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STEADY STATES OF GAS IONIZATION WITH SECONDARY EMISSION*

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Dedicated to Professor Ling Hsiao on the occasion of her 80th birthday

Abstract. We consider the steady states of a gas between two parallel plates that is ionized by a strong electric field so as to create a plasma. There can be a cascade of electrons due both to the electrons colliding with the gas molecules and to the ions colliding with the cathode (secondary emission). We use global bifurcation theory to prove that there is a one-parameter family $\mathcal{K}$ of such steady states with the following property. The curve $\mathcal{K}$ begins at the sparking voltage and either the particle density becomes unbounded or $\mathcal{K}$ ends at an anti-sparking voltage. These critical voltages are characterized explicitly.

Key words. ionization, gas discharge, secondary emission, sparking voltage, global bifurcation, plasma.

Mathematics Subject Classification. 35A01, 35M12, 70K50, 76X05, 82D10.

1. Introduction. This paper is concerned with a model for the ionization of a gas such as air due to a strong applied electric field. For instance, the strong electric field may be created when a capacitor discharges into a gap between electrodes. The high voltage thereby creates a plasma, which may possess very hot or bright electrical arcs. A century ago Townsend experimented with a pair of parallel plates to which he applied a strong voltage that produced cascades of free electrons and ions. This phenomenon is called the Townsend discharge or avalanche.

Such an avalanche primarily occurs due to free electrons colliding with gas molecules, thus liberating other electrons. This is called the α-mechanism. Another important contribution to an avalanche may be due to the impact of ions with the cathode, which then emits additional electrons. This is called the secondary emission or the γ-mechanism. In this paper we discuss a model that takes account of both mechanisms.

The model is as follows. Let $I = (0, L)$ be the distance between the planar parallel plates. Let us put the anode at $x = 0$ and the cathode at $x = L$. Let ρ_i be the density of positive ions, ρ_e the density of electrons, and $-\Phi$ the electrostatic potential. Let u_i and u_e be the ion and electron velocities. Then the equations within the region I are as follows.

$$\partial_t \rho_i + \partial_x(\rho_i u_i) = a \exp\left(-b|\partial_x \Phi|^{-1}\right) \rho_e \, |v_e| \,, \tag{1.1a}$$

$$\partial_t \rho_e + \partial_x(\rho_e u_e) = a \exp\left(-b|\partial_x \Phi|^{-1}\right) \rho_e \, |v_e| \,, \tag{1.1b}$$

$$\partial_x^2 \Phi = \rho_i - \rho_e, \tag{1.1c}$$

$$u_i := k_i \partial_x \Phi, \quad u_e := v_e - k_e \partial_x \rho_e / \rho_e, \quad v_e := -k_e \partial_x \Phi. \tag{1.1d}$$

Here k_i, k_e, a, and b are positive constants. The constitutive velocity relations (1.1d) are due to the ions being much heavier than the electrons. The right sides of (1.1a)

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and (1.1b) come from the α-mechanism. They express the number of ion–electron pairs generated per unit volume by the impacts of the electrons. Specifically, the coefficient $\alpha = a\exp\left(-b|\partial_x\Phi|^{-1}\right)$ is the first Townsend ionization coefficient.

The boundary conditions at the anode $x = 0$ are $\rho_i = \rho_e = \Phi = 0$, due to the assumption that the anode is a perfect conductor, so that the electrons are absorbed by the anode and the ions are repelled from the anode. We denote the voltage at the cathode $x = L$ by $V_c > 0$. The secondary emission at the cathode (or γ-mechanism) is expressed by

$$\rho_e u_e = -\gamma\rho_i u_i \tag{1.2}$$

where $\gamma > 0$ is average number of electrons ejected from the cathode by an ion impact.

In this paper we consider the steady state problem, where the unknowns do not depend on time, even though the individual particles can move rapidly. First of all, there are the completely trivial solutions $\rho_i \equiv 0, \rho_e \equiv 0$, $\Phi(x) = \frac{V_c}{L}x$, where V_c is an arbitrary constant. Avalanche does not occur unless the electric field is strong enough. In our model the ionization coefficient a or the secondary emission coefficient γ must be large enough, depending on b and L, in order to reach this threshold. Then the critical threshold value of the voltage is called the *sparking voltage* $V_c^\dagger$. Assuming that the sparking voltage does exist, we prove that there are many other steady solutions, in fact a whole global curve of them, for most choices of the parameters (a, b, γ).

Theorem 1.1. *Assume that the sparking voltage $V_c^\dagger$ exists. For almost every (a, b, γ), there exists a unique continuous one-parameter family $\mathcal{K}$ (that is, a curve) of steady solutions of the system of equations together with the boundary conditions written above with the following properties. Both densities are positive, $\rho_i \in C^1$, $\rho_e \in C^2$, $\Phi \in C^3$, the curve begins at the trivial solution with voltage $V_c^\dagger$ and "ends" with one of the following three alternatives:*

Either the density $|\rho_i| + |\rho_e|$ becomes unbounded along $\mathcal{K}$,

Or the potential Φ becomes unbounded along $\mathcal{K}$,

Or the curve ends at a different trivial solution with some voltage $V_c^\ddagger > V_c^\dagger$.

The sparking voltage $V_c^\dagger$ is the smallest positive root of a certain elementary function $D(\cdot)$, which we call the *sparking function.* We say that the sparking voltage *exists* for a given parameter triple (a, b, γ) if D has a positive root for any triple in a neighborhood of it. We call $V_c^\ddagger$ the *anti-sparking* voltage; it is a larger root of $D(\cdot)$. The explicit *sparking function* D is defined as follows. For brevity we first denote

$$\lambda = \frac{V_c}{L}, \quad h(\lambda) = a\lambda e^{-b/\lambda}, \quad g(\lambda L) = h(\lambda) - \lambda^2/4. \tag{1.3}$$

Then let $\mu = L\sqrt{-g(V_c)}$ and

$$D(V_c) = \frac{1}{2}\left(e^{\mu} + e^{-\mu}\right) + \frac{V_c}{4\mu}\left(e^{\mu} - e^{-\mu}\right) - \frac{\gamma}{1+\gamma}e^{\frac{V_c}{2}}. \tag{1.4}$$

Note that, even if $g(V_c)$ is positive, $D(V_c)$ is real. In case $g(V_c)$ vanishes, $D(V_c)$ is defined as the limit $\lim_{g(V_c)\to 0} D(V_c)$. Thus $D \in C((0,\infty);\mathbb{R})$. Depending on γ, a, b and L, the sparking function D may have no root, one root or several roots. If D has a root, the *sparking voltage* $V_c^\dagger > 0$ is defined as the smallest one:

$$V_c^\dagger := \inf\{V_c > 0\,;\, D(V_c) = 0\}. \tag{1.5}$$

Sufficient conditions for D to have one or more roots, or none, are given in Appendix A.

We prove Theorem 1.1 by a local, and then a global, bifurcation argument. In Section 2 we set up the notation used in the analysis. In Section 3 we apply the well-known local bifurcation theorem. In particular, we prove that the nullspace and the range of the linearized operator around any trivial solution is determined by the function D. A transversality condition is required in order to guarantee the local bifurcation. We prove in Lemma 3.4 that this condition is valid for almost every (a, b, γ). Then in Section 4 we apply a global bifurcation theorem to construct a *global* curve $\mathcal{K}$ of steady solutions (ρ_i, ρ_e, Φ). The general properties of this global curve are given in Theorem 4.4. The curve may include mathematical solutions with positive densities as well as solutions with negative "densities". In Section 5 we restrict our attention to positive densities. Further analysis of the possible ways that the curve may "terminate" is then provided. The main conclusion (as in Theorem 1.1) is given in Theorem 5.5. In case the voltage becomes unbounded, it is proven in Section 5 that the densities tend to zero.

Appendix A is devoted to the sparking function (1.4). It is shown that there is a sparking voltage if either a or γ is large enough. In Appendix B we discuss the location of the sparking voltage (1.5).

2. History and Notation. We now briefly summarize the history of the model. Many models have been proposed to describe this phenomenon [1, 9, 10, 11, 13, 14, 15, 16]. In 1985 Morrow [16] was perhaps the first to provide a model of its detailed mechanism in terms of particle densities. The model consists of continuity equations for the electrons and ions coupled to the Poisson equation for the electrostatic potential. For simplicity in this paper we consider only electrons and positive ions and we focus on the γ and α mechanisms. Various other mechanisms can occur, such as 'attachment' and 'recombination' as mentioned in Morrow's paper, which have a much smaller effect on the ionization.

The interesting article [9] of Degond and Lucquin-Desreux derives the model directly from the general Euler-Maxwell system by scaling assumptions, in particular by assuming a very small mass ratio between the electrons and ions. In an appropriate limit the Morrow model is obtained at the end of their paper in equations (160) and (163), which we have specialized to assume constant temperature and no neutral particles.

Suzuki and Tani in [20] gave the first mathematical analysis of the Morrow model. Typical shapes of the cathode and anode in physical and numerical experiments are a sphere or a plate. Therefore they proved the time-local solvability of an initial boundary value problem over domains with a pair of boundaries that are plates or spheres. In another paper [21] they did a deeper analysis of problem (1.1), proving that there exists a certain threshold of voltage at which the trivial solution transitions from stable to unstable. This fact means that gas discharge can occur and continue for a voltage greater than the threshold.

In [19] we considered the Morrow model with the α-mechanism but without the γ-mechanism. The boundary condition (1.2) was replaced by the condition that $\rho_e = 0$ at the cathode, which means that the electrons are simply repelled by the cathode. For that simpler model the sparking voltage $V_c^\dagger$ is the smallest root of the function g and the anti-sparking voltage $V_c^\ddagger$ is the other root if it exists. We proved similarly that there is a global curve of steady solutions that starts at $V_c^\dagger$ and either goes to infinity or is a half-loop that goes to $V_c^\ddagger$. In that case we eliminated the alternative that the voltage may be unbounded.

Now we describe some notation that we use in the rest of the paper. For mathematical convenience we rewrite the problem (1.1) in terms of the new unknown function

$$R_e := \rho_e e^{\frac{V_c}{2L}x}.$$

We decompose the electrostatic potential as

$$\Phi = V + \frac{V_c}{L}x.$$

Thus $\partial_x^2 V = \rho_i - e^{-\frac{V_c}{2L}x}R_e$ with the boundary conditions $V(0) = V(L) = 0$. As a result, from (1.1) we have the following system for stationary solutions:

$$k_i\partial_x\left\{\left(\partial_x V + \frac{V_c}{L}\right)\rho_i\right\} = k_e h\left(\partial_x V + \frac{V_c}{L}\right)e^{-\frac{V_c}{2L}x}R_e, \tag{2.1a}$$

$$-k_e\partial_x^2 R_e - k_e g(V_c)R_e = k_e f_e[V_c, R_e, V], \tag{2.1b}$$

$$\partial_x^2 V = \rho_i - e^{-\frac{V_c}{2L}x}R_e \tag{2.1c}$$

with the boundary conditions

$$\rho_i(0) = R_e(0) = V(0) = V(L) = 0, \tag{2.1d}$$

$$\partial_x R_e(L) + \left(\partial_x V(L) + \frac{V_c}{2L}\right)R_e(L) = \gamma\frac{k_i}{k_e}e^{\frac{V_c}{2L}L}\left(\partial_x V(L) + \frac{V_c}{L}\right)\rho_i(L), \tag{2.1e}$$

where the nonlinear term $f_e = f_e[V_c, R_e, V]$ is defined as

$$f_e = (\partial_x V)\partial_x R_e - \frac{V_c}{2L}R_e\partial_x V + R_e\partial_x^2 V - \left[h\left(\frac{V_c}{L}\right) - h\left(\partial_x V + \frac{V_c}{L}\right)\right]R_e.$$

It is convenient to draw the graph of $g(V_c)$, which of course depends on the physical parameters a, b, and L. The function g has at most one local maximum in $(0, \infty)$.

For the analysis in the rest of the paper it is convenient to write the system (2.1) as

$$\mathcal{F}_j(\lambda, \rho_i, R_e, V) = 0 \quad \text{for } j = 1, 2, 3, 4, \tag{2.2}$$

where we denote $\lambda = V_c/L$ and

$$\begin{aligned}
\mathcal{F}_1 :=& k_i\partial_x\left\{(\partial_x V + \lambda)\rho_i\right\} - k_e h\left(\partial_x V + \lambda\right)e^{-\frac{\lambda}{2}x}R_e,\\
\mathcal{F}_2 :=& -\partial_x^2 R_e - (\partial_x V)\partial_x R_e + \left\{\frac{\lambda}{2}\partial_x V - \partial_x^2 V + \frac{\lambda^2}{4} - h\left(\partial_x V + \lambda\right)\right\}R_e,\\
\mathcal{F}_3 :=& \partial_x^2 V - \rho_i + e^{-\frac{\lambda}{2}x}R_e,\\
\mathcal{F}_4 :=& \partial_x R_e(L) + \left(\partial_x V(L) + \frac{\lambda}{2}\right)R_e(L) - \gamma\frac{k_i}{k_e}e^{\frac{\lambda}{2}L}\left(\partial_x V(L) + \lambda\right)\rho_i(L).
\end{aligned}$$

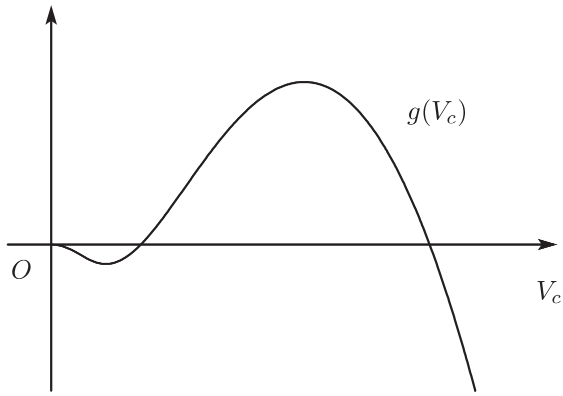

FIG. 1. *local max is positive*

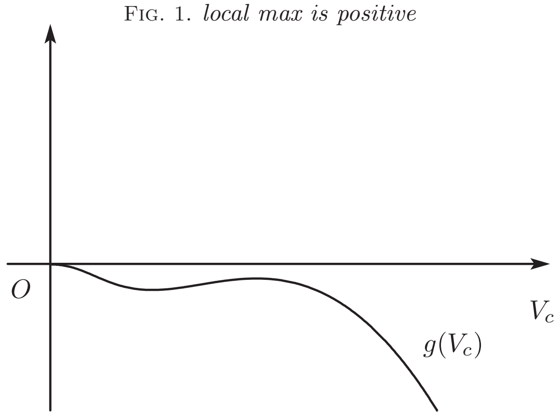

FIG. 2. *local max is negative*

3. Bifurcation. In this section we apply the following well-known theorem [5] on bifurcation from a simple eigenvalue. Let $N(\mathcal{L})$ and $R(\mathcal{L})$ denote the nullspace and range of any linear operator $\mathcal{L}$ between two Banach spaces.

