_1 = \sum_{k=1}^{\infty} k \big(\frac{1}{2}\big)^{k-1} < \infty. \tag{4.48}$$

Moreover, as proved in the appendix of [14], we have the following estimates:

$$\begin{aligned} \|\partial_t A_i\|_{\mathcal{B}_p^{\frac{3}{p}}} &\lesssim \|D v_i\|_{\mathcal{B}_p^{\frac{3}{p}}}, \quad i = 1, 2, \\ \|\delta A\|_{L_t^\infty(\mathcal{B}_p^{\frac{3}{p}})} &\lesssim \|D\delta v\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}})}, \\ \|A_i - \mathrm{Id}\|_{L_t^\infty(\mathcal{B}_p^{\frac{3}{p}})} &\lesssim \|D v_i\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}})}, \quad i = 1, 2, \\ \|\partial_t \delta A\|_{L_t^2(\mathcal{B}_p^{\frac{3}{p}-1})} &\lesssim \|v_1, v_2\|_{L_t^2(\mathcal{B}_p^{\frac{3}{p}})} \|D\delta v\|_{L_t^1(\mathcal{B}_p^{\frac{3}{p}})} + \|\delta v\|_{L_t^2(\mathcal{B}_p^{\frac{3}{p}})}. \end{aligned} \tag{4.49}$$

Set

$$\delta G(t) \overset{\text{def}}{=} \|\delta v\|_{\widetilde{L}^\infty_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})} + \|\delta v\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}+2})} + \|(\partial_t \delta v, \nabla \delta P)\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})}.$$

Then we deduce from Proposition 4.2 and (4.46) that

$$\begin{aligned} \delta G(t) &\lesssim \int_0^t \|\delta v\|_{\tilde{\mathcal{B}}_p^{\frac{3}{p}}} \,\mathrm{d}\tau + \|(\delta f_1, \delta f_2, \nabla \delta g, \delta R)\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}}))} \\ &\lesssim t\delta G(t) + \|(\delta f_1, \delta f_2, \nabla \delta g, \delta R)\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}}))}. \end{aligned} \tag{4.50}$$

To prove the uniqueness of the solutions, we shall prove $\delta G(t) = 0$ for small enough t.

Let us handle term by term on the right-hand side of (4.50). Indeed it follows from (4.49), product laws and interpolation in Besov spaces that

$$\begin{aligned} \|\delta f_1\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})} &\lesssim \big(1 + \|a_0\|_{\mathcal{B}_q^{\frac{3}{q}}}\big)\big(\|(\mathrm{Id} - A_2^\top)\nabla \delta P\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})} + \|\delta A \nabla P_1\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})}\big) \\ &\lesssim \|(\mathrm{Id} - A_2^\top)\|_{L^\infty_t(\mathcal{B}_p^{\frac{3}{p}})} \|\nabla \delta P\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})} + \|\delta A\|_{L^\infty_t(\mathcal{B}_p^{\frac{3}{p}})} \|\nabla P_1\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})} \\ &\lesssim \big(\|v_2\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+1})} + \|\nabla P_1\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})}\big)\delta G(t), \end{aligned}$$

and

$$\begin{aligned} \|\delta f_2\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})} &\lesssim \|(A_2 A_2^\top - \mathrm{Id})\nabla \delta v\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}+1})} + \|(A_2 A_2^\top - A_1 A_1^\top)\nabla v_1\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}+1})} \\ &\lesssim \|(\mathrm{Id} - A_2^\top, \mathrm{Id} - A_2)\|_{L^\infty_t(\mathcal{B}_p^{\frac{3}{p}})} \|\nabla \delta v\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}+1})} + \|\nabla v_1\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}})} \|\delta A\|_{L^\infty_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}+1})} \\ &\lesssim \big(\|v_1\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+1})} + \|v_2\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+1})}\big)\delta G(t). \end{aligned}$$

Along the same line, we have

$$\begin{aligned} \|\nabla \delta g\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})} &\lesssim \|(\mathrm{Id} - A_2) : D\delta v - \delta A : Dv_1\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}+1})} \\ &\lesssim \|v_2\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+1})} \|\nabla \delta v\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}+1})} + \|Dv_1\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}})} \|\delta A\|_{L^\infty_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}+1})} \\ &\lesssim \big(\|v_2\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+1})} + \|v_1\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+1})}\big)\delta G(t) \end{aligned}$$

and

$$\begin{aligned} \|\delta R\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})} &\lesssim \|\partial_t[(\mathrm{Id} - A_2)\delta v]\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})} + \|\partial_t[\delta A v_1]\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})} \\ &\lesssim \|v_2\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+1})} \|\delta v\|_{L^\infty_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})} + \|Dv_2\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}})} \|\partial_t \delta v\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})} \\ &\quad + \|v_1\|_{L^2_t(\mathcal{B}_p^{\frac{3}{p}})} \|\nabla \delta v\|_{L^2_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})} + \|\delta A\|_{L^\infty_t(\mathcal{B}_p^{\frac{3}{p}})} \|\partial_t v_1\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})} \\ &\lesssim \big(\|v_2\|_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+1})} + \|v_1\|^{\frac{1}{2}}_{L^1_t(\mathcal{B}_p^{\frac{3}{p}+1})} + \|\partial_t v_1\|_{L^1_t(\tilde{\mathcal{B}}_p^{\frac{3}{p}})}\big)\delta G(t). \end{aligned}$$

By substituting the above estimates into (4.50), we arrive at

$$\delta G(t) \lesssim \varphi(t)\delta G(t) \tag{4.51}$$

for some positive continuous function $\varphi(t)$ which tends to 0 as $t \to 0$. Thus, the uniqueness on $[0, T]$ can be obtained by a standard argument.

