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GLOBAL WELL-POSEDNESS AND LARGE TIME BEHAVIOR TO 2D BOUSSINESQ EQUATIONS FOR MHD CONVECTION*

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Dedicated to Professor Ling Hsiao on the Occasion of Her Eightieth Birthday

Abstract. We study the Cauchy problem for the 2D incompressible MHD-Boussinesq equations without thermal diffusion. We prove the global existence and uniqueness of the solutions for suitably regular initial data. To obtain large time decay properties of the solutions, we insert an artificial thermal damping term. By applying the classical Fourier splitting methods, we derive optimal large time decay rates of the solutions and their first-order derivatives.

Key words. MHD-Boussinesq equations, global well-posedness, large time behavior, Fourier splitting.

Mathematics Subject Classification. 35B40, 35Q35, 76D03.

1. Introduction.

This paper is concerned with the global well-posedness and large time decay properties of solutions to the Cauchy problem for the 2D incompressible MHD-Boussinesq system. The standard incompressible Boussinesq equations for MHD convection in $\mathbb{R}^n$ can be written as

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi - \nabla \cdot (\mu(\theta)\, \nabla \mathbf{u}) = \mathbf{b} \cdot \nabla \mathbf{b} + \theta e_n, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta - \nabla \cdot (\kappa(\theta)\, \nabla \theta) = 0, \\ \partial_t \mathbf{b} + \mathbf{u} \cdot \nabla \mathbf{b} - \nabla \cdot (\nu(\theta)\, \nabla \mathbf{b}) = \mathbf{b} \cdot \nabla \mathbf{u}, \\ \nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{b} = 0, \end{cases} \tag{1.1}$$

where the functions $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, $\theta = \theta(\mathbf{x}, t)$, $\pi = \pi(\mathbf{x}, t)$ and $\mathbf{b} = \mathbf{b}(\mathbf{x}, t)$ represent the velocity field, the temperature, the pressure and the magnetic field respectively. The parameter $\mu(\theta) \geq 0$ denotes the fluid viscosity, $\kappa(\theta) \geq 0$ the thermal diffusivity, $\nu(\theta) \geq 0$ the electrical resistivity of the fluid and $e_n = (0, \cdots, 0, 1)$ the unit vector in the n-th direction in $\mathbb{R}^n$.

Physically, the MHD-Boussinesq system may be used to model the convection of an incompressible flow driven by the buoyance effect of a thermal fluid and the Lorenz force generated by the magnetic field of the fluid. In fact, system (1.1) is a combination of the incompressible Boussinesq equations of fluid dynamics and the Maxwell's equations of electromagnetism where the displacement current can be neglected (see, e.g., [31]).

The MHD-Boussinesq system is closely related to many classical systems. When the temperature and magnetic effects are neglected, i.e., $\theta = 0$, $\mathbf{b} = \mathbf{0}$, system (1.1) reduces to the well-known incompressible Navier-Stokes equations which have been extensively studied by physicists and mathematicians. When $\theta = 0$, system (1.1) reduces to the incompressible magnetohydrodynamics (MHD) system which describes the motion of electrically conducting fluids and reflects the basic physical conservation

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laws, and has ample applications in applied sciences such as astrophysics, geophysics and plasma physics. Besides its physical applications, the mathematical study of the MHD system has also been widely pursued (see, e.g., [9, 12, 17, 21, 23, 24, 28, 29, 35, 36, 37, 41, 45, 50]). Finally, when the Lorentz force is neglected, that is, $\mathbf{b} = \mathbf{0}$, system (1.1) reduces to the Boussinesq equations which are the zeroth-order approximation to the coupling between the Navier-Stokes equations and the thermodynamic equations, and describe many geophysical phenomena in atmospheric and oceanographic sciences (see, e.g., [38, 40]). We remark that the global well-posedness of the Boussinesq system with full or partial viscosity coefficients has attracted extensive attention (see, e.g., [1, 8, 10, 11, 25, 27, 32, 34, 48, 49]). In particular, the global regularity issue for the Boussinesq system with zero viscosity and zero diffusion is still open.

For the 2D MHD-Boussinesq system (1.1) with positive and temperature-dependent viscosity coefficients $\mu(\theta)$, $\kappa(\theta)$ and $\nu(\theta)$, Bian and Gui [3], Bian and Liu [5] obtained the global well-posedness and the exponential time-decay rates for the Cauchy problem and the initial-boundary value problem respectively. Moreover, Bian and Gui [3], and Bian, et al. [4] justified the stability and instability respectively of such a 2D system in a fully nonlinear dynamical setting from mathematical points of view. On the other hand, for the 3D MHD-Boussinesq system with constant $\mu, \nu > 0$ and $\kappa = 0$, Larios and Pei [33] proved a Prodi-Serrin-type global regularity criterion in terms of only two velocity and two magnetic components. In addition, Larios and Pei [33] obtained the local well-posedness of solutions to the 3D MHD-Boussinesq system for the three cases: $\mu, \nu, \kappa > 0$; $\mu = \nu = \kappa = 0$; and $\mu, \nu > 0, \kappa = 0$.

In the 2D case, Bian [2] established the global well-posedness for the initial-boundary value problem of the MHD-Boussinesq system with constant $\mu, \nu > 0$ and $\kappa = 0$, i.e.,

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi - \mu \Delta \mathbf{u} = \mathbf{b} \cdot \nabla \mathbf{b} + \theta e_2, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta = 0, \\ \partial_t \mathbf{b} + \mathbf{u} \cdot \nabla \mathbf{b} - \nu \Delta \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{u}, \\ \nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{b} = 0. \end{cases} \tag{1.2}$$

Besides the basic well-posedness issue, other important properties such as the large time behavior and asymptotic structure of the solutions may also be relevant in physical applications. Such questions have been extensively studied for many important PDE models including the Navier-Stokes equations, the MHD system, and the Boussinesq equations, etc. (see, e.g., [6, 22, 26, 41, 49, 50]). However, for system (1.2), the large time behavior of its solutions remains unknown. Even for the Cauchy problem, there are no published results available on the well-posedness and large time decay properties of the solutions. Motivated by this, we are devoted to investigating this problem.

For completeness, we will first establish the global existence and uniqueness of solutions to the Cauchy problem for system (1.2) as follows.

THEOREM 1.1. *Let $(\mathbf{u}_0, \theta_0, \mathbf{b}_0) \in H^s(\mathbb{R}^2)$, $s > 2$ and $\nabla \cdot \mathbf{u}_0 = \nabla \cdot \mathbf{b}_0 = 0$. Then there exists a unique global solution $(\mathbf{u}, \theta, \mathbf{b})$ to system (1.2) such that $\mathbf{u}, \mathbf{b} \in C(0, T; H^s(\mathbb{R}^2)) \cap L^2(0, T; H^{s+1}(\mathbb{R}^2))$ and $\theta \in C(0, T; H^s(\mathbb{R}^2))$ for any $T > 0$.*

For the proof of Theorem 1.1, it is sufficient to establish the estimates of $\|(\nabla \mathbf{u}, \nabla \mathbf{b})\|_\infty$. To achieve this, we take the curl on $(1.2)_1$ and $(1.2)_3$ to obtain

$$\begin{cases} \partial_t \Omega + \mathbf{u} \cdot \nabla \Omega - \mu \Delta \Omega = \mathbf{b} \cdot \nabla j + \partial_1 \theta, \\ \partial_t j + \mathbf{u} \cdot \nabla j - \nu \Delta j = \mathbf{b} \cdot \nabla \Omega + 2[\partial_1 b_1 (\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1 (\partial_1 b_2 + \partial_2 b_1)], \end{cases} \tag{1.3}$$

where $\Omega = \nabla^{\perp} \cdot \mathbf{u} = \partial_1 u_2 - \partial_2 u_1$ and $j = \nabla^{\perp} \cdot \mathbf{b} = \partial_1 b_2 - \partial_2 b_1$ represent the scalar vorticity function and the current density function respectively. For system (1.3), we first establish the global bounds of $\|(\Omega, j)\|_p$ for $p \in [2, \infty)$ and then use the Brezis–Wainger inequality (see, e.g., [7]) to further establish the global bounds of $\|(\nabla\Omega, \nabla j)\|_p$. Next, by applying the Gagliardo–Nirenberg interpolation techniques (see, e.g., [39]) and the Calderón–Zygmund inequality, we derive the global bounds of $\|(\nabla\mathbf{u}, \nabla\mathbf{b})\|_\infty$, and then the regularity estimates of the solutions.

To obtain large time decay properties of the solutions, we study the 2D incompressible MHD-Boussinesq equations (1.2) with an artificial thermal damping term in the temperature equation

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi - \mu \Delta \mathbf{u} = \mathbf{b} \cdot \nabla \mathbf{b} + \theta e_2, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta + \tau^{-1}\theta = 0, \\ \partial_t \mathbf{b} + \mathbf{u} \cdot \nabla \mathbf{b} - \nu \Delta \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{u}, \\ \nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{b} = 0, \end{cases} \tag{1.4}$$

where $\tau > 0$ and the $\tau^{-1}\theta$ term in $(1.4)_2$ describes the thermal damping. Compared with system (1.2), the only difference in system (1.4) is the appearance of the term $\tau^{-1}\theta$, which will not change the conclusion of Theorem 1.1. The well-posedness of solutions to system (1.4) can be similarly established with minor modifications and we omit it to avoid repetition.

In the absence of the thermal damping term $\tau^{-1}\theta$ in (1.4), the temperature equation is then a pure transport equation. It is difficult to capture decay estimates of θ, and even if the dissipation terms $\Delta\mathbf{u}$ and $\Delta\mathbf{b}$ are present in the $\mathbf{u}$ and $\mathbf{b}$ equations, it is possible that $\|\mathbf{u}(\cdot, t)\|_2$ and $\|\mathbf{b}(\cdot, t)\|_2$ can grow monotonically (see, e.g., [1, 48]). To overcome this, we consider the large time behavior of solutions to MHD-Boussinesq system with thermal damping. Here, for simplicity, we set $\mu = \nu = \tau = 1$ in (1.4). Our main results can be stated as follows.

THEOREM 1.2. *Let* $(\mathbf{u}, \theta, \mathbf{b})$ *be the solution of system* (1.4) *with initial data* $(\mathbf{u}_0, \theta_0, \mathbf{b}_0) \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ *and* $\nabla \cdot \mathbf{u}_0 = \nabla \cdot \mathbf{b}_0 = 0$. *Then it holds that*

$$\|\mathbf{u}(\cdot, t)\|_2 + \|\mathbf{b}(\cdot, t)\|_2 \le C_1(1+t)^{-\frac{1}{2}}, \quad \text{for} \quad \text{all} \quad t \ge 0.$$

If in addition, $(\mathbf{u}_0, \mathbf{b}_0) \in H^1(\mathbb{R}^2)$, *then it also holds that*

$$\|\nabla\mathbf{u}(\cdot, t)\|_2 + \|\nabla\mathbf{b}(\cdot, t)\|_2 \le C_2(1+t)^{-1}, \quad \text{for} \quad \text{all} \quad t \ge 0.$$

Here the constant C_1 *depends on* $\|\mathbf{u}_0\|_1$, $\|\mathbf{b}_0\|_1$, $\|\theta_0\|_1$, $\|\mathbf{u}_0\|_2$, $\|\mathbf{b}_0\|_2$ *and* $\|\theta_0\|_2$ *only, the constant* C_2 *depends on* $\|\mathbf{u}_0\|_1$, $\|\mathbf{b}_0\|_1$, $\|\theta_0\|_1$, $\|\mathbf{u}_0\|_{H^1}$, $\|\mathbf{b}_0\|_{H^1}$ *and* $\|\theta_0\|_2$ *only.*

To establish the large time decay rates of the solutions and their first-order derivatives, we apply the Fourier splitting methods of Schonbek [42, 43, 44] and Wiegner [46], see also [13, 14, 15] for details. Due to the lack of a dissipation term $\Delta\theta$ for the temperature equation in system (1.4), we are not able to derive the decay estimates for θ similar to those for $\mathbf{u}$ and $\mathbf{b}$. In addition, we remark that the time decay rates $(1+t)^{-\frac{1}{2}}$ for $\|\mathbf{u}(\cdot, t)\|_2$ and $\|\mathbf{b}(\cdot, t)\|_2$ are optimal in the sense that they coincide with the decay rate of the solution to the heat equation.

The remainder of the paper is organized as follows. First we state some basic tools in Section 2. In Section 3, we establish the a priori estimates of solutions to system (1.2) and prove the global existence and uniqueness of solutions. In Section 4, we apply the Fourier splitting methods to establish the large time decay rates of the solutions to system (1.4).

2. Preliminaries. In this section, we state some useful tools which are needed in later sections.

Lemma 2.1 (Gagliardo-Nirenberg interpolation inequality [39]). *Let $1 \le p, q, r \le \infty$, and $0 \le j < m$ be nonnegative integers such that*

$$\frac{1}{p} - \frac{j}{n} = \alpha \left(\frac{1}{r} - \frac{m}{n} \right) + (1-\alpha)\frac{1}{q}, \qquad \frac{j}{m} \le \alpha \le 1,$$

then every function $f : \mathbb{R}^n \mapsto \mathbb{R}$ that lies in $L^q(\mathbb{R}^n)$ with m^{th} derivatives in $L^r(\mathbb{R}^n)$ also has j^{th} derivatives in $L^p(\mathbb{R}^n)$. Furthermore, it holds that

$$\|\mathrm{D}^j f\|_p \le C \|\mathrm{D}^m f\|_r^\alpha \|f\|_q^{1-\alpha},$$

where the constant C depends upon the indices n, m, j, q, r and α only.

