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SUPPRESSION OF BLOW UP BY MIXING IN AN AGGREGATION EQUATION WITH SUPERCRITICAL DISSIPATION*

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Dedicated to Professor Hsiao Ling on the occasion of her 80th birthday

Abstract. In this paper, we consider the Cauchy problem for an aggregation equation with supercritical dissipation and an additional mixing mechanism of advection by an incompressible flow, where the attractive kernel is a non-negative radial decreasing kernel with a Lipschitz point at the origin. Without advection, the solution of equation blows up in finite time. Under a suitable mixing condition on the advection, we show the global existence of the solution with large initial data. Firstly, we study enhanced dissipation effect of mixing mechanism of advection by a linear equation with fractional dissipation. The main idea of proof is based on the Gearhart-Prüss type theorem. Next, we establish the L^∞-criterion of solution and obtain the global L^∞ estimate. We give a new proof, which is based on a new observation for mixing mechanism and the RAGE theorem. Finally, the nonlinear maximum principle on tours is applied to get the L^∞ estimate of solution.

Key words. Aggregation equation, Mixing, Fractional dissipation, Suppression of blow up.

Mathematics Subject Classification. 35A01, 35B45, 35R11, 35Q92.

1. Introduction. We consider the following aggregation equation on tours $\mathbb{T}^d$ with fractional dissipation in the presence of an incompressible flow

$$\begin{cases} \partial_t u + Av\cdot\nabla u + \nabla\cdot(u\nabla K * u) + \nu(-\Delta)^{\frac{\alpha}{2}}u = 0, & t>0, x\in\mathbb{T}^d, \\ u(0,x) = u_0(x), & x\in\mathbb{T}^d. \end{cases} \tag{1.1}$$

Here $u(t,x)$ is a real valued function of t and x, $\mathbb{T}^d = [-\frac{1}{2},\frac{1}{2})^d$ is the periodic box with dimension $d\geq 2$, the quantity $v = v(x)$ is a divergence-free vector field which is an ambient flow, and A is a positive constant. The attractive kernel K is a periodic convolution kernel. The $\nu > 0$ and $0<\alpha<1$ are parameters controlling the strength of the dissipation term. The nonlocal operator $(-\Delta)^{\frac{\alpha}{2}}$ is known as the Laplacian of the order $\frac{\alpha}{2}$, which is defined by

$$(-\Delta)^{\frac{\alpha}{2}}\phi(x) = \mathcal{F}^{-1}(|\xi|^\alpha\hat{\phi}(\xi))(x), \quad x\in\mathbb{R}^d, \tag{1.2}$$

where $\mathcal{F}^{-1}$ is inverse Fourier transformation. The kernel K is a nonnegative radial decreasing kernel with a Lipschitz point at the origin. We consider

$$K(x) = e^{-|x|}, \quad x\in\mathbb{R}^d. \tag{1.3}$$

In this paper, we consider the tours $\mathbb{T}^d$, on which the definitions of $(-\Delta)^{\frac{\alpha}{2}}$ and the kernel K are different from those in (1.2) and (1.3). The fractional Laplacian operator needs a kernel representation. The details can be referred to Section 2. Similar to the discussion in [28], the nonlinear effect of equation (1.1) comes from the behavior of the attractive kernel K near the origin. We pose the following assumptions on the attractive kernel K

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(1) K is a periodic convolution kernel, which is smooth and converges to zero rapidly as $|x|$ increases away from the origin,
(2) $K = e^{-|x|}$ on $B_\varepsilon(0)$ for some $0 < \varepsilon \ll 1$.

Without advection, the equation (1.1) is an aggregation equation with fractional dissipation

$$\partial_t u + \nabla \cdot (u \nabla K * u) + \nu(-\Delta)^{\frac{\alpha}{2}} u = 0 \quad u(0,x) = u_0(x), \quad x \in \mathbb{R}^d. \tag{1.4}$$

There are three different ranges to the parameter α for the equation (1.4), namely, $0 \le \alpha < 1, \alpha = 1$ and $1 < \alpha \le 2$, known as the supercritical, critical and subcritical regimes respectively. Aggregation equations of the form (1.4) with more general kernels arise in many problems in biology, chemistry and population dynamics. For example, if $K = |x|^\beta, 1-d < \beta < 0, d \ge 2$, the equation (1.4) is a generalized Keller-Segel equation, which has been studied extensively and the solution can blow up in finite time for large initial data. One can refer to [7, 8, 9, 16, 27, 28, 32, 37, 38, 44, 45, 46]. If $K = e^{-|x|}, \nu > 0, d \ge 2$, Li and Rodrigo show that the solution of equation (1.4) blows up in finite time under the supercritical dissipation. For the critical case, the solution globally exists with small initial data, and for the subcritical case, there exists global solution for generalized initial data. See [34, 35, 36] for details. For more studies, one can also refer to [6, 33, 43].

As the kernel K is non-negative, decreasing, radial and with a Lipschitz point at the origin, we can rewrite the gradient of K as follows (see [34])

$$\nabla K = a\frac{x}{|x|} + S(x), \quad x \in \Omega, \quad a \neq 0. \tag{1.5}$$

If $\Omega = \mathbb{R}^d$, $K = e^{-|x|}$, then $S(x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is continuous at $x = 0$ with $S(0) = 0$ and $\nabla S(x) \in L^1(\mathbb{R}^d)$. For $\Omega = \mathbb{T}^d$, according to the assumptions of K, the properties of ∇K and $S(x)$ are similar to those in the case of $\mathbb{R}^d$. In $\mathbb{R}^d$, the ∇K scales as $\frac{x}{|x|}$ near the origin, so the equation (1.4) which is not a scaling invariant can be approximated by the homogeneous version

$$\partial_t u + \nabla \cdot (u\frac{x}{|x|} * u) + \nu(-\Delta)^{\frac{\alpha}{2}} u = 0, \quad x \in \mathbb{R}^d. \tag{1.6}$$

The solution of equation (1.6) is invariable under scaling

$$u_\lambda(t,x) = \lambda^{d+\alpha-1} u(\lambda^\alpha t, \lambda x). \tag{1.7}$$

Suppose that the critical space is L^{p_c}, in the sense that $\|u_\lambda(t,\cdot)\|_{L^{p_c}} = \|u(t,\cdot)\|_{L^{p_c}}$. One can get

$$p_c = \frac{d}{\alpha + d - 1}. \tag{1.8}$$

For $0 < \alpha < 1$, $p_c > 1$, the L^1 estimate of equation (1.6) is subcritical, and the solution blows up in finite time (see [34]).

Enhanced dissipation effect of mixing mechanism is an interesting phenomenon for the equations with dissipative and convective terms. It means that when convection $v \cdot \nabla$ is added to the equation, the dissipation effect will be enhanced and the solution of the equation has a faster decaying. For incompressible vector field v, there are two cases studied by many scholars recently. First, v is shear flow, we can refer to [2, 49], and some specific shear flows are considered to the linear Navier-Stokes equation. For example, Coutte flow (see [4]), Poiseuille flow (see [42]), Kolmogorov flow (see [39, 50, 51]). Secondly, Constantin, Kiselev, Ryzhik, and Zlatoš (see [13]) defined the relaxation enhancing flow, and proved that v is relaxation enhancing flow if and only if $v \cdot \nabla$ has no nontrivial $\dot{H}^1$ eigenfunction. Similarly Constantin, Hopf et.al (see [28]) defined the α-relaxation enhancing flow. The details are introduced in Section 2. Some other studies and examples of relaxation enhancing flow can be also referred to [22, 23, 31, 32].

Recently, the chemotaxis in fluid has been studied (see [12, 18, 19, 40, 41, 48, 52]). The possible effects result from the interaction of chemotactic and fluid transport process. Many people get interested in the suppression of blow up in the chemotactic model with mixing mechanism. Kiselev, Xu (see [32]), Hopf, Rodrigo (see [28]), and Shi, Wang (see [47]) obtained the global solution of Keller-Segel system by the mixing mechanism of relaxation enhancing flow. Bedrossian and He (see [3, 26]) showed that the enhanced dissipation effect and the global classical solution of Keller-Segel system with shear flow.

Our goal is to show that the blowup solution of (1.1) can be suppressed through the enhanced dissipation effect of relaxation enhancing flow. This question is motivated by the works of Kiselev et. al (see [32]), Hopf et. al (see [28]) and Shi et. al (see [47]). In this paper, we consider the equation (1.1) with supercritical dissipation ($0 < \alpha < 1$), the dimension is any $d \geq 2$ and we take $\nu = 1$ for the convenience of discussion. According to (1.7), in order to get the global classical solution of equation (1.1), we need to establish global $L^p(p > \frac{d}{d+\alpha-1})$ estimate of the solution. In fact, since $0 < \alpha < 1$, the L^2 estimate of solution is supercritical, and we can get global classical solution by the global L^2 estimate. However, in this paper, we consider the $L^p(p = \infty)$ estimate of solution for the convenience of discussion and to avoid the complex energy inequalities.

For the analysis of mixing mechanism to relaxation enhancing flow, the RAGE theorem (see Lemma 2.7) is an important technique, see [13, 28, 32, 47] for the details. Recently, Wei introduced a new method, which is based on the Gearhart-Prüss type theorem, the author considered the linear equation

$$\partial_t u - \Delta u + Av \cdot \nabla u = 0. \tag{1.9}$$

The main idea is to consider the resolvent estimate of operator $-\Delta + Av \cdot \nabla$ by contradiction, which is perfect and simple for analyzing the mixing mechanism of relaxation enhancing flow. The detail can be referred to [49] and Section 2. In this paper, we first consider linear equation with mixing mechanism and fractional dissipation, and get the enhanced dissipation effect by the same method as [49]. It can be seen as an extension of equation (1.9). For the L^2 estimate of solution to equation (1.1), we want to obtain the smallness by the Gearhart-Prüss type theorem, however there are some technical difficulties coming from nonlinear estimates. Based on the idea of contradiction in [49] and the RAGE theorem, we give a new proof, which is a new observation of the mixing mechanism and simplifies the analysis. The key idea is to define $\Phi(A)$ in (4.45), which describes the time and frequency of solution to equation (1.1), see Lemma 4.5 in Section 4 for more details. And we obtain the L^∞ estimate of the solution through nonlinear maximum principle on tours, which is introduced in [47]. The detail can be referred to Lemma 2.3 and appendix.