THEOREM 3.1. *Let X and Y be Banach spaces, $\mathcal{O}$ be an open subset of $\mathbb{R} \times X$ and $\mathcal{F} : \mathcal{O} \to Y$ be a C^2 function. Suppose that*
(H1) $(\lambda, 0) \in \mathcal{O}$ and $\mathcal{F}(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$;
(H2) for some $\lambda^* \in \mathbb{R}$, $N(\partial_u \mathcal{F}(\lambda^*, 0))$ and $Y \backslash R(\partial_u \mathcal{F}(\lambda^*, 0))$ are one-dimensional, with the null space generated by u^*, which satisfies the transversality condition

$$\partial_\lambda \partial_u \mathcal{F}(\lambda^*, 0)(1, u^*) \notin R(\partial_u \mathcal{F}(\lambda^*, 0)),$$

where ∂_u and $\partial_\lambda \partial_u$ denote Fréchet derivatives for $(\lambda, u) \in \mathcal{O}$.
Then there exists in $\mathcal{O}$ a continuous curve $\mathcal{K} = \{(\lambda(s), u(s)); s \in \mathbb{R}\}$ of solutions of the equation $\mathcal{F}(\lambda, u) = 0$ such that:
(C1) $(\lambda(0), u(0)) = (\lambda^*, 0)$;
(C2) $u(s) = su^* + o(s)$ in X as $s \to 0$;
(C3) there exists a neighborhood $\mathcal{W}$ of $(\lambda^*, 0)$ and $\varepsilon > 0$ sufficiently small such that

$$\{(\lambda, u) \in \mathcal{W}; u \neq 0 \text{ and } \mathcal{F}(\lambda, u) = 0\} = \{(\lambda(s), u(s)); 0 < |s| < \varepsilon\}.$$

In order to apply the theorem to our situation, we use the notation $u = (\rho_i, R_e, V)$

and we define the two spaces

$$\begin{aligned}
X : \; & \rho_i \in \{f \in C^1([0,L]); f(0) = 0\}, \quad R_e \in \{f \in C^2([0,L]); f(0) = 0\}, \\
& V \in \{f \in C^3([0,L]); f(0) = f(L) = 0\}; \\
Y : \; & \mathcal{F}_1 \in C^0([0,L]), \quad \mathcal{F}_2 \in C^0([0,L]), \quad \mathcal{F}_3 \in C^1([0,L]), \quad \mathcal{F}_4 \in \mathbb{R}
\end{aligned}$$

and the sets

$$\begin{aligned}
\mathcal{O} :=& \{(\lambda, \rho_i, R_e, V) \in (0,\infty) \times X;\ \partial_x V + \lambda > 0\} \; = \; \bigcup_{j \in \mathbb{N}} \mathcal{O}_j, \text{ where} \\
\mathcal{O}_j :=& \{(\lambda, \rho_i, R_e, V) \in (0,\infty) \times X;\ \lambda + \|(\rho_i, R_e, V)\|_X \le j,\ \lambda \ge \tfrac{1}{j},\ \partial_x V + \lambda \ge \tfrac{1}{j}\}.
\end{aligned}$$

Note that $\mathcal{O}$ is an open set and each $\mathcal{O}_j$ is a closed bounded subset of $\mathcal{O}$. Furthermore, the $\mathcal{F}_j$ are real-analytic operators because they are polynomials in $(\lambda, \rho_i, R_e, V)$ and their x-derivatives, except for the factor $h(\partial_x V + \lambda)$. However, $\partial_x V + \lambda > 0$ in $\mathcal{O}$ and the function $s \to h(s)$ is analytic for $s > 0$. Hypothesis $(H1)$ is obvious. The local bifurcation condition $(H2)$ is verified in Lemmas 3.2–3.4.

LEMMA 3.2. *Recall that* $\lambda = V_c/L$. *Let* $\mathcal{L} = \partial_{(\rho_i, R_e, V)} \mathcal{F}(\lambda, 0, 0, 0)$ *be the linearized operator around a trivial solution and let* $N(\mathcal{L})$ *be its nullspace. Then*
(a) $N(\mathcal{L})$ *is at most one-dimensional for any* $\lambda > 0$.
(b) $N(\mathcal{L})$ *is one-dimensional if and only if* $D(V_c) = 0$. *Thus the sparking voltage exists.*
(c) $N(\mathcal{L})$ *has a basis* $(\varphi_i, \varphi_e, \varphi_v)$ *with*

$$\varphi_i(x) > 0, \quad \varphi_e(x) > 0 \quad \text{for } x \in (0, L] \tag{3.1}$$

if and only if

$$D(V_c) = 0, \quad g(V_c) < \frac{\pi^2}{L^2}. \tag{3.2}$$

(d) $V_c^\dagger$ *defined in* (1.5) *satisfies* (3.2).

Proof. We remark that the positivity (3.1) will lead to the positivity of R_e and ρ_i in the local bifurcation proof.

(a) If $(S_i, S_e, W) \in N(\mathcal{L}) \subset X$, then (S_i, S_e, W) solves

$$\partial_{(\rho_i, R_e, V)} \mathcal{F}_1(\lambda, 0, 0, 0)[S_i, S_e, W] = k_i \lambda \partial_x S_i - k_e h(\lambda) e^{-\frac{\lambda}{2} x} S_e = 0, \tag{3.3}$$

$$\partial_{(\rho_i, R_e, V)} \mathcal{F}_2(\lambda, 0, 0, 0)[S_i, S_e, W] = -\partial_x^2 S_e - g(\lambda L) S_e = 0, \tag{3.4}$$

$$\partial_{(\rho_i, R_e, V)} \mathcal{F}_3(\lambda, 0, 0, 0)[S_i, S_e, W] = \partial_x^2 W - S_i + e^{-\frac{\lambda}{2} x} S_e = 0, \tag{3.5}$$

$$\partial_{(\rho_i, R_e, V)} \mathcal{F}_4(\lambda, 0, 0, 0)[S_i, S_e, W] = \partial_x S_e(L) + \frac{\lambda}{2} S_e(L) - \gamma \frac{k_i}{k_e} \lambda e^{\frac{\lambda}{2} L} S_i(L) = 0. \tag{3.6}$$

Solving (3.3) with $S_i(0) = 0$, we have

$$\frac{k_i}{k_e} \lambda S_i(x) = h(\lambda) \int_0^x e^{-\frac{\lambda}{2} y} S_e(y)\, dy. \tag{3.7}$$

By (3.7) with $x = L$, we rewrite the boundary condition (3.6) so that

$$\partial_x S_e(L) + \frac{\lambda}{2} S_e(L) = \gamma h(\lambda) e^{\frac{\lambda}{2} L} \int_0^L e^{-\frac{\lambda}{2} y} S_e(y)\, dy, \tag{3.8}$$

which is a closed equation with respect to S_e. Therefore, we have a differential equation for S_e with two boundary conditions. It suffices to solve it in order to obtain all elements of the nullspace. Indeed, S_i is obtained by (3.7) and S_e and W is obtained by solving (3.5) with $W(0) = W(L) = 0$. The general solutions of the second order equation (3.4) with $S_e(0) = 0$ are

$$S_e(x) = \begin{cases} A \sinh \sqrt{-g(\lambda L)}x & \text{if } g(\lambda L) < 0, \\ Ax & \text{if } g(\lambda L) = 0, \\ A \sin \sqrt{g(\lambda L)}x & \text{if } g(\lambda L) > 0, \end{cases} \tag{3.9}$$

where we have also used the the boundary condition $S_e(0) = 0$. This fact means that the null space $N(\partial_{(\rho_i, R_e, V)}\mathcal{F}(\lambda, 0, 0, 0))$ is at most one-dimensional for any $\lambda > 0$.

(b) We will show that equation (3.4) with $S_e(0) = 0$ and (3.8) admits nontrivial solutions if and only if $D(V_c) = 0$. We write $g = g(\lambda L)$ and first treat the case $g < 0$. To this end, we substitute the general solution into (3.8) and see that a necessary and sufficient condition for the existence of nontrivial solutions is

$$\begin{aligned} 0 =& \sqrt{-g} \cosh \sqrt{-g}L + \frac{\lambda}{2} \sinh \sqrt{-g}L - \gamma h(\lambda) e^{\frac{\lambda}{2}L} \int_0^L e^{-\frac{\lambda}{2}y} \sinh \sqrt{-g}y \, dy \\ =& (1+\gamma) \left\{ \sqrt{-g} \cosh \sqrt{-g}L + \frac{\lambda}{2} \sinh \sqrt{-g}L \right\} + 2\gamma\sqrt{-g}e^{\frac{\lambda}{2}L}. \end{aligned}$$

In deriving the last equality, we have also used the fact $g + \frac{\lambda^2}{4} = h(\lambda)$. This equality is equivalent to $D(V_c) = 0$. Now we consider the case $g = 0$. As above, we find the condition

$$0 = 1 + \frac{\lambda}{2}L - \gamma h(\lambda) e^{\frac{\lambda}{2}L} \int_0^L e^{-\frac{\lambda}{2}y} y \, dy = (1+\gamma)\left(1 + \frac{\lambda}{2}L\right) - \gamma e^{\frac{\lambda}{2}}.$$

This too is equivalent to $D(V_c) = 0$. For the case $g > 0$, we have

$$\begin{aligned} 0 =& \sqrt{g} \cos \sqrt{g}L + \frac{\lambda}{2} \sin \sqrt{g}L - \gamma h(\lambda) e^{\frac{\lambda}{2}L} \int_0^L e^{-\frac{\lambda}{2}y} \sin \sqrt{g}y \, dy \\ =& (1+\gamma)\left(\sqrt{g} \cos \sqrt{g}L + \frac{\lambda}{2} \sin \sqrt{g}L \right) - \gamma\sqrt{g}e^{\frac{\lambda}{2}L}. \end{aligned}$$

Once again this is equivalent to $D(V_c) = 0$. Thus we conclude in all three cases that $N(\mathcal{L})$ is one-dimensional if and only if $D(V_c) = 0$.

(c) Furthermore, it is seen from (3.9) that the null space $N(\mathcal{L})$ has a basis with (3.1) if and only if (3.2) holds.

(d) It remains to show that the sparking voltage $V_c^\dagger$ must satisfy (3.2). Suppose on the contrary that $g(V_c^\dagger) \geq \pi^2/L^2$ holds. Then the graph of g must be drawn as in Figure 1. Therefore, there exists a positive constant $V_c^* \leq V_c^\dagger$ such that $g(V_c) < \pi^2/L^2$ for all $V_c \in [0, V_c^*)$ and $g(V_c^*) = \pi^2/L^2$. Evaluating $D(V_c)$ at $V_c = V_c^*$, we see that

$$D(V_c^*) = \cos \sqrt{g(V_c^*)}L + \frac{V_c^*}{2\sqrt{g(V_c^*)}L} \sin \sqrt{g(V_c^*)}L - \frac{\gamma}{1+\gamma}e^{\frac{V_c^*}{2}} = -1 - \frac{\gamma}{1+\gamma}e^{\frac{V_c^*}{2}} < 0.$$

However, $\lim_{V_c \to 0} D(V_c) = 1/(1+\gamma) > 0$. These facts together with the intermediate value theorem means that there exists $0 < c_0 < V_c^*$ such that $D(c_0) = 0$, so that $V_c^\dagger$ is not the smallest root of D, which contradicts its definition. $\square$

In order to apply Theorem 3.1, we define $\lambda^* = V_c^\dagger/L$ and we let $u^* = (\varphi_i^\dagger, \varphi_e^\dagger, \varphi_v^\dagger)$ denote a basis of $N(\partial_{(\rho_i,R_e,V)}\mathcal{F}(V_c^\dagger/L,0,0,0))$ that satisfies (3.1).

Lemma 3.3. *The quotient space $Y\backslash R(\mathcal{L})$ is at most one-dimensional. Furthermore, it is one-dimensional if and only if $D(V_c) = 0$.*

Proof. Let us denote $\partial_{(\rho_i,R_e,V)}\mathcal{F}(V_c/L,0,0,0))$ by $\mathcal{L}$. We begin by representing the range as

$$R(\mathcal{L}) = \{(f_i, f_e, f_v, f_b) \in Y; (3.11)\}\,, \tag{3.10}$$

$$\int_0^L (f_i\psi_i + f_e\psi_e + f_v\psi_v)dx + f_b\psi_b = 0 \ \text{ for all } (\psi_i,\psi_e,\psi_v,\psi_b) \in N((\mathcal{L}^\bullet)^*). \tag{3.11}$$

Here $(\mathcal{L}^\bullet)^*$ is defined conveniently on a Hilbert space as follows. Let $X^\bullet$ be the same as X except that C^k is replaced by H^k for $k = 1,2,3$. Let $Y^\bullet$ be the same as Y except that C^k is replaced by H^k for $k = 0,1$. Define $\mathcal{L}^\bullet : X^\bullet \to Y^\bullet$ to be the unique linear extension of $\mathcal{L}$ to $X^\bullet$, and $(\mathcal{L}^\bullet)^*$ to be the adjoint operator of $\mathcal{L}^\bullet$. By standard operator theory,

$$R(\mathcal{L}^\bullet) = \{(f_i, f_e, f_v, f_b) \in Y^\bullet; (3.11)\}.$$

From this and the fact $Y \subset Y^\bullet$, (3.11) is necessary for the solvability of the problem $\mathcal{L}^\bullet(S_i,S_e,W) = (f_i,f_e,f_v,f_b) \in Y$. On the other hand, if $(f_i,f_e,f_v,f_b) \in \{f \in Y; (3.11)\}$, we have a unique solution $(S_i,S_e,W) \in X^\bullet$ to the problem $\mathcal{L}^\bullet(S_i,S_e,W) = (f_i,f_e,f_v,f_b) \in Y$. Then $(S_i,S_e,W) \in X$ by standard elliptic estimates. These facts lead to the representation (3.10).

It remains to prove that $N((\mathcal{L}^\bullet)^*)$ is at most one-dimensional, and it is one-dimensional if and only if $D(V_c) = 0$. We first claim that the operator $(\mathcal{L}^\bullet)^*$ is precisely given by

$$D((\mathcal{L}^\bullet)^*) := \{(\psi_i,\psi_e,\psi_v,\psi_b) \in H^1(I)\times H^2(I)\times H^3(I)\times\mathbb{R}; \text{(3.12e) holds}\}, \tag{3.12a}$$

$$(\mathcal{L}^\bullet)_1^*(\psi_i,\psi_e,\psi_v,\psi_b) := -k_i\lambda\partial_x\psi_i - \psi_v, \tag{3.12b}$$

$$(\mathcal{L}^\bullet)_2^*(\psi_i,\psi_e,\psi_v,\psi_b) := -\partial_x^2\psi_e - g(\lambda L)\psi_e - k_e h(\lambda)e^{-\frac{\lambda}{2}x}\psi_i + e^{-\frac{\lambda}{2}x}\psi_v, \tag{3.12c}$$

$$(\mathcal{L}^\bullet)_3^*(\psi_i,\psi_e,\psi_v,\psi_b) := \partial_x^2\psi_v, \tag{3.12d}$$

where

$$\begin{aligned}\psi_e(L) - \psi_b = \psi_e(0) = \psi_v(0) = \psi_v(L) &= k_i\lambda\psi_i(L) - \gamma\frac{k_i}{k_e}\lambda e^{\frac{\lambda}{2}L}\psi_b \\ &= \partial_x\psi_e(L) + \frac{\lambda}{2}\psi_b = 0.\end{aligned} \tag{3.12e}$$