Corollary 2.1. *Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$. Then it holds that*

(1) $\|f\|_4 \le C\,\|f\|_2^{\frac{1}{2}} \|\nabla f\|_2^{\frac{1}{2}}$, if $f \in H^1(\mathbb{R}^2)$;
(2) $\|\nabla f\|_4 \le C\,\|f\|_2^{\frac{1}{4}} \|\nabla^2 f\|_2^{\frac{3}{4}}$, if $f \in H^2(\mathbb{R}^2)$;
(3) $\|f\|_\infty \le C\,\|f\|_2^{\frac{1}{2}} \|\nabla^2 f\|_2^{\frac{1}{2}}$, if $f \in H^2(\mathbb{R}^2)$;
(4) $\|f\|_\infty \le C\,\|f\|_2^{\frac{2}{3}} \|\nabla^3 f\|_2^{\frac{1}{3}}$, if $f \in H^3(\mathbb{R}^2)$,
where the above constants C are independent of f.

Lemma 2.2 (Elliptic regularity [19, 20]). *Consider the elliptic equation $-\Delta f = g$ in $\mathbb{R}^2$. If $g \in L^p(\mathbb{R}^2)$, $p \in (1,\infty)$, then there exists a unique solution $f \in W^{2,p}(\mathbb{R}^2)$ satisfying $\int_{\mathbb{R}^2} f\,dx = 0$ and $\|\nabla^2 f\|_p \le C\|g\|_p$ where the constant C depends only on p.*

Lemma 2.3 (Fourier transform and Plancherel's identity [19]). *Let $v \in L^2(\mathbb{R}^n)$ and $\hat{v} = \mathcal{F}v$ be its Fourier transform defined by*

$$\hat{v}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\, v(x)\,dx, \quad \xi \in \mathbb{R}^n.$$

Then $\hat{v} \in L^2(\mathbb{R}^n)$ and $\|\hat{v}\|_2 = \|v\|_2$. Furthermore, v can be recovered from $\hat{v}$ by the inverse Fourier transform

$$v(x) = (\mathcal{F}^{-1}\hat{v})(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, \hat{v}(\xi)\,d\xi.$$

With the help of the Fourier transform, we can express

$$H^s(\mathbb{R}^n) = \Big\{ v \in L^2(\mathbb{R}^n) : (1+|\xi|^2)^{s/2}\,\hat{v}(\xi) \in L^2(\mathbb{R}^n) \Big\}, \quad s > 0.$$

Also we denote by $\Lambda^s = (-\Delta)^{s/2}$ where $\Lambda^s v = \mathcal{F}^{-1}(|\xi|^s\,\hat{v})$ for $v \in H^s(\mathbb{R}^2)$ and $[A, B] = AB - BA$ the standard commutator notation. Then we have the following commutator estimates (see, e.g., [16, 30, 47]).

Lemma 2.4. *Assume that p, p_2, $p_3 \in (1,\infty)$, $p_1, p_4 \in [1,\infty]$ with*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Let f, g be smooth functions such that $\nabla f \in L^{p_1}$, $\Lambda^{s-1} g \in L^{p_2}$, $\Lambda^s f \in L^{p_3}$, $g \in L^{p_4}$. Then it holds that

$$\|\Lambda^s(fg) - f\Lambda^s g\|_p \le C(\|\nabla f\|_{p_1}\|\Lambda^{s-1}g\|_{p_2} + \|\Lambda^s f\|_{p_3}\|g\|_{p_4}),$$

where $s > 0$ and the constant C depends on s, p, p_1, p_2, p_3 and p_4 only.

Corollary 2.2. *Let $f \in H^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $s > 0$, and let g be a divergence-free vector field that satisfies $\nabla g \in H^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Then it holds that*

$$\|[\Lambda^s, g \cdot \nabla] f\|_2 \le C(\|\nabla g\|_\infty \|f\|_{H^s} + \|\nabla g\|_{H^s}\|f\|_\infty), \quad \text{for } C \text{ independent of } f \text{ and } g.$$

If, in addition, $\nabla f \in L^\infty(\mathbb{R}^2)$, then it holds that

$$\|[\Lambda^s, g \cdot \nabla] f\|_2 \le C(\|\nabla g\|_\infty \|f\|_{H^s} + \|g\|_{H^s}\|\nabla f\|_\infty), \quad \text{for } C \text{ independent of } f \text{ and } g.$$

Lemma 2.5 (Brezis–Wainger inequality [7, 18]). *Let $f \in W^{s,q}(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ with $\|f\|_{W^{k,p}} \le 1$, where $k, s \in (0, \infty)$, $p \in (1, \infty)$, $q \in [1, \infty]$ and $kp = n < sq$. Then it holds that*

$$\|f\|_\infty \le C\left[1 + \log^{1/p'}(1 + \|f\|_{W^{s,q}})\right],$$

where $1/p' + 1/p = 1$ and the constant C depends on k, p, s, q and n only.

3. Global well-posedness. To prove the global existence and uniqueness of solutions to system (1.2) in Theorem 1.1, we first establish the following a priori estimates for system (1.2).

Proposition 3.1. *Assume that the initial data $(\mathbf{u}_0, \theta_0, \mathbf{b}_0) \in H^s(\mathbb{R}^2)$, $s > 2$ and $\nabla \cdot \mathbf{u}_0 = \nabla \cdot \mathbf{b}_0 = 0$. Let $(\mathbf{u}, \theta, \mathbf{b})$ be a smooth solution of system (1.2). Then for any $T > 0$, it holds that*

$$\|(\mathbf{u}, \theta, \mathbf{b})\|_{L^\infty(0,T;H^s)} + \|(\mathbf{u}, \mathbf{b})\|_{L^2(0,T;H^{s+1})} \le C,$$

where the constant C depends only on T, $\|\mathbf{u}_0\|_{H^s}$, $\|\mathbf{b}_0\|_{H^s}$ and $\|\theta_0\|_{H^s}$.

The proof of Proposition 3.1 is divided into six subsections.

3.1. Global H^1 estimates. We start with the basic energy estimates.

Lemma 3.1. *Let $\mathbf{u}$ and θ be sufficiently smooth functions satisfying $\partial_t\theta + \mathbf{u}\cdot\nabla\theta = 0$ with $\theta_0 \in L^p(\mathbb{R}^2)$, $1 \le p \le \infty$. Then for all $t \ge 0$, it holds that $\|\theta(\cdot, t)\|_p = \|\theta_0\|_p$.*

Proof. Consider the characteristics (Lagrangian coordinates) $\mathbf{x} = \mathbf{x}(\mathbf{y}; t)$ defined by

$$\begin{cases} \frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t), \\ \mathbf{x}|_{t=0} = \mathbf{y}. \end{cases}$$

Then we have, along the characteristics,

$$\frac{d\theta(\mathbf{x}(\mathbf{y}; t), t)}{dt} = \frac{\partial \theta}{\partial t} + \sum_{i=1}^n \frac{\partial \theta}{\partial x_i}\frac{\partial x_i}{\partial t} = \frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla\theta = 0.$$

Therefore we obtain

$$\theta(\mathbf{x}(\mathbf{y};t),t)=\theta(\mathbf{x}(\mathbf{y};0),0)=\theta(\mathbf{y},0)=\theta_0(\mathbf{y}),\quad \text{for}\quad \text{all}\quad t\geq 0.$$

Since $\theta(\mathbf{x},t)=\theta_0(\mathbf{y})$ for $\mathbf{x}=\mathbf{x}(\mathbf{y};t)$, it follows that $\|\theta(\cdot,t)\|_\infty=\|\theta_0\|_\infty$. Furthermore, we have for any $p\geq 1$,

$$\int_{\mathbb{R}^2}|\theta(\mathbf{x},t)|^p\,d\mathbf{x}=\int_{\mathbb{R}^2}|\theta(\mathbf{x}(\mathbf{y};t),t)|^p\left|\frac{\partial\mathbf{x}}{\partial\mathbf{y}}\right|\,d\mathbf{y}=\int_{\mathbb{R}^2}|\theta_0(\mathbf{y})|^p\left|\frac{\partial\mathbf{x}}{\partial\mathbf{y}}\right|\,d\mathbf{y}.$$

By using the volume–preserving property of incompressible flows, it is clear that the condition $\nabla\cdot\mathbf{u}=0$ implies that the Jacobian $\left|\frac{\partial\mathbf{x}}{\partial\mathbf{y}}\right|=1$. Thus we obtain

$$\int_{\mathbb{R}^2}|\theta(\mathbf{x},t)|^p\,d\mathbf{x}=\int_{\mathbb{R}^2}|\theta_0(\mathbf{y})|^p\,d\mathbf{y},$$

that is, $\|\theta(\cdot,t)\|_p=\|\theta_0\|_p$ for all $t\geq 0$. $\square$

Lemma 3.2. *Let $\mathbf{u}:\mathbb{R}^n\to\mathbb{R}^n$ with $\nabla\cdot\mathbf{u}=0$, $v,w:\mathbb{R}^n\to\mathbb{R}$ be sufficiently smooth functions. Then for any continuous function $g:\mathbb{R}\to\mathbb{R}$, it holds that*

$$\int_{\mathbb{R}^2}(\mathbf{u}\cdot\nabla v)w\,dx+\int_{\mathbb{R}^2}(\mathbf{u}\cdot\nabla w)v\,dx=0,\quad \int_{\mathbb{R}^2}(\mathbf{u}\cdot\nabla v)g(v)\,dx=0.$$

Furthermore, in the case of vector functions $\mathbf{v},\mathbf{w}:\mathbb{R}^n\to\mathbb{R}^n$, it holds that

$$\int_{\mathbb{R}^2}\mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{w}\,dx+\int_{\mathbb{R}^2}\mathbf{u}\cdot\nabla\mathbf{w}\cdot\mathbf{v}\,dx=0,\quad \int_{\mathbb{R}^2}\mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{v}\,dx=0,$$

where $\int_{\mathbb{R}^2}\mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{w}\,dx=\int_{\mathbb{R}^2}(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,dx=\sum\limits_{i,j=1}^{n}\int_{\mathbb{R}^2}u^i\partial_iv^jw^j\,dx$.

Lemma 3.3. *Let $(\mathbf{u},\theta,\mathbf{b})$ be a smooth solution of system (1.2). Under the assumptions of Lemma 3.1, if in addition, $(\mathbf{u}_0,\mathbf{b}_0)\in L^2(\mathbb{R}^2)$, then it holds that*

$$\sup_{0\leq t\leq T}(\|\mathbf{u}(\cdot,t)\|_2^2+\|\mathbf{b}(\cdot,t)\|_2^2)+2\mu\int_0^T\|\nabla\mathbf{u}(\cdot,s)\|_2^2\,ds+2\nu\int_0^T\|\nabla\mathbf{b}(\cdot,s)\|_2^2\,ds\leq C$$

where the constant C depends only on T, $\|\mathbf{u}_0\|_2$, $\|\mathbf{b}_0\|_2$ and $\|\theta_0\|_2$.

Proof. By taking the L^2-inner product of $(1.2)_1$ with $\mathbf{u}$ and $(1.2)_3$ with $\mathbf{b}$ respectively, we derive

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}\|_2^2+\|\mathbf{b}\|_2^2)+\mu\|\nabla\mathbf{u}\|_2^2+\nu\|\nabla\mathbf{b}\|_2^2=\int_{\mathbb{R}^2}\theta e_2\cdot\mathbf{u}\,dx,$$

where we have used, as consequences of the incompressibility conditions $\nabla\cdot\mathbf{u}=0$ and $\nabla\cdot\mathbf{b}=0$ and Lemma 3.2, the following relations

$$\int_{\mathbb{R}^2}\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}\,dx=\int_{\mathbb{R}^2}\mathbf{u}\cdot\nabla\mathbf{b}\cdot\mathbf{b}\,dx=0,\quad \int_{\mathbb{R}^2}\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{u}\,dx+\int_{\mathbb{R}^2}\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{b}\,dx=0.$$

By applying Hölder's inequality and Young's inequality, Lemma 3.1, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}\|_2^2+\|\mathbf{b}\|_2^2)+\mu\|\nabla\mathbf{u}\|_2^2+\nu\|\nabla\mathbf{b}\|_2^2\leq\|\theta\|_2\|\mathbf{u}\|_2\leq\frac{1}{2}\|\theta_0\|_2(\|\mathbf{u}\|_2^2+1+\|\mathbf{b}\|_2^2),$$

and hence by applying Grönwall's inequality, we now obtain

$$\begin{aligned}&\sup_{0\le t\le T}(\|\mathbf{u}\|_2^2+\|\mathbf{b}\|_2^2+1)+2\mu\int_0^T\|\nabla\mathbf{u}\|_2^2\,dt+2\nu\int_0^T\|\nabla\mathbf{b}\|_2^2\,dt\\&\le e^{T\|\theta_0\|_2}(\|\mathbf{u}_0\|_2^2+\|\mathbf{b}_0\|_2^2+1).\end{aligned}$$

This completes the proof of Lemma 3.3. □

Next, we establish the H^1 estimates of $\mathbf{u}$ and $\mathbf{b}$.

LEMMA 3.4. *Under the assumptions of Lemma 3.3, if in addition,* $(\mathbf{u}_0,\mathbf{b}_0)\in H^1(\mathbb{R}^2)$*, then it holds that*

$$\sup_{0\le t\le T}(\|\nabla\mathbf{u}(\cdot,t)\|_2^2+\|\nabla\mathbf{b}(\cdot,t)\|_2^2)+\mu\int_0^T\|\Delta\mathbf{u}(\cdot,s)\|_2^2\,ds+\nu\int_0^T\|\Delta\mathbf{b}(\cdot,s)\|_2^2\,ds\le C$$

where the constant C depends only on T, $\|\mathbf{u}_0\|_{H^1}$, $\|\mathbf{b}_0\|_{H^1}$ and $\|\theta_0\|_2$.

