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STABILITY OF RAREFACTION WAVES FOR THE TWO-SPECIES VLASOV-POISSON-BOLTZMANN SYSTEM WITH SOFT POTENTIALS*

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Dedicated to Professor Ling Hsiao on the occasion of her 80th birthday

Abstract. In this paper, we construct the global solutions near a local Maxwellian for the one-dimensional two-species Vlasov-Poisson-Boltzmann system with soft potentials. The macroscopic components of this local Maxwellian are the approximate rarefaction wave solutions to the associated one-dimensional compressible Euler system. Then we prove the stability of the rarefaction waves for the two-species Vlasov-Poisson-Boltzmann system in the weighted function space. Moreover, some time decay rates of the disparity between two species and the electric field are obtained.

Key words. Vlasov-Poisson-Boltzmann system, rarefaction waves, macro-micro decomposition, energy method.

Mathematics Subject Classification. 35Q20, 76P05, 35B35, 35B40.

1. Introduction. The dynamics of charged dilute particles (e.g., electrons and ions) in the absence of magnetic effects can be described by the two-species Vlasov-Poisson-Boltzmann(VPB) system

$$\begin{cases} \partial_t F_+ + v\cdot\nabla_x F_+ - \nabla_x\phi\cdot\nabla_v F_+ = Q(F_+,F_+) + Q(F_+,F_-), \\ \partial_t F_- + v\cdot\nabla_x F_- + \nabla_x\phi\cdot\nabla_v F_- = Q(F_-,F_+) + Q(F_-,F_-), \\ -\Delta\phi = \int_{\mathbb{R}^3}(F_+ - F_-)\,dv. \end{cases} \tag{1.1}$$

Here unknown functions $F_\pm = F_\pm(t,x,v) \geq 0$ are the number distribution functions for the ions (+) and electrons (-) with position $x = (x_1,x_2,x_3) \in \mathbb{R}^3$ and velocity $v = (v_1,v_2,v_3) \in \mathbb{R}^3$ at time $t \geq 0$, respectively. The self-consistent electric potential $\phi = \phi(t,x)$ is coupled with $F_\pm$ through the Poisson equation $(1.1)_3$. The Boltzmann collision operator $Q(\cdot,\cdot)$ in (1.1) is given by

$$Q(f,g) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(|v-v_*|,\vartheta)\{f(v_*')g(v') - f(v_*)g(v)\}\,d\omega dv_*, \tag{1.2}$$

where $f(v) = f(t,x,v)$, $\omega \in \mathbb{S}^2$, with $\mathbb{S}^2$ denoting the unit sphere $\mathbb{R}^3$ and the velocity pairs (v,v_*) before collision and (v',v_*') after collision are given by

$$v' = v - [(v-v_*)\cdot\omega]\omega, \quad v_*' = v_* + [(v-v_*)\cdot\omega]\omega,$$

in terms of the conservation laws of momentum and energy

$$v + v_* = v' + v_*', \quad |v|^2 + |v_*|^2 = |v'|^2 + |v_*'|^2, \tag{1.3}$$

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due to the elastic collision of two particles. Under the Grad's angular cutoff condition, the Boltzmann collision kernel $B(|v-v_*|,\vartheta)$ is assumed to satisfy

$$B(|v-v_*|,\vartheta)=B(\vartheta)|v-v_*|^\gamma,\quad 0<B(\vartheta)\le const.|\cos\vartheta|,\quad \cos\vartheta=\frac{(v-v_*)\cdot\omega}{|v-v_*|},$$

where the exponent $\gamma\in(-3,1]$ is determined by the potential of intermolecular force, which is classified into the soft potential case for $-3<\gamma<0$, the Maxwell molecular case for $\gamma=0$, and the hard potential case for $0<\gamma\le 1$ which includes the hard sphere model with $\gamma=1$ and $B(\vartheta)=const.|\cos\vartheta|$. For the soft potentials, the case $-2\le\gamma<0$ is called the moderately soft potentials while $-3<\gamma<-2$ is called the very soft potentials, cf. [26] by Villani. In this paper we focus on the case $-2\le\gamma<0$.