We now verify the claim. It suffices to check that

$$\begin{aligned}&\langle(\mathcal{L}_1^\bullet(S_i,S_e,W), \mathcal{L}_2^\bullet(S_i,S_e,W), \mathcal{L}_3^\bullet(S_i,S_e,W)), (\psi_i,\psi_e,\psi_v)\rangle + \mathcal{L}_4^\bullet(S_i,S_e,W)\psi_b \\ &= \langle(S_i,S_e,W), (\mathcal{L}^\bullet)^*(\psi_i,\psi_e,\psi_v,\psi_b)\rangle\end{aligned}$$

for all $(S_i,S_e,W) \in X^\bullet$ and $(\psi_i,\psi_e,\psi_v,\psi_b) \in D((\mathcal{L}^\bullet)^*)$, where $\langle\cdot,\cdot\rangle$ denotes the inner product of $L^2(I)$. We observe that

$$
\begin{aligned}
&\langle(\mathcal{L}_1^\bullet(S_i,S_e,W),\mathcal{L}_2^\bullet(S_i,S_e,W),\mathcal{L}_3^\bullet(S_i,S_e,W)),(\psi_i,\psi_e,\psi_v)\rangle+\mathcal{L}_4^\bullet(S_i,S_e,W)\psi_b\\
&=-\langle S_i,k_i\lambda\partial_x\psi_i\rangle+S_i(L)k_i\lambda\psi_i(L)-\langle S_e,k_eh(\lambda)e^{-\frac{\lambda}{2}x}\psi_i\rangle\\
&\quad-\langle S_e,\partial_x^2\psi_e\rangle-\partial_xS_e(L)\psi_e(L)+\partial_xS_e(0)\psi_e(0)+S_e(L)\partial_x\psi_e(L)-\langle S_e,g(\lambda L)\psi_e\rangle\\
&\quad+\langle W,\partial_x^2\psi_v\rangle+\partial_xW(L)\psi_v(L)-\partial_xW(0)\psi_v(0)-\langle S_i,\psi_v\rangle+\langle S_e,e^{-\frac{\lambda}{2}x}\psi_v\rangle\\
&\quad+\partial_xS_e(L)\psi_b+S_e(L)\frac{\lambda}{2}\psi_b-S_i(L)\gamma\frac{k_i}{k_e}\lambda e^{\frac{\lambda}{2}L}\psi_b,
\end{aligned}
$$

due to integration by parts and $S_i(0)=S_e(0)=W(0)=W(L)=0$. Grouping them with respect to S_i, $S_i(L)$, S_e, $\partial_xS_e(L)$, $\partial_xS_e(0)$, $S_e(L)$, W, $\partial_xW(L)$, and $\partial_xW(0)$, and also using the boundary conditions (3.12e), we have

$$
\begin{aligned}
&\langle(\mathcal{L}_1^\bullet(S_i,S_e,W),\mathcal{L}_2^\bullet(S_i,S_e,W),\mathcal{L}_3^\bullet(S_i,S_e,W)),(\psi_i,\psi_e,\psi_v)\rangle+\mathcal{L}_4^\bullet(S_i,S_e,W)\psi_b\\
&=-\langle S_i,k_i\lambda\partial_x\psi_i+\psi_v\rangle-\langle S_e,k_eh(\lambda)e^{-\frac{\lambda}{2}x}\psi_i+\partial_x^2\psi_e+g(\lambda L)\psi_e-e^{-\frac{\lambda}{2}x}\psi_v\rangle+\langle W,\partial_x^2\psi_v\rangle\\
&=\langle(S_i,S_e,W),(\mathcal{L}^\bullet)^*(\psi_i,\psi_e,\psi_v,\psi_b)\rangle.
\end{aligned}
$$

This proves the claim.

Next we compute $N((\mathcal{L}^\bullet)^*)$. To this end, we seek solutions $(\psi_i,\psi_e,\psi_v,\psi_b)\in D((\mathcal{L}^\bullet)^*)$ to the problem $(\mathcal{L}^\bullet)^*(\psi_i,\psi_e,\psi_v,\psi_b)=0$. From $(\mathcal{L}^\bullet)_3^*=0$ and boundary conditions $\psi_v(0)=\psi_v(L)=0$, we see that

$$\psi_v=0. \tag{3.13}$$

From this and $(\mathcal{L}^\bullet)_1^*=(\mathcal{L}^\bullet)_2^*=0$, we have the equations

$$\partial_x\psi_i=0, \tag{3.14a}$$

$$-\partial_x^2\psi_e-g(\lambda L)\psi_e-k_eh(\lambda)e^{-\frac{\lambda}{2}x}\psi_i=0. \tag{3.14b}$$

Owing to (3.12e) and substituting $\psi_b=\psi_e(L)$, the boundary conditions for this system are

$$\psi_i(L)-\frac{\gamma}{k_e}e^{\frac{\lambda}{2}L}\psi_e(L)=0, \tag{3.14c}$$

$$\psi_e(0)=0, \tag{3.14d}$$

$$\partial_x\psi_e(L)+\frac{\lambda}{2}\psi_e(L)=0. \tag{3.14e}$$

Now it remains to solve the problem (3.14) in order to check the dimension of $N((\mathcal{L}^\bullet)^*)$.

Let us reduce the problem (3.14) to a problem with a scalar equation for ψ_e alone. Integrating (3.14a) over $[x,L]$ and using (3.14c), we obtain

$$\psi_i(x)=\frac{\gamma}{k_e}e^{\frac{\lambda}{2}L}\psi_e(L). \tag{3.15}$$

Plugging this into (3.14b), we have the problem for ψ_e:

$$-\partial_x^2\psi_e-g(\lambda L)\psi_e=\gamma h(\lambda)e^{\frac{\lambda}{2}L}\psi_e(L)e^{-\frac{\lambda}{2}x}, \tag{3.16}$$

together with (3.14d) and (3.14e).

Then, regarding $\psi_e(L)$ on the left hand side of (3.16) as a given value, we have general solutions to (3.16):

$$\psi_e = \begin{cases} Ae^{\sqrt{-g(\lambda L)}x} + Be^{-\sqrt{-g(\lambda L)}x} - \gamma e^{\frac{\lambda}{2}L}\psi_e(L)e^{-\frac{\lambda}{2}x} & \text{if } g(\lambda L) < 0, \\ Ax + B - \gamma e^{\frac{\lambda}{2}L}\psi_e(L)e^{-\frac{\lambda}{2}x} & \text{if } g(\lambda L) = 0, \\ A\sin\sqrt{g(\lambda L)}x + B\cos\sqrt{g(\lambda L)}x - \gamma e^{\frac{\lambda}{2}L}\psi_e(L)e^{-\frac{\lambda}{2}x} & \text{if } g(\lambda L) > 0. \end{cases} \tag{3.17}$$

We do a separate but similar calculation in each case.

Case $g<0$. We write $g = g(\lambda L)$ and put $x = L$ in (3.17). Then we see that

$$\psi_e(L) = \frac{1}{1+\gamma}(Ae^{\sqrt{-g}L} + Be^{-\sqrt{-g}L}).$$

This and (3.14d) give

$$0 = \psi_e(0) = A + B - \frac{\gamma}{1+\gamma}e^{\frac{\lambda}{2}L}(Ae^{\sqrt{-g}L} + Be^{-\sqrt{-g}L}).$$

Furthermore, from (3.14e) and $\partial_x(e^{-\frac{\lambda}{2}x}) + \frac{\lambda}{2}e^{-\frac{\lambda}{2}x} = 0$, it must hold that

$$0 = \partial_x\psi_e(L) + \frac{\lambda}{2}\psi_e(L) = \sqrt{-g}\left(Ae^{\sqrt{-g}L} - Be^{-\sqrt{-g}L}\right) + \frac{\lambda}{2}\left(Ae^{\sqrt{-g}L} + Be^{-\sqrt{-g}L}\right).$$

Summarizing these two, we have a linear system for the pair (A,B):

$$M^-\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad M^- := \begin{bmatrix} 1 - \frac{\gamma}{1+\gamma}e^{\frac{\lambda}{2}L}e^{\sqrt{-g}L} & 1 - \frac{\gamma}{1+\gamma}e^{\frac{\lambda}{2}L}e^{-\sqrt{-g}L} \\ \sqrt{-g}e^{\sqrt{-g}L} + \frac{\lambda}{2}e^{\sqrt{-g}L} & -\sqrt{-g}e^{-\sqrt{-g}L} + \frac{\lambda}{2}e^{-\sqrt{-g}L} \end{bmatrix}.$$

It has nontrivial solutions if and only if $\det M^- = 0$. Then the kernel is one-dimensional since m_{21}^- is positive. On the other hand, it holds that

$$\begin{aligned} \det M^- &= \left(-\sqrt{-g}e^{-\sqrt{-g}L} + \frac{\lambda}{2}e^{-\sqrt{-g}L} - \sqrt{-g}e^{\sqrt{-g}L} - \frac{\lambda}{2}e^{\sqrt{-g}L}\right) \\ &\quad + \frac{\gamma}{1+\gamma}e^{\frac{\lambda}{2}L}\left(\sqrt{-g} - \frac{\lambda}{2} + \sqrt{-g} + \frac{\lambda}{2}\right) \\ &= -2\sqrt{-g}D(V_c). \end{aligned}$$

Hence we conclude that $N((\mathcal{L}^\bullet)^*)$ is at most one-dimensional, and it is one-dimensional if and only if $D(V_c) = 0$.

Case $g=0$. Putting $x = L$ in (3.17), we have $\psi_e(L) = AL + B - \gamma\psi_e(L)$. In the same way as above, using (3.14d) and (3.14e), we have

$$M^0\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad M^0 := \begin{bmatrix} -\frac{\gamma}{1+\gamma}e^{\frac{\lambda}{2}L}L & 1 - \frac{\gamma}{1+\gamma}e^{\frac{\lambda}{2}L} \\ 1 + \frac{\lambda}{2}L & \frac{\lambda}{2} \end{bmatrix}.$$

Note that $1 + \frac{\lambda}{2}L > 0$ and

$$\det M^0 = -\left(1 + \frac{\lambda}{2}L\right) + \frac{\gamma}{1+\gamma}e^{\frac{\lambda}{2}L}\left(-\frac{\lambda}{2}L + 1 + \frac{\lambda}{2}L\right) = -D(V_c).$$

Hence we conclude that $N((\mathcal{L}^\bullet)^*)$ is at most one-dimensional, and it is one-dimensional if and only if $D(V_c) = 0$.

Case $g > 0$. We write $g = g(\lambda L)$ and put $x = L$ in (3.17). Then

$$\psi_e(L) = \frac{1}{1+\gamma}(A \sin \sqrt{g}L + B \cos \sqrt{g}L).$$

This, (3.14d) and (3.14e) give us the identities

$$0 = \psi_e(0) = B - \frac{\gamma}{1+\gamma} e^{\frac{\lambda}{2}L}(A \sin \sqrt{g}L + B \cos \sqrt{g}L),$$

$$0 = \partial_x \psi_e(L) + \frac{\lambda}{2}\psi_e(L) = \sqrt{g}(A \cos \sqrt{g}L - B \sin \sqrt{g}L) + \frac{\lambda}{2}(A \sin \sqrt{g}L + B \cos \sqrt{g}L).$$

Summarizing these two, we have a linear equation for (A, B):

$$M^+ \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad M^+ := \begin{bmatrix} -\frac{\gamma}{1+\gamma} e^{\frac{\lambda}{2}L} \sin \sqrt{g}L & 1 - \frac{\gamma}{1+\gamma} e^{\frac{\lambda}{2}L} \cos \sqrt{g}L \\ \sqrt{g} \cos \sqrt{g}L + \frac{\lambda}{2} \sin \sqrt{g}L & -\sqrt{g} \sin \sqrt{g}L + \frac{\lambda}{2} \cos \sqrt{g}L \end{bmatrix}.$$

But note that

$$\begin{aligned} \det M^+ &= -\sqrt{g} \cos \sqrt{g}L + \frac{\lambda}{2} \sin \sqrt{g}L \\ &\quad + \frac{\gamma}{1+\gamma} e^{\frac{\lambda}{2}L} \left\{ \sin \sqrt{g}L \left(\sqrt{g} \sin \sqrt{g}L - \frac{\lambda}{2} \cos \sqrt{g}L \right) \right. \\ &\quad \left. + \cos \sqrt{g}L \left(\sqrt{g} \cos \sqrt{g}L + \frac{\lambda}{2} \sin \sqrt{g}L \right) \right\} \\ &= -\sqrt{g} D(V_c). \end{aligned}$$

Hence we conclude that $N((\mathcal{L}^\bullet)^*)$ is at most one-dimensional, and it is one-dimensional if and only if $D(V_c) = 0$. ∎

In order to clarify the variables in the next lemma, we denote $D(V_c) = D(V_c, a, b, \gamma)$. Let us also define

$$A = \{(a, b, \gamma) \in (\mathbb{R}_+)^3 \ ; \ \text{there exists a root of } D(V_c, a, b, \gamma) = 0\}.$$

Then by definition $V_c^\dagger = V_c^\dagger(a, b, \gamma)$ is the smallest root, for any $(a, b, \gamma) \in A$. Let A° be the interior of A. Let $(\varphi_i^\dagger, \varphi_e^\dagger, \varphi_v^\dagger)$ generate the one-dimensional nullspace of $\mathcal{L}$. We also explicitly denote $g(V_c) = g(V_c, a, b) = \frac{aV_c}{L} \exp \frac{-bL}{V_c} - \frac{V_c^2}{4L^2}$. Transversality is the condition that the tangent of the presumed local curve and the tangent of the trivial curve do not coincide.

Lemma 3.4. *The transversality condition*

$$\partial_\lambda \partial_{(\rho_i, R_e, V)} \mathcal{F}(V_c^\dagger/L, 0, 0, 0)[1, \varphi_i^\dagger, \varphi_e^\dagger, \varphi_v^\dagger] \notin R(\partial_{(\rho_i, R_e, V)} \mathcal{F}(V_c^\dagger/L, 0, 0, 0)) \qquad (3.18)$$

is valid for almost every $(a, b, \gamma) \in A^\circ$.

Proof. The first part of the proof is devoted to showing that various sets of points (a, b, γ) have measure zero in $\mathbb{R}^3$. It is easy to check that $g(V_c, a, b) = 0$ has no solution if $a < \frac{e}{4}b$, exactly one solution $W_0(a, b)$ if $a = \frac{e}{4}b$, and exactly two solutions

$W_1(a,b), W_2(a,b)$ if $a > \frac{e}{4}b$. The set $Z_0 = \{(a,b,\gamma) \in (\mathbb{R}_+)^3 \ ; \ a = \frac{e}{4}b\}$ obviously has measure zero. On its complement Z_0^c we calculate that $\frac{\partial}{\partial V_c} g(W_j(a,b),a,b) \neq 0$ for $j = 1,2$.

Denoting $\mu = L\sqrt{-g(V_c,a,b)}$ as before, recall the definition (1.4) of the sparking function:

$$D(V_c,a,b,\gamma) = \frac{1}{2}\left(e^{\mu} + e^{-\mu}\right) + \frac{V_c}{4\mu}\left(e^{\mu} - e^{-\mu}\right) - \frac{\gamma}{1+\gamma} e^{\frac{V_c}{2}}. \tag{3.19}$$

A short calculation shows that if both $g(W,a,b) = 0$ and $D(W,a,b,\gamma) = 0$, then

$$\gamma = \left[1 + \tfrac{W}{2}\right]\left[\exp(\tfrac{W}{2}) - 1 - \tfrac{W}{2}\right]^{-1}. \tag{3.20}$$

The set $Z_1 = \{(a,b,\gamma) \in (\mathbb{R}_+)^3 \ ; \ (3.20) \text{ holds, where } g(W,a,b) = 0\}$ obviously has measure zero. Thus it is clear that on the complementary set Z_1^c we have $g(V_c^\dagger(a,b,\gamma),a,b) \neq 0$. Within Z_1^c the implicit function theorem ensures that the functions $W_j(a,b)$ are continuous ($j = 1,2$).