4.3. Higher regularity part of Theorem 4.1. Let (a^n, u^n, π^n) be the approximate solution of (1.4) constructed in Step 4 of Section 4.1. Then for $0 < \tau < t_0 < t \le T^*$ with T^* being determined by (4.22), we deduce by a similar proof of (3.11) and (3.18) that

$$\begin{aligned}
&\|u^n\|_{\widetilde{L}^\infty([\tau,t];\mathcal{B}_p^{\frac{3}{p}})} + \|u^n\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}+2})} + \|\nabla\pi^n\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}})}\\
&\lesssim \|u(\tau)\|_{\mathcal{B}_p^{\frac{3}{p}}} + \|(a^n - \dot{S}_m a^n)(\Delta u^n - \nabla\pi^n)\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}})} + \|\pi^n\nabla\dot{S}_m a^n\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}})}\\
&\quad + \sum_{j\in\mathbb{Z}} 2^{\frac{3}{p}j}\big(\|[\mathbb{P}\dot{\Delta}_j, \dot{S}_m a^n]\Delta u^n\|_{L^p} + \|\nabla S_m a^n \cdot \dot{\Delta}_j\nabla u^n\|_{L^p}\big) + \|u^n\cdot\nabla u^n\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}})}.
\end{aligned} \tag{4.52}$$

It follows from the law of product in Besov spaces that

$$\begin{aligned}
\|u^n\cdot\nabla u^n\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}})} &\lesssim \int_\tau^t \|u^n\|_{\mathcal{B}_p^{\frac{3}{p}}}\|u^n\|_{\mathcal{B}_p^{\frac{3}{p}+1}}\,\mathrm{d}t'\\
\|(a^n - \dot{S}_m a^n)(\Delta u^n - \nabla\pi^n)\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}})} &\lesssim \|a^n - \dot{S}_m a^n\|_{L^\infty([\tau,t];\mathcal{B}_q^{\frac{3}{q}})}\|\Delta u^n - \nabla\pi^n\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}})}\\
\|\pi^n\nabla\dot{S}_m a^n\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}})} &\lesssim \|\nabla\dot{S}_m a^n\|_{L^\infty([\tau,t];\mathcal{B}_q^{\frac{3}{q}})}\|\nabla\pi^n\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}-1})}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} 2^{\frac{3}{p}j}\big(\|[\mathbb{P}\dot{\Delta}_j, \dot{S}_m a^n]\nabla u^n\|_{L^p} + \|\nabla S_m a^n\cdot\dot{\Delta}_j\nabla u^n\|_{L^p}\big)\\
&\qquad\qquad\lesssim \|\nabla\dot{S}_m a^n\|_{L^\infty([\tau,t];\mathcal{B}_q^{\frac{3}{q}})}\|u^n\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}+1})}.
\end{aligned}$$

Substituting the above estimate into (4.52) results in

$$\begin{aligned}
&\|u^n\|_{\widetilde{L}^\infty([\tau,t];\mathcal{B}_p^{\frac{3}{p}})} + \|u^n\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}+2})} + \|\nabla\pi^n\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}})}\\
&\lesssim \|u(\tau)\|_{\mathcal{B}_p^{\frac{3}{p}}} + 2^{(1+\frac{3}{q})m}\|a_0\|_{L^q}\big(\|\nabla\pi^n\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}-1})} + \|u^n\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}+1})}\big)\\
&\quad + \|a^n - \dot{S}_m a^n\|_{L^\infty([\tau,t];\mathcal{B}_q^{\frac{3}{q}})}\|\Delta u^n - \nabla\pi^n\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}})} + \int_\tau^t \|u^n\|_{\mathcal{B}_p^{\frac{3}{p}}}\|u^n\|_{\mathcal{B}_p^{\frac{3}{p}+1}}\,\mathrm{d}t'.
\end{aligned}$$

By applying smallness of (4.17), (4.1) and Gronwall's inequality, one has

$$\begin{aligned}
&\|u^n\|_{\widetilde{L}^\infty([\tau,t];\mathcal{B}_p^{\frac{3}{p}})} + \|u^n\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}+2})} + \|\nabla\pi^n\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}})}\\
&\qquad\lesssim \|a_0\|_{\mathcal{B}_q^{\frac{3}{q}}}\big(\|u(\tau)\|_{\mathcal{B}_p^{\frac{3}{p}}} + \|u(\tau)\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\big)\exp\Big(C\|u^n\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}+1})}\Big).
\end{aligned}$$