In order to study the nonlinear stability of the rarefaction waves along the x_1-direction, we assume the slab symmetry in space and hence consider the single spatial variable $x\in\mathbb{R}$. Therefore, we focus on the following one-dimensional bipolar VPB system:

$$\begin{cases}\partial_t F_+ + v_1\partial_x F_+ - \partial_x\phi\partial_{v_1}F_+ = Q(F_+,F_+)+Q(F_-,F_+),\\ \partial_t F_- + v_1\partial_x F_- + \partial_x\phi\partial_{v_1}F_- = Q(F_+,F_-)+Q(F_-,F_-),\\ -\partial_{xx}\phi=\int_{\mathbb{R}^3}(F_+-F_-)dv,\end{cases}\tag{1.4}$$

with initial values and the far field states satisfying

$$\begin{cases}F_+(0,x,v)=F_{+0}(x,v)\to M_{[\rho_\pm,u_\pm,\theta_\pm]}(v),\quad \text{as } x\to\pm\infty,\\ F_-(0,x,v)=F_{-0}(x,v)\to M_{[\rho_\pm,u_\pm,\theta_\pm]}(v),\quad \text{as } x\to\pm\infty,\\ \phi_x(0,x)=\phi_{x0}(x)\to 0,\quad \text{as } x\to\pm\infty,\end{cases}\tag{1.5}$$

where $(\rho_+,u_+,\theta_+)\neq(\rho_-,u_-,\theta_-)$ are two constant states with $\rho_\pm>0$, $u_\pm=(u_{1\pm},0,0)^t$, $\theta_\pm>0$ and $M_{[\rho_\pm,u_\pm,\theta_\pm]}(v)$ are global Maxwellians defined by (1.9). Firstly, we reformulate the bipolar VPB system (1.4). Motivated by some previous works [16, 20], we consider the sum and difference of F_+ and F_- and define

$$F_1=\frac{F_++F_-}{2}\quad\text{and}\quad F_2=\frac{F_+-F_-}{2}.$$

In terms of F_1 and F_2, the system (1.4) can be written as follows

$$\begin{cases}\partial_t F_1+v_1\partial_x F_1-\partial_x\phi\partial_{v_1}F_2=2Q(F_1,F_1),\\ \partial_t F_2+v_1\partial_x F_2-\partial_x\phi\partial_{v_1}F_1=2Q(F_1,F_2),\\ -\partial_{xx}\phi=2\int_{\mathbb{R}^3}F_2dv,\end{cases}\tag{1.6}$$

with the initial values and the far field states satisfying

$$\begin{cases}F_1(0,x,v)=F_{10}(x,v)\to M_{[\rho_\pm,u_\pm,\theta_\pm]}(v),\quad \text{as } x\to\pm\infty,\\ F_2(0,x,v)=F_{20}(x,v)\to 0,\quad \text{as } x\to\pm\infty,\\ \phi_x(0,x)=\phi_{x0}(x)\to 0,\quad \text{as } x\to\pm\infty.\end{cases}\tag{1.7}$$

Notice that the system (1.6) becomes the Boltzmann equation without external force if F_2 and ϕ are zero. The Boltzmann operator $Q(\cdot,\cdot)$ has five collision invariants $\Psi_i(v)$ which are given by

$$\Psi_0(v)=1,\quad \Psi_i(v)=v_i\ (i=1,2,3),\quad \Psi_4(v)=\frac{1}{2}|v|^2,$$

satisfying

$$\int_{\mathbb{R}^3} \Psi_i(v)Q(f,f)dv = 0, \quad \text{for } i = 0,1,2,3,4.$$

As in [17, 18], in terms of the solution F_1 to the equation $(1.6)_1$, we introduce the five conserved quantities, that is, the mass density $\rho = \rho(t,x)$, momentum $\rho u(t,x)$, and the total energy $\rho(e + \frac{1}{2}|u|^2)(t,x)$ defined by