Proof. Taking the inner products of $(1.2)_1$ with $-\Delta\mathbf{u}$ and $(1.2)_3$ with $-\Delta\mathbf{b}$ respectively, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{b}\|_2^2)+\mu\|\Delta\mathbf{u}\|_2^2+\nu\|\Delta\mathbf{b}\|_2^2\\&=-\int_{\mathbb{R}^2}\theta e_2\cdot\Delta\mathbf{u}\,dx+2\int_{\mathbb{R}^2}\mathbf{u}\cdot\nabla\mathbf{b}\cdot\Delta\mathbf{b}\,dx,\end{aligned}$$

where, similar to Lemma 3.2, the conditions $\nabla\cdot\mathbf{u}=0$ and $\nabla\cdot\mathbf{b}=0$ imply the relations

$$\begin{aligned}&\int_{\mathbb{R}^2}\mathbf{u}\cdot\nabla\mathbf{u}\cdot\Delta\mathbf{u}\,dx=0,\\&\int_{\mathbb{R}^2}\mathbf{u}\cdot\nabla\mathbf{b}\cdot\Delta\mathbf{b}\,dx=-\int_{\mathbb{R}^2}\mathbf{b}\cdot\nabla\mathbf{b}\cdot\Delta\mathbf{u}\,dx-\int_{\mathbb{R}^2}\mathbf{b}\cdot\nabla\mathbf{u}\cdot\Delta\mathbf{b}\,dx.\end{aligned}$$

By using Hölder's inequality, Corollary 2.1, Lemma 2.2 and Young's inequality, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{b}\|_2^2)+\mu\|\Delta\mathbf{u}\|_2^2+\nu\|\Delta\mathbf{b}\|_2^2\\&\le\int_{\mathbb{R}^2}|\theta e_2|\,|\Delta\mathbf{u}|\,dx+C\int_{\mathbb{R}^2}|\nabla\mathbf{u}|\,|\nabla\mathbf{b}|\,|\nabla\mathbf{b}|\,dx\\&\le\|\theta\|_2\|\Delta\mathbf{u}\|_2+C\|\nabla\mathbf{u}\|_2\|\nabla\mathbf{b}\|_4^2\\&\le\|\theta\|_2\|\Delta\mathbf{u}\|_2+C\|\nabla\mathbf{u}\|_2\|\nabla\mathbf{b}\|_2\|\Delta\mathbf{b}\|_2\\&\le\frac{\mu}{2}\|\Delta\mathbf{u}\|_2^2+\frac{\nu}{2}\|\Delta\mathbf{b}\|_2^2+C\|\theta\|_2^2+C\|\nabla\mathbf{u}\|_2^2\|\nabla\mathbf{b}\|_2^2,\end{aligned}\tag{3.1}$$

and by further applying Grönwall's inequality, we obtain

$$\begin{aligned}&\sup_{0\le t\le T}(\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{b}\|_2^2)+\mu\int_0^T\|\Delta\mathbf{u}\|_2^2\,dt+\nu\int_0^T\|\Delta\mathbf{b}\|_2^2\,dt\\&\le(\|\nabla\mathbf{u}_0\|_2^2+\|\nabla\mathbf{b}_0\|_2^2+CT\|\theta_0\|_2^2)\,e^{\{Ce^T(\|\mathbf{u}_0\|_2^2+\|\mathbf{b}_0\|_2^2+T\|\theta_0\|_2^2)\}}\le C(T).\end{aligned}$$

This completes the proof of Lemma 3.4. □

3.2. Global $W^{1,p}$ estimates for u and b. In this subsection, we derive the global L^p–estimates for the scalar vorticity function $\Omega = \partial_1 u_2 - \partial_2 u_1$ and the scalar current density function $j = \partial_1 b_2 - \partial_2 b_1$.

LEMMA 3.5. *Under the assumptions of Lemma 3.4, if in addition,* $(\mathbf{u}_0, \mathbf{b}_0) \in W^{1,p}(\mathbb{R}^2)$, $\theta_0 \in L^p(\mathbb{R}^2)$ *for any* $2 \le p < \infty$, *then it holds that*

$$\|\Omega\|_{L^\infty(0,T;L^p)} + \|j\|_{L^\infty(0,T;L^p)} \le C$$

where C *depends on* T, $\|\mathbf{u}_0\|_{W^{1,p}}$, $\|\mathbf{b}_0\|_{W^{1,p}}$ *and* $\|\theta_0\|_p$ *only.*

Proof. For $p = 2$, we obtain from Lemma 3.4 and Lemma 2.2 that

$$\sup_{0\le t\le T}(\|\Omega\|_2^2 + \|j\|_2^2) + \mu \int_0^T \|\nabla\Omega\|_2^2\, dt + \nu \int_0^T \|\nabla j\|_2^2\, dt \le C. \tag{3.2}$$

Next, we consider the case of $2 < p < \infty$, which is divided into the following three steps.

Step 1. Multiplying $(1.3)_1$ with $|\Omega|^{p-2}\Omega$ and integrating over $\mathbb{R}^2$, we obtain

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2} |\Omega|^p\, dx + (p-1)\mu \int_{\mathbb{R}^2} |\nabla\Omega|^2|\Omega|^{p-2}\, dx\\ &= \int_{\mathbb{R}^2} (\mathbf{b}\cdot\nabla j)|\Omega|^{p-2}\,\Omega\, dx + \int_{\mathbb{R}^2} (\partial_1\theta)|\Omega|^{p-2}\,\Omega\, dx.\end{aligned}$$

Then by integration by parts, Hölder's inequality, Young's inequality and Corollary 2.1, we obtain

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2} |\Omega|^p\, dx + (p-1)\mu \int_{\mathbb{R}^2} |\nabla\Omega|^2|\Omega|^{p-2}\, dx\\
&\le \frac{(p-1)\mu}{4}\int_{\mathbb{R}^2} |\nabla\Omega|^2|\Omega|^{p-2}\, dx + \frac{(p-1)}{\mu}\int_{\mathbb{R}^2} |\mathbf{b}|^2|j|^2|\Omega|^{p-2}\, dx\\
&\quad + \frac{(p-1)\mu}{4}\int_{\mathbb{R}^2} |\nabla\Omega|^2|\Omega|^{p-2}\, dx + \frac{(p-1)}{\mu}\int_{\mathbb{R}^2} |\theta|^2|\Omega|^{p-2}\, dx\\
&\le \frac{(p-1)\mu}{2}\int_{\mathbb{R}^2} |\nabla\Omega|^2|\Omega|^{p-2}\, dx + \frac{(p-1)}{\mu}\|\mathbf{b}\|_\infty^2\|j\|_p^2\|\Omega\|_p^{p-2} + \frac{(p-1)}{\mu}\|\theta\|_p^2\|\Omega\|_p^{p-2}\\
&\le \frac{(p-1)\mu}{2}\int_{\mathbb{R}^2} |\nabla\Omega|^2|\Omega|^{p-2}\, dx + C\|\mathbf{b}\|_2\|\Delta\mathbf{b}\|_2\|j\|_p^2\|\Omega\|_p^{p-2} + \frac{(p-1)}{\mu}\|\theta\|_p^2\|\Omega\|_p^{p-2}\end{aligned}$$

which, after dividing both sides by $\|\Omega\|_p^{p-2}$ and using Young's inequality, Lemma 3.1 and Lemma 3.3, implies that

$$\frac{d}{dt}\|\Omega\|_p^2 \le C\|\mathbf{b}\|_2\|\Delta\mathbf{b}\|_2\|j\|_p^2 + C\|\theta\|_p^2 \le C(1 + \|\Delta\mathbf{b}\|_2^2)\|j\|_p^2 + C\|\theta_0\|_p^2. \tag{3.3}$$

Step 2. Repeating the above procedures for the current density function j, we multiply $(1.3)_2$ with $|j|^{p-2}j$ to obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|j|^p\,dx+(p-1)\nu\int_{\mathbb{R}^2}|\nabla j|^2|j|^{p-2}\,dx\\
&=\int_{\mathbb{R}^2}(\mathbf{b}\cdot\nabla\Omega)\,|j|^{p-2}j\,dx+2\int_{\mathbb{R}^2}[\partial_1 b_1(\partial_1 u_2+\partial_2 u_1)-\partial_1 u_1(\partial_1 b_2+\partial_2 b_1)]\,|j|^{p-2}j\,dx\\
&\le\frac{(p-1)\nu}{2}\int_{\mathbb{R}^2}|\nabla j|^2|j|^{p-2}\,dx+\frac{p-1}{2\nu}\int_{\mathbb{R}^2}|\mathbf{b}|^2|\Omega|^2|j|^{p-2}\,dx+C\|\nabla\mathbf{u}\|_{2p}\|\nabla\mathbf{b}\|_{2p}\|j\|_p^{p-1}\\
&\le\frac{(p-1)\nu}{2}\int_{\mathbb{R}^2}|\nabla j|^2|j|^{p-2}\,dx+\frac{p-1}{2\nu}\|\mathbf{b}\|_\infty^2\|\Omega\|_p^2\|j\|_p^{p-2}+C\|\nabla\mathbf{u}\|_{2p}\|\nabla\mathbf{b}\|_{2p}\|j\|_p^{p-1}\\
&\le\frac{(p-1)\nu}{2}\int_{\mathbb{R}^2}|\nabla j|^2|j|^{p-2}\,dx+C\|\mathbf{b}\|_2\|\Delta\mathbf{b}\|_2\|\Omega\|_p^2\|j\|_p^{p-2}+C\|\mathbf{u}\|_{H^2}\|\mathbf{b}\|_{H^2}\|j\|_p^{p-1}.
\end{aligned}$$

Dividing both sides of the above inequality by $\|j\|_p^{p-2}$, and using Young's inequality, Lemma 2.1, Lemma 3.3, Lemma 3.4, we obtain

$$\begin{aligned}
\frac{d}{dt}\|j\|_p^2&\le C\|\mathbf{b}\|_2\|\Delta\mathbf{b}\|_2\|\Omega\|_p^2+C\|\mathbf{u}\|_{H^2}\|\mathbf{b}\|_{H^2}\|j\|_p\\
&\le C(\|\mathbf{b}\|_2^2+\|\Delta\mathbf{b}\|_2^2)\|\Omega\|_p^2+C(\|\mathbf{u}\|_2+\|\Delta\mathbf{u}\|_2)(\|\mathbf{b}\|_2+\|\Delta\mathbf{b}\|_2)(\|j\|_p^2+1)\\
&\le C(1+\|\Delta\mathbf{b}\|_2^2)\|\Omega\|_p^2+C(1+\|\Delta\mathbf{u}\|_2)(1+\|\Delta\mathbf{b}\|_2)(\|j\|_p^2+1)\\
&\le C(\|\Omega\|_p^2+\|j\|_p^2)(1+\|\Delta\mathbf{b}\|_2^2+\|\Delta\mathbf{u}\|_2^2)+C(1+\|\Delta\mathbf{b}\|_2^2+\|\Delta\mathbf{u}\|_2^2). \quad (3.4)
\end{aligned}$$

Step 3. By summing up the inequalities (3.3) and (3.4), we have

$$\begin{aligned}
&\frac{d}{dt}(\|\Omega\|_p^2+\|j\|_p^2)\\
&\le C(\|\Omega\|_p^2+\|j\|_p^2)(1+\|\Delta\mathbf{b}\|_2^2+\|\Delta\mathbf{u}\|_2^2)+C\|\theta_0\|_p^2+C(1+\|\Delta\mathbf{b}\|_2^2+\|\Delta\mathbf{u}\|_2^2).
\end{aligned}$$

By Grönwall's inequality, we obtain

$$\begin{aligned}
\sup_{0\le t\le T}(\|\Omega\|_p^2+\|j\|_p^2)\le e^{\left\{C\int_0^T(1+\|\Delta\mathbf{b}\|_2^2+\|\Delta\mathbf{u}\|_2^2)\,dt\right\}}[\|\Omega_0\|_p^2+\|j_0\|_p^2+CT\|\theta_0\|_p^2\\
+C\int_0^T(1+\|\Delta\mathbf{b}\|_2^2+\|\Delta\mathbf{u}\|_2^2)\,dt].
\end{aligned}$$

By Lemma 3.4, we have $\int_0^T(\|\Delta\mathbf{b}\|_2^2+\|\Delta\mathbf{u}\|_2^2)\,dt\le C(T)$. Therefore, it follows that $\sup_{0\le t\le T}(\|\Omega\|_p^2+\|j\|_p^2)\le C(T)$. This completes the proof of Lemma 3.5. □

Next, we establish the global L^∞-estimates for $\mathbf{u}$ and $\mathbf{b}$.

Lemma 3.6. *Under the assumptions of Lemma 3.5, it holds that*

$$\sup_{0\le t\le T}(\|\mathbf{u}(\cdot,t)\|_\infty+\|\mathbf{b}(\cdot,t)\|_\infty)\le C$$

where C depends only on T, $\|\mathbf{u}_0\|_{W^{1,p}}$, $\|\mathbf{b}_0\|_{W^{1,p}}$ and $\|\theta_0\|_p$, $2<p<\infty$.

Proof. First, by the Gagliardo-Nirenberg interpolation inequality, we obtain

$$\|\mathbf{u}\|_\infty\le C_p\,\|\mathbf{u}\|_2^{\frac{p-2}{2p-2}}\,\|\nabla\mathbf{u}\|_p^{\frac{p}{2p-2}},\quad 2<p<\infty. \quad (3.5)$$

By the Calderón-Zygmund inequality, we have

$$\|\nabla \mathbf{u}\|_p \le C_p \|\Omega\|_p, \quad 1 < p < \infty. \tag{3.6}$$

Then by inequalities (3.5) and (3.6), Lemma 3.3, Lemma 3.5, it follows that

$$\|\mathbf{u}\|_\infty \le C\|\mathbf{u}\|_2^{\frac{p-2}{2p-2}} \|\Omega\|_p^{\frac{p}{2p-2}} \le C, \quad 2 < p < \infty.$$

Similarly, we can obtain $\|\mathbf{b}\|_\infty \le C$. This completes the proof of Lemma 3.6. ∎

3.3. Global $W^{2,p}$ estimates for u, b. We first establish the global $W^{1,p}$–estimates for Ω and j as follows.

Lemma 3.7. *Under the assumptions of Lemma 3.5, if in addition, $(\mathbf{u}_0, \mathbf{b}_0) \in W^{2,p}(\mathbb{R}^2)$, $\theta_0 \in W^{1,p}(\mathbb{R}^2)$ for any $2 < p < \infty$, then it holds that*

$$\|\nabla\Omega\|_{L^\infty(0,T;L^p)} + \|\nabla j\|_{L^\infty(0,T;L^p)} + \|\nabla\theta\|_{L^\infty(0,T;L^p)} \le C$$

where C depends only on T, $\|\mathbf{u}_0\|_{W^{2,p}}$, $\|\mathbf{b}_0\|_{W^{2,p}}$ and $\|\theta_0\|_{W^{1,p}}$.