Clearly the set $\tilde{A} := A^\circ \cap Z_0^c \cap Z_1^c$ is open. Now let

$$Z_2 = \left\{(a,b,\gamma) \in \tilde{A} \ ; \ \frac{\partial D}{\partial V_c}(V_c^\dagger,a,b,\gamma) = 0\right\},$$

where $V_c^\dagger = V_c^\dagger(a,b,\gamma)$. We claim that $\tilde{A} \cap Z_2^c$ is an open set. In order to prove the claim, notice that both $g(V_c^\dagger,a,b) \neq 0$ (as shown above) and $\frac{\partial D}{\partial V_c}(V_c^\dagger,a,b,\gamma) \neq 0$ are true on $\tilde{A} \cap Z_2^c$. The sparking function $D(V_c,a,b,\gamma)$ is a real-analytic function of four variables except where $g(V_c,a,b)$ vanishes. So for each point $(a,b,\gamma) \in \tilde{A} \cap Z_2^c$, we can apply the real-analytic version of the implicit function theorem to the equation $D(V_c^\dagger,a,b,\gamma) = 0$. Hence there is a neighborhood of (a,b,γ) in which the function $V_c^\dagger$ is real-analytic and $\frac{\partial D}{\partial V_c}(V_c^\dagger,a,b,\gamma) \neq 0$. Thus $\tilde{A} \cap Z_2^c$ is open. Furthermore, $V_c^\dagger : \tilde{A} \cap Z_2^c \to \mathbb{R}$ is a real-analytic function for which $\frac{\partial D}{\partial V_c}(V_c^\dagger,a,b,\gamma)$ does not vanish.

Next we claim that the set Z_2 also has $\mathbb{R}^3$-measure zero. Within Z_2 both of the equations, $D = 0$ and $\frac{\partial D}{\partial V_c} = 0$, are satisfied by $(V_c^\dagger,a,b,\gamma)$. We calculate

$$\begin{aligned}\frac{\partial D}{\partial V_c} = \frac{-L^2 g'(V_c)}{2\mu}&\left\{(\frac{1}{2} - \frac{V_c}{4\mu^2})(e^{\mu} - e^{-\mu}) + \frac{V_c}{4\mu}(e^{\mu} + e^{-\mu})\right\} \\ &+ \frac{1}{4\mu}(e^{\mu} - e^{-\mu}) - \frac{1}{2}\frac{\gamma}{1+\gamma} e^{\frac{V_c}{2}}.\end{aligned} \tag{3.21}$$

The equation $D - 2\frac{\partial D}{\partial V_c} = 0$ contains no explicit γ. It is a single equation for $(V_c^\dagger,a,b)$. Thus, within Z_2, the function $V_c^\dagger$ depends only on (a,b). Hence, using (3.19) within Z_2, we see that the variable γ is determined uniquely by (a,b). So, due to the Fubini–Tonelli theorem, Z_2 has $\mathbb{R}^3$-measure zero.

Now we define the function

$$\begin{aligned}F(a,b,\gamma) := -\gamma e^{\frac{V_c^\dagger}{2}} \psi_e(L) &\int_0^L \left\{h'(V_c^\dagger/L) - \frac{x}{2} h(V_c^\dagger/L)\right\} e^{-\frac{V_c^\dagger}{2L}x} \varphi_e^\dagger(x)\, dx \\ &- L g'(V_c^\dagger) \int_0^L \psi_e(x) \varphi_e^\dagger(x)\, dx \\ &+ \frac{1}{2}\psi_e(L)\left\{\varphi_e^\dagger(L) - L\partial_x \varphi_e^\dagger(L) - \frac{V_c^\dagger}{2}\varphi_e^\dagger(L)\right\},\end{aligned} \tag{3.22}$$

where ψ_e is given in (3.17) and $\varphi_e^\dagger$ is equal to the function S_e in (3.9) with (3.2). In (3.22) the functions $V_c^\dagger, \psi_e$ and $\varphi_e^\dagger$ depend on the parameters (a,b,γ). Not only is $V_c^\dagger : \tilde{A} \cap Z_2^c \to \mathbb{R}$ real-analytic, but we observe from (3.9) and (3.17) that $\varphi_e^\dagger$ and ψ_e also depend analytically on (a,b,γ). It follows that the set $Z_3 = \{(a,b,\gamma) \in \tilde{A}\ ;\ F(a,b,\gamma) = 0\}$ also has measure zero because the zero set of any analytic function $\not\equiv 0$ must have measure zero. In the rest of the proof we will only consider the set $\mathcal{A} = \tilde{A} \cap Z_2^c \cap Z_3^c = A^\circ \cap Z_0^c \cap Z_1^c \cap Z_2^c \cap Z_3^c$. Because of the definition of Z_3, we know that $F(a,b,\gamma) \neq 0$ within $\mathcal{A}$.

By differentiating (3.3)–(3.6) with respect to λ, we see that

$$\partial_\lambda \partial_{(\rho_i,R_e,V)} \mathcal{F}_1(\lambda,0,0,0)[1,\varphi_i^\dagger,\varphi_e^\dagger,\varphi_v^\dagger] = k_i \partial_x \varphi_i^\dagger - k_e \left\{ h'(\lambda)\, e^{-\frac{\lambda}{2}x} - \frac{x}{2} h(\lambda)\, e^{-\frac{\lambda}{2}x} \right\} \varphi_e^\dagger, \tag{3.23}$$

$$\partial_\lambda \partial_{(\rho_i,R_e,V)} \mathcal{F}_2(\lambda,0,0,0)[1,\varphi_i^\dagger,\varphi_e^\dagger,\varphi_v^\dagger] = -Lg'(\lambda L)\varphi_e^\dagger, \tag{3.24}$$

$$\partial_\lambda \partial_{(\rho_i,R_e,V)} \mathcal{F}_3(\lambda,0,0,0)[1,\varphi_i^\dagger,\varphi_e^\dagger,\varphi_v^\dagger] = -\frac{x}{2} e^{-\frac{\lambda}{2}x} \varphi_e^\dagger, \tag{3.25}$$

$$\partial_\lambda \partial_{(\rho_i,R_e,V)} \mathcal{F}_4(\lambda,0,0,0)[1,\varphi_i^\dagger,\varphi_e^\dagger,\varphi_v^\dagger] = \frac{1}{2}\varphi_e^\dagger(L) - \gamma \frac{k_i}{k_e} \left(e^{\frac{\lambda}{2}L} + \frac{L}{2}\lambda e^{\frac{\lambda}{2}L} \right) \varphi_i^\dagger(L). \tag{3.26}$$

On the other hand, consider the range $R(\partial_{(\rho_i,R_e,V)}\mathcal{F}(V_c/L,0,0,0))$, which is given in (3.10) and (3.11). Owing to these formulas together with (3.12e), (3.13), and (3.15), the transversality condition (3.18) can be written as

$$\begin{aligned} &\frac{\gamma}{k_e} e^{\frac{V_c^\dagger}{2}} \psi_e(L) \int_0^L \left[k_i \partial_x \varphi_i^\dagger(x) - k_e \left\{ h'(V_c^\dagger/L) - \frac{x}{2} h(V_c^\dagger/L) \right\} e^{-\frac{V_c^\dagger}{2L}x} \varphi_e^\dagger(x) \right] dx \\ &-Lg'(V_c^\dagger) \int_0^L \psi_e(x)\varphi_e^\dagger(x)\,dx + \psi_e(L) \left\{ \frac{1}{2}\varphi_e^\dagger(L) - \gamma\frac{k_i}{k_e} \left(e^{\frac{V_c^\dagger}{2}} + \frac{V_c^\dagger}{2} e^{\frac{V_c^\dagger}{2}} \right) \varphi_i^\dagger(L) \right\} \neq 0. \end{aligned} \tag{3.27}$$

This is what we have to prove. However, the first and last terms in (3.27) add up to

$$\begin{aligned} &\frac{\gamma}{k_e} e^{\frac{V_c^\dagger}{2}} \psi_e(L) \int_0^L k_i \partial_x \varphi_i^\dagger(x)\,dx - \gamma \frac{k_i}{k_e} \left(e^{\frac{V_c^\dagger}{2}} + \frac{V_c^\dagger}{2} e^{\frac{V_c^\dagger}{2}} \right) \psi_e(L)\varphi_i^\dagger(L) \\ &= -\gamma \frac{k_i}{k_e} \frac{V_c^\dagger}{2} e^{\frac{V_c^\dagger}{2}} \psi_e(L)\varphi_i^\dagger(L) = -\frac{1}{2}\psi_e(L) \left\{ L\partial_x \varphi_e^\dagger(L) + \frac{V_c^\dagger}{2} \varphi_e^\dagger(L) \right\}. \end{aligned}$$

The last equality is due to (3.6) and the fact that $(\varphi_i,\varphi_e,\varphi_v) \in N(\partial_{(\rho_i,R_e,V)}\mathcal{F}(V_c^\dagger/L,0,0,0))$. Substituting this simple equality into (3.27) shows that the transversality condition (3.22) is precisely the same as $F(a,b,c) \neq 0$, which we have already shown is true within $\mathcal{A}$. We previously showed that the complement of $\mathcal{A}$ has measure zero. ◻

4. Global Bifurcation. In this section, we apply a functional-analytic global bifurcation theorem to the stationary problem (2.2). The theory of global bifurcation goes back to Rabinowitz [18] using topological degree. For a nice exposition see [12]. A different version using analytic continuation goes back to Dancer [8] with major improvements in [4] and a final improvement in [7]. The specific version that is most convenient to use here is Theorem 6 in [7], which is the following:

THEOREM 4.1 ([7]). *Let X and Y be Banach spaces, $\mathcal{O}$ be an open subset of $\mathbb{R} \times X$ and $\mathcal{F} : \mathcal{O} \to Y$ be a real-analytic function. Suppose that*

(H1) $(\lambda, 0) \in \mathcal{O}$ *and* $\mathcal{F}(\lambda, 0) = 0$ *for all* $\lambda \in \mathbb{R}$*;*
(H2) *for some* $\lambda^* \in \mathbb{R}$, $N(\partial_u \mathcal{F}(\lambda^*, 0))$ *and* $Y \backslash R(\partial_u \mathcal{F}(\lambda^*, 0))$ *are one-dimensional, with the null space generated by* u^**, which satisfies the transversality condition*

$$\partial_\lambda \partial_u \mathcal{F}(\lambda^*, 0)(1, u^*) \notin R(\partial_u \mathcal{F}(\lambda^*, 0)),$$

where ∂_u *and* $\partial_\lambda \partial_u$ *mean Fréchet derivatives for* $(\lambda, u) \in \mathcal{O}$*, and* $N(\mathcal{L})$ *and* $R(\mathcal{L})$ *denote the null space and range of a linear operator* $\mathcal{L}$ *between two Banach spaces;*
(H3) $\partial_u \mathcal{F}(\lambda, u)$ *is a Fredholm operator of index zero for any* $(\lambda, u) \in \mathcal{O}$ *that satisfies the equation* $\mathcal{F}(\lambda, u) = 0$*;*
(H4) *for some sequence* $\{\mathcal{O}_j\}_{j \in \mathbb{N}}$ *of bounded closed subsets of* $\mathcal{O}$ *with* $\mathcal{O} = \cup_{j \in \mathbb{N}} \mathcal{O}_j$*, the set* $\{(\lambda, u) \in \mathcal{O}; \mathcal{F}(\lambda, u) = 0\} \cap \mathcal{O}_j$ *is compact for each* $j \in \mathbb{N}$*.*

Then there exists in $\mathcal{O}$ *a continuous curve* $\mathcal{K} = \{(\lambda(s), u(s)); s \in \mathbb{R}\}$ *of* $\mathcal{F}(\lambda, u) = 0$ *such that:*
(C1) $(\lambda(0), u(0)) = (\lambda^*, 0)$*;*
(C2) $u(s) = su^* + o(s)$ *in* X *as* $s \to 0$*;*
(C3) *there exists a neighborhood* $\mathcal{W}$ *of* $(\lambda^*, 0)$ *and* $\varepsilon > 0$ *sufficiently small such that*

$$\{(\lambda, u) \in \mathcal{W}; u \neq 0 \text{ and } \mathcal{F}(\lambda, u) = 0\} = \{(\lambda(s), u(s)); 0 < |s| < \varepsilon\};$$

(C4) $\mathcal{K}$ *has a real-analytic reparametrization locally around each of its points;*
(C5) *one of the following two alternatives occurs:*
(I) *for every* $j \in \mathbb{N}$*, there exists* $s_j > 0$ *such that* $(\lambda(s), u(s)) \notin \mathcal{O}_j$ *for all* $s \in \mathbb{R}$ *with* $|s| > s_j$*;*
(II) *there exists* $T > 0$ *such that* $(\lambda(s), u(s)) = (\lambda(s+T), u(s+T))$ *for all* $s \in \mathbb{R}$*.*

Moreover, such a curve of solutions of $\mathcal{F}(\lambda, u) = 0$ *having the properties (C1)-(C5) is unique (up to reparametrization).*

Hypothesis $(H2)$ is the same local bifurcation condition as in Theorem 3.1, while $(H3)$ and $(H4)$ are the global ones. $(C1) - (C3)$ are local conclusions, $(C4)$ is a statement of regularity, which is a consequence of the real-analyticity of $\mathcal{F}$. $(C5)$ is the global conclusion which states that *either* the curve reaches the boundary of the set $\mathcal{O}_j$ *or* the curve is periodic (that is, forms a closed loop). The hypotheses $(H3)$ and $(H4)$ are validated in Lemmas 4.2 and 4.3, respectively. For that purpose, consider the linearized operator around an arbitrary triple of functions $(\rho_i^0, R_e^0, V^0) \in X$.

LEMMA 4.2. *For any* $(\lambda, \rho_i^0, R_e^0, V^0) \in \mathcal{O}$*, the Fréchet derivative* $\mathcal{L}^0 = \partial_{(\rho_i, R_e, V)} \mathcal{F}(\lambda, \rho_i^0, R_e^0, V^0)$ *is a linear Fredholm operator of index zero from* X *to* Y*.*

Proof. For any fixed choice of $(\lambda, \rho_i^0, R_e^0, V^0)$, we know that $\inf_x \partial_x V^0 + \lambda > 0$. The operator $\mathcal{L}^0 = (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4)$ acting linearly on the triple $(S_i, S_e, W) \in X$ has the form

$$\mathcal{L}_1 = \mathcal{L}_1(S_i, S_e, W) = k_i \partial_x(\{\partial_x V^0 + \lambda\} S_i) + b_1 \partial_x^2 W + b_2 S_i + b_3 S_e + b_4 \partial_x W, \tag{4.1}$$

$$\mathcal{L}_2 = \mathcal{L}_2(S_i, S_e, W) = -\partial_x^2 S_e + a_1 \partial_x S_e + b_5 S_e + b_6 \partial_x^2 W + b_7 \partial_x W, \tag{4.2}$$

$$\mathcal{L}_3 = \mathcal{L}_3(S_i, S_e, W) = -\partial_x^2 W + a_2 S_i + a_3 S_e, \tag{4.3}$$

$$\begin{aligned} \mathcal{L}_4 = \mathcal{L}_3(S_i, S_e, W) = {} & \partial_x S_e(L) + (\partial_x V^0(L) + \tfrac{\lambda}{2}) S_e(L) + \partial_x W(L) R_e^0(L) \\ & - \tfrac{\gamma k_i}{k_e} \exp(\tfrac{\lambda}{2} L)[\partial_x V^0(L) + \lambda) S_i(L) + \partial_x W(L) \rho_i^0(L)], \end{aligned} \tag{4.4}$$

where the coefficients $a_1 = -\partial_x V^0$, a_2 and a_3 belong to $C^1([0,L])$ and the coefficients $b_1, ..., b_7$ belong to $C^0([0,L])$.