Our main result is

THEOREM 1.1. *Let* $0 < \alpha < 1, d \geq 2$, *for any initial data* $u_0(x) \geq 0, u_0(x) \in$

$H^3(\mathbb{T}^d)\cap L^\infty(\mathbb{T}^d)$, there exists a smooth incompressible flow v and a positive constant $A_0 = A(\alpha, u_0, d)$, such that for $A \geq A_0$, the unique solution $u(t,x)$ of equation (1.1) is global in time and

$$u(t,x) \in C(\mathbb{R}^+, H^3(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)).$$

Remark 1. *The smooth incompressible flow v is weakly mixing (see Definition 2.5).*

In the following, we briefly state our main ideas of the proof. Firstly, we establish the L^∞-criterion of solution to equation (1.1). Namely, the global classical solution of equation (1.1) exists if the L^∞ norm of solution is uniformly bounded. Next, we obtain the L^∞ estimate of the solution to equation (1.1). The local L^2, L^p and L^∞ estimates of the solution are obtained by energy estimate, where we need to choose $p > \frac{d}{\alpha}$, since we only consider $\|\nabla K\|_{L^1}, \|\Delta K\|_{L^1}$ is bounded and apply the nonlinear maximum principle (see Lemma 2.3). Based on the RAGE theorem and contradiction, the local L^2 estimate of the solution is small by enhanced dissipation effect. Combining with the local L^2 and L^∞ estimate of the solution, we deduce that the local $L^p(p > \frac{d}{a})$ norm of the solution is controlled by its initial data. Using the nonlinear maximum principle, the local L^∞ norm is estimated by the initial data. Repeating the above process, the local L^2, L^p and L^∞ estimate of the solution are extended to all time. Thus, we get the uniform L^∞ estimate.

This paper is organized as follows. In Section 2, we introduce the properties of the nonlocal operator and the functional space. The mixing mechanism of relaxation enhancing flow is also introduced in this section. In Section 3, we get the enhanced dissipation effect of relaxation enhancing flow by linear equation with mixing mechanism and fractional dissipation. In Section 4, we give the proof of Theorem 1.1 by a new technique. First, we establish the local existence and L^∞-criterion. Next, we get the L^∞ estimate of solution to equation (1.1). In Section 5, we give the proofs of the Gearhart-Prüss type theorem and nonlinear maximum principle on tours as appendix.

Throughout the paper, C stands for universal constant that may change from line to line.

2. Preliminaries. In the follows, we provide some auxiliary results and notations.

2.1. Nonlocal operator. The fractional Laplacian is a nonlocal operator and it has the following kernel representation on $\mathbb{T}^d$ (see [10, 11])

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = C_{\alpha,d} \sum_{k\in\mathbb{Z}^d} P.V. \int_{\mathbb{T}^d} \frac{f(x)-f(y)}{|x-y+k|^{d+\alpha}} dy, \tag{2.1}$$

where

$$C_{\alpha,d} = \frac{2^\alpha \Gamma(\frac{d+\alpha}{2})}{\pi^{\frac{d}{2}} |\Gamma(-\frac{\alpha}{2})|}.$$

The following results are important lemmas.

LEMMA 2.1 (Positivity Lemma, see [15, 29]). *Suppose* $0 \le \alpha \le 2, \Omega = \mathbb{R}^d, \mathbb{T}^d$ *and* $f, (-\Delta)^{\frac{\alpha}{2}} f \in L^p$, *where* $p \ge 2$. *Then*

$$\frac{2}{p}\int_\Omega ((-\Delta)^{\frac{\alpha}{4}}|f|^{\frac{p}{2}})^2 dx \le \int_\Omega |f|^{p-2} f(-\Delta)^{\frac{\alpha}{2}} f dx.$$

LEMMA 2.2 (see [1]). *Suppose* $0 < \alpha < 2, \Omega = \mathbb{R}^d, \mathbb{T}^d$ *and* $f \in \mathcal{S}(\Omega)$. *Then*

$$\int_\Omega (-\Delta)^{\frac{\alpha}{2}} f(x) dx = 0.$$

LEMMA 2.3 (Nonlinear maximum principle, see [47]). *Let* $f \in \mathcal{S}(\mathbb{T}^d)$ *and denote by* $\overline{x}$ *the point such that*

$$f(\overline{x}) = \max_{x\in\mathbb{T}^d} f(x),$$

and $f(\overline{x}) > 0$, *then we have the following*

$$(-\Delta)^{\frac{\alpha}{2}} f(\overline{x}) \ge C(\alpha, d, p)\frac{f(\overline{x})^{1+\frac{p\alpha}{d}}}{\|f\|_{L^p}^{\frac{p\alpha}{d}}}, \tag{2.2}$$

or

$$f(\overline{x}) \le C(d, p)\|f\|_{L^p}. \tag{2.3}$$

REMARK 2. *Nonlinear maximum principle is an important technique for getting the* L^∞ *estimate of solution to equation (1.1), the detail can be refered to Section 4. And the Lemma 2.3 has been proved in [47], we will also give the proof in the appendix.*

2.2. Functional spaces and inequalities. We write $L^p(\mathbb{T}^d)$ for the usual Lebesgue space

$$L^p(\mathbb{T}^d) = \left\{ f \textit{ measurable s.t. } \int_{\mathbb{T}^d} |f(x)|^p dx < \infty \right\},$$

the norm for the L^p space is denoted as $\|\cdot\|_{L^p}$, it means

$$\|f\|_{L^p} = \left(\int_{\mathbb{T}^d} |f|^p dx\right)^{\frac{1}{p}},$$

with natural adjustment when $p = \infty$. The homogeneous Sobolev norm $\|\cdot\|_{\dot{H}^s}$,

$$\|f\|_{\dot{H}^s}^2 = \|(-\Delta)^{\frac{s}{2}} f\|_{L^2}^2 = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |k|^{2s}|\hat{f}(k)|^2, \tag{2.4}$$

and the non-homogeneous Sobolev norm $\|\cdot\|_{H^s}$,

$$\|f\|_{H^s}^2 = \|f\|_{L^2}^2 + \|f\|_{\dot{H}^s}^2.$$

For some standard inequalities, one can refer to [5, 21, 24].

2.3. Enhanced dissipation and mixing mechanism. Given an incompressible vector field $v = v(x)$, which is Lipschitz in spatial variables. If we define the trajectories map by (see [13, 32, 47])

$$\frac{d}{dt}\Phi_t(x) = v(\Phi_t(x)), \quad \Phi_0(x) = x.$$

Then define a unitary operator $U = U^t$ acting on $L^2(\mathbb{T}^d)$ as follows

$$U^t f(x) = f(\Phi_t^{-1}(x)), \;\; f(x) \in L^2(\mathbb{T}^d).$$

We take

$$\mathcal{G} = \left\{ f \in L^2(\mathbb{T}^d) \Big| \int_{\mathbb{T}^d} f(x)dx = 0, \; f \neq 0 \right\},$$

and for a fixed $t > 0$, denote

$$U = U^t. \tag{2.5}$$

First, we introduce the definition of α-relaxation enhancing flow, the details can be referred to [28].

Definition 2.4. *Let $0 < \alpha < 2$, a divergence-free Lipschitz vector field $v = v(x)$ is called α-relaxation enhancing if the corresponding unitary operator U does not have any non-constant eigenfunctions in $H^{\frac{\alpha}{2}}(\mathbb{T}^d)$.*

In particular, we recall the definition of weakly mixing (see [13, 32]).

Definition 2.5. *The incompressible flow v is called weakly mixing, if $v = v(x)$ is smooth and the spectrum of the operator U is purely continuous on $\mathcal{G}$.*

Remark 3. *In fact, the definition in [13] is 2-relaxation enhancing flow, and the weakly mixing is α-relaxation enhancing flow for any $0 < \alpha \leq 2$. Some examples can be referred to [13, 22, 23, 28, 32].*

Let us denote $\omega(t, x)$ is the unique solution of the equation

$$\partial_t \omega + Av \cdot \nabla\omega = 0, \quad \omega(0, x) = u_0(x), \tag{2.6}$$

there is the following lemma (see [28, 32]).

Lemma 2.6. *Suppose that $0 < \alpha < 2$, $v(x)$ is a smooth divergence-free vector field. Let $\omega(t, x)$ be the solution of (2.6). Then for every $t \geq 0$ and $\rho_0 \in \dot{H}^{\frac{\alpha}{2}}$, we have*

$$\|\omega(t,\cdot)\|_{\dot{H}^{\frac{\alpha}{2}}} \leq F(At)\|u_0\|_{\dot{H}^{\frac{\alpha}{2}}},$$

where

$$F(t) = \exp\left(\int_0^t D(v)ds\right),$$

and

$$D(v) \leq C\|(-\Delta)^{\frac{2\alpha+d+2}{4}} v\|_{L^2}.$$

We denote by $0 \le \lambda_1^{\frac{\alpha}{2}} \le \lambda_2^{\frac{\alpha}{2}} \le \cdots \le \lambda_n^{\frac{\alpha}{2}} \le \cdots$ the eigenvalues of $(-\Delta)^{\frac{\alpha}{2}}$ and by $e_1, e_2, \cdots, e_n, \cdots$ the corresponding orthogonal eigenvectors on $\mathbb{T}^d$. Let us also denote by P_N the orthogonal projection on the subspace spanned by the first N eigenvectors $e_1, e_2, \cdots, e_N$ and

$$S = \{\phi \in L^2 \big| \|\phi\|_{L^2} = 1\}. \tag{2.7}$$

The following lemma is an extension of the well-known RAGE theorem (see [13, 17, 32]).

LEMMA 2.7. *Let U^t be a unitary operator with purely continuous spectrum defined on $L^2(\mathbb{T}^d)$. Let $\mathcal{K} \subset S$ be a compact set. Then for every N and $\sigma > 0$, there exists $T_c = T(N, \sigma, \mathcal{K}, U^t)$ such that for all $T \ge T_c$ and every $\phi \in \mathcal{K}$, we have*

$$\frac{1}{T} \int_0^T \|P_N U^t \phi\|_{L^2}^2 dt \le \sigma.$$

Next, we introduce enhanced dissipation effect of relaxation enhancing flow via resolvent estimate (see [49]). Let X be a Hilbert space, we denote by $\|\cdot\|$ the norm and by $\langle , \rangle$ the inner product. Let H be a linear operator in X with the domain $D(H)$, it is defined as follows (see [30, 49])

DEFINITION 2.8. *A closed operator H in a Hilbert space X is called m-accretive if the left open half-plane is contained in the resolvent set $\rho(H)$ with*

$$(H + \lambda)^{-1} \in \mathcal{B}(X), \quad \|(H + \lambda)^{-1}\| \le (Re\lambda)^{-1}, \quad Re\lambda > 0,$$

where $\mathcal{B}(X)$ is the set of bounded linear operators on X.