Proof. **Step 1**. By taking the first-order partial differential operator $\partial_i = \partial/\partial_{x_i}$, $i = 1, 2$, on $(1.3)_1$, and multiplying the resulting equation with $|\partial_i\Omega|^{p-2}\partial_i\Omega$, we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2} |\partial_i\Omega|^p\,dx + (p-1)\mu\int_{\mathbb{R}^2} |\nabla\partial_i\Omega|^2\,|\partial_i\Omega|^{p-2}\,dx \\
&= -\int_{\mathbb{R}^2} [\partial_i(\mathbf{u}\cdot\nabla\Omega)]\,\partial_i\Omega\,|\partial_i\Omega|^{p-2}\,dx + \int_{\mathbb{R}^2} [\partial_i(\mathbf{b}\cdot\nabla j)]\,\partial_i\Omega\,|\partial_i\Omega|^{p-2}\,dx \\
&\quad + \int_{\mathbb{R}^2} [\partial_i(\partial_1\theta)]\,\partial_i\Omega\,|\partial_i\Omega|^{p-2}\,dx \\
&= (p-1)\int_{\mathbb{R}^2} (\mathbf{u}\cdot\nabla\Omega)\,\partial_i^2\Omega\,|\partial_i\Omega|^{p-2}\,dx - (p-1)\int_{\mathbb{R}^2} (\mathbf{b}\cdot\nabla j)\,\partial_i^2\Omega\,|\partial_i\Omega|^{p-2}\,dx \\
&\quad -(p-1)\int_{\mathbb{R}^2} (\partial_1\theta)\,\partial_i^2\Omega\,|\partial_i\Omega|^{p-2}\,dx.
\end{aligned}$$

By using Hölder's inequality, Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2} |\partial_i\Omega|^p\,dx + (p-1)\mu\int_{\mathbb{R}^2} |\nabla\partial_i\Omega|^2\,|\partial_i\Omega|^{p-2}\,dx \\
&\le \frac{(p-1)\mu}{8}\int_{\mathbb{R}^2} |\partial_i^2\Omega|^2\,|\partial_i\Omega|^{p-2}\,dx + \frac{2(p-1)}{\mu}\int_{\mathbb{R}^2} |\mathbf{u}|^2\,|\nabla\Omega|^2\,|\partial_i\Omega|^{p-2}\,dx \\
&\quad + \frac{(p-1)\mu}{8}\int_{\mathbb{R}^2} |\partial_i^2\Omega|^2\,|\partial_i\Omega|^{p-2}\,dx + \frac{2(p-1)}{\mu}\int_{\mathbb{R}^2} |\mathbf{b}|^2\,|\nabla j|^2\,|\partial_i\Omega|^{p-2}\,dx \\
&\quad + \frac{(p-1)\mu}{4}\int_{\mathbb{R}^2} |\partial_i^2\Omega|^2\,|\partial_i\Omega|^{p-2}\,dx + \frac{(p-1)}{\mu}\int_{\mathbb{R}^2} |\partial_1\theta|^2\,|\partial_i\Omega|^{p-2}\,dx \\
&\le \frac{(p-1)\mu}{2}\int_{\mathbb{R}^2} |\nabla\partial_i\Omega|^2\,|\partial_i\Omega|^{p-2}\,dx + \frac{2(p-1)}{\mu}\|\mathbf{u}\|_\infty^2\|\nabla\Omega\|_p^p \\
&\quad + \frac{2(p-1)}{\mu}\|\mathbf{b}\|_\infty^2\|\nabla j\|_p^2\|\nabla\Omega\|_p^{p-2} + \frac{(p-1)}{\mu}\|\nabla\theta\|_p^2\|\nabla\Omega\|_p^{p-2} \\
&\le \frac{(p-1)\mu}{2}\int_{\mathbb{R}^2} |\nabla\partial_i\Omega|^2\,|\partial_i\Omega|^{p-2}\,dx + \frac{2(p-1)}{\mu}\|\mathbf{u}\|_\infty^2\|\nabla\Omega\|_p^p
\end{aligned}$$

$$+\frac{4(p-1)}{p\mu}\|\mathbf{b}\|_\infty^2\|\nabla j\|_p^p+\frac{2(p-1)(p-2)}{p\mu}\|\mathbf{b}\|_\infty^2\|\nabla\Omega\|_p^p$$
$$+\frac{2(p-1)}{p\mu}\|\nabla\theta\|_p^p+\frac{(p-1)(p-2)}{p\mu}\|\nabla\Omega\|_p^p.$$

Then by summing over i, and using Lemma 3.6, it follows that

$$\frac{1}{p}\frac{d}{dt}\|\nabla\Omega\|_p^p+\frac{(p-1)\mu}{2}\int_{\mathbb{R}^2}|\nabla^2\Omega|^2\,|\nabla\Omega|^{p-2}\,dx\ \le C(\|\nabla\Omega\|_p^p+\|\nabla j\|_p^p+\|\nabla\theta\|_p^p). \quad (3.7)$$

Step 2. Taking the first-order partial differential operator ∂_i on $(1.3)_2$, and taking the inner product of the equation with $|\partial_i j|^{p-2}\partial_i j$, we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|\partial_i j|^p\,dx+(p-1)\nu\int_{\mathbb{R}^2}|\nabla\partial_i j|^2\,|\partial_i j|^{p-2}\,dx\\
&=-\int_{\mathbb{R}^2}[\partial_i(\mathbf{u}\cdot\nabla j)]\,\partial_i j\,|\partial_i j|^{p-2}\,dx+\int_{\mathbb{R}^2}[\partial_i(\mathbf{b}\cdot\nabla\Omega)]\,\partial_i j\,|\partial_i j|^{p-2}\,dx\\
&\quad+2\int_{\mathbb{R}^2}\partial_i[\partial_1 b_1(\partial_1 u_2+\partial_2 u_1)-\partial_1 u_1(\partial_1 b_2+\partial_2 b_1)]\,\partial_i j\,|\partial_i j|^{p-2}\,dx\\
&=(p-1)\int_{\mathbb{R}^2}(\mathbf{u}\cdot\nabla j)\,\partial_i^2 j\,|\partial_i j|^{p-2}\,dx-(p-1)\int_{\mathbb{R}^2}(\mathbf{b}\cdot\nabla\Omega)\,\partial_i^2 j\,|\partial_i j|^{p-2}\,dx\\
&\quad-2(p-1)\int_{\mathbb{R}^2}[\partial_1 b_1(\partial_1 u_2+\partial_2 u_1)-\partial_1 u_1(\partial_1 b_2+\partial_2 b_1)]\,\partial_i^2 j\,|\partial_i j|^{p-2}\,dx\\
&\le\frac{(p-1)\nu}{8}\int_{\mathbb{R}^2}|\partial_i^2 j|^2\,|\partial_i j|^{p-2}\,dx+\frac{2(p-1)}{\nu}\int_{\mathbb{R}^2}|\mathbf{u}|^2\,|\nabla j|^2\,|\partial_i j|^{p-2}\,dx\\
&\quad+\frac{(p-1)\nu}{8}\int_{\mathbb{R}^2}|\partial_i^2 j|^2\,|\partial_i j|^{p-2}\,dx+\frac{2(p-1)}{\nu}\int_{\mathbb{R}^2}|\mathbf{b}|^2\,|\nabla\Omega|^2\,|\partial_i j|^{p-2}\,dx\\
&\quad+\frac{(p-1)\nu}{4}\int_{\mathbb{R}^2}|\partial_i^2 j|^2\,|\partial_i j|^{p-2}\,dx+\frac{(p-1)}{\nu}\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2\,|\nabla\mathbf{b}|^2\,|\partial_i j|^{p-2}\,dx\\
&\le\frac{(p-1)\nu}{2}\int_{\mathbb{R}^2}|\nabla\partial_i j|^2\,|\partial_i j|^{p-2}\,dx+\frac{2(p-1)}{\nu}\|\mathbf{u}\|_\infty^2\|\nabla j\|_p^p\\
&\quad+\frac{2(p-1)}{\nu}\|\mathbf{b}\|_\infty^2\|\nabla\Omega\|_p^2\|\nabla j\|_p^{p-2}+\frac{(p-1)}{\nu}\|\nabla\mathbf{u}\|_{2p}^2\|\nabla\mathbf{b}\|_{2p}^2\|\nabla j\|_p^{p-2}\\
&\le\frac{(p-1)\nu}{2}\int_{\mathbb{R}^2}|\nabla\partial_i j|^2\,|\partial_i j|^{p-2}\,dx+\frac{2(p-1)}{\nu}\|\mathbf{u}\|_\infty^2\|\nabla j\|_p^p\\
&\quad+\frac{4(p-1)}{p\nu}\|\mathbf{b}\|_\infty^2\|\nabla\Omega\|_p^p+\frac{2(p-1)(p-2)}{p\nu}\|\mathbf{b}\|_\infty^2\|\nabla j\|_p^p\\
&\quad+\frac{2(p-1)}{p\nu}\|\nabla\mathbf{u}\|_{2p}^p\|\nabla\mathbf{b}\|_{2p}^p+\frac{(p-1)(p-2)}{p\nu}\|\nabla j\|_p^p.
\end{aligned}$$

Then by summing over i, and using the Calderón-Zygmund inequality, Lemma 3.5 and Lemma 3.6, we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|\nabla j\|_p^p+\frac{(p-1)\nu}{2}\int_{\mathbb{R}^2}|\nabla^2 j|^2\,|\nabla j|^{p-2}\,dx\\
&\le C(\|\nabla j\|_p^p+\|\nabla\Omega\|_p^p)(\|\mathbf{u}\|_\infty^2+\|\mathbf{b}\|_\infty^2+1)+C\|\nabla\mathbf{u}\|_{2p}^p\|\nabla\mathbf{b}\|_{2p}^p\\
&\le C(\|\nabla j\|_p^p+\|\nabla\Omega\|_p^p)(\|\mathbf{u}\|_\infty^2+\|\mathbf{b}\|_\infty^2+1)+C\|\Omega\|_{2p}^p\|j\|_{2p}^p\\
&\le C(\|\nabla j\|_p^p+\|\nabla\Omega\|_p^p+1).
\end{aligned} \quad (3.8)$$

Step 3. Repeating the above procedures for θ, we take the first-order partial $\partial_i = \partial/\partial_{x_i}$, $i = 1, 2$, on $(1.2)_2$, take the inner product of the resulting equation with $|\partial_i\theta|^{p-2}\partial_i\theta$, and sum over i to obtain

$$\frac{1}{p}\frac{d}{dt}\|\nabla\theta\|_p^p \leq \|\nabla\mathbf{u}\|_\infty\|\nabla\theta\|_p^p. \tag{3.9}$$

Summing up the inequalities (3.7), (3.8) and (3.9), we obtain

$$\begin{aligned}&\frac{d}{dt}(\|\nabla\Omega\|_p^p + \|\nabla j\|_p^p + \|\nabla\theta\|_p^p)\\ &\leq C(\|\nabla\mathbf{u}\|_\infty + 1)(\|\nabla\Omega\|_p^p + \|\nabla j\|_p^p + \|\nabla\theta\|_p^p + 1).\end{aligned} \tag{3.10}$$

By the Brezis-Wainger inequality, for $f \in L^2(\mathbb{R}^2) \cap W^{1,p}(\mathbb{R}^2)$, it holds [10] that

$$\|f\|_\infty \leq C(1 + \|\nabla f\|_2)\left[1 + \log^{\frac{1}{2}}(1 + \|\nabla f\|_p)\right] + C\|f\|_2, \quad 2 < p < \infty. \tag{3.11}$$

Then we obtain from (3.10)–(3.11) and Lemma 3.5 that

$$\begin{aligned}&\frac{d}{dt}(\|\nabla\Omega\|_p^p + \|\nabla j\|_p^p + \|\nabla\theta\|_p^p)\\ &\leq C(1 + \|\nabla\mathbf{u}\|_2 + \|\nabla^2\mathbf{u}\|_2)\left[1 + \log^{\frac{1}{2}}(1 + \|\nabla^2\mathbf{u}\|_p)\right](\|\nabla\Omega\|_p^p + \|\nabla j\|_p^p + \|\nabla\theta\|_p^p + 1)\\ &\leq C(1 + \|\Omega\|_2 + \|\nabla\Omega\|_2)\left[1 + \log^{\frac{1}{2}}(1 + \|\nabla\Omega\|_p^p)\right](\|\nabla\Omega\|_p^p + \|\nabla j\|_p^p + \|\nabla\theta\|_p^p + 1)\\ &\leq C(1 + \|\nabla\Omega\|_2^2)\left[1 + \log(1 + \|\nabla\Omega\|_p^p + \|\nabla j\|_p^p + \|\nabla\theta\|_p^p)\right](\|\nabla\Omega\|_p^p + \|\nabla j\|_p^p\\ &\quad + \|\nabla\theta\|_p^p + 1).\end{aligned}$$

Setting $X(t) = \|\nabla\Omega\|_p^p + \|\nabla j\|_p^p + \|\nabla\theta\|_p^p + 1$ and $\phi(t) = 1 + \|\nabla\Omega\|_2^2$, we have

$$\frac{dX}{dt} \leq C\phi(t)(1 + \log X)X.$$

By Grönwall's inequality, we obtain

$$1 + \log X(t) \leq (1 + \log X_0)\, e^{C\int_0^T \phi(t)\,dt}.$$

By inequality (3.2), it is clear that $\phi(t) \in L^1(0, T)$. Then it follows that $X \leq C(T)$, and hence we have $\sup\limits_{0\leq t\leq T}(\|\nabla\Omega\|_p^p + \|\nabla j\|_p^p + \|\nabla\theta\|_p^p) \leq C(T)$. This completes the proof of Lemma 3.7. □

Next, we establish the global $W^{1,\infty}$-estimates for $\mathbf{u}$ and $\mathbf{b}$.