Let us first show that the linear operator $\mathcal{L}^0$ has a finite-dimensional nullspace and a closed range. By [22, Theorem 12.12] or [3, Exercise 6.9.1], it is equivalent to prove that $\mathcal{L}^0$ satisfies the estimate

$$C\|(S_i, S_e, W)\|_X \leq \|\mathcal{L}^0(S_i, S_e, W)\|_Y + \|(S_i, S_e, W)\|_Z \tag{4.5}$$

for all $(S_i, S_e, W) \in X$ and for some constant C depending only on $(\lambda, \rho_i^0, R_e^0, V^0)$, where

$$Z := C^0([0,L]) \times C^0([0,L]) \times C^1([0,L]).$$

Keeping in mind that $\partial_x V^0 + \lambda \geq 1/j$, we see from (4.1) and (4.4) that S_i can be estimated by

$$\begin{aligned}
&\|\partial_x S_i\|_{C^0} \\
&= \|(\partial_x V^0 + \lambda)^{-1} \left(\{\partial_x(\partial_x V^0 + \lambda)\} S_i + b_1 \partial_x^2 W + b_2 S_i + b_3 S_e + b_4 \partial_x W - \mathcal{L}_1\right) \|_{C^0} \\
&\leq C(\|S_i\|_{C^0} + \|S_e\|_{C^0} + \|W\|_{C^2} + \|\mathcal{L}_1\|_{C^0}) \\
&\leq C\|\mathcal{L}^0(S_i, S_e, W)\|_Y + C\|(S_i, S_e, W)\|_Z.
\end{aligned} \tag{4.6}$$

Next, (4.4) leads to the required estimate of W as follows:

$$\|\partial_x^2 W\|_{C^1} = \|a_2 S_i + a_3 S_e - \mathcal{L}_3\|_{C^1} \leq C\|\mathcal{L}^0(S_i, S_e, W)\|_Y + C\|(S_i, S_e, W)\|_Z. \tag{4.7}$$

We also have $\|\partial_x W\|_{C^0} \leq L\|\partial_x^2 W\|_{C^0}$ because $\int_0^L \partial_x W(x)dx = 0$.

Finally, we estimate S_e as follows. Due to the bounds on S_i and W, the equation (4.2) implies that $\partial_x^2 S_e + (\partial_x V^0)\partial_x S_e$ is bounded by the right side of (4.5). Furthermore, $S_e(0) = 0$ and $\partial_x S_e(L) + (\partial_x V^0(L) + \frac{\lambda}{2})S_e(L)$ is also bounded. Thus $\partial_x\{\partial_x S_e + (\partial_x V^0)S_e\}$ is also bounded. Integrating from x to L, we find that

$$\partial_x S_e(x) + \partial_x V^0(x) S_e(x) - \partial_x S_e(L) + \partial_x V^0(L) S_e(L)$$

is also bounded, whence $\partial_x S_e(x)$ is bounded as well. The preceding estimates on S_i, W and S_e prove (4.5).

Owing to the fact $\lim_{V_c \to 0} D(V_c) > 0$, we can find a constant $V_c' > 0$ such that $D(V_c') > 0$. The preceding lemmas state that the nullspace of $\partial_{(\rho_i, R_e, V)}\mathcal{F}(V_c'/L, 0, 0, 0)$ has dimension zero and the codimension of its range is also zero, so that its index is zero. Because $\mathcal{O}$ is connected and the index is a topological invariant [2, Theorem 4.51, p166], $\mathcal{L}^0$ also has index zero. This means that the codimension of $\mathcal{L}^0$ is also finite. This completes the proof of Lemma 4.2. ☐

LEMMA 4.3. *For each $j \in \mathbb{N}$, the set $K_j = \{(\lambda, \rho_i, R_e, V) \in \mathcal{O}_j;\ \mathcal{F}(\lambda, \rho_i, R_e, V) = 0\}$ is compact in $\mathbb{R} \times X$.*

Proof. Let $\{(\lambda_n, \rho_{in}, R_{en}, V_n)\}$ be any sequence in K_j. It suffices to show that it has a convergent subsequence whose limit also belongs to K_j. By the assumed bound $|\lambda_n| + \|(\rho_{in}, R_{en}, V_n)\|_X \leq j$, there exists a subsequence, still denoted by $\{(\lambda_n, \rho_{in}, R_{en}, V_n)\}$, and $(\lambda, \rho_i, R_e, V)$ such that

$$\left\{\begin{array}{llll}
\lambda_n & \to & \lambda & \text{in } \mathbb{R}, \\
\rho_{in} & \to & \rho_i & \text{in } C^0([0,L]), \\
R_{en} & \to & R_e & \text{in } C^1([0,L]), \\
V_n & \to & V & \text{in } C^2([0,L]).
\end{array}\right. \tag{4.8}$$

Furthermore,

$$\partial_x V + \lambda \geq \tfrac{1}{j}.$$

Since $\mathcal{O}_j$ is closed in X, it remains to show that

$$\mathcal{F}_j(\lambda, \rho_i, R_e, V) = 0 \quad \text{for } j = 1, 2, 3, 4,$$
$$\rho_{in} \to \rho_i \text{ in } C^1([0, L]), \quad R_{en} \to R_e \text{ in } C^2([0, L]), \quad V_n \to V \text{ in } C^3([0, L]).$$

Now the first equation $\mathcal{F}_1(\lambda_n, \rho_{in}, R_{en}, V_n) = 0$ with $\rho_{in}(0) = 0$ is equivalent to

$$\rho_{in}(x) = \frac{k_e}{k_i}(\partial_x V_n(x) + \lambda_n)^{-1} \int_0^x h(\partial_x V_n(y) + \lambda_n) e^{-\frac{\lambda_n}{2} y} R_{en}(y)\, dy.$$

Taking the limit and using (4.8), we see that

$$\rho_i(x) = \frac{k_e}{k_i}(\partial_x V(x) + \lambda)^{-1} \int_0^x h(\partial_x V(y) + \lambda) e^{-\frac{\lambda}{2} y} R_e(y)\, dy,$$

where the right hand side converges in $C^1([0, L])$. Hence, we see that $\mathcal{F}_1(\lambda, \rho_i, R_e, V) = 0$ and $\rho_{in} \to \rho_i$ in $C^1([0, L])$.

Taking the limit using (4.8) in the third equation $\mathcal{F}_3(\lambda_n, \rho_{in}, R_{en}, V_n) = 0$ immediately leads to

$$\partial_x^2 V = \rho_i - e^{-\frac{\lambda}{2} x} R_e.$$

Hence $\mathcal{F}_3(\lambda, \rho_i, R_e, V) = 0$ and $V_n \to V$ in $C^3([0, L])$.

The second equation $\mathcal{F}_2(\lambda_n, \rho_{in}, R_{en}, V_n) = 0$ can be written as

$$\partial_x\{\partial_x R_{en} - (\partial_x V_n) R_{en}\} = \{\tfrac{\lambda_n}{2} + \tfrac{\lambda_n{}^2}{4} - h(\partial_x V_n + \lambda_n)\} R_{en}.$$

Because the right side converges in $C^1([0, L])$, we see that $\{\partial_x R_{en} - (\partial_x V_n) R_{en}\}$ converges in $C^2([0, L])$. But $(\partial_x V_n) R_{en}$ converges in $C^1([0, L])$. Hence $\partial_x R_{en}$ converges in $C^1([0, L])$, which means that R_{en} converges to R in $C^2([0, L])$.

It is obvious from (4.8) and $\mathcal{F}_4(\lambda_n, \rho_{in}, R_{en}, V_n) = 0$ that $\mathcal{F}_4(\lambda, \rho_i, R_e, V) = 0$ holds. $\square$

As we have checked all conditions in Theorem 4.1, the following conclusion is valid.

Theorem 4.4. *Assume that the sparking voltage $V_c^\dagger$, defined by* (1.5), *exists. There exists in the open set $\mathcal{O}$ a continuous curve $\mathcal{K} = \{(\lambda(s), \rho_i(s), R_e(s), V(s)); s \in \mathbb{R}\} \subset \mathbb{R} \times X$ of stationary solutions to problem* (2.2) *such that*

(C1) $(\lambda(0), \rho_i(0), R_e(0), V(0)) = (V_c^\dagger/L, 0, 0, 0)$, *where $V_c^\dagger$ is defined in* (1.5);

(C2) $(\rho_i(s), R_e(s), V(s)) = s(\varphi_i^\dagger, \varphi_e^\dagger, \varphi_v^\dagger) + o(s)$ *in the space X as $s \to 0$, where $(\varphi_i^\dagger, \varphi_e^\dagger, \varphi_v^\dagger)$ is a basis with* (3.1) *of $N(\partial_{(\rho_i, R_e, V)} \mathcal{F}(V_c^\dagger/L, 0, 0, 0))$.*

(C3) there exists a neighborhood $\mathcal{W}$ of $(V_c^\dagger/L, 0, 0, 0)$ and $\varepsilon < 1$ such that

$$\{(\lambda, \rho_i, R_e, V) \in \mathcal{W}; (\rho_i, R_e, V) \neq (0, 0, 0),\ \mathcal{F}(\lambda, \rho_i, R_e, V) = 0\}$$
$$= \{(\lambda(s), \rho_i(s), R_e(s), V(s)); 0 < |s| < \varepsilon\};$$

(C4) $\mathcal{K}$ has a real-analytic reparametrization locally around each of its points;

(C5) at least one of the following four alternatives occurs:
(a) $\underline{\lim}_{s\to\infty} \lambda(s) = 0$;
(b) $\underline{\lim}_{s\to\infty} (\inf_{x\in I} \partial_x V(x,s) + \lambda(s)) = 0$;
(c) $\overline{\lim}_{s\to\infty} (\|\rho_i\|_{C^1} + \|R_e\|_{C^2} + \|V\|_{C^3} + \lambda)(s) = \infty$;
(d) there exists $T > 0$ *such that*

$$(\lambda(s), \rho_i(s), R_e(s), V(s)) = (\lambda(s+T), \rho_i(s+T), R_e(s+T), V(s+T))$$

for all $s \in \mathbb{R}$.

Moreover, such a curve of solutions to problem (2.2) *having the properties (C1)-(C5) is unique (up to reparametrization).*

Conditions (C1)-(C3) are an expression of the local bifurcation, while (C4)-(C5) are assertions about the global curve $\mathcal{K}$. Alternative (c) asserts that $\mathcal{K}$ may be unbounded. Alternative (d) asserts that $\mathcal{K}$ may form a closed curve (a 'loop').

5. Positive Densities. Of course, we should keep in mind that for the physical problem ρ_i and R_e are densities of particles and so they should be non-negative. In this section we investigate the part of the curve $\mathcal{K}$ that corresponds to such densities. We will often suppress the variable x, as in $\rho_i(s) = \rho_i(s,\cdot), R_e(s) = R_e(s,\cdot), V(s) = V(s,\cdot)$.

A basic observation is the following theorem, which states that *either (i)* ρ_i and R_e remain positive *or (ii)* the curve of positive solutions forms a half-loop going from $V_c^\dagger$ to some other voltage $V_c^\ddagger$. Here $V_c^\dagger$ is defined in (1.5) and $V_c^\ddagger$ is a voltage with (3.2) and $V_c^\dagger < V_c^\ddagger$. We remark that the curve $\mathcal{K}$ is never the half-loop unless a voltage $V_c^\ddagger > V_c^\dagger$ exists satisfying (3.2).

Theorem 5.1. *Assume the sparking voltage* $V_c^\dagger$ *exists. For the global bifurcation curve* $\mathcal{K} = (\lambda(s), \rho_i(s), R_e(s), V(s))$ *in Theorem 4.4, one of the following two alternatives occurs.*
(i) $\rho_i(s,x) > 0$ *and* $R_e(s,x) > 0$ *for all* $0 < s < \infty$ *and* $x \in (0,L]$.
(ii) there exists a voltage $V_c^\ddagger$ *satisfying* (3.2) *and* $V_c^\dagger < V_c^\ddagger$ *and a finite parameter value* $s^\ddagger > 0$ *such that*
(1) $\rho_i(s,x) > 0$ *and* $R_e(s,x) > 0$ *for all* $s \in (0,s^\ddagger)$ *and* $x \in (0,L]$;
(2) $(\lambda(s^\ddagger), \rho_i(s^\ddagger), R_e(s^\ddagger), V(s^\ddagger)) = (V_c^\ddagger/L, 0, 0, 0)$;
(3) $(\rho_i(s), R_e(s)) = (s^\ddagger - s)(\varphi_i^\ddagger, \varphi_e^\ddagger) + o(|s - s^\ddagger|)$ *as* $s \nearrow s^\ddagger$, *where* $(\varphi_i^\ddagger, \varphi_e^\ddagger)$ *is a basis with* (3.1) *of* $N(\partial_{(\rho_i,R_e,V)}\mathcal{F}(V_c^\ddagger/L, 0,0,0))$;
(4) $\rho_i(s,x) < 0$ *and* $R_e(s,x) < 0$ *for* $0 < s - s^\ddagger \ll 1$ *and* $x \in (0,L]$.

Proof. First let us define

$$s^\ddagger := \inf\{s > 0 : R_e(s,x_0) = 0 \text{ for some } x_0 \in (0,L]\}. \tag{5.1}$$

Clearly $R_e > 0$ in $(0,s^\ddagger) \times (0,L]$. By (C2) in Theorem 4.4, $s^\ddagger > 0$. If $s^\ddagger = \infty$, then $R_e > 0$ in $(0,\infty) \times (0,L]$. Also $\partial_x V + \lambda$ is positive owing to $(\lambda(s), \rho_i(s), R_e(s), V(s)) \in \mathcal{O}$. Then the following formula from (2.1a) also yields $\rho_i > 0$.

$$\rho_i(x) = \frac{k_e}{k_i}(\partial_x V(x) + \lambda)^{-1} \int_0^x h(\partial_x V(y) + \lambda) e^{-\frac{\lambda}{2}y} R_e(y)\, dy. \tag{5.2}$$

Thus alternative *(i)* is valid.

Assuming that $s^\ddagger < \infty$, we will show that *(ii)* happens. First we will show that $R_e(s^\ddagger,\cdot)$ vanishes identically. Certainly $R_e(s^\ddagger,\cdot)$ takes the value zero, which is its minimum, at some point $x_0 \in \bar{I} = [0, L]$. In case $x_0 \in I$, $\partial_x R_e(s^\ddagger, x_0) = 0$ also holds. Solving $\mathcal{F}_2(\lambda, \rho_i, R_e, V) = 0$ with $R_e(s^\ddagger, x_0) = \partial_x R_e(s^\ddagger, x_0) = 0$, we see by uniqueness that $R_e(s^\ddagger) \equiv 0$. Secondly, in case $x_0 = 0$, by (5.1) there exists a sequence $\{(s_n, x_n)\}_{n\in\mathbb{N}}$ such that $R_e(s_n, x_n) = 0$ with $s_n \searrow s^\ddagger$ and $x_n \searrow 0$. Rolle's theorem ensures that there also exists some $y_n \in (0, x_n)$ such that $\partial_x R_e(s_n, y_n) = 0$. Letting $n \to \infty$, we see that $y_n \to 0$ and thus $\partial_x R_e(s^\ddagger, 0) = 0$. Hence we again deduce by uniqueness that $R_e \equiv 0$. Thirdly, in case $x_0 = L$, it is obvious that $\partial_x R_e(s^\ddagger, L) \le 0$. On the other hand, we see from $\mathcal{F}_4 = 0$ and (5.2) that

$$\partial_x R_e(s^\ddagger, L) = \gamma \frac{k_i}{k_e} e^{\frac{\lambda}{2}L} \left(\partial_x V(L) + \lambda\right) \rho_i(s^\ddagger, L) \ge 0.$$

This leads to $\partial_x R_e(s^\ddagger, L) = 0$ so that $R_e \equiv 0$ once again. Therefore we conclude that $R_e \equiv 0$ in every case. By (5.2), we also have $\rho_i \equiv 0$ and thus $V \equiv 0$. Hence $(\rho_i, R_e, V)(s^\ddagger) = (0, 0, 0)$ is the trivial solution. So (1) and (2) in the theorem are valid.