$$\begin{cases} \rho(t,x) = \int_{\mathbb{R}^3} \Psi_0(v) F_1 dv, \\ \rho u_i(t,x) = \int_{\mathbb{R}^3} \Psi_i(v) F_1 dv, \quad \text{for } i = 1,2,3, \\ \rho(e + \frac{1}{2}|u|^2)(t,x) = \int_{\mathbb{R}^3} \Psi_4(v) F_1 dv. \end{cases} \tag{1.8}$$

Here $e(t,x) > 0$ is the internal energy which is related to the temperature θ by $e = \frac{3}{2}R\theta = \theta$ with the gas constant R taken to be $\frac{2}{3}$ in this paper for convenience, and $u = u(t,x)$ is the fluid velocity.

We define the local Maxwellian M associated with the solution F_1 to the equation $(1.6)_1$ in terms of the fluid quantities of F_1 as in (1.8) by

$$M = M_{[\rho,u,\theta]}(t,x,v) = \frac{\rho(t,x)}{(2\pi R\theta(t,x))^{3/2}} \exp\Big(- \frac{|v-u(t,x)|^2}{2R\theta(t,x)} \Big). \tag{1.9}$$

We denote an $L^2_v(\mathbb{R}^3)$ inner product as $\langle h, g\rangle = \int_{\mathbb{R}^3} h(v)g(v)\,dv$. Then, the macroscopic kernel space is spanned by the following five pair wise orthogonal base

$$\begin{cases} \chi_0(v) = \frac{1}{\sqrt{\rho}} M, \quad \chi_i(v) = \frac{v_i - u_i}{\sqrt{R\rho\theta}} M, \quad \text{for } i = 1,2,3, \\ \chi_4(v) = \frac{1}{\sqrt{6\rho}} (\frac{|v-u|^2}{R\theta} - 3) M, \quad \langle \chi_i, \frac{\chi_j}{M} \rangle = \delta_{ij}, \quad i,j = 0,1,2,3,4. \end{cases} \tag{1.10}$$

In light of (1.10), we define

$$P_0 h = \sum_{i=0}^{4} \langle h, \frac{\chi_i}{M} \rangle \chi_i, \quad P_1 h = h - P_0 h, \tag{1.11}$$

where P_0 and P_1 are called the macroscopic projection and the microscopic projection, respectively. A function $h(v)$ is called microscopic or non-fluid if

$$\int_{\mathbb{R}^3} h(v)\Psi_i(v)\,dv = 0, \quad \text{for } i = 0,1,2,3,4. \tag{1.12}$$

For a non-trivial solution profile connecting two different global Maxwellians at $x = \pm\infty$, we decompose the equation $(1.6)_1$ and its solution with respect to the local Maxwellian as

$$F_1 = M + G, \quad P_0 F_1 = M, \quad P_1 F_1 = G, \tag{1.13}$$

where the local Maxwellian M as (1.9) and $G = G(t,x,v)$ represent the macroscopic and microscopic component in the solution respectively. Then the equation $(1.6)_1$ becomes

$$\begin{aligned} &\partial_t(M+G) + v_1\partial_x(M+G) - \partial_x\phi\partial_{v_1}F_2 \\ &= 2Q(G,M) + 2Q(M,G) + 2Q(G,G), \end{aligned} \tag{1.14}$$

due to $Q(M,M)=0$. Multiplying (1.14) by the collision invariants $\Psi_i(v)$ $(i=0,1,2,3,4)$ and integrating the resulting equations with respect to v over $\mathbb{R}^3$, one gets the following macroscopic system

$$\begin{cases}\rho_t+(\rho u_1)_x=0,\\ (\rho u_1)_t+(\rho u_1^2+p)_x+\partial_x\phi\int_{\mathbb{R}^3}F_2dv=-\int_{\mathbb{R}^3}v_1^2G_xdv,\\ (\rho u_i)_t+(\rho u_1u_i)_x=-\int_{\mathbb{R}^3}v_1v_iG_xdv,\quad i=2,3,\\ (\rho(\theta+\frac{|u|^2}{2}))_t+(\rho u_1(\theta+\frac{|u|^2}{2})+pu_1)_x+\partial_x\phi\int_{\mathbb{R}^3}v_1F_2dv\\ \quad=-\int_{\mathbb{R}^3}\frac{1}{2}v_1|v|^2G_xdv.\end{cases}\tag{1.15}$$