REMARK 4. *An m-accretive operator H is accretive and densely defined (see [30]), namely, $D(H)$ is dense in X and $Re\langle Hf, f\rangle \ge 0$ for $f \in D(H)$.*

If $-H$ is a generator of a semigroup e^{-tH}, we denote

$$\Psi(H) = \inf\{\|(H - i\lambda)f\| : f \in D(H), \lambda \in \mathbb{R}, \|f\| = 1\}. \tag{2.8}$$

The following result is the Gearchart-Prüss type theorem for accretive operators (see [49]).

LEMMA 2.9. *Let H be an m-accretive operator in a Hilbert space X. Then for any $t \ge 0$, we have*

$$\|e^{-tH}\| \le e^{-t\Psi(H) + \frac{\pi}{2}}.$$

REMARK 5. *For the convenience of reading, the proof of Lemma 2.9 is introduced as an appendix in Section 5. We can also see [20, 49] for more details.*

Consider the operator

$$H = H_A = -\Delta + Av \cdot \nabla, \tag{2.9}$$

and denote $\Psi_1(A) = \Psi(H)$, if v is relaxation enhancing flow, then

$$\lim_{A \to +\infty} \Psi_1(A) = +\infty,$$

the details can be referred to [49].

3. Enhanced dissipation effect of α-relaxation enhancing flow. Consider linear equation with mixing mechanism and fractional dissipation

$$\partial_t u + (-\Delta)^{\frac{\alpha}{2}} u + Av \cdot \nabla u = 0, \quad u(0,x) = u_0(x), \quad t \geq 0, \quad x \in \mathbb{T}^d, \tag{3.1}$$

where $v = v(x)$ is α-relaxation enhancing flow . We denote

$$H = H_A = (-\Delta)^{\frac{\alpha}{2}} + Av \cdot \nabla, \quad 0 < \alpha < 2, \tag{3.2}$$

and

$$D(H) = H^2 \cap \mathcal{G}.$$

And define

$$\Psi_2(A) = \Psi(H). \tag{3.3}$$

Remark 6. *According to the Definition 2.8, the H in (3.2) is a m-accretive operator.*

First, let us state the enhanced dissipation effect of α-relaxation enhancing flow.

Theorem 3.1. *Let $0 < \alpha < 2$, suppose that $u(t,x)$ is solution of equation (3.1) with $u_0(x) \in D(H)$, v is a α-relaxation enhancing flow. For any $\epsilon_0 > 0, t_0 > 0$, there exists a $A_0 = A(\alpha, u_0, d, \epsilon_0, t_0)$, such that for $A \geq A_0$, we have*

$$\|u(t_0, \cdot)\|_{L^2} \leq \epsilon_0 \|u_0\|_{L^2}.$$

Remark 7. *The enhanced dissipation effect of α-relaxation enhancing flow has been studied through the RAGE theorem (see [13, 28]). In this paper, we studied it by using the method of the resolvent estimate of operator H in (3.2), we can also refer to [49].*

Next, we give an important lemma, it is as follows

Lemma 3.2. *Let $0 < \alpha < 2$, v is α-relaxation enhancing flow, then*

$$\lim_{A \to +\infty} \Psi_2(A) = +\infty,$$

where $\Psi_2(A)$ is defined in (3.3).

Proof. We only claim that $\liminf_{A \to +\infty} \Psi_2(A) < +\infty$ implies that U has a nonzero eigenfunction in $H^{\frac{\alpha}{2}}(\mathbb{T}^d) \cap \mathcal{G}$, where U is defined in (2.5). In this case, there exists a sequence $\{A_m\}_{m=1}^{\infty}$ and constant $\delta_0 \in \mathbb{R}^+$, such that $\lim_{m \to +\infty} A_m = +\infty$ and

$$\Psi_2(A_m) < \delta_0, \quad m = 1, 2, \cdots . \tag{3.4}$$

Combining (2.8), (3.3) and (3.4), we imply that for any fixed m, there exists $\lambda_m \in \mathbb{R}, f_m \in D(H)$, such that $\|f_m\|_{L^2} = 1$ and

$$\|(H_{A_m} - i\lambda_m) f_m\|_{L^2} < \delta_0. \tag{3.5}$$

According to Cauchy-Schwartz inequality and (3.5), one has

$$\|\Lambda^{\frac{\alpha}{2}} f_m\|_{L^2}^2 = Re\langle f_m, g_m\rangle \le \|f_m\|_{L^2}\|g_m\|_{L^2} < \delta_0, \tag{3.6}$$

where

$$g_m = (H_{A_m} - i\lambda_m) f_m. \tag{3.7}$$

Thus there exists $f_0 \in H^{\frac{\alpha}{2}}(\mathbb{T}^d)$ and a subsequence of $\{f_m\}_{m=1}^{\infty}$, still denote by $\{f_m\}_{m=1}^{\infty}$, such that

$$f_m \to f_0 \quad in \;\; L^2(\mathbb{T}^d) \tag{3.8}$$

is strongly convergence and one has

$$\|f_0\|_{L^2} = 1, \quad f_0 \in \mathcal{G}.$$

According to the definition of (3.7) and the property of fractional Laplacian, we know that for any $f \in H^1(\mathbb{T}^d)$, one has

$$\langle g_m, f\rangle = \langle \Lambda^{\frac{\alpha}{2}} f_m, \Lambda^{\frac{\alpha}{2}} f\rangle + A_m\langle v\cdot\nabla f_m, f\rangle - i\lambda_m\langle f_m, f\rangle. \tag{3.9}$$

As $f \in H^{\frac{\alpha}{2}}(\mathbb{T}^d)$ and $\lim\limits_{m\to\infty} A_m = \infty$, we deduce by (3.5), (3.8) and (3.9) that

$$\langle v\cdot\nabla f_m, f\rangle - i\frac{\lambda_n}{A_m}\langle f_m, f\rangle = \frac{\langle g_m, f\rangle - \langle \Lambda^{\frac{\alpha}{2}} f_m, \Lambda^{\frac{\alpha}{2}} f\rangle}{A_m} \to 0, \quad m\to\infty, \tag{3.10}$$

where we use the

$$\begin{aligned} |\langle g_m, f\rangle - \langle \Lambda^{\frac{\alpha}{2}} f_m, \Lambda^{\frac{\alpha}{2}} f\rangle| &\le \|g_m\|_{L^2}\|f\|_{L^2} + \|\Lambda^{\frac{\alpha}{2}} f_m\|_{L^2}\|\Lambda^{\frac{\alpha}{2}} f\|_{L^2} \\ &\le \delta_0\|f\|_{L^2} + \delta_0^{\frac{1}{2}}\|\Lambda^{\frac{\alpha}{2}} f\|_{L^2}. \end{aligned}$$

As v is divergence-free and (3.8), one has

$$\langle v\cdot\nabla f_m, f\rangle = -\langle f_m, v\cdot\nabla f\rangle \to -\langle f_0, v\cdot\nabla f\rangle = \langle v\cdot\nabla f_0, f\rangle, \quad m\to\infty, \tag{3.11}$$

and

$$\langle f_m, f\rangle \to \langle f_0, f\rangle, \quad m\to\infty.$$

Notes that the (3.11) indicates that $v\cdot\nabla f_0$ is well defined in the sense of inner product. Combining (3.10) and (3.11), one has

$$\lim_{m\to+\infty} i\frac{\lambda_m}{A_m}\langle f_m, f\rangle = \langle v\cdot\nabla f_0, f\rangle. \tag{3.12}$$

Since $H^{\frac{\alpha}{2}}(\mathbb{T}^d)$ is dense in $H^1(\mathbb{T}^d)$, then there exist $f_1 \in H^1(\mathbb{T}^d)$, such that

$$|\langle f_0 - f_1, f_0\rangle| = |\langle f_0, f_0\rangle - \langle f_0, f_1\rangle| \le \frac{1}{2},$$

without loss of generality, we assume that $\langle f_0, f_1\rangle = \frac{1}{2}$. If we take $f = f_1$, one has

$$\langle f_m, f_1\rangle \to \langle f_0, f_1\rangle = \langle f_0, f_1\rangle = \frac{1}{2} \neq 0, \quad m\to\infty.$$

Therefore we deduce by (3.12) that

$$\frac{1}{2} \times i \frac{\lambda_m}{A_m} \to \langle v \cdot \nabla f_0, f_1 \rangle \doteq i\lambda.$$

Then for every $f \in H^1(\mathbb{T}^d)$, one has

$$i2\lambda \langle f_0, f \rangle = \lim_{m \to +\infty} i \frac{\lambda_m}{A_m} \langle f_m, f \rangle = \langle v \cdot \nabla f_0, f \rangle.$$

If $\lambda = 0$, one can know that f_0 is a constant, this is contradictory to $f_0 \in \mathcal{G}$. Then

$$v \cdot \nabla f_0 = i2\lambda f_0, \quad f_0 \neq 0.$$

Thus f_0 is a nonzero eigenfunction of operator U in $H^{\frac{\alpha}{2}}(\mathbb{T}^d) \cap \mathcal{G}$, this contradicts the definition that v is α-relaxation enhancing flow. So we have

$$\lim_{A \to +\infty} \Psi_2(A) = +\infty.$$

This completes the proof of Lemma 3.2. □

REMARK 8. *The proof idea of Lemma 3.2 comes from [49].*

The proof of Theorem 3.1. Combining Lemma 2.9 and Lemma 3.2, if A is large enough, then

$$\|u(t_0, \cdot)\|_{L^2} \leq Ce^{-t_0 \Psi_2(A)} \|u_0\|_{L^2} \leq \epsilon_0 \|u_0\|_{L^2}.$$

This completes the proof of Theorem 3.1. □

4. The proof of Theorem 1.1 . In this section, we consider an aggregation equation with supercritical dissipation ($0 < \alpha < 1$) and weakly mixing. The existence of global classical solution to equation (1.1) with large initial data is proved.

4.1. Local existence and continuation criterion. We state our results starting with an existence of local solution and continuation. In this case we have the following

THEOREM 4.1. *Let $0 < \alpha < 1, d \geq 2$. Given initial data $u_0 \geq 0, u_0 \in H^3(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$, there exists a time $T^* = T(u_0) > 0$ such that the non-negative solution of equation (1.1)*

$$u(t, x) \in C([0, T^*], H^3(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)).$$

Moreover, if for a given T, the solution verifies the following bound

$$\lim_{t \to T} \sup_{0 \leq \tau \leq t} \|u(\tau, \cdot)\|_{L^\infty} < \infty,$$

then it may be extended up to time $T + \delta$ for small enough $\delta > 0$. Furthermore, if $u_0 \in L^1(\mathbb{T}^d)$, then the L^1 norm of the solution of equation (1.1) is preserved for all time, namely $\|u\|_{L^1} = \|u_0\|_{L^1}$.