LEMMA 3.8. *Under the assumptions of Lemma 3.7, it holds that*

$$\sup_{0\leq t\leq T}(\|\nabla\mathbf{u}(\cdot, t)\|_\infty + \|\nabla\mathbf{b}(\cdot, t)\|_\infty) \leq C$$

where C depends only on T, $\|\mathbf{u}_0\|_{W^{2,p}}$, $\|\mathbf{b}_0\|_{W^{2,p}}$ and $\|\theta_0\|_{W^{1,p}}$, $2 < p < \infty$.

Proof. Similar to the proof of Lemma 3.6, by applying the Gagliardo-Nirenberg interpolation inequality and the Calderón-Zygmund inequality, we can derive

$$\|\nabla\mathbf{u}\|_\infty \leq C\|\nabla\mathbf{u}\|_2^{\frac{p-2}{2p-2}}\|\nabla^2\mathbf{u}\|_p^{\frac{p}{2p-2}} \leq C\|\Omega\|_2^{\frac{p-2}{2p-2}}\|\nabla\Omega\|_p^{\frac{p}{2p-2}}, \quad 2 < p < \infty.$$

Then by using Lemma 3.5 and Lemma 3.7, it follows that $\sup\limits_{0\leq t\leq T}\|\nabla\mathbf{u}\|_\infty \leq C$. Similarly, we can obtain $\sup\limits_{0\leq t\leq T}\|\nabla\mathbf{b}\|_\infty \leq C$. This completes the proof of Lemma 3.8. □

3.4. H^s estimates. This subsection is devoted to obtaining the global H^s-bound for $\mathbf{u}$, $\mathbf{b}$ and θ, namely, completing the proof of Proposition 3.1.

Proof of Proposition 3.1. Applying the operator $\Lambda^s = (-\Delta)^{s/2}$ to both sides of (1.2) and then multiplying the equations with $\Lambda^s\mathbf{u}$, $\Lambda^s\theta$, $\Lambda^s\mathbf{b}$ respectively, we get the following energy estimates

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}\|_{H^s}^2 + \|\mathbf{b}\|_{H^s}^2 + \|\theta\|_{H^s}^2) + \mu\|\nabla\mathbf{u}\|_{H^s}^2 + \nu\|\nabla\mathbf{b}\|_{H^s}^2\\
&= \int_{\mathbb{R}^2}[\Lambda^s, \mathbf{u}\cdot\nabla]\mathbf{u}\cdot\Lambda^s\mathbf{u}\,dx - \int_{\mathbb{R}^2}[\Lambda^s, \mathbf{b}\cdot\nabla]\mathbf{b}\cdot\Lambda^s\mathbf{u}\,dx + \int_{\mathbb{R}^2}\theta e_2\cdot(-\Delta)^s\mathbf{u}\,dx\\
&\quad + \int_{\mathbb{R}^2}[\Lambda^s, \mathbf{u}\cdot\nabla]\mathbf{b}\cdot\Lambda^s\mathbf{b}\,dx - \int_{\mathbb{R}^2}[\Lambda^s, \mathbf{b}\cdot\nabla]\mathbf{u}\cdot\Lambda^s\mathbf{b}\,dx + \int_{\mathbb{R}^2}[\Lambda^s, \mathbf{u}\cdot\nabla]\theta\cdot\Lambda^s\theta\,dx\\
&= \sum_{i=1}^{6} I_i, \qquad (3.12)
\end{aligned}$$

where we used the equalities

$$\int_{\mathbb{R}^2}\mathbf{u}\cdot\nabla\Lambda^s\mathbf{u}\cdot\Lambda^s\mathbf{u}\,dx = 0, \quad \int_{\mathbb{R}^2}\mathbf{u}\cdot\nabla\Lambda^s\mathbf{b}\cdot\Lambda^s\mathbf{b}\,dx = 0, \quad \int_{\mathbb{R}^2}\mathbf{u}\cdot\nabla\Lambda^s\theta\cdot\Lambda^s\theta\,dx = 0,$$

and

$$\int_{\mathbb{R}^2}\mathbf{b}\cdot\nabla\Lambda^s\mathbf{b}\cdot\Lambda^s\mathbf{u}\,dx + \int_{\mathbb{R}^2}\mathbf{b}\cdot\nabla\Lambda^s\mathbf{u}\cdot\Lambda^s\mathbf{b}\,dx = 0.$$

In the following, we will frequently apply the commutator estimates in Corollary 2.2.

We now estimate the above six terms one by one. By applying the Hölder inequality, Corollary 2.2 and the Young's inequality, we obtain

$$\begin{aligned}
I_1 + I_2 &\le \|[\Lambda^s, \mathbf{u}\cdot\nabla]\mathbf{u}\|_2\|\Lambda^s\mathbf{u}\|_2 + \|[\Lambda^s, \mathbf{b}\cdot\nabla]\mathbf{b}\|_{L^2}\|\Lambda^s\mathbf{u}\|_2\\
&\le C(\|\nabla\mathbf{u}\|_\infty\|\mathbf{u}\|_{H^s} + \|\mathbf{u}\|_{H^s}\|\nabla\mathbf{u}\|_\infty)\|\mathbf{u}\|_{H^s} + C(\|\nabla\mathbf{b}\|_\infty\|\mathbf{b}\|_{H^s}\\
&\quad + \|\mathbf{b}\|_{H^s}\|\nabla\mathbf{b}\|_\infty)\|\mathbf{u}\|_{H^s}\\
&\le C\|\nabla\mathbf{u}\|_\infty\|\mathbf{u}\|_{H^s}^2 + C\|\nabla\mathbf{b}\|_\infty\|\mathbf{b}\|_{H^s}\|\mathbf{u}\|_{H^s}\\
&\le C(\|\nabla\mathbf{u}\|_\infty + \|\nabla\mathbf{b}\|_\infty)(\|\mathbf{u}\|_{H^s}^2 + \|\mathbf{b}\|_{H^s}^2), \qquad (3.13)\\
I_3 &\le \|\theta\|_{H^s}\|\mathbf{u}\|_{H^s} \le \frac{1}{2}\|\theta\|_{H^s}^2 + \frac{1}{2}\|\mathbf{u}\|_{H^s}^2, \qquad (3.14)\\
I_4 + I_5 &\le \|[\Lambda^s, \mathbf{u}\cdot\nabla]\mathbf{b}\|_2\|\Lambda^s\mathbf{b}\|_2 + \|[\Lambda^s, \mathbf{b}\cdot\nabla]\mathbf{u}\|_2\|\Lambda^s\mathbf{b}\|_2\\
&\le C(\|\nabla\mathbf{u}\|_\infty\|\mathbf{b}\|_{H^s} + \|\mathbf{u}\|_{H^s}\|\nabla\mathbf{b}\|_\infty)\|\mathbf{b}\|_{H^s} + C(\|\nabla\mathbf{b}\|_\infty\|\mathbf{u}\|_{H^s}\\
&\quad + \|\mathbf{b}\|_{H^s}\|\nabla\mathbf{u}\|_\infty)\|\mathbf{b}\|_{H^s}\\
&\le C\|\nabla\mathbf{u}\|_\infty\|\mathbf{b}\|_{H^s}^2 + C\|\nabla\mathbf{b}\|_\infty\|\mathbf{u}\|_{H^s}\|\mathbf{b}\|_{H^s}\\
&\le C(\|\nabla\mathbf{u}\|_\infty + \|\nabla\mathbf{b}\|_\infty)(\|\mathbf{u}\|_{H^s}^2 + \|\mathbf{b}\|_{H^s}^2), \qquad (3.15)\\
I_6 &\le \|[\Lambda^s, \mathbf{u}\cdot\nabla]\theta\|_2\|\Lambda^s\theta\|_2\\
&\le C(\|\nabla\mathbf{u}\|_\infty\|\theta\|_{H^s} + \|\nabla\mathbf{u}\|_{H^s}\|\theta\|_\infty)\|\theta\|_{H^s}\\
&\le C\|\nabla\mathbf{u}\|_\infty\|\theta\|_{H^s}^2 + \frac{\mu}{2}\|\nabla\mathbf{u}\|_{H^s}^2 + C\|\theta\|_\infty^2\|\theta\|_{H^s}^2. \qquad (3.16)
\end{aligned}$$

Then substituting (3.13)–(3.16) into (3.12), we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|\mathbf{u}\|_{H^s}^2 + \|\mathbf{b}\|_{H^s}^2 + \|\theta\|_{H^s}^2) + \mu\|\nabla\mathbf{u}\|_{H^s}^2 + 2\nu\|\nabla\mathbf{b}\|_{H^s}^2\\
&\le C(\|\mathbf{u}\|_{H^s}^2 + \|\mathbf{b}\|_{H^s}^2 + \|\theta\|_{H^s}^2)(\|\nabla\mathbf{u}\|_\infty + \|\nabla\mathbf{b}\|_\infty + \|\theta\|_\infty^2 + 1). \qquad (3.17)
\end{aligned}$$

By Grönwall's inequality, Lemma 3.1 and Lemma 3.8, it follows that

$$\sup_{0\le t\le T}(\|\mathbf{u}\|_{H^s}^2+\|\mathbf{b}\|_{H^s}^2+\|\theta\|_{H^s}^2)+\mu\int_0^T\|\nabla\mathbf{u}\|_{H^s}^2\,dt+2\nu\int_0^T\|\nabla\mathbf{b}\|_{H^s}^2\,dt$$
$$\le(\|\mathbf{u}_0\|_{H^s}^2+\|\mathbf{b}_0\|_{H^s}^2+\|\theta_0\|_{H^s}^2)\,e^{C\int_0^T\left(\|\nabla\mathbf{u}\|_\infty+\|\nabla\mathbf{b}\|_\infty+\|\theta_0\|_\infty^2+1\right)dt}\le C(T).$$

This completes the proof of Proposition 3.1. □

3.5. Existence of strong solutions.

THEOREM 3.1. *Suppose that the initial data* $(\mathbf{u}_0,\theta_0,\mathbf{b}_0)\in H^s(\mathbb{R}^2)$, $s>2$ *and* $\nabla\cdot\mathbf{u}_0=\nabla\cdot\mathbf{b}_0=0$. *Then there exist strong solutions* $(\mathbf{u},\theta,\mathbf{b})$ *of system* (1.2) *globally in time such that* $\mathbf{u},\mathbf{b}\in C(0,T;H^s(\mathbb{R}^2))\cap L^2(0,T;H^{s+1}(\mathbb{R}^2))$ *and* $\theta\in C(0,T;H^s(\mathbb{R}^2))$ *for any* $T>0$.

Proof. The proof of the existence of solutions is based on the Friedrichs method, which is also known as the "modified Galerkin method". For any small $\epsilon>0$, let j be a positive radial compactly supported smooth function whose integral equals 1 and J_ϵ be a Friedrichs mollifier defined by

$$J_\epsilon=j_\epsilon*\mathbf{u},\quad\text{where}\ \ j_\epsilon=\frac{1}{\epsilon^2}\,j(\frac{x}{\epsilon}).$$

Let $\mathcal{P}$ denote the Leray projector onto divergence-free vector field. In addition, it is known that

$$J_\epsilon^2=J_\epsilon,\quad\mathcal{P}^2=\mathcal{P},\quad\mathcal{P}J_\epsilon=J_\epsilon\mathcal{P}.\tag{3.18}$$

Next, we consider the following regularized MHD-Boussinesq equations

$$\begin{cases}\partial_t\mathbf{u}_\epsilon+\mathcal{P}J_\epsilon(J_\epsilon\mathbf{u}_\epsilon\cdot\nabla J_\epsilon\mathbf{u}_\epsilon)-\mu\Delta\mathcal{P}J_\epsilon\mathbf{u}_\epsilon=\mathcal{P}J_\epsilon(J_\epsilon\mathbf{b}_\epsilon\cdot\nabla J_\epsilon\mathbf{b}_\epsilon)+\mathcal{P}J_\epsilon(\theta_\epsilon e_2),\\ \partial_t\mathbf{b}_\epsilon+\mathcal{P}J_\epsilon(J_\epsilon\mathbf{u}_\epsilon\cdot\nabla J_\epsilon\mathbf{b}_\epsilon)-\nu\Delta\mathcal{P}J_\epsilon\mathbf{b}_\epsilon=\mathcal{P}J_\epsilon(J_\epsilon\mathbf{b}_\epsilon\cdot\nabla J_\epsilon\mathbf{u}_\epsilon),\\ \partial_t\theta_\epsilon+J_\epsilon(J_\epsilon\mathbf{u}_\epsilon\cdot\nabla J_\epsilon\theta_\epsilon)=0,\\ (\mathbf{u}_\epsilon,\mathbf{b}_\epsilon,\theta_\epsilon)(x,0)=J_\epsilon(\mathbf{u}_0,\mathbf{b}_0,\theta_0).\end{cases}\tag{3.19}$$