Applying the projection operator P_1 to (1.14), one gets the following microscopic system

$$G_t+P_1(v_1G_x)+P_1(v_1M_x)-P_1(\partial_x\phi\partial_{v_1}F_2)=L_MG+2Q(G,G).\tag{1.16}$$

Here L_M is the linearized collision operator with respect to the local Maxwellian M by

$$L_MG=2Q(G,M)+2Q(M,G),$$

and the null space $\mathcal{N}$ of L_M is spanned by the macroscopic variables χ_i, $(i=0,1,2,3,4)$. It follows by (1.16) that

$$G=L_M^{-1}[P_1(v_1M_x)]+L_M^{-1}\Theta,\tag{1.17}$$

and

$$\Theta:=G_t+P_1(v_1G_x)-P_1(\partial_x\phi\partial_{v_1}F_2)-2Q(G,G).\tag{1.18}$$

We denote a given local Maxwellian

$$\overline{M}=M_{[\bar\rho,\bar u,\bar\theta]}(v)=\frac{\bar\rho(t,x)}{(2\pi R\bar\theta(t,x))^{3/2}}\exp\Big(-\frac{|v-\bar u(t,x)|^2}{2R\bar\theta(t,x)}\Big),\tag{1.19}$$

where its macroscopic variables $(\bar\rho,\bar u,\bar\theta)=(\bar\rho,\bar u,\bar\theta)(t,x)$ is the approximate rarefaction wave defined by (2.9) in Section 2. For simplicity, we choose $(\rho_\pm,u_\pm,\theta_\pm)$ in (1.5) to be close enough to the state $(1,0,\frac{3}{2})$. As in [19], we will use a global Maxwellian $\mu=M_{[1,0,\frac{3}{2}]}=(2\pi)^{-\frac{3}{2}}\exp\{-|v|^2/2\}$, which satisfies that there exists constant $\eta_0>0$ small enough such that for all (t,x)

$$\begin{aligned}&|\rho(t,x)-1|+|u(t,x)|+|\theta(t,x)-\frac{3}{2}|<\eta_0,\\ &\frac{1}{2}\sup_{t\ge0,x\in\mathbb{R}}\theta(t,x)<\frac{3}{2}<\inf_{t\ge0,x\in\mathbb{R}}\theta(t,x).\end{aligned}\tag{1.20}$$

In what follows, we define the perturbation $(\widetilde\rho,\widetilde u,\widetilde\theta)=(\widetilde\rho,\widetilde u,\widetilde\theta)(t,x)$ and $\widetilde G=\widetilde G(t,x,v)$ as

$$\begin{cases}(\widetilde\rho,\widetilde u,\widetilde\theta)=(\rho-\bar\rho,u-\bar u,\theta-\bar\theta)(t,x),\\ \widetilde G=(G-\overline{G})(t,x,v),\quad \widetilde G=\sqrt{\mu}\mathbf{g}(t,x,v),\quad F_2=\sqrt{\mu}\mathbf{f}(t,x,v).\end{cases}\tag{1.21}$$

Here the term $\overline{G} = \overline{G}(t, x, v)$ is defined as

$$\overline{G} = L_M^{-1} P_1 v_1 M\Big\{\frac{|v-u|^2\bar{\theta}_x}{2R\theta^2} + \frac{(v-u)\cdot\bar{u}_x}{R\theta}\Big\}. \tag{1.22}$$