Proof. The proofs of local existence and the $u \geq 0$ are standard method, the L^1 norm conservation is obviously. In this paper, we only prove the continuous criterion, namely, we derive a priori bound on the higher order derivatives in terms of L^∞ norm of the solution. Assume $u(t,x)$ is the solution of equation (1.1) and $\|u(t,\cdot)\|_{L^\infty}$ is bounded. Let us multiply both sides of (1.1) by $(-\Delta)^3 u$ and integrate over $\mathbb{T}^d$, to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{\dot H^3}^2 + A\int_{\mathbb{T}^d} v\cdot\nabla u(-\Delta)^3 u dx \\ + \int_{\mathbb{T}^d}(-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^3 u dx + \int_{\mathbb{T}^d}\nabla\cdot(u\nabla K*u)(-\Delta)^3 u dx = 0.\end{aligned} \tag{4.1}$$

For the second term of the left-hand side of (4.1), we use step-by-step integration and the incompressibility of v to obtain

$$\left|A\int_{\mathbb{T}^d} v\cdot\nabla u(-\Delta)^3 u dx\right| \le CA\|v\|_{C^3}\|u\|_{\dot H^3}^2. \tag{4.2}$$

And the third term of the left-hand side of (4.1) is equal to

$$\int_{\mathbb{T}^d}(-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^3 u dx = \|u\|_{\dot H^{3+\frac{\alpha}{2}}}^2. \tag{4.3}$$

For the fourth term of the left-hand side of (4.1), we split it into two pieces

$$\begin{aligned}\int_{\mathbb{T}^d}\nabla\cdot(u\nabla K*u)(-\Delta)^3 u dx = \int_{\mathbb{T}^d}\nabla u\cdot\nabla K*u(-\Delta)^3 u dx \\ + \int_{\mathbb{T}^d} u\Delta K*u(-\Delta)^3 u dx.\end{aligned} \tag{4.4}$$

Integrating by parts, the first term of the right-hand side of (4.4) is expressed as follows

$$\int_{\mathbb{T}^d}\nabla u\cdot\nabla K*u(-\Delta)^3 u dx \sim \sum_{l=0}^{3}\int_{\mathbb{T}^d} D^l(\nabla u)\cdot D^{3-l}(\nabla K*u)D^3 u dx, \tag{4.5}$$

where $l=0,1,2,3$ and D denotes any partial derivative. According to the definition of K in Section 1, we know that $\nabla K\in L^1(\mathbb{T}^d), \Delta K\in L^1(\mathbb{T}^d)$, then there exists constant $C_0>0$, one has

$$\|\nabla K\|_{L^1}\le C_0, \quad \|\Delta K\|_{L^1}\le C_0. \tag{4.6}$$

When $l=0,1$, for $1\le q_1,q_2<\infty$, we deduce by Young's, Hölder's inequality and Gagliardo-Nirenberg inequality that

$$\begin{aligned}\sum_{l=0}^{1}\int_{\mathbb{T}^d} D^l(\nabla u)\cdot D^{3-l}(\nabla K*u)D^3 u dx &\le C\|Du\|_{L^{p_1}}\|D^2 u\|_{L^q}\|u\|_{\dot H^3} \\ &\le C\|u\|_{L^{q_1}}^{1-\theta_1}\|u\|_{L^{q_2}}^{1-\theta_2}\|u\|_{\dot H^3}^{1+\theta_1+\theta_2} \\ &\le C\|u\|_{\dot H^3}^{1+\theta_1+\theta_2},\end{aligned}$$

where

$$\frac{1}{p_1}+\frac{1}{q}=\frac{1}{2}, \quad \theta_1=\frac{\frac{1}{p_1}-\frac{1}{d}-\frac{1}{q_1}}{\frac{1}{2}-\frac{3}{d}-\frac{1}{q_1}}, \quad \theta_2=\frac{\frac{1}{q}-\frac{2}{d}-\frac{1}{q_2}}{\frac{1}{2}-\frac{3}{d}-\frac{1}{q_2}}.$$

Then for $l = 2$, we get

$$\int_{\mathbb{T}^d} D^2(\nabla u)\cdot D(\nabla K * u)D^3 u dx \le \|\Delta K * u\|_{L^\infty}\|u\|_{\dot H^3}^2 \le C_0\|u\|_{L^\infty}\|u\|_{\dot H^3}^2,$$

and when $l = 3$, one has

$$\int_{\mathbb{T}^d} D^3(\nabla u)\cdot(\nabla K * u)D^3 u dx = -\frac{1}{2}\int_{\mathbb{T}^d}(D^3 u)^2\Delta K * u dx.$$

Therefore, we have

$$\int_{\mathbb{T}^d} \nabla u\cdot\nabla K * u(-\Delta)^3 u dx \le C(\|u\|_{\dot H^3}^{1+\theta_1+\theta_2} + \|u\|_{L^\infty}\|u\|_{\dot H^3}^2). \tag{4.7}$$

For the second term of the right-hand side of (4.1), similar to (4.5), one has

$$\int_{\mathbb{T}^d} u\Delta K * u(-\Delta)^3 u dx \sim \sum_{l=0}^{3}\int_{\mathbb{T}^d} D^l u D^{3-l}(\Delta K * u)D^3 u dx. \tag{4.8}$$

When $l = 1, 2$, the similar with above, to obtain

$$\sum_{l=1}^{2}\int_{\mathbb{T}^d} D^l u D^{3-l}(\Delta K * u)D^3 u dx \le C\|u\|_{\dot H^3}^{1+\theta_1+\theta_2},$$

and $l = 0$, for $2 < p_2 < \infty$, we deduce by Hölder's inequality and Young's inequality that

$$\int_{\mathbb{T}^d} u D^3(\Delta K * u)D^3 u dx \le C\|u\|_{\dot H^3}^2.$$

When $l = 3$, one has

$$\int_{\mathbb{T}^d} D^3 u(\Delta K * u)D^3 u dx \le \|\Delta K * u\|_{L^\infty}\|u\|_{\dot H^3}^2 \le C\|u\|_{\dot H^3}^2.$$

Then we have

$$\int_{\mathbb{T}^d} u\Delta K * u(-\Delta)^3 u dx \le C(\|u\|_{\dot H^3}^{1+\theta_1+\theta_2} + \|u\|_{\dot H^3}^2). \tag{4.9}$$

According to (4.7) and (4.9), to obtain

$$\int_{\mathbb{T}^d} \nabla\cdot(u\nabla K * u)(-\Delta)^3 u dx \le C(\|u\|_{\dot H^3}^{1+\theta_1+\theta_2} + \|u\|_{\dot H^3}^2). \tag{4.10}$$

Combining (4.1), (4.2), (4.3) and (4.10), we have

$$\frac{d}{dt}\|u\|_{\dot H^3}^2 \le -2\|u\|_{\dot H^{3+\frac{\alpha}{2}}}^2 + C(A\|v\|_{C^3}+1)\|u\|_{\dot H^3}^2 + C\|u\|_{\dot H^3}^{1+\theta_1+\theta_2}. \tag{4.11}$$

For any $1 \le p_2 < \infty$, as $\|u\|_{L^{p_1}}$ is bounded, then by Gagliardo-Nirenberg inequality, one has

$$-\|u\|_{\dot H^{3+\frac{\alpha}{2}}}^2 \le -C_4^{-1}\|u\|_{\dot H^3}^\gamma \le -C\|u\|_{\dot H^3}^\gamma, \tag{4.12}$$

where

$$\gamma = \frac{\frac{4d}{p_2} + 12 - 2d + 2\alpha}{\frac{2d}{p_2} + 6 - d}.$$

Combining (4.11) and (4.12), we have

$$\frac{d}{dt}\|u\|_{\dot{H}^3}^2 \le -C\|u\|_{\dot{H}^3}^{\gamma} + C(A\|v\|_{C^3} + 1)\|u\|_{\dot{H}^3}^2 + C\|u\|_{\dot{H}^3}^{1+\theta_1+\theta_2}, \tag{4.13}$$

and choose p_2, such that

$$\gamma > \max\{2, 1 + \theta_1 + \theta_2\}.$$

By the differential inequality (4.13), then the conclusion can easily be deduced. This completes the proof of Theorem 4.1. □

Remark 9. *In fact, there exists a constant $C = C(A, t)$ such that $\|u\|_{\dot{H}^3} \le C$ if A is large enough.*

4.2. Local estimates of the solution to equation (1.1). We establish the local $L^2, L^p(p > \frac{d}{\alpha})$ and L^∞ estimates of the solution to equation (1.1). A useful lemma is as follows

Lemma 4.2. *Let $0 < \alpha < 1, d \ge 2$, $u(t,x)$ is the local solution of equation (1.1) with initial data $u_0(x) \ge 0$. If $u \in L^p(\mathbb{T}^d)$, $\quad 1 \le p < \infty$, and denote*

$$\widetilde{u}(t) = u(t, \overline{x}_t) = \max_{x\in\mathbb{T}^d} u(t,x). \tag{4.14}$$

Then we have

$$\widetilde{u}(t) \le C(d,p)\|u\|_{L^p}, \tag{4.15}$$

or

$$\frac{d}{dt}\widetilde{u} \le C_0\widetilde{u}^2 - C(\alpha,d,p)\frac{\widetilde{u}^{1+\frac{p\alpha}{d}}}{\|u\|_{L^p}^{\frac{p\alpha}{d}}}. \tag{4.16}$$

Proof. For any fixed $t \ge 0$, using the vanishing of a derivative at the point of maximum, we see that

$$\partial_t u(t, \overline{x}_t) = \frac{d}{dt}\widetilde{u}(t), \quad Av \cdot \nabla u(t, \overline{x}_t) = 0, \tag{4.17}$$

and

$$\nabla \cdot (u\nabla K * u)(t, \overline{x}_t) = \nabla u \cdot \nabla K * u(t, \overline{x}_t) + u\Delta K * u(t, \overline{x}_t). \tag{4.18}$$

For the first term of the right-hand side in (4.18), one has

$$\nabla u \cdot \nabla K * u(t, \overline{x}_t) = 0, \tag{4.19}$$

and for the second term of the right-hand side of (4.18), we deduce by Young's inequality that

$$\begin{aligned} u\Delta K * u(t,\overline{x}_t) &\le \|u\Delta K * u\|_{L^\infty} \\ &\le \|u\|_{L^\infty}\|\Delta K * u\|_{L^\infty} \\ &\le \|u\|_{L^\infty}^2\|\Delta K\|_{L^1} = \widetilde{u}^2\|\Delta K\|_{L^1}. \end{aligned} \tag{4.20}$$