According to the Cauchy–Lipschitz theorem, we can obtain the existence of a unique smooth solution $(\mathbf{u}_\epsilon,\theta_\epsilon,\mathbf{b}_\epsilon)$ in short time. Thanks to (3.18), $(\mathcal{P}\mathbf{u}_\epsilon,\mathcal{P}\mathbf{b}_\epsilon,\theta_\epsilon)$ and $(J_\epsilon\mathbf{u}_\epsilon, J_\epsilon\mathbf{b}_\epsilon, J_\epsilon\theta_\epsilon)$ are also solutions of system (3.19) with the same initial data. By uniqueness, the solution to system (3.19) also solves the following system

$$\begin{cases}\partial_t\mathbf{u}_\epsilon+\mathcal{P}J_\epsilon(\mathbf{u}_\epsilon\cdot\nabla\mathbf{u}_\epsilon)-\mu\Delta\mathcal{P}\mathbf{u}_\epsilon=\mathcal{P}J_\epsilon(\mathbf{b}_\epsilon\cdot\nabla\mathbf{b}_\epsilon)+\mathcal{P}J_\epsilon(\theta_\epsilon e_2),\\ \partial_t\mathbf{b}_\epsilon+\mathcal{P}J_\epsilon(\mathbf{u}_\epsilon\cdot\nabla\mathbf{b}_\epsilon)-\nu\Delta\mathcal{P}\mathbf{b}_\epsilon=\mathcal{P}J_\epsilon(\mathbf{b}_\epsilon\cdot\nabla\mathbf{u}_\epsilon),\\ \partial_t\theta_\epsilon+J_\epsilon(\mathbf{u}_\epsilon\cdot\nabla\theta_\epsilon)=0,\\ \nabla\cdot\mathbf{u}_\epsilon=\nabla\cdot\mathbf{b}_\epsilon=0,\\ (\mathbf{u}_\epsilon,\mathbf{b}_\epsilon,\theta_\epsilon)(x,0)=J_\epsilon(\mathbf{u}_0,\mathbf{b}_0,\theta_0).\end{cases}\tag{3.20}$$

Since J_ϵ and $\mathcal{P}J_\epsilon$ are orthogonal projectors in L^2, similar to Proposition 3.1, we get

$$\mathbf{u}_\epsilon,\mathbf{b}_\epsilon\in L^\infty(0,T;H^s)\cap L^2(0,T;H^{s+1}),\quad\theta_\epsilon\in L^\infty(0,T;H^s).\tag{3.21}$$

Then there exists a subsequence, still denoted by $(\mathbf{u}_\epsilon,\mathbf{b}_\epsilon,\theta_\epsilon)$, and a triplet $(\mathbf{u},\mathbf{b},\theta)$ satisfying

$$\mathbf{u},\mathbf{b}\in L^\infty(0,T;H^s)\cap L^2(0,T;H^{s+1}),\quad\theta\in L^\infty(0,T;H^s).\tag{3.22}$$

By using the Hölder inequalities and the Sobolev embeddings inequalities ($H^1 \hookrightarrow L^4$), we can get $\mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon, \mathbf{b}_\epsilon \otimes \mathbf{b}_\epsilon, \mathbf{u}_\epsilon \otimes \mathbf{b}_\epsilon, \mathbf{b}_\epsilon \otimes \mathbf{u}_\epsilon, \mathbf{u}_\epsilon \theta_\epsilon \in L^2(0,T;L^2)$. Then we obtain from system (3.20) that $(\partial_t \mathbf{u}_\epsilon, \partial_t \mathbf{b}_\epsilon, \partial_t \theta_\epsilon) \in L^2(0,T;H^{-1})$. Noticing that the embedding $L^2 \hookrightarrow H^{-1}$ is locally compact, by the Aubin-Lions compactness lemma, we further get that as $\epsilon \to 0$,

$$\begin{aligned}
&\mathbf{u}_\epsilon \to \mathbf{u} \ \text{ in } \ C(0,T;H^{-1}) \cap L^2(0,T;L^2),\\
&\mathbf{b}_\epsilon \to \mathbf{b} \ \text{ in } \ C(0,T;H^{-1}) \cap L^2(0,T;L^2),\\
&\theta_\epsilon \to \theta \ \text{ in } \ C(0,T;H^{-1}) \cap L^2(0,T;L^2),
\end{aligned}$$

where $\to$ denotes the strong convergence. This enables us to pass to the limit to obtain a global solution $(\mathbf{u}, \mathbf{b}, \theta)$ in $\mathbb{R}^2 \times [0,T]$ for any $T > 0$. From standard arguments depending on the time continuity of $(\mathbf{u}, \mathbf{b}, \theta)$ in H^{-1}-norms, (3.21) and (3.22), it is easy to prove that $(\mathbf{u}, \mathbf{b}, \theta)$ is weakly continuous. This completes the proof of Theorem 3.1. □

3.6. Uniqueness of strong solutions. To finish the proof of Theorem 1.1, it remains to prove the uniqueness of strong solutions. Suppose that $(\mathbf{u}_1, \theta_1, \mathbf{b}_1)$ and $(\mathbf{u}_2, \theta_2, \mathbf{b}_2)$ are two solutions of system (1.2) with the regularity specified in Theorem 3.1. Setting

$$\mathbf{U} = \mathbf{u}_1 - \mathbf{u}_2, \quad \Theta = \theta_1 - \theta_2, \quad \mathbf{B} = \mathbf{b}_1 - \mathbf{b}_2, \quad \Pi = \pi_1 - \pi_2,$$

then $(\mathbf{U}, \Theta, \mathbf{B}, \Pi)$ satisfies

$$\begin{cases}
\mathbf{U}_t + \mathbf{u}_1 \cdot \nabla \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{u}_2 + \nabla \Pi - \mu \Delta \mathbf{U} = \mathbf{b}_1 \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{b}_2 + \Theta e_2,\\
\Theta_t + \mathbf{u}_1 \cdot \nabla \Theta + \mathbf{U} \cdot \nabla \theta_2 = 0,\\
\mathbf{B}_t + \mathbf{u}_1 \cdot \nabla \mathbf{B} + \mathbf{U} \cdot \nabla \mathbf{b}_2 - \nu \Delta \mathbf{B} = \mathbf{b}_1 \cdot \nabla \mathbf{U} + \mathbf{B} \cdot \nabla \mathbf{u}_2,\\
\nabla \cdot \mathbf{U} = 0, \quad \nabla \cdot \mathbf{B} = 0,\\
(\mathbf{U}, \Theta, \mathbf{B})(x,0) = 0.
\end{cases} \tag{3.23}$$

Multiplying the first three equations of (3.23) with $\mathbf{U}, \Theta, \mathbf{B}$ respectively and integrating over $\mathbb{R}^2$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\mathbf{U}\|_2^2 + \|\Theta\|_2^2 + \|\mathbf{B}\|_2^2) + \mu \|\nabla \mathbf{U}\|_2^2 + \nu \|\nabla \mathbf{B}\|_2^2\\
&= -\int_{\mathbb{R}^2} \mathbf{U} \cdot \nabla \mathbf{u}_2 \cdot \mathbf{U}\, dx + \int_{\mathbb{R}^2} \mathbf{B} \cdot \nabla \mathbf{b}_2 \cdot \mathbf{U}\, dx + \int_{\mathbb{R}^2} \Theta e_2 \cdot \mathbf{U}\, dx\\
&\quad - \int_{\mathbb{R}^2} \mathbf{U} \cdot \nabla \mathbf{b}_2 \cdot \mathbf{B}\, dx + \int_{\mathbb{R}^2} \mathbf{B} \cdot \nabla \mathbf{u}_2 \cdot \mathbf{B}\, dx - \int_{\mathbb{R}^2} (\mathbf{U} \cdot \nabla \theta_2) \Theta\, dx\\
&= \sum_{i=1}^{6} I_i.
\end{aligned} \tag{3.24}$$

For the six terms I_i, $i = 1, 2, \cdots, 6$ above, by using Hölder's inequality, the Sobolev embeddings inequalities ($H^s \hookrightarrow W^{1,\infty}$ with $s > 2$) and Young's inequality, we obtain

$$\begin{aligned}\sum_{i=1}^{6} I_i &\le \|\nabla \mathbf{u}_2\|_\infty \|\mathbf{U}\|_2^2 + 2\|\nabla \mathbf{b}_2\|_\infty \|\mathbf{B}\|_2 \|\mathbf{U}\|_2 + \|\Theta\|_2 \|\mathbf{U}\|_2 \\ &\quad + \|\nabla \mathbf{u}_2\|_\infty \|\mathbf{B}\|_2^2 + \|\nabla \theta_2\|_\infty \|\mathbf{U}\|_2 \|\Theta\|_2 \\ &\le C\|\mathbf{u}_2\|_{H^s} \|\mathbf{U}\|_2^2 + C\|\mathbf{b}_2\|_{H^s}(\|\mathbf{B}\|_2^2 + \|\mathbf{U}\|_2^2) + \frac{1}{2}\|\Theta\|_2^2 + \frac{1}{2}\|\mathbf{U}\|_2^2 \\ &\quad + C\|\mathbf{u}_2\|_{H^s}\|\mathbf{B}\|_2^2 + C\|\theta_2\|_{H^s}(\|\mathbf{U}\|_2^2 + \|\Theta\|_2^2) \\ &\le C(\|\mathbf{U}\|_2^2 + \|\Theta\|_2^2 + \|\mathbf{B}\|_2^2)(\|\mathbf{u}_2\|_{H^s} + \|\mathbf{b}_2\|_{H^s} + \|\theta_2\|_{H^s} + 1).\end{aligned}$$

Plugging the above estimates into (3.24), one gets

$$\begin{aligned}&\frac{d}{dt}(\|\mathbf{U}\|_2^2 + \|\Theta\|_2^2 + \|\mathbf{B}\|_2^2) + 2\mu\|\nabla \mathbf{U}\|_2^2 + 2\nu\|\nabla \mathbf{B}\|_2^2 \\ &\le C(\|\mathbf{U}\|_2^2 + \|\Theta\|_2^2 + \|\mathbf{B}\|_2^2)(\|\mathbf{u}_2\|_{H^s} + \|\mathbf{b}_2\|_{H^s} + \|\theta_2\|_{H^s} + 1),\end{aligned}$$

which implies, after applying Grönwall's inequality and Theorem 3.1, that

$$\begin{aligned}&\|\mathbf{U}\|_2^2 + \|\Theta\|_2^2 + \|\mathbf{B}\|_2^2 + 2\mu\int_0^t \|\nabla \mathbf{U}\|_2^2\,dt + 2\nu\int_0^t \|\nabla \mathbf{B}\|_2^2\,dt \\ &\le (\|\mathbf{U}_0\|_2^2 + \|\Theta_0\|_2^2 + \|\mathbf{B}_0\|_2^2)\, e^{C\int_0^t(\|\mathbf{u}_2\|_{H^s} + \|\mathbf{b}_2\|_{H^s} + \|\theta_2\|_{H^s} + 1)\,dt} = 0\end{aligned}$$

for any $t \in [0,T]$. Thus we obtain the uniqueness $\mathbf{U} = \Theta = \mathbf{B} \equiv 0$. This completes the proof of Theorem 1.1.

4. Large-time decay of solutions. In this section, we prove Theorem 1.2 by applying Schonbek's Fourier splitting methods that decompose the frequency space into two time-dependent sub-domains [42, 43, 44, 46]. To this end, we first derive some auxiliary results which are needed later.

LEMMA 4.1. *Let* $\mathbf{u}$ *and* θ *be sufficiently smooth functions satisfying* $\partial_t\theta + \mathbf{u}\cdot\nabla\theta + \theta = 0$ *and* $\theta_0 \in L^p(\mathbb{R}^2)$ *with* $1 \le p \le \infty$. *Then for all* $t \ge 0$, *it holds that* $\|\theta(\cdot,t)\|_p = e^{-t}\|\theta_0\|_p$.

Proof. By using an integrating factor e^t, we get $\partial_t(e^t\theta) + \mathbf{u}\cdot\nabla(e^t\theta) = 0$. By Lemma 3.1, we obtain

$$\|e^t\,\theta(\cdot,t)\|_p = \|\theta_0\|_p, \quad \|e^t\,\theta(\cdot,t)\|_\infty = \|\theta_0\|_\infty,$$

which implies that

$$\|\theta(\cdot,t)\|_p = e^{-t}\|\theta_0\|_p, \quad \|\theta(\cdot,t)\|_\infty = e^{-t}\|\theta_0\|_\infty.$$

This completes the proof of Lemma 4.1. □

LEMMA 4.2. *Suppose that the initial data* $(\mathbf{u}_0, \theta_0, \mathbf{b}_0) \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ *with* $\nabla\cdot\mathbf{u}_0 = \nabla\cdot\mathbf{b}_0 = 0$, *and* $(\mathbf{u}, \theta, \mathbf{b})$ *is the solution of system* (1.4). *Then it holds that*

$$|\widehat{\mathbf{u}}(\xi,t)| + |\widehat{\mathbf{b}}(\xi,t)| \le C + C|\xi|\int_0^t (\|\mathbf{u}\|_2^2 + \|\mathbf{b}\|_2^2)\,ds, \quad \text{for all } \xi \in \mathbb{R}^2 \text{ and } t > 0.$$

Proof. We can rewrite $(1.4)_1$ and $(1.4)_3$ as

$$\partial_t\mathbf{u} - \Delta\mathbf{u} = \mathcal{P}(-\nabla\cdot(\mathbf{u}\otimes\mathbf{u}) + \nabla\cdot(\mathbf{b}\otimes\mathbf{b}) + \theta e_2), \tag{4.1}$$

and

$$\partial_t \mathbf{b} - \Delta \mathbf{b} = \nabla \cdot (-\mathbf{u} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{u}), \tag{4.2}$$

where $\mathcal{P}$ is the Leray projection operator with $\mathcal{P}f = f - \nabla\Delta^{-1}\nabla \cdot f$ for vector f.