Since the term $P_1(v_1 M_x)$ in (1.16) contains $\|(\bar{u}_x, \bar{\theta}_x)\|^2$, which is not integrable about t, we need subtract $\overline{G}$ from G to cancel this term as in [19]. Recall $G = \overline{G} + \sqrt{\mu}\mathbf{g}$, we can rewrite the equation (1.16) as

$$\begin{aligned}\partial_t\mathbf{g} + v_1\partial_x\mathbf{g} - \mathcal{L}_1\mathbf{g} = {}& \Gamma(\mathbf{g}, \frac{M-\mu}{\sqrt{\mu}}) + \Gamma(\frac{M-\mu}{\sqrt{\mu}}, \mathbf{g}) + \Gamma(\frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}})\\ &+ \frac{P_0(v_1\sqrt{\mu}\partial_x\mathbf{g})}{\sqrt{\mu}} - \frac{1}{\sqrt{\mu}}P_1 v_1 M\Big\{\frac{|v-u|^2\widetilde{\theta}_x}{2R\theta^2} + \frac{(v-u)\cdot\widetilde{u}_x}{R\theta}\Big\}\\ &- \frac{P_1(v_1\partial_x\overline{G})}{\sqrt{\mu}} - \frac{\partial_t\overline{G}}{\sqrt{\mu}} + \frac{P_1(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}}.\end{aligned} \tag{1.23}$$

Here Γ and $\mathcal{L}_1$ are defined by

$$\Gamma(f, g) = \frac{2}{\sqrt{\mu}}Q(\sqrt{\mu}f, \sqrt{\mu}g), \quad \mathcal{L}_1 f = \frac{2}{\sqrt{\mu}}\{Q(\mu, \sqrt{\mu}f) + Q(\sqrt{\mu}f, \mu)\}, \tag{1.24}$$

and we have used the facts that

$$\begin{aligned}\mathcal{L}_1\mathbf{g} &= \Gamma(\sqrt{\mu}, \mathbf{g}) + \Gamma(\mathbf{g}, \sqrt{\mu}),\\ P_1(v_1 M_x) &= P_1 v_1 M\Big\{\frac{|v-u|^2\widetilde{\theta}_x}{2R\theta^2} + \frac{(v-u)\cdot\widetilde{u}_x}{R\theta}\Big\} + L_M\overline{G},\end{aligned} \tag{1.25}$$

and

$$\frac{1}{\sqrt{\mu}}L_M(\sqrt{\mu}\mathbf{g}) = \frac{2}{\sqrt{\mu}}\{Q(M, \sqrt{\mu}\mathbf{g}) + Q(\sqrt{\mu}\mathbf{g}, M)\} = \mathcal{L}_1\mathbf{g} + \Gamma(\mathbf{g}, \frac{M-\mu}{\sqrt{\mu}}) + \Gamma(\frac{M-\mu}{\sqrt{\mu}}, \mathbf{g}).$$

By the decomposition $F_1 = M + \overline{G} + \sqrt{\mu}\mathbf{g}$ and $F_2 = \sqrt{\mu}\mathbf{f}$, we can rewrite the equation $(1.6)_2$ as

$$\begin{aligned}\partial_t\mathbf{f} + v_1\partial_x\mathbf{f} + \partial_x\phi v_1\sqrt{\mu} - \mathcal{L}_2\mathbf{f} = {}& \Gamma(\frac{M-\mu}{\sqrt{\mu}}, \mathbf{f}) + \Gamma(\frac{\overline{G}}{\sqrt{\mu}}, \mathbf{f}) + \Gamma(\mathbf{g}, \mathbf{f})\\ &+ \frac{\partial_x\phi\partial_{v_1}(M-\mu)}{\sqrt{\mu}} + \frac{\partial_x\phi\partial_{v_1}\overline{G}}{\sqrt{\mu}} + \frac{\partial_x\phi\partial_{v_1}(\sqrt{\mu}\mathbf{g})}{\sqrt{\mu}}.\end{aligned} \tag{1.26}$$