Integrating the above inequality for τ over $[0, t_0]$, and then dividing the resulting inequality by t_0 and (4.1) leads to

$$\begin{aligned}
&\|u^n\|_{\widetilde{L}^\infty([t_0,t];\mathcal{B}_p^{\frac{3}{p}})} + \|u^n\|_{L^1([t_0,t];\mathcal{B}_p^{\frac{3}{p}+2})} + \|\nabla\pi^n\|_{L^1([\tau,t];\mathcal{B}_p^{\frac{3}{p}})}\\
&\lesssim \frac{\|a_0\|_{\mathcal{B}_q^{\frac{3}{q}}}}{t_0}\Big(\int_0^{t_0}\|u(\tau)\|_{\mathcal{B}_p^{\frac{3}{p}}}\,\mathrm{d}\tau + \int_0^{t_0}\|u(\tau)\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\,\mathrm{d}\tau\Big)\exp\Big(C\|u^n\|_{L^1([0,t];\mathcal{B}_p^{\frac{3}{p}+1})}\Big)\\
&\lesssim \|a_0\|_{\mathcal{B}_q^{\frac{3}{q}}}\|u_0\|_{B_{p,1}^{\frac{3}{p}-1}}\big(1 + t_0^{-\frac{1}{2}}\big)\exp\Big(C\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\Big).
\end{aligned}$$

Then along with the compactness argument in Step 3 of Section 4.1 implies (4.2), and we thus complete the proof of Theorem 4.1.

5. Proof of Theorem 1.2: Global existence. This section is devoted to the global existence part of Theorem 1.2. The main idea of the proof is similar to that of [3], so that we only outline its proof here.

We first deduce from Theorem 4.1 that: given $a_0 \in B_q^{\frac{3}{q}}(\mathbb{R}^3)$ and $u_0 \in \mathcal{B}_p^{\frac{3}{p}-1}(\mathbb{R}^3)$ with $\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}$ being sufficiently small, the system (1.4) admits a unique local solution (a, u) satisfying $a \in C([0, T^*); B_q^{\frac{3}{q}}(\mathbb{R}^3))$ and $u \in C([0, T^*); \mathcal{B}_p^{\frac{3}{p}-1}) \cap L^1_{loc}([0, T^*); \mathcal{B}_p^{\frac{3}{p}+1})$ for some $T^* > 1$. Moreover there exists $t_1 \in (0, 1)$ so that

$$\|u(t_1)\|_{\dot{B}_{p,1}^{\frac{3}{p}-1} \cap \dot{B}_{p,1}^{\frac{3}{p}+2}} \lesssim \|u_0\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}}. \tag{5.1}$$

Let us decompose the velocity $u = v + w$, where v satisfies the classical Navier-Stokes equations

$$\begin{cases} \partial_t v + v \cdot \nabla v - \Delta v + \nabla \pi_v = 0, \\ \text{div} v = 0, \\ v|_{t=t_1} = u(t_1), \end{cases} \tag{5.2}$$

and (ρ, w) solves the equations

$$\begin{cases} \partial_t \rho + \text{div } (\rho(v + w))) = 0, \\ \rho \partial_t w + \rho(v + w) \cdot \nabla w - \Delta w + \nabla \pi_w = (1 - \rho)(\partial_t v + v \cdot \nabla v) - \rho w \cdot \nabla v, \\ \text{div} w = 0, \\ \rho|_{t=t_1} = \rho(t_1), \ w|_{t=t_1} = 0. \end{cases} \tag{5.3}$$

Thanks to (5.1), we deduce from the classical theory of Navier-Stokes equations [7] that (5.2) has a unique global solution $v \in C([t_1, +\infty); \mathcal{B}_p^{\frac{3}{p}-1}) \cap L^1([t_1, +\infty; \mathcal{B}_p^{\frac{3}{p}+1})$ satisfying

$$\begin{aligned} \|v\|_{\widetilde{L}^\infty([t_1,\infty);\mathcal{B}_p^{\frac{3}{p}-1})} &+ \|v\|_{L^1([t_1,\infty);\mathcal{B}_p^{\frac{3}{p}+1})} \\ &+ \|(\partial_t v, \nabla \pi_v)\|_{L^1([t_1,\infty);\mathcal{B}_p^{\frac{3}{p}-1})} \lesssim \|u(t_1)\|_{\mathcal{B}_p^{\frac{3}{p}-1}}. \end{aligned} \tag{5.4}$$

We also recall the following lemma from [3]:

LEMMA 5.1. *Let* (v, π_v) *be the unique global solution of* (4.26)*. Then for* $s_1 \in [\frac{3}{p}, \frac{3}{p} + 2]$ *and* $s_2 \in [\frac{3}{p} - 1, \frac{3}{p}]$*, there hold*

$$\|v\|_{L^\infty([t_1,\infty);\mathcal{B}_p^{s_1})} + \|(\Delta v, \nabla \pi_v)\|_{L^1([t_1,\infty);\mathcal{B}_p^{s_1})} \lesssim \|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}},$$

$$\|\partial_t v\|_{\widetilde{L}^\infty([t_1,\infty);\mathcal{B}_p^{s_2})} + \|(\partial_t \Delta v, \partial_t \nabla \pi_v)\|_{L^1([t_1,\infty);\mathcal{B}_p^{s_1})} \lesssim \|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}},$$

$$\|v\|_{L^2([t_1,\infty);L^\infty)} + \|\Delta v - \nabla \pi_v)\|_{L^2([t_1,\infty);L^\infty)} \lesssim \|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}.$$

Then the proof of the global existence part of Theorem 1.2 reduces to the proof of the global existence of solutions to (5.3). For simplicity, in what follows, we just present the *a priori* estimates for smooth enough solutions of (5.3) on $[0, T^*)$.