First, applying the Fourier transform to (4.1), we get

$$\partial_t \widehat{\mathbf{u}}(\xi,t) + |\xi|^2 \widehat{\mathbf{u}}(\xi,t) = \left(1 - \frac{\xi \otimes \xi}{|\xi|^2}\right)\{-\xi \cdot (\widehat{\mathbf{u} \otimes \mathbf{u}})(\xi,t) + \xi \cdot (\widehat{\mathbf{b} \otimes \mathbf{b}})(\xi,t) + \widehat{\theta} e_2(\xi,t)\},$$

and hence we have

$$\partial_t (e^{|\xi|^2 t} \widehat{\mathbf{u}}(\xi,t)) = e^{|\xi|^2 t}\left(1 - \frac{\xi \otimes \xi}{|\xi|^2}\right)\{-\xi \cdot (\widehat{\mathbf{u} \otimes \mathbf{u}})(\xi,t) + \xi \cdot (\widehat{\mathbf{b} \otimes \mathbf{b}})(\xi,t) + \widehat{\theta} e_2(\xi,t)\}.$$

Then it follows that

$$\begin{aligned}
|\widehat{\mathbf{u}}(\xi,t)| &\le e^{-|\xi|^2 t}|\widehat{\mathbf{u}}_0(\xi)| + \int_0^t e^{-|\xi|^2(t-s)}\left\{|\xi||\widehat{\mathbf{u} \otimes \mathbf{u}}(\xi,s)| + |\xi||\widehat{\mathbf{b} \otimes \mathbf{b}}(\xi,s)| + |\widehat{\theta} e_2|\right\} ds \\
&\le \|\mathbf{u}_0\|_1 + \int_0^t \left\{|\xi|\|\widehat{\mathbf{u} \otimes \mathbf{u}}\|_{L^\infty_\xi} + |\xi|\|\widehat{\mathbf{b} \otimes \mathbf{b}}\|_{L^\infty_\xi} + \|\widehat{\theta} e_2\|_{L^\infty_\xi}\right\} ds \\
&\le \|\mathbf{u}_0\|_1 + \int_0^t \{|\xi|\|\mathbf{u} \otimes \mathbf{u}\|_1 + |\xi|\|\mathbf{b} \otimes \mathbf{b}\|_1 + \|\theta\|_1\} \, ds \\
&\le \|\mathbf{u}_0\|_1 + C\int_0^t |\xi|(\|\mathbf{u}\|_2^2 + \|\mathbf{b}\|_2^2)\, ds + \int_0^t e^{-s}\|\theta_0\|_1 \, ds \\
&\le \|\mathbf{u}_0\|_1 + C\int_0^t |\xi|(\|\mathbf{u}\|_2^2 + \|\mathbf{b}\|_2^2)\, ds + \|\theta_0\|_1 \\
&\le C + C|\xi|\int_0^t (\|\mathbf{u}\|_2^2 + \|\mathbf{b}\|_2^2)\, ds.
\end{aligned}$$

Next, applying the Fourier transform to (4.2), we get

$$\partial_t \widehat{\mathbf{b}}(\xi,t) + |\xi|^2\, \widehat{\mathbf{b}}(\xi,t) = -\xi \cdot \widehat{\mathbf{u} \otimes \mathbf{b}}(\xi,t) + \xi \cdot \widehat{\mathbf{b} \otimes \mathbf{u}}(\xi,t),$$

which implies

$$\begin{aligned}
|\widehat{\mathbf{b}}(\xi,t)| &\le e^{-|\xi|^2 t}|\widehat{\mathbf{b}}_0(\xi)| + \int_0^t e^{-|\xi|^2(t-s)}\{|\xi||\widehat{\mathbf{u} \otimes \mathbf{b}}(\xi,s)| + |\xi||\widehat{\mathbf{b} \otimes \mathbf{u}}(\xi,s)|\} \, ds \\
&\le \|\mathbf{b}_0\|_1 + \int_0^t |\xi|(\|\mathbf{u} \otimes \mathbf{b}\|_1 + \|\mathbf{b} \otimes \mathbf{u}\|_1)\, ds \\
&\le C + C|\xi|\int_0^t (\|\mathbf{u}\|_2^2 + \|\mathbf{b}\|_2^2)\, ds.
\end{aligned}$$

This completes the proof of Lemma 4.2. □

4.1. L^2 decay of solutions. In this subsection, we establish the L^2 decay rates of solutions to system (1.4).

THEOREM 4.1. *Let $(\mathbf{u}, \theta, \mathbf{b})$ be the solution of system (1.4). Assume that the initial data $(\mathbf{u}_0, \theta_0, \mathbf{b}_0) \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ with $\nabla \cdot \mathbf{u}_0 = \nabla \cdot \mathbf{b}_0 = 0$. Then for all $t \ge 0$, it holds that*

$$\|\mathbf{u}(\cdot,t)\|_2 + \|\mathbf{b}(\cdot,t)\|_2 \le C(1+t)^{-\frac{1}{2}}$$

where the constant C depends on $\|\mathbf{u}_0\|_1$, $\|\mathbf{b}_0\|_1$, $\|\theta_0\|_1$, $\|\mathbf{u}_0\|_2$, $\|\mathbf{b}_0\|_2$ and $\|\theta_0\|_2$ only.

Proof. Multiplying $(1.4)_1$ with $\mathbf{u}$ and $(1.4)_3$ with $\mathbf{b}$ respectively, and using Lemma 4.1, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}\|_2^2+\|\mathbf{b}\|_2^2)+\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{b}\|_2^2 = \int_{\mathbb{R}^2}\theta e_2\cdot\mathbf{u}\,dx &\le \|\theta\|_2\|\mathbf{u}\|_2\\ &\le \frac{1}{2}e^{-t}\|\theta_0\|_2(\|\mathbf{u}\|_2^2+1+\|\mathbf{b}\|_2^2). \quad (4.3)\end{aligned}$$

By Grönwall's inequality, we can derive

$$\|\mathbf{u}\|_2^2+\|\mathbf{b}\|_2^2+1\le(\|\mathbf{u}_0\|_2^2+\|\mathbf{b}_0\|_2^2+1)\,e^{\|\theta_0\|_2}.$$

Substituting the above estimates into the right-hand side of (4.3), we obtain

$$\frac{d}{dt}(\|\mathbf{u}\|_2^2+\|\mathbf{b}\|_2^2)+\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{b}\|_2^2\le Ce^{-t} \quad \text{for C independent of t.} \quad (4.4)$$

Applying Lemma 2.3 to (4.4), we get

$$\frac{d}{dt}(\|\widehat{\mathbf{u}}(\cdot,t)\|_2^2+\|\widehat{\mathbf{b}}(\cdot,t)\|_2^2)+\int_{\mathbb{R}^2}|\xi|^2(|\widehat{\mathbf{u}}|^2+|\widehat{\mathbf{b}}|^2)\,d\xi\le Ce^{-t}. \quad (4.5)$$

By applying the Fourier splitting method, we decompose the whole space $\mathbb{R}^2$ into two time-dependent subdomains: $\mathbb{R}^2=S(t)\cup S(t)^c$ where

$$S(t):=\left\{\xi\in\mathbb{R}^2:|\xi|^2\le r^2(t)=\frac{g'(t)}{g(t)}\right\}, \quad (4.6)$$

and $g(t)$ is a continuous function of t with $g(0)=1$, $g(t)>0$ and $g'(t)>0$. Then by multiplying $g(t)$ on both sides of (4.5), we obtain

$$\begin{aligned}&\frac{d}{dt}\left[g(t)(\|\widehat{\mathbf{u}}(\cdot,t)\|_2^2+\|\widehat{\mathbf{b}}(\cdot,t)\|_2^2)\right]+g(t)\int_{\mathbb{R}^2}|\xi|^2(|\widehat{\mathbf{u}}|^2+|\widehat{\mathbf{b}}|^2)\,d\xi\\ &\le g'(t)\int_{\mathbb{R}^2}(|\widehat{\mathbf{u}}(\xi,t)|^2+|\widehat{\mathbf{b}}(\xi,t)|^2)\,d\xi+Cg(t)e^{-t}. \quad (4.7)\end{aligned}$$

Thanks to (4.6), we have

$$\begin{aligned}&g(t)\int_{\mathbb{R}^2}|\xi|^2(|\widehat{\mathbf{u}}|^2+|\widehat{\mathbf{b}}|^2)\,d\xi\\ &=g(t)\int_{S(t)}|\xi|^2(|\widehat{\mathbf{u}}|^2+|\widehat{\mathbf{b}}|^2)\,d\xi+g(t)\int_{S(t)^c}|\xi|^2(|\widehat{\mathbf{u}}|^2+|\widehat{\mathbf{b}}|^2)\,d\xi\\ &\ge g(t)\int_{S(t)^c}|\xi|^2(|\widehat{\mathbf{u}}|^2+|\widehat{\mathbf{b}}|^2)\,d\xi\\ &\ge g'(t)\int_{S(t)^c}(|\widehat{\mathbf{u}}|^2+|\widehat{\mathbf{b}}|^2)\,d\xi\\ &=g'(t)\int_{\mathbb{R}^2}(|\widehat{\mathbf{u}}|^2+|\widehat{\mathbf{b}}|^2)\,d\xi-g'(t)\int_{S(t)}(|\widehat{\mathbf{u}}|^2+|\widehat{\mathbf{b}}|^2)\,d\xi,\end{aligned}$$

then we obtain from (4.7) that

$$\frac{d}{dt}\left[g(t)(\|\widehat{\mathbf{u}}(\cdot,t)\|_2^2+\|\widehat{\mathbf{b}}(\cdot,t)\|_2^2)\right]\le g'(t)\int_{S(t)}(|\widehat{\mathbf{u}}|^2+|\widehat{\mathbf{b}}|^2)\,d\xi+Cg(t)e^{-t}. \quad (4.8)$$

By Lemma 4.2, we have

$$\begin{aligned}\int_{S(t)} (|\widehat{\mathbf{u}}|^2 + |\widehat{\mathbf{b}}|^2)\, d\xi &\le C \int_{S(t)} \left\{ 1 + |\xi|^2 \left[\int_0^t (\|\mathbf{u}\|_2^2 + \|\mathbf{b}\|_2^2)\, ds \right]^2 \right\} d\xi \\ &\le C \int_{S(t)} d\xi + C \left(\int_{S(t)} |\xi|^2\, d\xi \right) \left(\int_0^t (\|\mathbf{u}\|_2^2 + \|\mathbf{b}\|_2^2)\, ds \right)^2 \\ &\le C \int_0^{r(t)} \rho\, d\rho + C \left(\int_0^{r(t)} \rho^2 \rho\, d\rho \right) \left(\int_0^t (\|\mathbf{u}\|_2^2 + \|\mathbf{b}\|_2^2)\, ds \right)^2 \\ &\le Cr^2(t) + Cr^4(t) \left(\int_0^t (\|\mathbf{u}\|_2^2 + \|\mathbf{b}\|_2^2)\, ds \right)^2 .\end{aligned}$$

Then substituting the above estimates into (4.8) yields

$$\begin{aligned}&\frac{d}{dt} \left[g(t)(\|\widehat{\mathbf{u}}(\cdot,t)\|_2^2 + \|\widehat{\mathbf{b}}(\cdot,t)\|_2^2) \right] \\ &\le C \left[\frac{(g'(t))^2}{g(t)} + \frac{(g'(t))^3}{(g(t))^2} \left(\int_0^t (\|\mathbf{u}\|_2^2 + \|\mathbf{b}\|_2^2)\, ds \right)^2 + g(t)e^{-t} \right]. \end{aligned} \tag{4.9}$$

Let $\varepsilon(t) = \|\widehat{\mathbf{u}}(\xi,t)\|_2^2 + \|\widehat{\mathbf{b}}(\xi,t)\|_2^2 = \|\mathbf{u}(x,t)\|_2^2 + \|\mathbf{b}(x,t)\|_2^2$. Then it follows that

$$g(t)\, \varepsilon(t) \le \varepsilon(0) + C \int_0^t \left[\frac{(g'(s))^2}{g(s)} + \frac{(g'(s))^3}{(g(s))^2} \left(\int_0^s \varepsilon(\tau)\, d\tau \right)^2 + g(s)e^{-s} \right] ds. \tag{4.10}$$

Taking $g(t) = [\log(e+t)]^3$ in (4.10), and using Lemma 3.3, we get

$$\begin{aligned}[\log(e+t)]^3\, \varepsilon(t) &\le \varepsilon(0) + C \int_0^t \left[\frac{\log(e+s)}{(e+s)^2} + \frac{1}{(e+s)^3} + [\log(e+s)]^3 e^{-s} \right] ds \\ &\le \varepsilon(0) + C \int_0^t \left[\frac{e+s}{(e+s)^2} + \frac{(e+s)^2}{(e+s)^3} + (e+s)^3 e^{-s} \right] ds \\ &\le \varepsilon(0) + C \int_0^t \left[\frac{1}{e+s} + (e+s)^3 e^{-s} \right] ds \\ &\le C \log(e+t).\end{aligned}$$

Therefore, we have

$$\varepsilon(t) \le C[\log(e+t)]^{-2}. \tag{4.11}$$

Next, taking $g(t) = (1+t)^2$ in (4.10), and using Hölder's inequality, we obtain

$$\begin{aligned}(1+t)^2\, \varepsilon(t) &\le \varepsilon(0) + C \int_0^t \left[1 + \frac{1}{1+s} \left(\int_0^s \varepsilon(\tau)\, d\tau \right)^2 + (1+s)^2 e^{-s} \right] ds \\ &\le C(1+t) + C \int_0^t (1+s)^{-1} \left(\int_0^s \varepsilon(\tau)\, d\tau \right)^2 ds \\ &\le C(1+t) + C \int_0^t \int_0^s \varepsilon^2(\tau)\, d\tau\, ds \\ &\le C(1+t) + C(1+t) \int_0^t \varepsilon^2(s)\, ds.\end{aligned}$$

Therefore, due to (4.11), we get

$$\begin{aligned}(1+t)\,\varepsilon(t) &\le C + C\int_0^t \varepsilon^2(s)\,ds\\ &\le C + C\int_0^t \varepsilon(s)(1+s)\left\{(1+s)^{-1}[\log(e+s)]^{-2}\right\}ds. \qquad (4.12)\end{aligned}$$

It is clear that

$$\int_0^\infty (1+s)^{-1}[\log(e+s)]^{-2}\,ds < \infty.$$

Then by Grönwall's inequality, we obtain

$$(1+t)\,\varepsilon(t) \le Ce^{\left\{\int_0^\infty (1+s)^{-1}[\log(e+s)]^{-2}\,ds\right\}} \le C,$$

and thus we have

$$\|\mathbf{u}(\cdot,t)\|_2^2 + \|\mathbf{b}(\cdot,t)\|_2^2 \le C(1+t)^{-1}.$$

This completes the proof of Theorem 4.1. □

4.2. Decay of the first-order derivatives of the solutions. We now turn to the large time behavior of the first-order derivatives of solutions to system (1.4).