Thus, combining (1.1), (4.6), (4.17), (4.18), (4.19) and (4.20), we implies that the evolution of $\widetilde{u}$ follows

$$\frac{d}{dt}\widetilde{u} + (-\Delta)^{\frac{\alpha}{2}}\widetilde{u} - C_0\widetilde{u}^2 \le 0. \tag{4.21}$$

According to Lemma 2.3, one has

$$\widetilde{u}(t) \le C(d,p)\|u\|_{L^p},$$

if not, we obtain

$$\frac{d}{dt}\widetilde{u} \le C_0\widetilde{u}^2 - C(\alpha,d,p)\frac{\widetilde{u}^{1+\frac{p\alpha}{d}}}{\|u\|_{L^p}^{\frac{p\alpha}{d}}}.$$

This completes the proof of Lemma 4.2. □

If the initial data u_0 satisfies

$$u_0 \in H^3(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d), u_0 \ge 0. \tag{4.22}$$

Without loss generality, suppose that

$$\|u_0\|_{L^2} \le B_0, \quad \|u_0 - \overline{u}\|_{L^p} \le D_0, \quad \|u_0\|_{L^\infty} \le C_\infty, \tag{4.23}$$

where

$$\overline{u} = \frac{1}{|\mathbb{T}^d|}\int_{\mathbb{T}^d} u_0(x)dx, \quad p > \frac{d}{\alpha}, \quad C_\infty \ge 2C(d,p)(D_0 + \overline{u}), \tag{4.24}$$

and $C(d,p)$ is defined in Lemma 2.3. And we denote

$$B_1 = \min\left\{(B_0^2 - \overline{u}^2)^{\frac{1}{2}}, \left(\frac{D_0}{(2C_\infty + \overline{u})^{1-\frac{2}{p}}}\right)^{\frac{p}{2}}\right\}. \tag{4.25}$$

Remark 10. *According to the L^1 norm of solution to equation (1.1) is conservation, then $B_0^2 - \overline{u}^2 \ge 0$.*

Firstly, we establish the local estimate of solution to equation (1.1).

Lemma 4.3. *Let $0 < \alpha < 1, d \ge 2$, $u(t,x)$ is the local solution of equation (1.1) with initial data $u_0(x)$. Suppose that $u_0(x)$ satisfies (4.22)-(4.24). Then there exists a time $\tau_1 > 0$, for any $0 \le t \le \tau_1$, such that*

$$\|u(t,\cdot) - \overline{u}\|_{L^2} \le 2(B_0^2 - \overline{u}^2)^{\frac{1}{2}}, \quad \|u(t,\cdot)\|_{L^p} \le 2(D_0 + \overline{u}), \qquad \|u(t,\cdot)\|_{L^\infty} \le 2C_\infty.$$

Proof. Combining (4.21) and $(-\Delta)^{\frac{\alpha}{2}}\widetilde{u} \geq 0$, one has

$$\frac{d}{dt}\widetilde{u} \leq C_0 \widetilde{u}^2. \tag{4.26}$$

Define

$$\tau_0 = \min\left\{\frac{1}{2C_0 C_\infty}, T^*\right\},$$

where T^* is defined in Theorem 4.1. As $\|u_0\|_{L^\infty} \leq C_\infty$, by solving (4.26), then for any $0 \leq t \leq \tau_0$, we get

$$\|u(t,\cdot)\|_{L^\infty} \leq 2C_\infty, \quad 0 \leq t \leq \tau_0. \tag{4.27}$$

Let us multiply both sides of (1.1) by $|u|^{p-2}u$ and integrate over $\mathbb{T}^d$, to obtain

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\|u\|_{L^p}^p &+ A\int_{\mathbb{T}^d} v\cdot\nabla u|u|^{p-2}u dx \\ &+ \int_{\mathbb{T}^d}(-\Delta)^{\frac{\alpha}{2}}u|u|^{p-2}udx + \int_{\mathbb{T}^d}\nabla(u\nabla K * u)|u|^{p-2}udx = 0. \end{aligned} \tag{4.28}$$

When $0 \leq t \leq \tau_0$, the fourth term of the left-hand side of (4.28) can be estimated as

$$\begin{aligned} \int_{\mathbb{T}^d}\nabla(u\nabla K * u)|u|^{p-2}\rho dx &\leq \frac{p-1}{p}\|\Delta K * u\|_{L^\infty}\|u\|_{L^p}^p \\ &\leq \frac{2C_0C_\infty(p-1)}{p}\|u\|_{L^p}^p. \end{aligned} \tag{4.29}$$

Thus, we deduce by the standard energy estimate, (4.28) and (4.29) that

$$\frac{d}{dt}\|u\|_{L^p}^p \leq -2\||u|^{\frac{p}{2}}\|_{\dot{H}^{\frac{\alpha}{2}}}^2 + 2C_0C_\infty(p-1)\|u\|_{L^p}^p. \tag{4.30}$$

Let us multiply both sides of (1.1) by $u-\overline{u}$ and integrate over $\mathbb{T}^d$, to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u-\overline{u}\|_{L^2}^2 &+ A\int_{\mathbb{T}^d} v\cdot\nabla u(u-\overline{u})dx \\ &+ \int_{\mathbb{T}^d}(-\Delta)^{\frac{\alpha}{2}}u(u-\overline{u})dx + \int_{\mathbb{T}^d}\nabla(u\nabla K * u)(u-\overline{u})dx = 0. \end{aligned} \tag{4.31}$$

For any $0 \leq t \leq \tau_0$, the fourth term of the left-hand side of (4.31) can be estimated as

$$\begin{aligned} &\int_{\mathbb{T}^d}\nabla(u\nabla K * u)(u-\overline{u})dx \\ =& -\frac{1}{2}\int_{\mathbb{T}^d}(u-\overline{u})^2\Delta K * udx + \int_{\mathbb{T}^d}u\Delta K*(u-\overline{u})(u-\overline{u})dx \\ \leq& \frac{1}{2}\|\Delta K\|_{L^1}\|u\|_{L^\infty}\|u-\overline{u}\|_{L^2}^2 + \|\Delta K\|_{L^1}\|u\|_{L^\infty}\|u-\overline{u}\|_{L^2}^2 \\ \leq& 3C_0C_\infty\|u-\overline{u}\|_{L^2}^2. \end{aligned} \tag{4.32}$$

Then

$$\frac{d}{dt}\|u-\overline{u}\|_{L^2}^2 \leq -2\|u\|_{\dot{H}^{\frac{\alpha}{2}}}^2 + 6C_0C_\infty\|u-\overline{u}\|_{L^2}^2. \tag{4.33}$$

If denote

$$\tau_1 = \min\left\{\tau_0, \frac{p\ln 2}{2C_0C_\infty(p-1)}, \frac{\ln 2}{3C_0C_\infty}\right\}, \tag{4.34}$$

then for any $0 < t \le \tau_1$, we deduce by (4.30) and (4.33) that

$$\|u(t,\cdot) - \overline{u}\|_{L^2} \le 2(B_0^2 - \overline{u}^2)^{\frac{1}{2}}, \quad \|u(t,\cdot)\|_{L^p} \le 2(D_0 + \overline{u}).$$

According to (4.27) and the definition of τ_1, for any $0 \le t \le \tau_1$, we have

$$\|u(t,\cdot)\|_{L^\infty} \le 2C_\infty, \quad 0 \le t \le \tau_1.$$

This completes the proof of Lemma 4.3. □

Next, an approximation lemma is as follows.

LEMMA 4.4. *Let $0 < \alpha < 1, d \ge 2$, the vector field $v(x)$ is smooth incompressible flow. Let $u(t,x), \omega(t,x)$ be the solutions of equation (1.1) and (2.6) respectively with $u_0 \in H^3(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d), u_0 \ge 0$. Then for every $t \in [0,T]$, we have*

$$\begin{aligned}\frac{d}{dt}\|u-\omega\|_{L^2}^2 \le& -\|u\|_{\dot H^{\frac{\alpha}{2}}}^2 + F(At)^2\|u_0\|_{\dot H^{\frac{\alpha}{2}}}^2 + C_0\|u\|_{L^\infty}\|u\|_{L^2}^2 \\ &+ 2C_0(\|\nabla u\|_{L^2}\|u\|_{L^2} + \|u\|_{L^2}^2)\|u_0\|_{L^\infty}.\end{aligned}$$

where $F(t)$ be defined in Lemma 2.6, and $F(t) \in L^\infty_{loc}[0,\infty)$.

Proof. By (1.1) and (2.6), to obtain the equation

$$\partial_t(u-\omega) + Av\cdot\nabla(u-\omega) + (-\Delta)^{\frac{\alpha}{2}}u + \nabla\cdot(u\nabla K * u) = 0. \tag{4.35}$$

Let us multiply both sides of (4.35) by $u-\omega$ and integrate over $\mathbb{T}^d$, then

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u-\omega\|_{L^2}^2 + A\int_{\mathbb{T}^d} v\cdot\nabla(u-\omega)(u-\omega)dx \\ &\quad + \int_{\mathbb{T}^d}(-\Delta)^{\frac{\alpha}{2}}u(u-\omega)dx + \int_{\mathbb{T}^d}\nabla\cdot(u\nabla K * u)(u-\omega)dx = 0.\end{aligned} \tag{4.36}$$

As the incompressibility of v, the second term of the left-hand side of (4.36) is

$$A\int_{\mathbb{T}^d} v\cdot\nabla(u-\omega)(u-\omega)dx = 0. \tag{4.37}$$

The third term of the left-hand side of (4.36) can be estimated as

$$\begin{aligned}\int_{\mathbb{T}^d}(-\Delta)^{\frac{\alpha}{2}}u(u-\omega)dx &= \int_{\mathbb{T}^d}(-\Delta)^{\frac{\alpha}{2}}uudx - \int_{\mathbb{T}^d}(-\Delta)^{\frac{\alpha}{2}}u\omega dx \\ &= \|u\|_{\dot H^{\frac{\alpha}{2}}}^2 - \int_{\mathbb{T}^d}(-\Delta)^{\frac{\alpha}{4}}u(-\Delta)^{\frac{\alpha}{4}}\omega dx,\end{aligned} \tag{4.38}$$

and by Hölder's inequality, to obtain

$$\int_{\mathbb{T}^d}(-\Delta)^{\frac{\alpha}{2}}u\omega dx \le \|u\|_{\dot H^{\frac{\alpha}{2}}}\|\omega\|_{\dot H^{\frac{\alpha}{2}}} \le \frac{1}{2}\|u\|_{\dot H^{\frac{\alpha}{2}}}^2 + \frac{1}{2}\|\omega\|_{\dot H^{\frac{\alpha}{2}}}^2. \tag{4.39}$$