LEMMA 5.2. *Let* $1 < p, q < 6$ *with* $\max\{\frac{1}{2}-\frac{1}{q}, \frac{1}{q}-\frac{1}{3}\} < \frac{1}{p} < \frac{1}{3}+\frac{1}{q}$ *and* u_0, a_0 *satisfy the assumption of Theorem 1.2, there exist two positive constants* c_2, c_3 *so that for* $t_1 < t < T^*$

$$\|w\|_{L^\infty([t_1,t];L^2)} + \|\nabla w\|_{L^2([t_1,t];L^2)} \leq C\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}, \tag{5.5}$$

$$\|\nabla w\|^2_{L^\infty([t_1,t];L^2)} + \int_{t_1}^t \big(c_2\|\partial_t w\|^2_{L^2} + c_3\|\nabla^2 w\|^2_{L^2} + \|\nabla \pi_w\|^2_{L^2}\big)\,dt' \leq C\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}, \tag{5.6}$$

where C *is independent of* t.

Proof. The proof of this lemma is similar to Lemmas 5.1 to 5.3 of [3]. For convenience of the readers, we present some details below. Thanks to (1.6), one deduce the transport equation of (5.3) that

$$\big(1+\|a_0\|_{\mathcal{B}_q^{\frac{3}{q}}}\big)^{-1} \leq \rho(t,x) \leq \underline{b}^{-1}. \tag{5.7}$$

Thanks to (5.7), we get by taking the L^2 inner product of the w equation of (5.3) with w, we find

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}w\|^2_{L^2} + \|\nabla w\|^2_{L^2}\\
&\quad = \int_{\mathbb{R}^3} ((1-\rho)(\partial_t v + v\cdot\nabla v) + \rho w\cdot\nabla v)\cdot w\,\mathrm{d}x\\
&\quad \leq C\|a(\partial_t v + v\cdot\nabla v)\|_{L^2}\|\sqrt{\rho}w\|_{L^2} + C\|\nabla v\|_{L^\infty}\|\sqrt{\rho}w\|^2_{L^2}.
\end{aligned} \tag{5.8}$$

It is easy to observe that

$$\begin{aligned}
\|a(\partial_t v + v\cdot\nabla v)\|_{L^2} &\leq \begin{cases} \|a_0\|_{L^q\cap L^\infty}\|\partial_t v + v\cdot\nabla v\|_{\mathcal{B}_p^{\frac{3}{p}}}, & \text{if} \quad q\leq 2\\ \|a_0\|_{L^q}\|\partial_t v + v\cdot\nabla v\|_{\mathcal{B}^0_{\frac{2q}{q-2}}}, & \text{if} \quad q>2 \end{cases}\\
&\leq \begin{cases} \|a_0\|_{B_q^{\frac{3}{q}}}\|\Delta v - \nabla\pi_v\|_{\mathcal{B}_p^{\frac{3}{p}}}, & \text{if} \quad q\leq 2\\ \|a_0\|_{B_q^{\frac{3}{q}}}\|\Delta v - \nabla\pi_v\|_{\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}+\frac{3}{q}}}, & \text{if} \quad q>2 \end{cases}.
\end{aligned} \tag{5.9}$$

Applying the assumptions of p, q in Theorem 1.2, we observe that for $q > 2$

$$\left.\begin{array}{c} 2<q<6\\ \frac{1}{2}-\frac{1}{q}<\frac{1}{p}<\frac{1}{3}+\frac{1}{q} \end{array}\right\} \Longrightarrow \begin{cases} \dfrac{2q}{q-2} > p,\\ \dfrac{3}{p}-\dfrac{1}{2}+\dfrac{3}{q} \in [\dfrac{3}{p}, \dfrac{3}{p}+1],\\ \dfrac{3}{p}-\dfrac{3}{2}+\dfrac{3}{q} \in [\dfrac{3}{p}-1, \dfrac{3}{p}]. \end{cases} \tag{5.10}$$

Therefore, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}w\|^2_{L^2} + \|\nabla w\|^2_{L^2} \leq C\|\nabla v\|_{L^\infty}\|\sqrt{\rho}w\|^2_{L^2} + C\Big(\chi_{[q\leq 2]}(q)\|\Delta v - \nabla\pi_v\|_{\mathcal{B}_p^{\frac{3}{p}}}\\
+ \chi_{[q>2]}(q)\|\Delta v - \nabla\pi_v\|_{\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}+\frac{3}{q}}}\Big)\|\sqrt{\rho}w\|_{L^2}.
\end{aligned}$$