Continuing to assume that $s^\ddagger < \infty$, we now know that ρ_i, R_e and V are identically zero at $s = s^\ddagger$. We define $V_c^\ddagger = L\,\lambda(s^\ddagger)$. By the simple bifurcation theorem of [5], the nullspace $\mathcal{N} = N[\partial_{(\rho_i,R_e,V)}\mathcal{F}(\lambda(s^\ddagger), 0, 0, 0)]$ is non-trivial because the curve $\mathcal{K}$ crosses the trivial curve transversely at $s = s^\ddagger$. So by Lemma 3.2, we have $D(V_c^\ddagger) = 0$. It remains to prove (3) and (4) and also that $V_c^\ddagger > V_c^\dagger$ and $g(V_c^\ddagger) \le \frac{\pi^2}{L^2}$.

Suppose on the contrary that $g(V_c^\ddagger) > \frac{\pi^2}{L^2}$. Then as in the proof of Lemma 3.2, the nullspace $\mathcal{N}$ has a basis $(\varphi_i, \varphi_e, \varphi_v)$ with

$$\varphi_e(x) = \sin\sqrt{g(V_c^\ddagger)}x, \quad \sqrt{g(V_c^\ddagger)} > \frac{\pi}{L}.$$

In that case the function φ_e has a node (changes its sign) in the interval I. Therefore $R_e(s, \cdot)$ also has a node for s near $s^\ddagger$, which contradicts the positivity. Thus $g(V_c^\ddagger) \le \frac{\pi^2}{L^2}$ so that the basis of $\mathcal{N}$ is positive, due to Lemma 3.2. Thus (3) and (4) are valid.

Finally, suppose that $V_c^\ddagger = V_c^\dagger$. Then $\lambda(s^\ddagger) = V_c^\dagger/L$, so that the curve $\mathcal{K}$ goes from the point $P = (V_c^\dagger/L, 0, 0, 0)$ at $s = 0$ to the same point P at $s = s^\ddagger$. By (C3) and (C4) of Theorem 4.4, $\mathcal{K}$ is a simple curve at P and is real-analytic. So the only way $\mathcal{K}$ could go from P to P would be if it were a loop with the part with s approaching $s^\ddagger$ from below coinciding with the part with s approaching 0 from below ($s < 0$). By (C2) of Theorem 4.4, $\rho_i(s, \cdot)$ and $R_e(s, \cdot)$ would be negative for $-1 \ll s - s^\ddagger < 0$, which would contradict their positivity. Hence $V_c^\ddagger > V_c^\dagger$. □

Since ρ_i and $R_e e^{-V_c x/2L}$ are the densities of the ions and electrons, respectively, we are interested only in the positive solutions. Let us investigate in detail the case that the global positivity alternative (i) in Theorem 5.1 occurs. More precisely, the next two lemmas show that if either one of the alternatives (a) or (b) in Theorem 4.4 occurs, then alternative (c) also occurs. In these proofs, we use the written boundary condition from (2.1e) and (5.2):

$$\partial_x R_e(L) = -\left(\partial_x V(L) + \frac{\lambda}{2}\right) R_e(L) + \gamma e^{\frac{\lambda}{2}L} \int_0^L h\left(\partial_x V(x) + \lambda\right) e^{-\frac{\lambda}{2}x} R_e(x)\, dx \quad (5.3)$$

and the elementary Poincaré inequality

$$\|u\|_{L^2} \le \sqrt{L}\|\partial_x u\|_{L^2} \quad \text{for } u \in \{f \in H^1(I);\ f(0) = 0\}. \quad (5.4)$$

LEMMA 5.2. *Assume alternative (i) in Theorem 5.1. If* $\underline{\lim}_{s\to\infty}\lambda(s)=0$*, then* $\sup_{s>0}\|V(s)\|_{C^2}$ *is unbounded.*

Proof. On the contrary suppose that $\sup_{s>0}\|V(s)\|_{C^2}$ is bounded. Because $\underline{\lim}_{s\to\infty}\lambda(s)=0$ and $(\partial_x V+\lambda)(s,x)>0$, there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ and limits (λ^*,V^*) such that

$$\begin{cases} \lambda(s_n) & \to & 0 & \text{in} & \mathbb{R}, \\ V(s_n) & \to & V^* & \text{in} & C^1([0,L]), \end{cases} \tag{5.5}$$

$$V^*(0)=V^*(L)=0, \tag{5.6}$$

$$\partial_x V^* \geq 0. \tag{5.7}$$

The boundary condition (5.6) means that $\int_0^L \partial_x V^*(x)\,dx=0$. This together with (5.7) implies $\partial_x V^*\equiv 0$. Using (5.6) again, we have $V^*\equiv 0$.

It follows that for suitably large n the three expressions $\|h(\partial_x V(s_n)+\lambda(s_n))\|_{C^0}$, $|\lambda(s_n)|$ and $\|V(s_n)\|_{C^2}$, are arbitrarily small. Multiplying $\mathcal{F}_2(\lambda(s_n),\rho_i(s_n),R_e(s_n),V(s_n))=0$ by $R_e(s_n)$ leads to

$$\begin{aligned}(\partial_x R_e)^2(s_n) &= \partial_x\left\{R_e(s_n)\partial_x R_e(s_n)+\partial_x V(s_n)R_e^2(s_n)\right\}-3\partial_x V(s_n)R_e(s_n)\partial_x R_e(s_n)\\ &\quad+\left\{\frac{\lambda(s_n)}{2}\partial_x V(s_n)+\frac{\lambda^2(s_n)}{4}-h\left(\partial_x V(s_n)+\lambda(s_n)\right)\right\}R_e^2(s_n).\end{aligned}$$

Then integrating this by parts over $[0,L]$, using $R_e(s_n,0)=0$, and rewriting $\partial_x R_e(s_n,L)$ by (5.3), we have

$$\begin{aligned}&\int_0^L(\partial_x R_e)^2(s_n)\,dx\\ &=-\left(\partial_x V(s_n,L)+\frac{\lambda(s_n)}{2}\right)R_e^2(s_n,L)\\ &\quad+\gamma e^{\frac{\lambda(s_n)}{2}L}R_e(s_n,L)\int_0^L h\left(\partial_x V(s_n)+\lambda(s_n)\right)e^{-\frac{\lambda(s_n)}{2}x}R_e(s_n)\,dx\\ &\quad-3\int_0^L\partial_x V(s_n)R_e(s_n)\partial_x R_e(s_n)\,dx+\partial_x V(s_n,L)R_e^2(s_n,L)\\ &\quad-\int_0^L\left\{\frac{\lambda(s_n)}{2}\partial_x V(s_n)+\frac{\lambda^2(s_n)}{4}-h\left(\partial_x V(s_n)+\lambda(s_n)\right)\right\}R_e^2(s_n)\,dx\\ &\leq\frac{1}{2}\int_0^L(\partial_x R_e)^2(s_n)\,dx,\end{aligned}$$

where we also have used Sobolev's and Poincaré's inequalities and taken n suitably large in deriving the last inequality. Hence $\partial_x R_e(s_n)\equiv 0$. Since R_e vanishes at $x=0$, we conclude that $R_e(s_n)\equiv 0$, which contradicts the assumed positivity. $\square$

LEMMA 5.3. *Assume alternative (i) in Theorem 5.1. If* $\underline{\lim}_{s\to\infty}\{\inf_{x\in I}\partial_x V(s,x)+\lambda(s)\}=0$*, then* $\sup_{s>0}\{\|\rho_i(s)\|_{C^0}+\|R_e(s)\|_{C^2}+\|V(s)\|_{C^2}+\lambda(s)\}$ *is unbounded.*

Proof. On the contrary, suppose that $\sup_{s>0}\{\|\rho_i(s)\|_{C^0}+\|R_e(s)\|_{C^2}+\|V(s)\|_{C^2}+\lambda(s)\}$ is bounded. We see from $\underline{\lim}_{s\to\infty}\{\inf_{x\in I}\partial_x V(s,x)+\lambda(s)\}=0$ that there exist a sequence $\{s_n\}_{n\in\mathbb{N}}$ and a quadruple $(\lambda^*, \rho_i^*, R_e^*, V^*)$ with $\lambda^* < \infty$ such that

$$\begin{cases} \lambda(s_n) & \to & \lambda^* & \text{in} & \mathbb{R}, & \\ \rho_i(s_n) & \rightharpoonup & \rho_i^* & \text{in} & L^\infty(0,L) & \text{weakly-star}, \\ R_e(s_n) & \to & R_e^* & \text{in} & C^1([0,L]), & \\ \partial_x^2 R_e(s_n) & \rightharpoonup & \partial_x^2 R_e^* & \text{in} & L^\infty(0,L) & \text{weakly-star}, \\ V(s_n) & \to & V^* & \text{in} & C^1([0,L]), & \\ \partial_x^2 V(s_n) & \rightharpoonup & \partial_x^2 V^* & \text{in} & L^\infty(0,L) & \text{weakly-star}, \end{cases} \tag{5.8}$$

$$R_e^*(0) = V^*(0) = V^*(L) = 0, \tag{5.9}$$

$$\rho_i^* \geq 0, \quad R_e^* \geq 0, \tag{5.10}$$

$$\inf_{x\in[0,L]} (\partial_x V^* + \lambda^*)(x) = 0. \tag{5.11}$$

We shall show that

$$\mathcal{F}_j(\lambda^*, \rho_i^*, R_e^*, V^*) = 0 \quad \text{for a.e. } x \text{ and } j = 1, 2, 3.$$

The equation $\mathcal{F}_1(\lambda(s_n), \rho_i(s_n), R_e(s_n), V(s_n)) = 0$ with $\rho_i(s_n, 0) = 0$ is equivalent to

$$(\partial_x V(s_n) + \lambda(s_n))\rho_i(s_n) = \frac{k_e}{k_i} \int_0^x h(\partial_x V(s_n) + \lambda(s_n)) e^{-\frac{\lambda(s)}{2} y} R_e(s_n)\, dy.$$

Multiplying by a test function $\varphi \in C^0([0, L])$ and integrating over $[0, L]$, we obtain

$$\begin{aligned} &\int_0^L (\partial_x V(s_n) + \lambda(s_n))\rho_i(s_n)\varphi\, dx \\ &= \int_0^L \frac{k_e}{k_i} \left(\int_0^x h(\partial_x V(s_n) + \lambda(s_n)) e^{-\frac{\lambda(s_n)}{2} y} R_e(s_n)\, dy \right) \varphi\, dx. \end{aligned} \tag{5.12}$$

We note that

$$\begin{aligned} &\left| \int_0^L \{(\partial_x V(s_n) + \lambda(s_n))\rho_i(s_n) - (\partial_x V^* + \lambda^*)\rho_i^*\} \varphi\, dx \right| \\ &\leq \left| \int_0^L \{\partial_x V(s_n) + \lambda(s_n) - \partial_x V^* - \lambda^*\} \rho_i(s_n)\varphi\, dx \right| \\ &\quad + \left| \int_0^L (\rho_i(s_n) - \rho_i^*)(\partial_x V^* + \lambda^*)\varphi\, dx \right|. \end{aligned}$$

So passing to the limit $n \to \infty$ in (5.12) and using (5.8), we obtain

$$\int_0^L (\partial_x V^* + \lambda^*)\rho_i^* \varphi\, dx = \int_0^L \frac{k_e}{k_i} \left(\int_0^x h(\partial_x V^* + \lambda^*) e^{-\frac{\lambda^*}{2} y} R_e^*\, dy \right) \varphi\, dx$$
$$\text{for any } \varphi \in C^0([0, L]).$$

This immediately gives

$$(\partial_x V^* + \lambda^*)\rho_i^* = \frac{k_e}{k_i} \int_0^x h(\partial_x V^* + \lambda^*) e^{-\frac{\lambda^*}{2} y} R_e^*\, dy \quad \textit{a.e.}, \tag{5.13}$$

which is equivalent to $\mathcal{F}_1(\lambda^*, \rho_i^*, R_e^*, V^*) = 0$ a.e.

We can write $\mathcal{F}_2(\lambda(s_n),\rho_i(s_n),R_e(s_n),V(s_n))=0$ and $R_e(s_n,0)=0$ weakly as

$$\int_0^L \partial_x R_e(s_n)\partial_x\varphi\,dx+\frac{(\lambda(s_n))^2}{4}\int_0^L R_e(s_n)\varphi\,dx$$
$$=-\int_0^L G_{2n}\varphi\,dx \quad \text{for any } \varphi\in H_0^1(0,L),$$

where

$$G_{2n}:=-\partial_x V(s_n)\partial_x R_e(s_n)$$
$$+\left\{\frac{\lambda(s_n)}{2}\partial_x V(s_n)-\partial_x^2 V(s_n)-h\left(\partial_x V(s_n)+\lambda(s_n)\right)\right\}R_e(s_n).$$

Noting that

$$\left|\int_0^L \{\partial_x^2 V(s_n)R_e(s_n)-(\partial_x^2 V^*)R_e^*\}\varphi\,dx\right|$$
$$\le\left|\int_0^L \partial_x^2 V(s_n)(R_e(s_n)-R_e^*)\varphi\,dx\right|+\left|\int_0^L (\partial_x^2 V(s_n)-\partial_x^2 V^*)R_e^*\varphi\,dx\right|,$$

taking the limit $n\to\infty$ in the weak form, and using (5.8), we have

$$\int_0^L (\partial_x R_e^*)(\partial_x\varphi)\,dx+\frac{\lambda^2}{4}\int_0^L R_e^*\varphi\,dx=-\int_0^L G_2^*\varphi\,dx \quad \text{for any } \varphi\in H_0^1(0,L),$$

where

$$G_2^*:=-(\partial_x V^*)\partial_x R_e^*+\left\{\frac{\lambda}{2}\partial_x V^*-\partial_x^2 V^*-h\left(\partial_x V^*+\lambda^*\right)\right\}R_e^*\in L^2(0,L).$$

This and (5.8) mean that $R_e^*\in C^1([0,L])\cap W^{2,\infty}(0,L)$ satisfies $\mathcal{F}_2=0$. Similarly we can show $\mathcal{F}_3(\lambda^*,\rho_i^*,R_e^*,V^*)=0$.

We now set

$$x_*:=\inf\{x\in[0,L];(\partial_x V^*+\lambda^*)(x)=0\}.$$

We divide our proof into two cases $x_*=0$ and $x_*>0$.

We first consider the case $x_*>0$. The equation (5.13), which holds for a sequence $x_\nu\to x_*$, yields the inequality

$$0=(\partial_x V^*+\lambda^*)\|\rho_i\|_{L^\infty(I)}\ge\frac{k_e}{k_i}\int_0^{x_*} h(\partial_x V^*+\lambda^*)e^{-\frac{\lambda^*}{2}y}R_e^*\,dy.$$

Together with the nonnegativity (5.10) this implies that $(h(\partial_x V^*+\lambda^*)e^{-\frac{\lambda^*}{2}\cdot}R_e^*)(x)=0$ for $x\in[0,x_*]$. From the definition of x_*, we see that

$$(\partial_x V^*+\lambda^*)(x)>0 \quad \text{for } x\in[0,x_*), \tag{5.14}$$

so that $h(\partial_x V^*+\lambda^*)>0$ on $[0,x_*)$. Therefore, $R_e^*(x)\equiv 0$ in $[0,x_*)$. Hence from (5.13) and (5.14), $\rho_i^*=0$ a.e. in $[0,x_*)$. Now from the equation $\mathcal{F}_3(\lambda^*,\rho_i^*,R_e^*,V^*)=0$ we see that $\partial_x V^*$ is a constant in $(0, x_*)$. Thus $\partial_x V^* + \lambda^* = 0$ in $[0, x_*]$. This contradicts the definition of x_*.