Here we have denoted

$$\Gamma(f, g) = \frac{2}{\sqrt{\mu}}Q(\sqrt{\mu}f, \sqrt{\mu}g), \quad \mathcal{L}_2 f = \frac{2}{\sqrt{\mu}}Q(\mu, \sqrt{\mu}f), \tag{1.27}$$

and used the fact that

$$\frac{2}{\sqrt{\mu}}Q(F_1, F_2) = \mathcal{L}_2\mathbf{f} + \Gamma(\frac{M-\mu}{\sqrt{\mu}}, \mathbf{f}) + \Gamma(\frac{\overline{G}}{\sqrt{\mu}}, \mathbf{f}) + \Gamma(\mathbf{g}, \mathbf{f}).$$

Recalling that $F_2 = \sqrt{\mu}\mathbf{f}$, then the Poisson equation $(1.6)_3$ can be rewritten as

$$-\partial_{xx}\phi = 2\int_{\mathbb{R}^3} \sqrt{\mu}\mathbf{f}dv. \tag{1.28}$$

Notice that the null space $\mathcal{N}_1$ of $\mathcal{L}_1$ is spanned by the functions $\{\sqrt{\mu}, v_i\sqrt{\mu}, |v|^2\sqrt{\mu}\}$ and the null space $\mathcal{N}_2$ of $\mathcal{L}_2$ is spanned by the single element $\{\sqrt{\mu}\}$, cf. [16]. We define orthogonal projection P_2 from $L^2_v(\mathbb{R}^3)$ to $\ker\mathcal{N}_2$, then $\mathbf{f} = P_2\mathbf{f} + (\mathbf{I} - P_2)\mathbf{f}$.

To present the result in this paper, the following notations are needed. We shall use $\langle\cdot,\cdot\rangle$ to denote the standard L^2 inner product in $\mathbb{R}^3_v$ with its corresponding L^2 norm $|\cdot|_2$. We also use $(\cdot,\cdot)$ to denote L^2 inner product in $\mathbb{R}_x$ or $\mathbb{R}_x \times \mathbb{R}^3_v$ with its corresponding L^2 norm $\|\cdot\|$. Let α and β be nonnegative integer and a multi-indices $\beta = [\beta_1, \beta_2, \beta_3]$, respectively. Denote a high order derivative

$$\partial^\alpha_\beta = \partial^\alpha_x \partial^{\beta_1}_{v_1}\partial^{\beta_2}_{v_2}\partial^{\beta_3}_{v_3}.$$

If each component of β is not greater than the corresponding one of $\overline{\beta}$, we use the standard notation $\beta \le \overline{\beta}$. And $\beta < \overline{\beta}$ means that $\beta \le \overline{\beta}$ and $|\beta| < |\overline{\beta}|$. $C^{\overline{\beta}}_\beta$ is the usual binomial coefficient. And C denotes some generic positive (generally large) constant and λ denotes some generic positive (generally small) constant, where C and λ may take different values in different places. The notation $\langle v\rangle = \sqrt{1+|v|^2}$ and $A \approx B$ is used to denote that there exists constant $c_0 > 1$ such that $c_0^{-1}B \le A \le c_0 B$. Motivated by [23, 6, 30], we introduce the following time-velocity weight function

$$w := w(\beta)(t,v) = \langle v\rangle^{|\gamma|(l-|\beta|)} e^{\frac{\langle v\rangle^2}{2}(q_1 + \frac{q_2}{(1+t)^{q_3}})}, \quad l \ge |\beta|, \quad q = (q_1, q_2, q_3), \tag{1.29}$$

where $0 \le q_1, q_2 < 1$, and $q_3 \ge 0$ will be chosen later .