Applying Gronwall's inequalities yields

$$
\begin{aligned}
\|w\|_{L^\infty([t_1,t];L^2)} + \|\nabla w\|^2_{L^2([t_1,t];L^2)} &\le C \exp\Big(\int_{t_1}^t \|\nabla v(\tau)\|_{L^\infty}\,\mathrm{d}\tau\Big) \\
&\quad \times \Big(\chi_{[q\le 2]}(q)\|\Delta v - \nabla\pi_v\|_{L^1([t_1,t];\mathcal{B}_p^{\frac{3}{p}})} + \chi_{[q>2]}(q)\|\Delta v - \nabla\pi_v\|_{L^1([t_1,t];\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}+\frac{3}{q}})}\Big) \\
&\le \big(\|u(t_1)\|_{\mathcal{B}_p^{\frac{3}{p}-1}} + \|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\big)\exp\Big(C\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\Big) \\
&\le C\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}},
\end{aligned}
$$

where we used the of Lemma 5.1 and (5.4) that

$$
\begin{aligned}
\chi_{[q\le 2]}(q)\|\Delta v - \nabla\pi_v\|_{L^1([t_1,t];\mathcal{B}_p^{\frac{3}{p}})} &+ \chi_{[q>2]}(q)\|\Delta v - \nabla\pi_v\|_{L^1([t_1,t];\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}+\frac{3}{q}})} \\
&\le C\big(\|u(t_1)\|_{\mathcal{B}_p^{\frac{3}{p}-1}} + \|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}\big).
\end{aligned}
$$

This finishes the proof of (5.5).

To prove (5.6), we get, by taking the L^2 inner product of the w equation of (5.3) with $\frac{1}{\rho}\Delta w$ and using (5.7), that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|^2_{L^2} + \Big\|\frac{\Delta w}{\sqrt{\rho}}\Big\|^2_{L^2} \\
&= \int_{\mathbb{R}^3} \big((1-\rho)(\partial_t v + v\cdot\nabla v) + \rho w\cdot\nabla v - \rho(v+w)\cdot\nabla w - \nabla\pi_w\big)\cdot\frac{\Delta w}{\rho}\,\mathrm{d}x \\
&\le \Big\|\frac{\Delta w}{\sqrt{\rho}}\Big\|_{L^2}\big(\|a(\partial_t v + v\cdot\nabla v)\|_{L^2} + \|\nabla w\|_{L^2}\|v\|_{L^\infty} \\
&\qquad + \|w\|_{L^2}\|\nabla v\|_{L^\infty} + \|\nabla w\|_{L^6}\|w\|_{L^3} + \|\nabla\pi_w\|_{L^2}\big).
\end{aligned}
$$

Notice that we already have the estimate of $\|a(\partial_t v + v\cdot\nabla v)\|_{L^2}$, and the remaining terms can be handled as the ones in Lemma 5.2 of [3]. Then along the same line to the proof of Lemma 5.2 of [3], we obtain (5.6). This completes the proof of this lemma. □

Lemma 5.3. *Let* $1 < p, q < 6$ *with* $\max\{\frac{1}{2}-\frac{1}{q}, \frac{1}{q}-\frac{1}{3}\} < \frac{1}{p} < \frac{1}{6}+\frac{1}{q}$ *and* u_0, a_0 *satisfy the assumption of Theorem 1.2, for* $t_1 < t < T^*$, *there holds*

$$
\|\nabla^2 w\|_{L^\infty([t_1,t];L^2)} + \|\nabla w_t\|_{L^2([t_1,t];L^2)} + \|\nabla^2 w\|_{L^2([t_1,t];L^6)} \le C \tag{5.11}
$$

with C *being independent of* t.

Proof. By applying ∂_t to the w equation of (5.3) and use $1-\rho = a\rho$, we write

$$
\begin{aligned}
&\rho w_{tt} + \rho(v+w)\cdot\nabla w_t - \Delta w_t + (\nabla\pi_w)_t \\
&\quad = [(1-\rho)(\partial_t v + v\cdot\nabla v)]_t - [\rho w\cdot\nabla v]_t - \rho_t(v+w)\cdot\nabla w - \rho(v+w)_t\cdot\nabla w - \rho_t w_t.
\end{aligned}
$$

By taking the L^2 inner product of the above equation with $\partial_t w$, we get

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}w_t\|^2_{L^2} + \|\nabla w_t\|^2_{L^2} &= \int_{\mathbb{R}^3}[a\rho(\partial_t v + v\cdot\nabla v)]_t w_t\,\mathrm{d}x - \int_{\mathbb{R}^3}[\rho w\cdot\nabla v]_t w_t\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3}\big(\rho_t(v+w)\cdot\nabla w - \rho(v+w)_t\cdot\nabla w + \rho_t w_t\big)w_t\,\mathrm{d}x.
\end{aligned} \tag{5.12}
$$

Due to $a_t = -\mathrm{div}\,(a(v+w))$ and $\rho_t = -\mathrm{div}\,(\rho(v+w))$, by using integration by parts, we find

$$\int_{\mathbb{R}^3} [a\rho(\partial_t v + v\cdot\nabla v)]_t w_t \,\mathrm{d}x = \int_{\mathbb{R}^3} a\rho(v+w)\cdot\nabla(\partial_t v + v\cdot\nabla v) w_t \,\mathrm{d}x$$
$$+ \int_{\mathbb{R}^3} a\rho(\partial_t v + v\cdot\nabla v)(v+w)\cdot\nabla w_t \,\mathrm{d}x + \int_{\mathbb{R}^3} a\rho\partial_t(\partial_t v + v\cdot\nabla v) w_t \,\mathrm{d}x.$$