THEOREM 4.2. *Under the assumptions of Theorem 4.1, if in addition,* $(\mathbf{u}_0,\mathbf{b}_0)\in H^1(\mathbb{R}^2)$*, then for all* $t\ge 0$*, it holds that*

$$\|\nabla\mathbf{u}(\cdot,t)\|_2 + \|\nabla\mathbf{b}(\cdot,t)\|_2 \le C(1+t)^{-1}$$

where the constant C *depends on* $\|\mathbf{u}_0\|_1$, $\|\mathbf{b}_0\|_1$, $\|\theta_0\|_1$, $\|\mathbf{u}_0\|_{H^1}$, $\|\mathbf{b}_0\|_{H^1}$ *and* $\|\theta_0\|_2$ *only.*

Proof. **Step 1**. Multiplying $(1.4)_1$ with $\mathbf{u}$ and $(1.4)_3$ with $\mathbf{b}$ respectively and integrating over $\mathbb{R}^2$, we obtain the estimate as follows, which is the same as (4.3)

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}\|_2^2+\|\mathbf{b}\|_2^2) + \|\nabla\mathbf{u}\|_2^2 + \|\nabla\mathbf{b}\|_2^2 \le \|\theta\|_2\|\mathbf{u}\|_2. \qquad (4.13)$$

Multiplying both sides of (4.13) by $(1+t)^\delta$ with δ to be determined later, we have

$$\begin{aligned}&\frac{d}{dt}\left[(1+t)^\delta(\|\mathbf{u}\|_2^2+\|\mathbf{b}\|_2^2)\right] + 2(1+t)^\delta(\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{b}\|_2^2)\\ &\le \delta(1+t)^{\delta-1}(\|\mathbf{u}\|_2^2+\|\mathbf{b}\|_2^2) + 2(1+t)^\delta\|\mathbf{u}\|_2\|\theta\|_2.\end{aligned}$$

By using Lemma 4.1 and Theorem 4.1, we derive

$$\begin{aligned}&\frac{d}{dt}\left[(1+t)^\delta(\|\mathbf{u}\|_2^2+\|\mathbf{b}\|_2^2)\right] + 2(1+t)^\delta(\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{b}\|_2^2)\\ &\le C(1+t)^{\delta-2} + Ce^{-t}(1+t)^{\delta-\frac{1}{2}}.\end{aligned}$$

After integration over $[0,t]$, we obtain

$$\begin{aligned}&(1+t)^\delta(\|\mathbf{u}(\cdot,t)\|_2^2+\|\mathbf{b}(\cdot,t)\|_2^2) + 2\int_0^t (1+s)^\delta(\|\nabla\mathbf{u}(\cdot,s)\|_2^2+\|\nabla\mathbf{b}(\cdot,s)\|_2^2)\,ds\\ &\le \|\mathbf{u}_0\|_2^2+\|\mathbf{b}_0\|_2^2 + C\int_0^t (1+s)^{\delta-2}\,ds + C\int_0^t e^{-s}(1+s)^{\delta-\frac{1}{2}}\,ds. \qquad (4.14)\end{aligned}$$

In order for the right-hand side of (4.14) to be bounded by C independent of t, we take $0<\delta<1$ to make the following estimates hold

$$\int_0^\infty (1+s)^{\delta-2}\,ds \le C, \quad \int_0^\infty e^{-s}(1+s)^{\delta-\frac{1}{2}}\,ds \le C.$$

Then it follows that

$$\begin{aligned}&(1{+}t)^{\delta}(\|\mathbf{u}(\cdot,t)\|_2^2+\|\mathbf{b}(\cdot,t)\|_2^2)\\&+2\int_0^t(1+s)^{\delta}(\|\nabla\mathbf{u}(\cdot,s)\|_2^2+\|\nabla\mathbf{b}(\cdot,s)\|_2^2)\,ds \le C\end{aligned}\tag{4.15}$$

where $0<\delta<1$ and the constant C is independent of t.

Step 2. Multiplying $(1.4)_1$ with $-\Delta\mathbf{u}$ and $(1.4)_3$ with $-\Delta\mathbf{b}$ respectively and integrating over $\mathbb{R}^2$, we can obtain the following estimate, which is the same as (3.1)

$$\begin{aligned}&\frac{d}{dt}(\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{b}\|_2^2)+\|\Delta\mathbf{u}\|_2^2+\|\Delta\mathbf{b}\|_2^2\\&\le C(\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{b}\|_2^2)\|\nabla\mathbf{b}\|_2^2+C\|\theta\|_2^2.\end{aligned}\tag{4.16}$$

Multiplying both sides of (4.16) by $(1+t)^{1+\delta}$, $0<\delta<1$, we have

$$\begin{aligned}&\frac{d}{dt}\left[(1+t)^{1+\delta}(\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{b}\|_2^2)\right]+(1+t)^{1+\delta}(\|\Delta\mathbf{u}\|_2^2+\|\Delta\mathbf{b}\|_2^2)\\&\le(1+\delta)(1+t)^{\delta}(\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{b}\|_2^2)+C(1+t)^{1+\delta}(\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{b}\|_2^2)\|\nabla\mathbf{b}\|_2^2\\&\quad+C(1+t)^{1+\delta}\|\theta\|_2^2.\end{aligned}$$

By Grönwall's inequality, we obtain

$$\begin{aligned}&(1+t)^{1+\delta}(\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{b}\|_2^2)\\&\le\Bigg(\|\nabla\mathbf{u}_0\|_2^2+\|\nabla\mathbf{b}_0\|_2^2+(1+\delta)\int_0^t(1+s)^{\delta}(\|\nabla\mathbf{u}(s)\|_2^2+\|\nabla\mathbf{b}(s)\|_2^2)\,ds\\&\quad+C\int_0^t(1+s)^{1+\delta}\|\theta(s)\|_2^2\,ds\Bigg)e^{\left\{C\int_0^t\|\nabla\mathbf{b}(s)\|_2^2\,ds\right\}}.\end{aligned}$$

Then by using the inequality (4.15) and Lemma 4.1, it follows that

$$\begin{aligned}&(1+t)^{1+\delta}(\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{b}\|_2^2)\\&\le(\|\nabla\mathbf{u}_0\|_2^2+\|\nabla\mathbf{b}_0\|_2^2+C+C\int_0^t e^{-s}(1+s)^{1+\delta}\,ds)e^{\left\{C\int_0^t(1+s)^{-\delta}(1+s)^{\delta}\|\nabla\mathbf{b}(s)\|_2^2\,ds\right\}}\\&\le(\|\nabla\mathbf{u}_0\|_2^2+\|\nabla\mathbf{b}_0\|_2^2+C)e^{\left\{C\int_0^t(1+s)^{\delta}\|\nabla\mathbf{b}(s)\|_2^2\,ds\right\}}\le C.\end{aligned}$$

Therefore, we obtain

$$\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{b}\|_2^2\le C(1+t)^{-(1+\delta)}, \quad 0<\delta<1.\tag{4.17}$$

Step 3. Recalling the inequality (4.16), using the inequality (4.17) and Lemma 4.1, we derive

$$\begin{aligned}&\frac{d}{dt}(\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{b}\|_2^2)+\|\Delta\mathbf{u}\|_2^2+\|\Delta\mathbf{b}\|_2^2\\&\le C(\|\nabla\mathbf{u}\|_2^2+\|\nabla\mathbf{b}\|_2^2)\|\nabla\mathbf{b}\|_2^2+C\|\theta\|_2^2\le C(1+t)^{-(2+2\delta)}+Ce^{-t}.\end{aligned}\tag{4.18}$$

Applying Lemma 2.3 to (4.18), we get

$$\frac{d}{dt}\left[\int_{\mathbb{R}^2}|\xi|^2(|\widehat{\mathbf{u}}(\xi,t)|^2+|\widehat{\mathbf{b}}(\xi,t)|^2)\,d\xi\right]+\int_{\mathbb{R}^2}|\xi|^4(|\widehat{\mathbf{u}}(\xi,t)|^2+|\widehat{\mathbf{b}}(\xi,t)|^2)\,d\xi$$
$$\le C(1+t)^{-(2+2\delta)}+Ce^{-t}. \tag{4.19}$$

By applying Schonbek's splitting method, we decompose the whole Fourier space $\mathbb{R}^2$ into two time-dependent subdomains: $\mathbb{R}^2 = B(t)\cup B(t)^c$ where

$$B(t) := \left\{\xi\in\mathbb{R}^2 : |\xi|^2 \le \frac{k}{1+t}\right\}, \tag{4.20}$$

with k to be determined later.

Subsequently, we get from (4.19)

$$\begin{aligned}
&\frac{d}{dt}\left[\int_{\mathbb{R}^2}|\xi|^2(|\widehat{\mathbf{u}}(\xi,t)|^2+|\widehat{\mathbf{b}}(\xi,t)|^2)\,d\xi\right]\\
&\le -\int_{\mathbb{R}^2}|\xi|^4(|\widehat{\mathbf{u}}(\xi,t)|^2+|\widehat{\mathbf{b}}(\xi,t)|^2)\,d\xi + C(1+t)^{-(2+2\delta)}+Ce^{-t}\\
&\le -\int_{B(t)^c}|\xi|^4(|\widehat{\mathbf{u}}(\xi,t)|^2+|\widehat{\mathbf{b}}(\xi,t)|^2)\,d\xi + C(1+t)^{-(2+2\delta)}+Ce^{-t}\\
&\le -\frac{k}{1+t}\int_{B(t)^c}|\xi|^2(|\widehat{\mathbf{u}}(\xi,t)|^2+|\widehat{\mathbf{b}}(\xi,t)|^2)\,d\xi + C(1+t)^{-(2+2\delta)}+Ce^{-t}\\
&= -\frac{k}{1+t}\int_{\mathbb{R}^2}|\xi|^2(|\widehat{\mathbf{u}}(\xi,t)|^2+|\widehat{\mathbf{b}}(\xi,t)|^2)\,d\xi + \frac{k}{1+t}\int_{B(t)}|\xi|^2(|\widehat{\mathbf{u}}(\xi,t)|^2+|\widehat{\mathbf{b}}(\xi,t)|^2)\,d\xi\\
&\quad + C(1+t)^{-(2+2\delta)}+Ce^{-t}\\
&\le -\frac{k}{1+t}\int_{\mathbb{R}^2}|\xi|^2(|\widehat{\mathbf{u}}(\xi,t)|^2+|\widehat{\mathbf{b}}(\xi,t)|^2)\,d\xi + \frac{k^2}{(1+t)^2}\int_{B(t)}(|\widehat{\mathbf{u}}(\xi,t)|^2+|\widehat{\mathbf{b}}(\xi,t)|^2)\,d\xi\\
&\quad + C(1+t)^{-(2+2\delta)}+Ce^{-t},
\end{aligned}$$

and hence

$$\frac{d}{dt}\left[\int_{\mathbb{R}^2}|\xi|^2(|\widehat{\mathbf{u}}(\xi,t)|^2+|\widehat{\mathbf{b}}(\xi,t)|^2)\,d\xi\right]+\frac{k}{1+t}\int_{\mathbb{R}^2}|\xi|^2(|\widehat{\mathbf{u}}(\xi,t)|^2+|\widehat{\mathbf{b}}(\xi,t)|^2)\,d\xi$$
$$\le C(1+t)^{-2}\int_{\mathbb{R}^2}(|\widehat{\mathbf{u}}(\xi,t)|^2+|\widehat{\mathbf{b}}(\xi,t)|^2)\,d\xi + C(1+t)^{-(2+2\delta)}+Ce^{-t}.$$

Multiplying $(1+t)^k$ on both sides, using Lemma 2.3 and Theorem 4.1, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[(1+t)^k\int_{\mathbb{R}^2}|\xi|^2(|\widehat{\mathbf{u}}(\xi,t)|^2+|\widehat{\mathbf{b}}(\xi,t)|^2)\,d\xi\right]\\
&\le C(1+t)^{k-2}\int_{\mathbb{R}^2}(|\widehat{\mathbf{u}}(\xi,t)|^2+|\widehat{\mathbf{b}}(\xi,t)|^2)\,d\xi + C(1+t)^{k-(2+2\delta)}+C(1+t)^ke^{-t}\\
&\le C(1+t)^{k-3}+C(1+t)^{k-(2+2\delta)}+C(1+t)^ke^{-t}.
\end{aligned}$$

We take $2<k<2\delta+1$ to guarantee that the following estimates hold,

$$\int_0^\infty (1+s)^{k-(2+2\delta)}\,ds\le C,\qquad \int_0^\infty (1+s)^k e^{-s}\,ds\le C,$$

where the constants C are independent of t. Then after integration over $[0,t]$, we obtain

$$\begin{aligned}(1+t)^k(\|\nabla \mathbf{u}\|_2^2+\|\nabla \mathbf{b}\|_2^2) &\le \|\nabla \mathbf{u}_0\|_2^2+\|\nabla \mathbf{b}_0\|_2^2+C\int_0^t (1+s)^{k-3}\,ds\\ &\quad +C\int_0^t (1+s)^{k-(2+2\delta)}\,ds+C\int_0^t(1+s)^k e^{-s}\,ds\\ &\le \|\nabla \mathbf{u}_0\|_2^2+\|\nabla \mathbf{b}_0\|_2^2+C(1+t)^{k-2}\\ &\quad +C\int_0^\infty (1+s)^{k-(2+2\delta)}\,ds+C\int_0^\infty(1+s)^k e^{-s}\,ds\\ &\le C+C(1+t)^{k-2},\end{aligned}$$

and hence we have

$$\|\nabla \mathbf{u}\|_2^2+\|\nabla \mathbf{b}\|_2^2\le C(1+t)^{-k}+C(1+t)^{-2}\le C(1+t)^{-2}.$$

Therefore, the proof of Theorem 1.2 is completed. □

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