Denote weighted L^2 norms as

$$|g|^2_{2,w} \equiv \int_{\mathbb{R}^3} w^2|g|^2 dv, \quad \|g\|^2_{2,w} \equiv \int_{\mathbb{R}}\int_{\mathbb{R}^3} w^2|g|^2 dvdx.$$

Note that the Boltzmann collision frequency is

$$\nu(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(\vartheta)|v - v_*|^\gamma \mu(v_*)d\omega dv_* \approx \langle v\rangle^\gamma. \tag{1.30}$$

With (1.30), we define the weighted dissipation norms as

$$|g|^2_{\nu,w} \equiv \int_{\mathbb{R}^3} \nu(v)w^2|g(v)|^2 dv, \quad \|g\|^2_{\nu,w} \equiv \int_{\mathbb{R}}\int_{\mathbb{R}^3} \nu(v)w^2|g(v)|^2 dvdx.$$

And we also write $|g|_2 = |g|_{2,1}$, $|g|_\nu = |g|_{\nu,1}$, $\|g\| = \|g\|_{2,1}$ and $\|g\|_\nu = \|g\|_{\nu,1}$.

Now, we define the following instant energy functional $\mathcal{E}_{N,l,q}(t)$ as

$$\begin{aligned}\mathcal{E}_{N,l,q}(t) = \sum_{|\alpha|\le N} \|\partial^\alpha(\widetilde{\rho}, \widetilde{u}, \widetilde{\theta})(t)\|^2 + \sum_{|\alpha|+|\beta|\le N} \|\partial^\alpha_\beta \mathbf{g}(t)\|^2_{2,w(\beta)} \\ + \sum_{|\alpha|+|\beta|\le N} \|\partial^\alpha_\beta \mathbf{f}(t)\|^2_{2,w(\beta)} + \sum_{|\alpha|\le N} \|\partial^\alpha\partial_x\phi(t)\|^2.\end{aligned} \tag{1.31}$$

Correspondingly, the dissipation rate functional $\mathcal{D}_{N,l,q}(t)$ is given by

$$\mathcal{D}_{N,l,q}(t) = \sum_{1\le|\alpha|\le N} \|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})(t)\|^2 + \sum_{|\alpha|+|\beta|\le N} \|\partial^\alpha_\beta \mathbf{g}(t)\|^2_{\nu,w(\beta)} + \sum_{|\alpha|+|\beta|\le N} \|\partial^\alpha_\beta \mathbf{f}(t)\|^2_{\nu,w(\beta)} + \sum_{|\alpha|\le N} \|\partial^\alpha\partial_x\phi(t)\|^2. \tag{1.32}$$

Throughout this paper we assume the Sobolev index $N \ge 6$.

With the above preparation, the main result of the paper can be stated as follows.

THEOREM 1.1. *Assume that* (1.20) *holds and* $(\rho^r,u^r,\theta^r)(\frac{x}{t})$ *be the Riemann solution of Euler system* (2.1)-(2.2) *consists of one 3-rarefaction wave given by* (2.6). *Let* $-2\le\gamma<0$, $l\ge\max\{N,\frac{1}{2}+\frac{1}{|\gamma|}\}$, $q_3\in(0,l-3)$, *any* $q_1\ge0$ *and* $q_2>0$ *small enough in* (1.29) *and* $\delta=|(\rho_+-\rho_-,u_+-u_-,\theta_+-\theta_-)|$ *be the wave strength with* $\delta>0$ *be small enough. There exist sufficient small constant* $\varepsilon_0>0$ *and given constant* $C_0>0$ *such that if* $\mathcal{E}_{N,l,q}(0)+C_0\delta^{\frac{1}{6}}\le\varepsilon_0$ *for* $N\ge6$, *then the Cauchy problem* (1.6)-(1.7) *admits a unique global solution* $(F_1,F_2,\partial_x\phi)$. *Moreover, it holds that* $F_\pm(t,x,v)\ge0$ *for the Cauchy problem* (1.4)-(1.5) *provided that it is so initially.*