Then we deduce by (4.38) and (4.39) that

$$\int_{\mathbb{T}^d} (-\Delta)^{\frac{\alpha}{2}} u(u-\omega)dx \geq \|u\|^2_{\dot{H}^{\frac{\alpha}{2}}} - \frac{1}{2}\|u\|^2_{\dot{H}^{\frac{\alpha}{2}}} - \frac{1}{2}\|\omega\|^2_{\dot{H}^{\frac{\alpha}{2}}}. \tag{4.40}$$

The fourth term of the left-hand side of (4.36) can be written as

$$\begin{aligned}&\int_{\mathbb{T}^d} \nabla\cdot(u\nabla K * u)(u-\omega)dx \\ &= \int_{\mathbb{T}^d} \nabla\cdot(u\nabla K * u)udx - \int_{\mathbb{T}^d} \nabla\cdot(u\nabla K * u)\omega dx.\end{aligned} \tag{4.41}$$

For the first term of the right-hand side of (4.41), one has

$$\begin{aligned}\int_{\mathbb{T}^d} \nabla\cdot(u\nabla K * u)udx &= -\frac{1}{2}\int_{\mathbb{T}^d} u^2\Delta K * udx \\ &\leq \frac{1}{2}\|\Delta K\|_{L^1}\|u\|_{L^\infty}\|u\|^2_{L^2} \\ &\leq \frac{C_0}{2}\|u\|_{L^\infty}\|u\|^2_{L^2},\end{aligned} \tag{4.42}$$

and for the second term of the right-hand side of (4.41), we get

$$\begin{aligned}\int_{\mathbb{T}^d} \nabla\cdot(u\nabla K * u)\omega dx &\leq \|\nabla\cdot(u\nabla K * u)\|_{L^1}\|\omega\|_{L^\infty} \\ &\leq (\|\nabla u\|_{L^2}\|\nabla K * u\|_{L^2} + \|u\|_{L^2}\|\Delta K * u\|_{L^2})\|\omega\|_{L^\infty} \\ &\leq C_0(\|\nabla u\|_{L^2}\|u\|_{L^2} + \|u\|^2_{L^2})\|\omega\|_{L^\infty},\end{aligned} \tag{4.43}$$

where $\|\nabla K\|_{L^1}, \|\Delta K\|_{L^1}$ is bounded. By Young's inequality, (4.41), (4.42) and (4.43), to obtain

$$\begin{aligned}\int_{\mathbb{T}^d} \nabla\cdot(u\nabla K * u)(u-\omega)dx \leq\ & \frac{C_0}{2}\|u\|_{L^\infty}\|u\|^2_{L^2} \\ &+ C_0(\|\nabla u\|_{L^2}\|u\|_{L^2} + \|u\|^2_{L^2})\|\omega\|_{L^\infty}.\end{aligned} \tag{4.44}$$

Combining (4.36), (4.37), (4.40), (4.44) and Lemma 2.6, we have

$$\begin{aligned}\frac{d}{dt}\|u-\omega\|^2_{L^2} \leq & -\|u\|^2_{\dot{H}^{\frac{\alpha}{2}}} + F(At)^2\|u_0\|^2_{\dot{H}^{\frac{\alpha}{2}}} + C_0\|u\|_{L^\infty}\|u\|^2_{L^2} \\ &+ 2C_0(\|\nabla u\|_{L^2}\|u\|_{L^2} + \|u\|^2_{L^2})\|u_0\|_{L^\infty}.\end{aligned}$$

This completes the proof of Lemma 4.4. □

The mixing mechanism of relaxation enhancing flow have been studied by Kiselev et. al and Wei. Motivated by their work, we consider an aggregation equation with fractional dissipation and α-relaxation enhancing flow. A new proof is introduced, the main is based on RAGE theorem and contradiction. Let us denote $u(t,x)$ is a solution of equation (1.1) with initial data $u_0(x)$. Define

$$\Phi(A) = \inf_{t\geq 0} \frac{\|\Lambda^{\frac{\alpha}{2}}(u-\overline{u})\|^2_{L^2}}{\|u-\overline{u}\|^2_{L^2}} = \inf_{t\geq 0} \frac{\|\Lambda^{\frac{\alpha}{2}}u\|^2_{L^2}}{\|u-\overline{u}\|^2_{L^2}}. \tag{4.45}$$

We establish an important estimate of $\Phi(A)$, this is a new observation for mixing mechanism.

LEMMA 4.5. *Let $0 < \alpha < 1, d \geq 2$, for any initial data $u_0(x) \geq 0, u_0(x) \in H^3(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$. Suppose that $u(t,x)$ is solution of equation (1.1) with $u_0(x)$. If v is weakly mixing, then for fixed ε_0 is small, there exists $\Sigma(t) \subset [0,\tau_1]$ and $|\Sigma(t)| \geq \tau_1 - \varepsilon_0$, such that for any $t \in \Sigma(t)$, we have*

$$\lim_{A\to+\infty} \Phi(A) = +\infty, \tag{4.46}$$

where $\Phi(A)$ is defined in (4.45), $|\Sigma(t)|$ is measure of $\Sigma(t)$, and $0 < \varepsilon_0 < \frac{\tau_1}{2}$.

Proof. If (4.46) is not true, then there exists $\Sigma_1(t) \subset [0,\tau_1]$ and $|\Sigma_1(t)| \geq \varepsilon_0$, such that there exists a sequence $\{A_n\}_{n=1}^\infty$ and constant $\delta_1 \in \mathbb{R}^+$, such that $\lim_{n\to+\infty} A_n = +\infty$, for any $A_n, n = 1, 2, \cdots$, one has

$$\sup_{t\in\Sigma_1(t)} \frac{\|\Lambda^{\frac{\alpha}{2}} u\|_{L^2}^2}{\|u-\overline{u}\|_{L^2}^2} \leq \delta_1. \tag{4.47}$$

Without loss of generality, we assume that $\Sigma_1(t) = [0,\varepsilon_0]$ and for any $t \in [0,\varepsilon_0]$, one has

$$\|u-\overline{u}\|_{L^2} \geq B_1, \tag{4.48}$$

where B_1 is defined in (4.25). According to (4.47), to obtain

$$\|\Lambda^{\frac{\alpha}{2}} u\|_{L^2}^2 \leq \delta_1 \|u-\overline{u}\|_{L^2}^2. \tag{4.49}$$

We choose N, such that

$$\lambda_N^{\frac{\alpha}{2}} > 4\delta_1. \tag{4.50}$$

If $\|(I-P_N)(u-\overline{u})\|_{L^2}^2 \geq \frac{1}{4}\|u-\overline{u}\|_{L^2}^2$, we deduce by (4.50) that

$$\|\Lambda^{\frac{\alpha}{2}} u\|_{L^2}^2 \geq \|\Lambda^{\frac{\alpha}{2}}(I-P_N)u\|_{L^2}^2 \geq \lambda_N^{\frac{\alpha}{2}}\|(I-P_N)(u-\overline{u})\|_{L^2}^2 > \delta_1\|u-\overline{u}\|_{L^2}^2. \tag{4.51}$$

This is contradictory to (4.49), then

$$\|(I-P_N)(u-\overline{u})\|_{L^2}^2 \leq \frac{1}{4}\|u-\overline{u}\|_{L^2}^2. \tag{4.52}$$

Combining (4.48) and (4.52), we have

$$\|P_N(u-\overline{u})\|_{L^2}^2 \geq \frac{3}{4}\|u-\overline{u}\|_{L^2}^2 \geq \frac{3}{4}B_1^2. \tag{4.53}$$

Define the compact set $\mathcal{K} \subset S$ by

$$\mathcal{K} = \{\phi \in S \big| \|\phi\|_{\dot{H}^{\frac{\alpha}{2}}}^2 \leq \lambda_N^{\frac{\alpha}{2}}\}, \tag{4.54}$$

where S is defined in (2.7). Let U^t is the unitary operator associated with weakly mixing flow u in the Definition 2.5. Fix $\sigma = \frac{B_1^2}{8(B_0^2-\overline{u}^2)}$, then we get $T_c = T_c(N,\sigma,\mathcal{K},U)$, which is the time provided by Lemma 2.7. As $\lim_{n\to+\infty} A_n = +\infty$, there exists a constant $N_0 = N(T_c,\varepsilon_0)$, when $n \geq N_0$, define τ as follows

$$\tau = \frac{T_c}{A_n} \leq \varepsilon_0, \tag{4.55}$$

and $u_0(x)$ satisfies (4.23). Then for the solution $u(t,x)$ of equation (1.1), we deduce by Lemma 4.3 that

$$\|u(t,\cdot)-\overline{u}\|_{L^2} \le 2(B_0^2-\overline{u}^2)^{\frac{1}{2}}, \quad \|u(t,\cdot)\|_{L^\infty} \le 2C_\infty, \quad 0\le t\le \tau_1, \tag{4.56}$$

where τ_1 is defined in (4.34). For the equation (2.6), we deduce by the definition of U^t that

$$\omega(t,x)-\overline{u} = U^{A_n t}(u_0(x)-\overline{u}).$$

Combining the definition of $\mathcal{K}$ in (4.54), (4.49) and (4.50), to obtain

$$\frac{u_0-\overline{u}}{\|u_0-\overline{u}\|_{L^2}} \in \mathcal{K},$$

then we deduce by the Lemma 2.7 and the definition of τ in (4.55) that

$$\begin{aligned}
\frac{1}{\tau}\int_0^\tau \|P_N(\omega-\overline{u})\|_{L^2}^2 dt &= \frac{1}{\tau}\int_0^\tau \|P_N U^{A_n t}(u_0-\overline{u})\|_{L^2}^2 dt \\
&= \frac{\|u_0-\overline{u}\|_{L^2}^2}{\tau}\int_0^\tau \|P_N U^{A_n t}\frac{(u_0-\overline{u})}{\|u_0-\overline{u}\|_{L^2}}\|_{L^2}^2 dt \\
&= \frac{\|u_0-\overline{u}\|_{L^2}^2}{A_n\tau}\int_0^\tau \|P_N U^{A_n t}\frac{(u_0-\overline{u})}{\|u_0-\overline{u}\|_{L^2}}\|_{L^2}^2 dA_n t \\
&= \frac{\|u_0-\overline{u}\|_{L^2}^2}{T_c}\int_0^{T_c} \|P_N U^{s}\frac{(u_0-\overline{u})}{\|u_0-\overline{u}\|_{L^2}}\|_{L^2}^2 ds \\
&\le \sigma\|u_0-\overline{u}\|_{L^2}^2 \le \frac{1}{4}B_1^2.
\end{aligned} \tag{4.57}$$