As a result, it comes out

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho} w_t\|_{L^2}^2 + \|\nabla w_t\|_{L^2}^2 = I_1 + I_2 + I_3 + I_4 + I_5 \tag{5.13}$$

with

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^3} a\rho(v+w)\cdot\nabla(\partial_t v + v\cdot\nabla v) w_t \,\mathrm{d}x,\\
I_2 &= \int_{\mathbb{R}^3} a\rho(\partial_t v + v\cdot\nabla v)(v+w)\cdot\nabla w_t \,\mathrm{d}x,\\
I_3 &= \int_{\mathbb{R}^3} a\rho\partial_t(\partial_t v + v\cdot\nabla v) w_t \,\mathrm{d}x,\\
I_4 &= -\int_{\mathbb{R}^3} \rho_t w_t\cdot(w_t + (w+v)\cdot\nabla w + w\cdot\nabla v)\,\mathrm{d}x,\\
I_5 &= -\int_{\mathbb{R}^3} \rho w_t\cdot((w+v)_t\cdot\nabla w + w_t\cdot\nabla v + w\cdot\nabla v_t)\,\mathrm{d}x.
\end{aligned}$$

Let us handle term by term above. We first observe that

$$\begin{aligned}
|I_1| &\le \|\sqrt{\rho} w_t\|_{L^2}\|(v+w)\|_{L^6}\|a\,\nabla(\Delta v - \nabla\pi_v)\|_{L^3}\\
&\le \|\sqrt{\rho} w_t\|_{L^2}\big(\|v\|_{\mathcal{B}_p^{\frac{3}{p}-\frac{1}{2}}} + \|\nabla w\|_{L^2}\big)\|a\,\nabla(\Delta v - \nabla\pi_v)\|_{L^3},
\end{aligned}$$

and

$$\|a\,\nabla(\Delta v - \nabla\pi_v)\|_{L^3} \le \begin{cases} \|a\|_{L^3}\|\nabla(\Delta v - \nabla\pi_v)\|_{L^\infty}, & \text{if} \quad q\le 3\\ \|a\|_{L^q}\|\nabla(\Delta v - \nabla\pi_v)\|_{L^{\frac{3q}{q-3}}}, & \text{if} \quad q>3 \end{cases}$$
$$\le \begin{cases} \|a_0\|_{B_q^{\frac{3}{q}}}\|\Delta v - \nabla\pi_v\|_{\mathcal{B}_p^{\frac{3}{p}+1}}, & \text{if} \quad q\le 3\\ \|a_0\|_{B_q^{\frac{3}{q}}}\|\Delta v - \nabla\pi_v\|_{\mathcal{B}_p^{\frac{3}{p}+\frac{3}{q}-1}}, & \text{if} \quad q>3 \end{cases},$$

where we use the fact for $q>3$

$$\left.\begin{aligned} 3<q<6\\ \frac{1}{2}-\frac{1}{q}<\frac{1}{p}<\frac{1}{6}+\frac{1}{q}\end{aligned}\right\} \Longrightarrow \begin{cases} \dfrac{3q}{q-3} > p,\\ \dfrac{3}{p}+\dfrac{3}{q}-1 \in [\dfrac{3}{p}-1, \dfrac{3}{p}). \end{cases}$$

This gives rise to

$$\begin{aligned}
|I_1| &\le C\|\sqrt{\rho} w_t\|_{L^2}\big(\|v\|_{\mathcal{B}_p^{\frac{3}{p}-\frac{1}{2}}} + \|\nabla w\|_{L^2}\big)\\
&\quad\times\big(\chi_{[q\le 3]}(q)\|\Delta v - \nabla\pi_v\|_{\mathcal{B}_p^{\frac{3}{p}+1}} + \chi_{[q>3]}(q)\|\Delta v - \nabla\pi_v\|_{\mathcal{B}_p^{\frac{3}{p}+\frac{3}{q}-1}}\big).
\end{aligned} \tag{5.14}$$

For I_2, we have

$$|I_2| \leq C\|\nabla w_t\|_{L^2}\|\Delta v - \nabla\pi_v\|_{L^\infty}\left(\|a_0\|_{L^\infty}\|w\|_{L^2} + \|a\,v\|_{L^2}\right),$$

and

$$\begin{aligned}\|a\,v\|_{L^2} &\leq \begin{cases}\|a\|_{L^2}\|v\|_{L^\infty}, & \text{if} \quad q \leq 2\\ \|a\|_{L^q}\|v\|_{L^{\frac{2q}{q-2}}}, & \text{if} \quad q > 2\end{cases}\\ &\leq \begin{cases}\|a_0\|_{B_q^{\frac{3}{q}}}\|v\|_{\mathcal{B}_p^{\frac{3}{p}}}, & \text{if} \quad q \leq 2\\ \|a_0\|_{B_q^{\frac{3}{q}}}\|v\|_{\mathcal{B}_p^{\frac{3}{p}+\frac{3}{q}-\frac{3}{2}}}, & \text{if} \quad q > 2\end{cases}\\ &\leq C,\end{aligned}$$

where we use the condition (5.10). Then we deduce from Lemma 5.2 and Young's inequality that

$$|I_2| \leq \frac{1}{16}\|\nabla w_t\|_{L^2}^2 + C\|\Delta v - \nabla\pi_v\|_{L^\infty}^2. \tag{5.15}$$