Now consider the other case $x_* = 0$. We first suppose that there exists $y_0 > 0$ such that $(\partial_x V^* + \lambda^*)(y_0) > 0$. Let us set

$$y^* := \sup\{x < y_0; (\partial_x V^* + \lambda^*)(x) = 0\}.$$

Note that $y^* \in [0, y_0)$ and $(\partial_x V^* + \lambda^*)(y^*) = 0$. On the other hand, integrating $\mathcal{F}_1(\lambda^*, \rho_i^*, R_e^*, V^*) = 0$ *a.e.* over $[y^*, y]$ for any $y \in [y^*, y_0]$ and using $\mathcal{F}_3(\lambda^*, \rho_i^*, R_e^*, V^*) = 0$, we have

$$\begin{aligned}&(\partial_x V^* + \lambda^*)(\partial_x^2 V^* + e^{-\frac{\lambda^*}{2}y} R_e^*)(y) \\ &\leq \int_{y^*}^{y} \frac{k_e}{k_i} h(\partial_x V^* + \lambda^*) e^{-\frac{\lambda^*}{2}z} R_e^* \, dz \quad \text{for a.e. } y \in [y^*, y_0].\end{aligned} \tag{5.15}$$

By (5.10) and (5.11), the left hand side is estimated from below as

$$(\partial_x V^* + \lambda^*)(\partial_x^2 V^* + e^{-\frac{\lambda^*}{2}y} R_e^*) \geq (\partial_x V^* + \lambda^*)\partial_x^2 V^* = \frac{1}{2}\partial_x \left\{(\partial_x V^* + \lambda^*)^2\right\} \ \textit{a.e.}$$

since $\partial_x V^*$ is absolutely continuous. The integrand on the right hand side of (5.15) is estimated from above by $Ce^{-b(\partial_x V^* + \lambda^*)^{-1}}$, due to the behavior of h; see (1.3). Consequently, substituting these expressions into (5.15), integrating the result over $[y^*, x]$, and using $(\partial_x V^* + \lambda^*)(y^*) = 0$, we have

$$(\partial_x V^* + \lambda^*)^2 (x) \leq C \int_{y^*}^{x} \int_{y^*}^{y} e^{-b(\partial_x V^*(z) + \lambda^*)^{-1}} \, dz dy \quad \text{for } x \in [y^*, y_0]. \tag{5.16}$$

Now let us define x_n by

$$x_n := \inf\left\{x \leq y_0; \ \ \partial_x V^*(x) + \lambda^* = \frac{1}{n}\right\}.$$

Notice that $y^* < x_n$ and $(\partial_x V^* + \lambda^*)(x) \leq 1/n$ for any $x \in [y^*, x_n]$, since the continuous function $(\partial_x V^* + \lambda^*)$ vanishes at $x = y^*$. Then we evaluate (5.16) at $x = x_n$ to obtain

$$\frac{1}{n^2} \leq C \int_{y^*}^{x_n} \int_{y^*}^{y} e^{-b(\partial_x V^*(z) + \lambda^*)^{-1}} \, dz dy \leq Ce^{-bn}.$$

For suitably large n, this clearly does not hold. So once again we have a contradiction.

The remaining case is that $x_* = 0$ and $\partial_x V^* + \lambda^* \equiv 0$. In this case, $\partial_x^2 V^* \equiv 0$ and so the equation $\mathcal{F}_2(\lambda^*, \rho_i^*, R_e^*, V^*) = 0$ yields $\partial_x^2(e^{-\lambda^* x/2} R_e^*) = e^{-\lambda^* x/2}(\partial_x^2 R_e^* - \lambda \partial_x R_e^* + \frac{\lambda^2}{4} R_e^*) = 0$. This means that $e^{-\lambda^* x/2} R_e^*(x) = cx + d$ for some constants c and d. Furthermore, $d = 0$ also follows from (5.9). On the other hand, (5.3) holds for any $s_n > 0$ and then using (5.8) and $(\partial_x V^* + \lambda^*) \equiv 0$, we have

$$\begin{aligned}\partial_x R_e^*(L) &= -\left(\partial_x V^*(L) + \frac{\lambda^*}{2}\right) R_e^*(L) + \gamma e^{\frac{\lambda^*}{2}L} \int_0^L h\left(\partial_x V^*(x) + \lambda^*\right) e^{-\frac{\lambda^*}{2}x} R_e^*(x) \, dx \\ &= \frac{\lambda^*}{2} R_e^*(L).\end{aligned}$$

Substituting $R_e^*(x) = cxe^{\lambda^* x/2}$, we find $c = 0$. Consequently, $R_e^* \equiv 0$. Then we obtain $\rho_i^* \equiv 0$ from $\mathcal{F}_3(\lambda^*, \rho_i^*, R_e^*, V^*) = 0$. Solving $\mathcal{F}_3(\lambda^*, \rho_i^*, R_e^*, V^*) = 0$ with (5.9) and $\rho_i^* \equiv R_e^* \equiv 0$, we also have $V^* \equiv 0$. Consequently $\lambda^* = 0$ holds and $\underline{\lim}_{s\to 0} \lambda(s) = 0$. This contradicts Lemma 5.2, since $\sup_{s>0} \|V(s)\|_{C^2}$ is bounded. ∎

Next, we reduce Condition (c) in Theorem 4.4 to a simpler condition. We write the result directly in terms of the ion density ρ_i and the electron density $\rho_e = R_e e^{-\lambda x/2}$.

Lemma 5.4. *Assume the global positivity alternative (i) in Theorem 5.1. If* $\sup_{s>0}\{\|\rho_i(s)\|_{C^0} + \|\rho_e(s)\|_{C^0} + \lambda(s)\}$ *is bounded, then* $\sup_{s>0}\{\|\rho_i(s)\|_{C^1} + \|R_e(s)\|_{C^2} + \|V(s)\|_{C^3}\}$ *is bounded.*

Proof. It is clear from $\mathcal{F}_3 = 0$ together with the definition $\rho_e = R_e e^{-\lambda x/2}$, that

$$\sup_{s>0} \|V(s)\|_{C^2} \le C \sup_{s>0}\{\|\rho_i(s)\|_{C^0} + \|\rho_e(s)\|_{C^0}\} < +\infty.$$

From this, the equation $\mathcal{F}_2 = 0$, and $\sup_{s>0} \lambda(s) < +\infty$, we also deduce that $\sup_{s>0} \|R_e(s)\|_{C^2} < +\infty$. Now Lemma 5.3 implies that $\underline{\lim}_{s\to 0}\{\inf_x(\partial_x V + \lambda)(s,x)\}\} \ne 0$. Together with (5.2), this result leads to $\sup_{s>0} \|\rho_i(s)\|_{C^1} < +\infty$. Finally the bound $\sup_{s>0} \|\partial_x^3 V(s)\|_{C^0} < +\infty$ follows from $\mathcal{F}_3(\lambda(s), \rho_i(s), R_e(s), V(s)) = 0$. ∎

We conclude with the following *main result.*

Theorem 5.5. *Assume that the sparking voltage exists (that is, D vanishes somewhere), and the transversality condition* (3.22) *holds. Then one of the following two alternatives occurs:*

(A) Both $\rho_i(s,x)$ and $\rho_e(s,x) = (R_e e^{-\lambda \cdot/2})(s,x)$ are positive for any $s \in (0,\infty)$ and $x \in I$. Furthermore, $\overline{\lim}_{s\to\infty}\{\|\rho_i(s)\|_{C^0} + \|\rho_e(s)\|_{C^0} + \lambda(s)\} = \infty$.

(B) there exists a finite s-value $s^\ddagger > 0$ and a voltage $V_c^\ddagger > V_c^\dagger$ such that
  - *(1) $D(V^\ddagger) = 0$, $g(V_c^\ddagger) \le \pi^2/L^2$;*
  - *(2) $\rho_i(s,x) > 0$ and $\rho_e(s,x) > 0$ for all $s \in (0, s^\ddagger)$ and $x \in (0, L]$;*
  - *(3) $(\lambda(s^\ddagger), \rho_i(s^\ddagger), R_e(s^\ddagger), V(s^\ddagger)) = (V_c^\ddagger/L, 0, 0, 0)$;*
  - *(4) $\rho_i(s,x) < 0$ and $\rho_e(s,x) < 0$ for $0 < s - s^\ddagger \ll 1$ and $x \in (0, L]$.*

Proof. Suppose that (B), which is the same as the second alternative (*ii*) in Theorem 5.1, does *not* hold. We will prove (A). Then the first alternative (*i*) in Theorem 5.1 must hold. Now in Theorem 4.4 there are four alternatives. Alternative (d) cannot happen because ρ_i and R_e are negative on part of the loop. Lemmas 5.2 and 5.3 assert that either (a) or (b) implies that $\sup_{s>0}\{\|\rho_i(s)\|_{C^0} + \|R_e(s)\|_{C^2} + \|V(s)\|_{C^2} + \lambda(s)\}$ is unbounded. Then Lemma 5.4 implies that $\sup_{s>0}\{\|\rho_i(s)\|_{C^0} + \|\rho_e(s)\|_{C^0} + \lambda(s)\}$ must also be unbounded. This means that (A) holds. ∎

This concludes the proof of Theorem 1.1. We remark that (B) never occurs unless a voltage $V_c^\ddagger > V_c^\dagger$ exists satisfying (3.2).

6. Bounded Densities. It is of interest to know how the global bifurcation curve behaves for the case that the densities are bounded but λ is unbounded. We see from (2.2) that (ρ_i, ρ_e, V) solves

$$\partial_x \{(\partial_x V + \lambda)\rho_i\} = \frac{k_e}{k_i} h(\partial_x V + \lambda)\rho_e, \tag{6.1a}$$

$$-\partial_x\{(\partial_x V + \lambda)\rho_e + \partial_x \rho_e\} = h(\partial_x V + \lambda)\rho_e, \tag{6.1b}$$

$$\partial_x^2 V = \rho_i - \rho_e, \tag{6.1c}$$

$$(\partial_x V(L) + \lambda)\rho_e(L) + \partial_x \rho_e(L) = \gamma \frac{k_i}{k_e}(\partial_x V(L) + \lambda)\rho_i(L) \tag{6.1d}$$

with boundary conditions

$$\rho_i(0) = \rho_e(0) = V(0) = V(L) = 0. \tag{6.1e}$$

LEMMA 6.1. *Assume $\gamma(1+\gamma)^{-1} \neq e^{-aL}$ and that there is a sparking voltage* [1]. *Also assume alternative (A) in Theorem 5.5. Furthermore, suppose that there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} s_n = \infty, \quad \sup_{n\geq 1}(\|\rho_i(s_n)\|_{C^0} + \|\rho_e(s_n)\|_{C^0}) < +\infty, \quad \lim_{n\to\infty} \lambda(s_n) = \infty. \tag{6.2}$$

Then $\lim_{n\to\infty}(\|\rho_i(s_n)\|_{C^0} + \|\rho_e(s_n)\|_{L^1}) = 0$.

Proof. First, it is clear from (6.1c) and (6.1e) that

$$\sup_{n\geq 1} \|V(s_n)\|_{C^2} \leq C \sup_{n\geq 1}\{\|\rho_i(s_n)\|_{C^0} + \|\rho_e(s_n)\|_{C^0}\} < +\infty. \tag{6.3}$$

Solve (6.1a) for $\partial_x \rho_i$ and write h explicitly from (1.3) to obtain

$$\partial_x \rho_i = \frac{k_e}{k_i} a \exp\left(\frac{-b}{|\partial_x V + \lambda|}\right) \frac{|\partial_x V + \lambda|}{\partial_x V + \lambda}\rho_e - \frac{\partial_x^2 V}{\partial_x V + \lambda}\rho_i.$$

From this, (6.2), and (6.3), we see that $\sup_{n\geq 1}\|\rho_i(s_n)\|_{C^1} < +\infty$ and thus there exist a subsequence [still denoted by s_n] and $(\rho_i^*, \rho_e^*, V^*)$ such that

$$\left\{\begin{array}{llll} \lambda(s_n) & \to & \infty & \text{in } \mathbb{R}, \\ \rho_i(s_n) & \to & \rho_i^* & \text{in } C^0([0,L]), \\ \rho_e(s_n) & \rightharpoonup & \rho_e^* & \text{in } L^\infty(0,L) \quad \text{weakly-star}, \\ V(s_n) & \to & V^* & \text{in } C^1([0,L]), \end{array}\right. \tag{6.4}$$

$$\rho_i^*(0) = V^*(0) = V^*(L) = 0, \tag{6.5}$$

$$\rho_i^* \geq 0, \quad \rho_e^* \geq 0. \tag{6.6}$$

For the completion of the proof, we claim that it suffices to prove the identity

$$-a\gamma \int_0^L \rho_e^*(y)\,dy + \rho_e^*(x) = a\int_x^L \rho_e^*(y)\,dy \quad a.e. \tag{6.7}$$

In order to prove this claim, first note that (6.7) implies that ρ_e^* is a continuous function. Now multiplying the identity by e^{ax}, we have

$$\partial_x\left(e^{ax}\int_x^L \rho_e^*(y)\,dy\right) = -a\gamma\int_0^L \rho_e^*(y)\,dy\, e^{ax} \quad a.e.$$

[1] Lemma A.2 ensures that we can have a sparking voltage $V_c^\dagger$ under the inequality $\gamma(1+\gamma)^{-1} > e^{-aL}$.

Then integration over $[0, L]$ leads to

$$\int_0^L \rho_e^*(y)\, dy \, \{1 - \gamma(e^{aL} - 1)\} = 0,$$

which together with the assumption $\gamma(1+\gamma)^{-1} \neq e^{-aL}$ means that $\|\rho_e^*\|_{L^1} = 0$. We also see from (6.4) and $\rho_e(s_n) \geq 0$ that

$$\|\rho_e(s_n)\|_{L^1} = \int_0^L 1 \cdot \rho_e(s_n, x)\, dx \to \int_0^L 1 \cdot \rho_e^*(x)\, dx = 0 \quad \text{as } n \to \infty. \tag{6.8}$$

It follows that $\|\rho_e(s_n)\|_{L^1} \to 0$ *for the whole original sequence*. Furthermore, solving (6.1a) with (6.5), we have

$$\rho_i(s_n, x) = a\frac{k_e}{k_i} \int_0^x K_n(y)\rho_e(s_n, y)\, dy \leq C\|\rho_e(s_n)\|_{L^1}, \tag{6.9}$$

where

$$K_n(y) := \exp\left(\frac{-b}{|\partial_x V(s_n, y) + \lambda(s_n)|}\right) \frac{|\partial_x V(s_n, y) + \lambda(s_n)|}{\partial_x V(s_n, x) + \lambda(s_n)}.$$

Here we have used (6.3) in derving the last inequality. Together with (6.8) this completes the proof of the lemma.