Since $(u_0-\overline{u})/\|u_0-\overline{u}\|_{L^2} \in \mathcal{K}$, by the definition of $\mathcal{K}$, one has

$$\|u_0\|_{\dot{H}^{\frac{\alpha}{2}}}^2 \le \lambda_N^{\frac{\alpha}{2}}\|u_0-\overline{u}\|_{L^2}^2 \le \lambda_N^{\frac{\alpha}{2}}(B_0^2-\overline{u}^2). \tag{4.58}$$

For any $t\in[0,\tau_1]$, as $\|u(t,\cdot)\|_{L^2}, \|u(t,\cdot)\|_{L^\infty}$ are bounded and $\|u(t,\cdot)\|_{L^1}$ is conservation, then for any $1\le p_0<\infty$, the $\|u(t,\cdot)\|_{L^{p_0}}$ is bounded by interpolation inequality. According to Theorem 4.1, we choose suitable p_0 and by Gagliardo-Nirenberg inequality, we deduce that there exists $C_1 = C(A,t)<\infty$, such that

$$\|\nabla u\|_{L^2} \le C\|u\|_{L^{p_0}}^{1-\theta_3}\|u\|_{\dot{H}^3}^{\theta_3} \le C_1, \quad t\in[0,\tau_1], \tag{4.59}$$

where

$$\theta_3 = \frac{\frac{1}{2}-\frac{1}{d}-\frac{1}{p_0}}{\frac{1}{2}-\frac{3}{d}-\frac{1}{p_0}}, \quad 0<\theta_3<1.$$

Combining (4.23), (4.56), (4.58), (4.59) and Lemma 4.4, for any $0<t\le\tau_1$, to obtain

$$\begin{aligned}
\frac{d}{dt}\|u-\omega\|_{L^2}^2 &\le \lambda_N^{\frac{\alpha}{2}}F(A_n t)^2(B_0^2-\overline{u}^2) + 8C_0C_\infty B_0^2 + 8C_0B_0^2(C_\infty C_1+B_0) \\
&\le \lambda_N^{\frac{\alpha}{2}}F(A_n t)^2(B_0^2-\overline{u}^2) + 8C_0B_0^2(C_\infty + C_\infty C_1+B_0).
\end{aligned} \tag{4.60}$$

As $F(t)$ is a locally bounded function, then there exists a constant $N_1 = N(T_c, N, d, u_0) \geq N_0$, when $n \geq N_1$, and according to (4.55), we have

$$\begin{aligned}
&\int_0^{\tau} \lambda_N^{\frac{\alpha}{2}} F(A_n t)^2 (B_0^2 - \overline{u}^2) + 8C_0 B_0^2 (C_\infty + C_\infty C_1 + B_0) dt \\
&\leq \frac{\lambda_N^{\frac{\alpha}{2}} (B_0^2 - \overline{\rho}^2)}{A_n} \int_0^{T_c} F(t)^2 dt + 8C_0 B_0^2 (C_\infty + C_\infty C_1 + B_0)\tau \\
&\leq \frac{B_1^2}{8}.
\end{aligned}$$

Therefore, integrating from 0 to t on both sides of (4.60), where $0 \leq t \leq \tau$, to obtain

$$\|u(t,\cdot) - \omega(t,\cdot)\|_{L^2}^2 \leq \frac{B_1^2}{8}. \tag{4.61}$$

Furthermore, by the estimates (4.57) and (4.61), one has

$$\begin{aligned}
\frac{1}{\tau} \int_0^{\tau} \|P_N(u(t,\cdot) - \overline{u})\|_{L^2}^2 dt &\leq \frac{2}{\tau} \int_0^{\tau} \|P_N(\omega(t,\cdot) - \overline{u})\|_{L^2}^2 dt \\
&\quad + \frac{2}{\tau} \int_0^{\tau} \|P_N(u(t,\cdot) - \omega(t,\cdot))\|_{L^2}^2 dt \\
&\leq \frac{B_1^2}{2}.
\end{aligned} \tag{4.62}$$

We deduce by (4.53) that

$$\frac{1}{\tau} \int_0^{\tau} \|P_N(u(t,\cdot) - \overline{u})\|_{L^2}^2 dt \geq \frac{3}{4} B_1^2 \frac{1}{\tau} \int_0^{\tau} 1 dt = \frac{3}{4} B_1^2. \tag{4.63}$$

Contradiction between (4.62) and (4.63), so we have

$$\lim_{A \to +\infty} \Phi(A) = +\infty, \quad t \in \Sigma(t).$$

This completes the proof of Lemma 4.5. □

REMARK 11. *For (4.48), if there exists $t_2 \in [0, \varepsilon_0], t_2 > 0$, such that*

$$\|u(t_2,\cdot) - \overline{u}\|_{L^2} \leq B_1.$$

then the local solution can be extended to $[0, \tau_1 + t_2]$, the detail can be referred to the proof of Proposition 4.6.

REMARK 12. *In this paper, we only consider the $\Sigma(t) = [\varepsilon_0, \tau_1]$ for convenience of discussion. The result of general case is the same as $[\varepsilon_0, \tau_1]$, and the proof is also similar.*

4.3. Global L^∞ estimate. We establish global L^∞ estimate of the solution to equation (1.1) by mixing mechanism.

PROPOSITION 4.6 (Global L^∞ estimate). *Let $0 < \alpha < 1, d \geq 2$, suppose $u(t,x)$ is the solution of equation (1.1) with initial data $u_0 \geq 0, u_0 \in H^3(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$, and v*

is a weakly mixing. Then there exists positive constant $A_0 = A(u_0, d)$ *and* C_{L^∞}*, such that for* $A \geq A_0$*, we have*

$$\|u(t,\cdot)\|_{L^\infty} \leq C_{L^\infty}, \quad t \in [0, +\infty].$$

Proof. As $u_0 \in H^3(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$, without loss of generality, we assume $u_0(x)$ satisfies (4.22)-(4.24). Combining (4.33) and the definition of $\Phi(A)$ in (4.45), one has

$$\frac{d}{dt}\|u-\overline{u}\|_{L^2}^2 \leq -2\Phi(A)\|u-\overline{u}\|_{L^2}^2 + 6C_0C_\infty\|u-\overline{u}\|_{L^2}^2, \quad t \in [\varepsilon_0, \tau_1]. \tag{4.64}$$

According to Lemma 4.5, without loss of generality, we can assume that

$$\Sigma(t) = [\varepsilon_0, \tau_1].$$

By solving the (4.64), one has

$$\|u(\tau_1,\cdot)-\overline{u}\|_{L^2} \leq e^{-(\Phi(A)-3C_0C_\infty)(t-\varepsilon_0)}\|u(\varepsilon_0,\cdot)-\overline{u}\|_{L^2}. \tag{4.65}$$

As

$$\lim_{A\to+\infty}\Phi(A) = +\infty, \quad t \in [\varepsilon_0, \tau_1],$$

then there exists a positive constant A_0, when $A \geq A_0$, we deduce by Lemma 4.3 and (4.65) that

$$\|u(\tau_1,\cdot)-\overline{u}\|_{L^2} \leq B_1. \tag{4.66}$$

Combining Lemma 4.3, (4.66) and interpolation inequality, to obtain

$$\|u(\tau_1,\cdot)-\overline{u}\|_{L^p} \leq \|u-\overline{u}\|_{L^2}^{\frac{2}{p}}\|u-\overline{u}\|_{L^\infty}^{1-\frac{2}{p}} \leq D_0,$$

then

$$\|u(\tau_1,\cdot)\|_{L^p} \leq D_0 + \overline{u}. \tag{4.67}$$

According to $\|u_0-\overline{u}\|_{L^p} \leq D_0$ and Lemma 4.3, for any $0 \leq t \leq \tau_1$, one has

$$\|u(t,\cdot)\|_{L^p} \leq 2(D_0+\overline{u}).$$

Denote

$$\widetilde{u}(t) = u(t, \overline{x}_t) = \max_{x\in\mathbb{T}^d} u(t,x),$$

then by nonlinear maximum principle, if $t \in [0, \tau_1]$ and $\widetilde{u}(t)$ satisfies (4.15), it means

$$\widetilde{u}(t) \leq C(d,p)\|u(t,\cdot)\|_{L^p} \leq 2C(d,p)(D_0+\overline{u}) \leq C_\infty. \tag{4.68}$$

If not, then $\widetilde{u}(t) \geq 2C(d,p)(D_0+\overline{u})$, and $\widetilde{u}(t)$ satisfies (4.16), then

$$\frac{d}{dt}\widetilde{u} \leq \widetilde{u}^2 - \widetilde{u} - C_3\widetilde{u}^{1+\frac{p\alpha}{d}}, \quad t \in [0, \tau_1], \tag{4.69}$$

where

$$C_3 = \frac{C(\alpha, d, p)}{(2D_0 + 2\overline{u})^{\frac{p\alpha}{d}}}.$$

We set

$$\eta = \max\{x | x^2 - x - C_3 x^{1+\frac{p\alpha}{d}} = 0\},$$

and denote

$$C_{L^\infty} = \max\left\{2C(d,p)(D_0 + \overline{u}), \eta, \|u_0\|_{L^\infty}\right\},$$

as $\alpha > \frac{d}{p}$, then

$$1 + \frac{p\alpha}{d} > 2.$$

Solving the differential inequality of (4.69), we deduce that

$$\widetilde{u}(t) \leq C_{L^\infty}, \quad t \in\in [0, \tau_1]. \tag{4.70}$$

Combining with (4.68) and (4.70), for any $t \in\in [0, \tau_1]$, one has

$$\|u(t, \cdot)\|_{L^\infty} \leq C_{L^\infty}.$$

For the solution $u(t, x)$ of equation (1.1), by the same argument as above, we deduce that for any $n \in \mathbb{Z}^+$, one has

$$\|u(\tau_1, \cdot)\|_{L^p} \leq D_0 + \overline{u}.$$

By the similar to (4.67) and (4.70), we obtain

$$\|u(t, \cdot)\|_{L^\infty} \leq C_{L^\infty}.$$

This completes the proof of Proposition 4.6. □

Remark 13. *Without loss of generality, we can assume $C_\infty = C_{L^\infty}$ for the completeness of proof.*

The proof of Theorem 1.1. According to Theorem 4.1 and Proposition 4.6, we finish the proof of Theorem 1.1. □

5. Appendix. In this paper, the Gearhart-Prüss type theorem (see, Lemma 2.9) and nonlinear maximum principle on tours (see Lemma 2.3) are two important technique, and have been proved in reference of [49] and [47]. For the convenience of reading, we give the proof as the appendix.