While by repeating the argument as in (5.9) and (5.10), we find

$$\begin{aligned}|I_3| \leq& C\|\sqrt{\rho}w_t\|_{L^2}\|a\,(\partial_t\Delta v - \partial_t\nabla\pi_v)\|_{L^2}\\ \leq& C\|\sqrt{\rho}w_t\|_{L^2}\Big(\ \chi_{[q\leq 2]}(q)\|(\partial_t\Delta v - \partial_t\nabla\pi_v)\|_{\mathcal{B}_p^{\frac{3}{p}}}\\ &\qquad + \chi_{[q>2]}(q)\|(\partial_t\Delta v - \partial_t\nabla\pi_v)\|_{\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}+\frac{3}{q}}}\Big).\end{aligned} \tag{5.16}$$

To deal with I_4, we use the transport equation of (5.3) and integration by parts to get

$$\begin{aligned}I_4 =& -\int_{\mathbb{R}^3}\rho(v+w)^i w_t\cdot\big(\partial_i(w+v)\cdot\nabla w + (w+v)\cdot\nabla\partial_i w + \partial_i w\cdot\nabla v + w\cdot\nabla\partial_i v\big)\,\mathrm{d}x\\ &-2\int_{\mathbb{R}^3}\rho w_t\cdot\big((w+v)\cdot\nabla w_t\big)\,\mathrm{d}x - \int_{\mathbb{R}^3}\rho(v+w)^i\partial_i w_t\cdot\big((w+v)\cdot\nabla w + w\cdot\nabla v\big)\,\mathrm{d}x\\ =& I_{41} + I_{42} + I_{43}.\end{aligned}$$

It is easy to observe that

$$\begin{aligned}|I_{41}| \leq& \|w_t\|_{L^6}\|v+w\|_{L^6}\big(\|\nabla(v+w)\|_{L^6}\|\nabla w\|_{L^2} + \|v+w\|_{L^6}\|\nabla^2 w\|_{L^2}\\ &+ \|\nabla v\|_{L^6}\|\nabla w\|_{L^2} + \|\nabla^2 v\|_{L^6}\|w\|_{L^2}\big)\\ \leq& \frac{1}{8}\|\nabla w_t\|_{L^2}^2 + C\big(\|\Delta w\|_{L^2}^2 + \|\nabla v\|_{L^6}^2 + \|\nabla^2 v\|_{L^6}^2\big),\\ |I_{42}| \leq& \|\sqrt{\rho}w_t\|_{L^2}\|v+w\|_{L^\infty}\|\nabla w_t\|_{L^2}\\ \leq& \frac{1}{16}\|\nabla w_t\|_{L^2}^2 + C\|\sqrt{\rho}w_t\|_{L^2}^2\big(\|v\|_{L^\infty}^2 + \|\nabla w\|_{L^2}\|\nabla^2 w\|_{L^2}\big),\\ |I_{43}| \leq& \|\nabla w_t\|_{L^2}\big(\|v+w\|_{L^\infty}^2\|\nabla w\|_{L^2} + \|v+w\|_{L^6}\|\nabla v\|_{L^6}\|w\|_{L^6}\big)\\ \leq& \frac{1}{16}\|\nabla w_t\|_{L^2}^2 + C\big(\|v\|_{L^\infty}^4 + \|\nabla v\|_{L^6}^2 + \|\Delta w\|_{L^2}^2\big).\end{aligned}$$

As a result, it comes out

$$
\begin{aligned}
|I_4| \leq \frac{1}{4}\|\nabla w_t\|_{L^2}^2 &+ C\big(\|v\|_{L^\infty}^4 + \|\nabla v\|_{L^6}^2 + \|\Delta w\|_{L^2}^2 + \|\nabla^2 v\|_{L^6}^2\big) \\
&+ C\|\sqrt{\rho}w_t\|_{L^2}^2\big(\|v\|_{L^\infty}^2 + \|\nabla w\|_{L^2}\|\nabla^2 w\|_{L^2}\big).
\end{aligned} \tag{5.17}
$$

Finally let us turn to the estimate of I_5, which can be dealt with as follows

$$
\begin{aligned}
|I_5| &\leq \|\sqrt{\rho}w_t\|_{L^2}\big(\|v_t\|_{L^\infty}\|\nabla w\|_{L^2} + \|w_t\|_{L^6}\|\nabla w\|_{L^3} \\
&\qquad + \|\sqrt{\rho}w_t\|_{L^2}\|\nabla v\|_{L^\infty} + \|\nabla v_t\|_{L^6}\|w\|_{L^3}\big) \\
&\leq \frac{1}{16}\|\nabla w_t\|_{L^2}^2 + C\|\sqrt{\rho}w_t\|_{L^2}^2\big(\|\nabla w\|_{L^2}\|\nabla^2 w\|_{L^2} + \|\nabla v\|_{L^\infty}\big) \\
&\qquad + C\|\sqrt{\rho}w_t\|\big(\|v_t\|_{\mathcal{B}_p^{\frac{3}{p}}} + \|v_t\|_{\mathcal{B}_p^{\frac{3}{p}+\frac{1}{2}}}\big).
\end{aligned} \tag{5.18}
$$