It remains to prove (6.7). Integrating (6.1b) over $[x, L]$, using (6.1d), and multiplying the result by λ^{-1}, we obtain

$$\begin{aligned}
&-\gamma\frac{k_i}{k_e}\left(\frac{\partial_x V(L)}{\lambda} + 1\right)\rho_i(L) + \left(\frac{\partial_x V(x)}{\lambda} + 1\right)\rho_e(x) + \frac{1}{\lambda}\partial_x \rho_e(x) \\
&= a\int_x^L \exp\left(\frac{-b}{|\partial_x V(y) + \lambda|}\right)\left|\frac{\partial_x V(y)}{\lambda} + 1\right| \rho_e(y)\, dy.
\end{aligned}$$

We take this identity at $s = s_n$ and look at the behavior of each term as $s_n \to \infty$. We multiply it by a test function $\phi \in C_c^\infty((0, L))$, integrate it over $(0, L)$, and let $n \to \infty$. Then we notice from (6.2)–(6.4) that

$$\begin{gathered}
-\gamma\frac{k_i}{k_e}\left(\frac{\partial_x V(s_n, L)}{\lambda(s_n)} + 1\right)\rho_i(s_n, L)\int_0^L \phi(x)\, dx \to -\gamma\frac{k_i}{k_e}\rho_i^*(L)\int_0^L \phi(x)\, dx, \\
\int_0^L \left(\frac{\partial_x V(s_n, x)}{\lambda(s_n)} + 1\right)\rho_e(s_n, x)\phi(x)\, dx \to \int_0^L \rho_e^*(x)\phi(x)\, dx, \\
\int_0^L \frac{1}{\lambda}\partial_x \rho_e(s_n, x)\phi(x)\, dx = -\int_0^L \frac{1}{\lambda}\rho_e(s_n, x)\partial_x \phi(x)\, dx \to 0.
\end{gathered}$$

Furthermore, there holds that

$$\begin{aligned}
&\int_0^L \left[a\int_x^L \exp\left(\frac{-b}{|\partial_x V(s_n, y) + \lambda(s_n)|}\right)\left|\frac{\partial_x V(s_n, y)}{\lambda(s_n)} + 1\right| \rho_e(s_n, y)\, dy\right]\phi(x)\, dx \\
&= I_{1,n} + I_{2,n},
\end{aligned}$$

where

$$I_{1,n} := \int_0^L \left[a \int_x^L \left\{ \exp\left(\frac{-b}{|\partial_x V(s_n,y) + \lambda(s_n)|} \right) \times \left| \frac{\partial_x V(s_n,y)}{\lambda(s_n)} + 1 \right| - 1 \right\} \rho_e(s_n,y)\, dy \right] \phi(x)\, dx,$$

$$I_{2,n} := \int_0^L \left[a \int_x^L \rho_e(s_n,y)\, dy \right] \phi(x)\, dx.$$

Then it is also seen from (6.2)–(6.4) that

$$|I_{1,n}| \le \|\rho_e\|_{L^1} \|\phi\|_{L^1} \sup_{y\in[0,L]} \left| \exp\left(\frac{-b}{|\partial_x V(s_n,y) + \lambda(s_n)|} \right) \left| \frac{\partial_x V(s_n,y)}{\lambda(s_n)} + 1 \right| - 1 \right| \to 0,$$

$$I_{2,n} = a \int_0^L \rho_e(s_n,y) \left[\int_0^y \phi(x)\, dx \right] dy$$
$$\to a \int_0^L \rho_e^*(y) \left[\int_0^y \phi(x)\, dx \right] dy = \int_0^L \left[a \int_x^L \rho_e^*(y)\, dy \right] \phi(x)\, dx.$$

Therefore, we conclude that

$$-\gamma \frac{k_i}{k_e} \rho_i^*(L) + \rho_e^*(x) = a \int_x^L \rho_e^*(y)\, dy \quad a.e. \tag{6.10}$$

Comparing with (6.7), it is left to show that

$$\rho_i^*(L) = a \frac{k_e}{k_i} \int_0^L \rho_e^*(y)\, dy. \tag{6.11}$$

Indeed, plugging (6.11) into (6.10) leads to (6.7). Evaluating (6.9) at $x = L$, we have

$$\rho_i(s_n, L) = a \frac{k_e}{k_i} \int_0^L K_n(y) \rho_e(s_n,y)\, dy.$$

Now $K_n \to 1$ uniformly and $\rho_e(s_n) \rightharpoonup \rho_e^*$ in L^∞ weakly-star. Therefore, letting $n \to \infty$, we get (6.11) in the limit. ∎

Appendix A. Roots of the Sparking Function D. In this appendix we investigate the roots of $D(V_c)$. The first lemma means that in the case of Figure 3, which we discussed in our first paper [19], D always has at least one root.

Lemma A.1. *(i) If* $\max_{V_c>0} g(V_c) > \pi^2/L^2$*, then* $D(V_c)$ *has at least one root* V_c *that satisfies* $g(V_c^\dagger) < \pi^2/L^2$ *in the interval* $(0, V_c^*)$*. (ii) In addition, if*

$$a > 4^{-1}eb + e\pi^2 b^{-1}, \tag{A.1}$$

then $\max_{V_c>0} g(V_c) > \pi^2/L^2$*.*

Proof. (i) Since $\max g > 0$, the function g has exactly two positive roots. Define V_c^* by

$$g(V_c^*) = \frac{\pi^2}{L^2}, \quad g'(V_c^*) > 0 \tag{A.2}$$

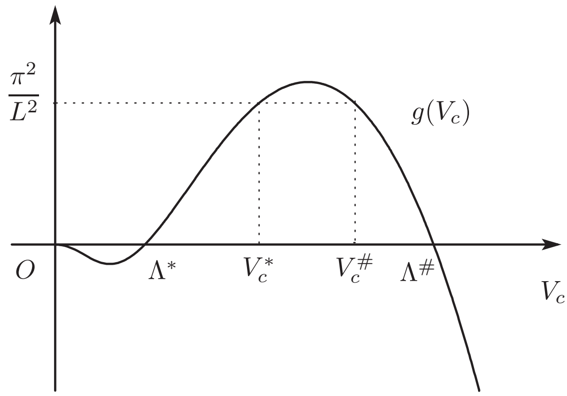

FIG. 3. *local max is greater than* π^2/L^2

as in Figure 3. We have

$$D(V_c^*) = -1 - \frac{\gamma}{1+\gamma} e^{V_c^*/2} < 0.$$

In addition, $\lim_{V_c \to 0} D(V_c) = \frac{1}{1+\gamma} > 0$. So we see that $D(V_c)$ has at least one root V_c that satisfies (3.2) on the interval $(0, V_c^*)$. For (ii) we simply note that (A.1) implies that $g(b) > \pi^2/L^2$. ∎

We also can find a sufficient condition for the existence of roots of D that is caused by the γ-mechanism. In this case it does not matter whether or not $\max_{V_c>0} g(V_c) > \pi^2/L^2$ holds.

LEMMA A.2. *Suppose that*

$$\gamma(1+\gamma)^{-1} > e^{-aL}. \tag{A.3}$$

Then $D(V_c)$ *has at least one root.*

Proof. First $\lim_{V_c \to 0} D(V_c) = \frac{1}{1+\gamma} > 0$ holds. We also see that $\mu = L\sqrt{g(-V_c)} = V_c/2 - aL + O(V_c^{-1})$ as $V_c \to \infty$. Thus

$$\lim_{V_c \to \infty} \frac{D(V_c)}{e^{V_c/2}} = e^{-aL} - \frac{\gamma}{1+\gamma} < 0,$$

which means $\lim_{V_c \to \infty} D(V_c) = -\infty$. Hence D has a positive root. ∎

We remark the roots in Lemmas A.1 and A.2 are sparking voltages. Indeed for a fixed triple (a, b, γ) in the open set $\{(a, b, \gamma) \in (\mathbb{R}_+)^3 \text{ ; either (A.1) or (A.3) holds}\}$, the sparking function D has a positive root for any triple in a neighborhood of it.

In the next lemma, we find a candidate of the *anti-sparking voltage* $V_c^\ddagger$. Therefore alternative (B) in Theorem 5.5 is an actual possibility.

LEMMA A.3. *Let* (A.1) *hold. There exists a positive constant* γ_0 *such that if* $\gamma < \gamma_0$, *then* $D(V_c)$ *has at least two roots* $V_c^\dagger$ *and* $V_c^\ddagger$ *with* (3.2).

Proof. We know from Lemma A.1 and its proof that a root V_c with (3.2) exists in the open interval $(0, V_c^*)$. Let us seek another root $V_c^\ddagger$. The graph of g is sketched in Figure 3 and thus $V_c^{\#}$ is the unique value such that $g(V_c^{\#}) = \frac{\pi^2}{L^2}$ and $g'(V_c^{\#}) < 0$.

The function g has three roots 0, Λ^*, and $\Lambda^\#$ such that $0 < \Lambda^* < V_c^* < V_c^\# < \Lambda^\#$. We emphasize that g is independent of γ. Now consider the function D on the interval $[V_c^\#, \Lambda^\#]$. Evaluating $D(V_c)$ at the point $V_c = V_c^\#$, we have

$$D(V_c^\#) = -1 - \frac{\gamma}{1+\gamma} e^{V_c^\#/2} < 0.$$

On the other hand, evaluating $D(V_c)$ at the point $V_c = \Lambda^\#$, we have

$$D(\Lambda^\#) = 1 + \frac{\Lambda^\#}{2} - \frac{\gamma}{1+\gamma} e^{\Lambda^\#/2} > 0,$$

where the last inequality is valid for suitably small γ. Thus there must be a root in between; that is, D has a root with (3.2) in the open interval $(V_c^\#, \Lambda^\#)$. $\square$

The next lemma states a sufficient condition for the absence of any root of D.

Lemma A.4. *If $a < 4^{-1}eb$ and $\gamma(1+\gamma)^{-1} \le e^{-2aL}$ hold, then $D(V_c)$ has no root.*

Proof. We first claim that $g(\lambda L)$ is negative for any $\lambda = V_c/L \in (0, \infty)$ if and only if $a < 4^{-1}eb$. Indeed, $g(\lambda L) < 0$ holds if and only if $ae^{-b/\lambda} < \frac{\lambda}{4}$ holds. By taking logarithms, we see that $g(\lambda L) < 0$ is equivalent to $G(\lambda) := -\log\lambda - \frac{b}{\lambda} + \log a + \log 4 < 0$. It is straightforward to check that G attains a maximum at $\lambda = b$. Furthermore, the maximum is less than zero if and only if $a < 4^{-1}eb$. This proves the claim.

For any $\lambda = V_c/L > 0$, the negativity of $g(\lambda L)$ implies that

$$\begin{aligned} D(\lambda L) &= \cosh(L\sqrt{-g(\lambda L)}) + \frac{\lambda}{2\sqrt{-g(\lambda L)}} \sinh(L\sqrt{-g(\lambda L)}) - \frac{\gamma}{1+\gamma} e^{\frac{\lambda}{2}L} \\ &> e^{L\sqrt{-g(\lambda L)}} - \frac{\gamma}{1+\gamma} e^{\frac{\lambda}{2}L} \end{aligned} \tag{A.4}$$

because

$$\frac{\lambda}{2\sqrt{-g(\lambda L)}} > \frac{\lambda}{2\sqrt{\lambda^2/4}} = 1$$

and $\cosh z + \sinh z = e^z$. Now we note that

$$\sqrt{-g(\lambda L)} - \tfrac{\lambda}{2} = -h(\lambda)\left(\sqrt{-g(\lambda L)} + \tfrac{\lambda}{2}\right)^{-1} > -h(\lambda)(\tfrac{\lambda}{2})^{-1} > -2a.$$

So the right side of (A.4) is greater than

$$e^{\frac{\lambda}{2}L}\{e^{-2aL} - \tfrac{\gamma}{1+\gamma}\} \ge 0$$

by hypothesis. $\square$

Independently, it can be shown numerically that $D(V_c)$ has a unique root or many roots for suitable choices of L, a, b, and γ. To illustrate this, the graphs of $(1+\gamma)e^{-V_c/2}D(V_c)$ are sketched in Figures 4 and 5 for the case $L = 1$, $a = 3$, $b = 4$, $\gamma = 5$ and for the case $L = 1$, $a = 70$, $b = 0.1$, $\gamma = 0.1$, respectively.

Appendix B. The Sparking Voltage $V_c^\dagger$. In this brief appendix we illustrate the location of the sparking voltage if γ is very small or very large. Let V_c^* be defined in (A.2).

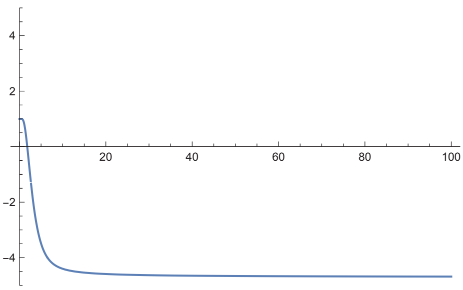

FIG. 4. *unique root*

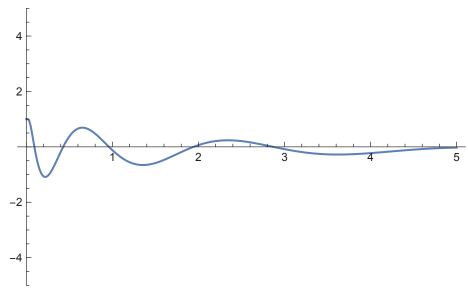

FIG. 5. *many roots*

LEMMA B.1. *Suppose that* $\max_{V_c>0} g(V_c) > \pi^2/L^2$ *(see Figure 3). If* γ *is sufficiently small, then* $V_c^\dagger < V_c^*$ *and* $\frac{\pi^2}{4L^2} < g(V_c^\dagger) < \frac{\pi^2}{L^2}$.

Proof. We know from Lemma A.1 that $V_c^\dagger < V_c^*$ and $g(V_c^\dagger) < \frac{\pi^2}{L^2}$. It only remains to show that $g(V_c^\dagger) > \frac{\pi^2}{4L^2}$. By continuity it suffices to prove the strict inequalities of the conclusion in case $\gamma = 0$. We begin by proving that $g(V_c^\dagger) > 0$. On the contrary, suppose that $g(V_c^\dagger) \le 0$. This assumption and $\gamma = 0$ lead to $D(V_c^\dagger) > 0$, which contradicts to the fact that $V_c^\dagger$ is the sparking voltage, that is, $D(V_c^\dagger) = 0$. Now let us suppose that $0 < g(V_c^\dagger) \le \frac{\pi^2}{4L^2}$. We see from $D(V_c^\dagger) = 0$ that

$$\frac{V_c^\dagger}{2L\sqrt{g(V_c^\dagger)}} = -\cot\left(L\sqrt{g(V_c^\dagger)}\right).$$

The signs are contradictory. Thus we conclude that $g(V_c^\dagger) > \frac{\pi^2}{4L^2}$. ◻

LEMMA B.2. *Suppose that* $\max_{V_c>0} g(V_c) > 0$ *(see Figure 1). There exists* $\Gamma > 0$ *such that for* $\gamma > \Gamma$*, we have* $V_c^\dagger \in (0, \Lambda^*)$*, where* Λ^* *is the smallest positive root of* $g(V_c) = 0$.

Proof. We first see that $\lim_{V_c \to 0} D(V_c) = \frac{1}{1+\gamma} > 0$. Evaluating D at $V_c = \Lambda^*$ and using $g(\Lambda^*) = 0$, we have

$$D(\Lambda^*) = 1 + \frac{\Lambda^*}{2} - \frac{\gamma}{1+\gamma} e^{\frac{\Lambda^*}{2}} < \frac{1}{1+\gamma}\left(1 + \frac{\Lambda^*}{2}\right) - \frac{1}{2}\frac{\gamma}{1+\gamma}\left(\frac{\Lambda^*}{2}\right)^2 < 0.$$

In deriving the last inequality, we have taken γ suitably large. Therefore, the intermediate value theorem gives $V_c^\dagger \in (0, \Lambda^*)$. ◻

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