The proof of Lemma 2.9. Let $\Psi = \Psi(H)$. Since $D(H)$ is dense in X, we only need to prove that

$$\|e^{-tH} f\| \leq e^{-t\Psi + \frac{\pi}{2}} \|f\|, \quad \forall f \in D(H), \ \ t \geq 0. \tag{5.1}$$

For $f \in D(H), t \geq 0$, let $g(t) = \|e^{-tH} f\|^2$. Since H is accretive, $g(t)$ is decreasing for $t \geq 0$, and we only need to prove (5.1) for $t\Psi > \frac{\pi}{2}$. In this case, $\Psi > 0$. Denote

$$t_1 = \frac{\pi}{4\Psi}, \quad t_2 = t - \frac{\pi}{4\Psi}, \quad t_3 = t + \frac{\pi}{4\Psi}, \quad l = t + \frac{\pi}{2\Psi}.$$

For $\chi \in C^1[0,l], \chi(0) = \chi(l) = 0$, set

$$f_1(s) = \chi(s)e^{-sH}f, \quad f_2(s) = \chi'(s)e^{-sH}f.$$

Then for any $t \in [0,l]$, one has

$$\partial_t f_1 + Hf_1 = f_2.$$

Take Fourier transform in t

$$\hat{f}_j(\lambda) = \int_0^l f_j(s)e^{-i\lambda s}ds, \quad j = 1,2, \quad \lambda \in \mathbb{R}.$$

Then

$$\hat{f}_2(\lambda) = (i\lambda + H)\hat{f}_1(\lambda).$$

By the definition of Ψ in (2.7), to obtain

$$\|\hat{f}_2(\lambda)\| \geq \Psi\|\hat{f}_1(\lambda)\|.$$

We use Plancherel's Theorem to conclude

$$\|f_2\|_{L^2([0,l],X)} = (2\pi)^{-\frac{1}{2}}\|\hat{f}_2\|_{L^2(\mathbb{R},X)} \geq (2\pi)^{-\frac{1}{2}}\Psi\|\hat{f}_1\|_{L^2(\mathbb{R},X)} = \Psi\|f_1\|_{L^2([0,l],X)}. \quad (5.2)$$

By the definition of f_1, f_2 and g, the (5.2) become

$$\int_0^l \chi'(s)^2 g(s)ds \geq \Psi^2 \int_0^l \chi(s)^2 g(s)ds. \quad (5.3)$$

Now we choose χ as follows

$$\chi(s) = \begin{cases} \sin \Psi s, & 0 \leq s \leq t_1, \\ e^{\Psi s - \frac{\pi}{4}}/\sqrt{2}, & t_1 \leq s \leq t_2, \\ e^{\Psi l - \pi}\sin(\Psi(l-s)), & t_2 \leq s \leq l. \end{cases}$$

Set

$$h(s) = \chi'(s)^2 - \Psi^2\chi(s)^2.$$

According to the (5.3), one has

$$\int_0^l h(s)g(s)ds \geq 0, \quad (5.4)$$

and

$$h(s) = \begin{cases} \Psi^2\cos(\Psi s), & 0 \leq s \leq t_1, \\ 0, & t_1 \leq s \leq t_2, \\ \Psi^2 e^{2\Psi l - 2\pi}\cos(2\Psi(l-s)), & t_2 \leq s \leq l. \end{cases}$$

Therefore

$$\begin{aligned} &h(s) \geq 0, \quad 0 \leq s \leq t_1 \ or \ t_3 \leq s \leq l, \\ &h(s) = 0 \quad t_1 \leq s \leq t_2, \\ &h(s) \leq 0 \quad t_2 \leq s \leq t_3. \end{aligned}$$

Since $g(t)$ is decreasing and $t \in [t_2, t_3]$, then

$$\begin{aligned}
&h(s)g(s) \le h(s)g(0) \quad 0 \le s \le t_1,\\
&h(s)g(s) = 0 \quad t_1 \le s \le t_2,\\
&h(s)g(s) \le h(s)g(t) \quad t_2 \le s \le t,\\
&h(s)g(s) \le 0 \quad t \le s \le t_3,\\
&h(s)g(s) \le h(s)g(t_3), \quad t_3 \le s \le l.
\end{aligned}$$

So

$$\begin{aligned}
0 \le \int_0^l h(s)g(s)ds &= \int_0^{t_1} h(s)g(s)ds + \int_{t_2}^l h(s)g(s)ds\\
&\le \int_0^{t_1} h(s)g(0)ds + \int_{t_2}^t h(s)g(t)ds + \int_{t_3}^l h(s)g(t_3)ds\\
&\le \frac{\Psi}{2}g(0) - \frac{\Psi}{2}e^{2\Psi l - 2\pi}g(t) + 0.
\end{aligned} \tag{5.5}$$

Therefore,

$$g(t) \le e^{-2\Psi l + 2\pi} g(0),$$

which implies that

$$\|e^{-tH} f\| \le e^{-\Psi l + \pi}\|f\| = e^{-t\Psi + \frac{\pi}{2}}\|f\|.$$

This completes the proof of Lemma 2.9. □

Next, we give the proof of Lemma 2.3, some other nonlinear maximum principle can be referred to [10, 14, 25].

The proof of Lemma 2.3. We take $R > 0$ a positive number and define

$$N_1(R) = \big\{\lambda \in B(0,R) \big| f(\overline{x}) - f(\overline{x} - \lambda) > \frac{f(\overline{x})}{2}\big\}.$$

and

$$M = \min_{y \in \partial\mathbb{T}^d} |\overline{x} - y|,$$

where $\partial\mathbb{T}^d$ represents the boundary of the periodic box $\mathbb{T}^d$. Without loss of generality, we assume that $M \ge \frac{1}{4}$. If

$$R \le M, \tag{5.6}$$

then, we have

$$B(0,R) \subset \mathbb{T}^d.$$

If we denote

$$N_2(R) = B(0,R) - N_1(R),$$

then

$$N_2(R) = \{\lambda \in B(0,R) \big| f(\overline{x}) - f(\overline{x}-\lambda) \le \frac{f(\overline{x})}{2}\},$$

and

$$\|f\|_{L^p}^p \ge \int_{\mathbb{T}^d} |f(\overline{x}-\lambda)|^p d\lambda \ge \int_{N_2(R)} |f(\overline{x}-\lambda)|^p d\lambda \ge \left(\frac{|f(\overline{x})|}{2}\right)^p |N_2(R)|,$$

thus, we obtain

$$|N_2(R)| \le \left(\frac{2\|f\|_{L^p}}{f(\overline{x})}\right)^p. \tag{5.7}$$

According to the definition of (2.1), we have

$$\begin{aligned}
(-\Delta)^{\frac{\alpha}{2}} f(\overline{x}) &\ge C_{\alpha,d} P.V. \int_{\mathbb{T}^d} \frac{f(\overline{x}) - f(\overline{x}-\lambda)}{|\lambda|^{d+\alpha}} d\lambda \\
&\ge C_{\alpha,d} P.V. \int_{N_1(R)} \frac{f(\overline{x}) - f(\overline{x}-\lambda)}{|\lambda|^{d+\alpha}} d\lambda \\
&\ge C_{\alpha,d} \frac{f(\overline{x})}{2} \frac{1}{R^{d+\alpha}} |N_1(R)|.
\end{aligned}$$

We deduce by (5.7), the definition of $N_1(R)$ and $N_2(R)$ that

$$|N_1(R)| = |B(0,R)| - |N_2(R)| \ge \omega_d R^d - \left(\frac{2\|f\|_{L^p}}{f(\overline{x})}\right)^p,$$

where ω_d is the volume per sphere, then we get

$$(-\Delta)^{\frac{\alpha}{2}} f(\overline{x}) \ge C_{\alpha,d} \frac{f(\overline{x})}{2R^{d+\alpha}} \left(\omega_d R^d - \left(\frac{2\|f\|_{L^p}}{f(\overline{x})}\right)^p\right). \tag{5.8}$$

We take R such that

$$\omega_d R^d = 2\left(\frac{2\|f\|_{L^p}}{f(\overline{x})}\right)^p,$$

thus

$$R = \left(\frac{2}{\omega_d}\left(\frac{2\|f\|_{L^p}}{f(\overline{x})}\right)^p\right)^{\frac{1}{d}} = \left(\frac{2}{\omega_d}\right)^{\frac{1}{d}} \left(\frac{2\|f\|_{L^p}}{f(\overline{x})}\right)^{\frac{p}{d}}. \tag{5.9}$$

By (5.8) and (5.9), we have

$$\begin{aligned}
(-\Delta)^{\frac{\alpha}{2}} f(\overline{x}) &\ge C_{\alpha,d} \frac{f(\overline{x})}{2R^{d+\alpha}} \left(\omega_d R^d - \left(\frac{2\|f\|_{L^p}}{f(\overline{x})}\right)^p\right) = C_{\alpha,d} \frac{f(\overline{x})}{2R^{d+\alpha}} \left(\frac{2\|f\|_{L^p}}{f(\overline{x})}\right)^p \\
&= \frac{C_{\alpha,d} 2^p}{2\left(\frac{2}{\omega_d}\right)^{\frac{d+\alpha}{d}} 2^{\frac{p(d+\alpha)}{d}}} \frac{\|f\|_{L^p}^p f(\overline{x})^{\frac{p(d+\alpha)}{d}} f(\overline{x})}{(\|f\|_{L^p})^{\frac{p(d+\alpha)}{d}} f(\overline{x})^p} \\
&= C(\alpha,d,p) \frac{f(\overline{x})^{1+\frac{p\alpha}{d}}}{\|f\|_{L^p}^{\frac{p\alpha}{d}}}.
\end{aligned}$$

If R does not fulfill (5.6), then

$$\left(\frac{2}{\omega_d}\right)^{\frac{1}{d}}\left(\frac{2\|f\|_{L^p}}{f(\overline{x})}\right)^{\frac{p}{d}} > M,$$

so we conclude that

$$f(\overline{x}) \leq \frac{2}{M^{\frac{d}{p}}(\frac{\omega_d}{2})^{\frac{1}{p}}}\|f\|_{L^p} \leq C(d,p)\|f\|_{L^p}.$$

This completes the proof of Lemma 2.3. □

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