By inserting the estimates (5.14-5.18) into (5.13), we arrive at

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}w_t\|_{L^2}^2 + \|\nabla w_t\|_{L^2}^2 \leq C\|\sqrt{\rho}w_t\|_{L^2}^2 f_1(t) + C\|\sqrt{\rho}w_t\|_{L^2} f_2(t) + Cf_3(t) \tag{5.19}
$$

with

$$
\begin{aligned}
f_1(t) &= \|\nabla w\|_{L^2}\|\nabla^2 w\|_{L^2} + \|\nabla v\|_{L^\infty} + \|v\|_{L^\infty}^2, \\
f_2(t) &= \|v_t\|_{\mathcal{B}_p^{\frac{3}{p}}} + \chi_{[q\leq 2]}(q)\|(\partial_t\Delta v - \partial_t\nabla\pi_v)\|_{\mathcal{B}_p^{\frac{3}{p}}} + \chi_{[q>2]}(q)\|(\partial_t\Delta v - \partial_t\nabla\pi_v)\|_{\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}+\frac{3}{q}}} \\
&\quad + \|v_t\|_{\dot{B}_{p,1}^{\frac{3}{p}+\frac{1}{2}}} + \chi_{[q\leq 3]}(q)\|\Delta v - \nabla\pi_v\|_{\mathcal{B}_p^{\frac{3}{p}+1}} + \chi_{[q>3]}(q)\|\Delta v - \nabla\pi_v\|_{\mathcal{B}_p^{\frac{3}{p}+\frac{3}{q}-1}}, \\
f_3(t) &= \|\Delta v - \nabla\pi_v\|_{L^\infty}^2 + \|v\|_{L^\infty}^4 + \|\nabla v\|_{L^6}^2 + \|\Delta w\|_{L^2}^2 + \|\nabla^2 v\|_{L^6}^2.
\end{aligned}
$$

Applying Gronwall's inequality to (5.19) gives

$$
\begin{aligned}
&\|\sqrt{\rho}w_t(t)\|_{L^2}^2 + \int_{t_1}^t \|\nabla w_t(t')\|_{L^2}^2 \mathrm{d}t' \\
&\quad \leq C\exp\Big(C\int_{t_1}^t f_1(t')\mathrm{d}t'\Big)\big(\|\sqrt{\rho}w_t(t_1)\|_{L^2}^2 + \int_{t_1}^t [\|\sqrt{\rho}w_t(t')\|_{L^2} f_2(t') + f_3(t')]\,\mathrm{d}t'\big)
\end{aligned} \tag{5.20}
$$

Yet it follows from (5.4), Lemma 5.1 and 5.2 that

$$
\int_t^{t_1} [f_1(t') + f_2(t') + f_3(t')]\,\mathrm{d}t' \leq C
$$

with C being independent of t.

Whereas by taking L^2 inner product of the w equation of (5.3) with w_t at $t = t_1$, and then in view the proof of (5.9)

$$
\begin{aligned}
\|(\sqrt{\rho}w_t)(t_1)\|_{L^2} &\leq C\|a(t_1)(\partial_t v + v\cdot\nabla v)(t_1)\|_{L^2} \\
&\leq C\big(\chi_{[q\leq 2]}(q)\|(\Delta v - \nabla\pi_v)(t_1)\|_{\mathcal{B}_p^{\frac{3}{p}}} + \chi_{[q>2]}(q)\|\Delta v - \nabla\pi_v\|_{\mathcal{B}_p^{\frac{3}{p}-\frac{3}{2}+\frac{3}{q}}}\big) \\
&\leq C.
\end{aligned}
$$

As a consequence, we deduce from (5.20) that

$$\sup_{t\in[t_1,T^*)} \|\sqrt{\rho}w_t(t)\|_{L^2}^2 + \int_{t_1}^{t} \|\nabla w_t(t')\|_{L^2}^2 \, \mathrm{d}t' \le C. \tag{5.21}$$

At last, we also can recover the second space derivative estimate of w which is same the Lemma 5.3 in [3], and we omit the detail here. We thus finish the proof of this lemma. □

As a conclusion, we obtain the following theorem:

THEOREM 5.4. *Under the assumption of Theorem 1.2 with* $2 \le p < 6$, (1.4) *has a unique global solution* $(a, u, \nabla\pi)$ *satisfying* (1.8) *provided that* $\|u_0\|_{\mathcal{B}_p^{\frac{3}{p}-1}}$ *is sufficiently small.*

The proof of this theorem is same to [33], and we omit it here, and finally by combining this theorem with Theorem 4.1, we complete the proof of Theorem 1.2.

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