In addition, the following time-asymptotic behaviors holds true:

$$\lim_{t\to+\infty}\Big\|\frac{F_1(t,x,v)-M_{[\rho^r,u^r,\theta^r](x/t)}(v)}{\sqrt{\mu}}\Big\|_{L^\infty_x L^2_v}=0. \tag{1.33}$$

Furthermore, there exists a constant $c>0$ *such that*

$$\sum_{|\alpha|\le N-1}\|\partial^\alpha\mathbf{f}\|^2+\sum_{|\alpha|\le N}\|\partial^\alpha\partial_x\phi\|^2\le C\varepsilon_0(1+t)^{-(2l-4)},\quad \text{for}\quad q_1\ge0, \tag{1.34}$$

and

$$\sum_{|\alpha|\le N-1}\|\partial^\alpha\mathbf{f}\|^2+\sum_{|\alpha|\le N}\|\partial^\alpha\partial_x\phi\|^2\le C\varepsilon_0 e^{-c(1+t)^{\frac{2}{2+|\gamma|}}},\quad \text{for}\quad q_1>0. \tag{1.35}$$

In what follows we shall review some previous works related to this paper. There have been extensive studies on the existence and stability of wave patterns for the one-dimensional Boltzmann equation. Around 1980, under the angular cutoff condition, Caflisch and Nicolaenko constructed the shock profile solutions of the Boltzmann equation in [2] for hard potentials. The stability of shock profile was proved by Liu-Yu [17] with the zero total macroscopic mass condition by the energy method based on the micro-macro decomposition and this result was later generalized to the case under without zero mass condition in [35]. The nonlinear stability of rarefaction waves was proved in [19, 34] for hard potentials and in [31] for soft potentials. And the stability of contact discontinuities for hard potentials was proved in [14, 15]. Recently, there were some works on the existence and stability of wave patterns for the VPB system. The stability of rarefaction waves for the unipolar VPB system with hard sphere model was proved in [4], and this result was generalized to the bipolar VPB system in [5]. And the nonlinear stability of viscous shock waves and rarefaction waves for the bipolar VPB system with hard sphere model was obtained in [20] and this result was later generalized to the weighted function space in [32, 33] for the case of general hard potentials. Here we would like to mention that the stability of rarefaction waves for the bipolar Vlasov-Poisson-Landau system was proved in [7]. In addition to the above works, there are some other works, cf. [21, 28, 27] and the references therein. The Boltzmann equation around global Maxwellians has been extensively studied in [3, 6, 8, 9, 10, 13, 23, 25, 30] and the references therein.

Although, the stability of rarefaction waves for the VPB system with hard potentials has been heavily studied as mentioned above. But the case of general soft potentials has remained open. Inspired by our previous works [32, 33], we further consider the one-dimensional bipolar VPB system with soft potentials near a local Maxwellian in this paper. We construct the global solutions near a local Maxwellian for the one-dimensional bipolar VPB system (1.6) with moderately soft potentials and prove that the nonlinear large time-asymptotic stability of the rarefaction waves to the solution F_1 and some time decay rates of the solution (F_2, ϕ_x). Moreover, we generalize the results with hard potentials in [20, 32, 33] to the case of general moderately soft potentials.

Finally, we would like to mention our the proof methods in this paper. The two different sets of decompositions of the solutions are crucially applied in this paper. For the energy analysis of the first species F_1, our proof is based on the decomposition of the solutions for the Boltzmann equation with respect to the local Maxwellian that was initiated by Liu-Yu [17] and developed by Liu-Yang-Yu [18]. We thus can make use of the macro-micro decomposition to rewrite the nonlinear VPB system as the form of the compressible Navier-Stokes-type system so that the analysis in the context of the viscous conservation laws can be applied. As mentioned in [7, 31], since we study the Boltzmann equation around a local Maxwellian and both the linearized operators $\mathcal{L}_1$ defined as (1.24) and $\mathcal{L}_2$ defined as (1.27) have no a spectral gap as (3.5), it gives rises to more analytic difficulties than the study on the perturbation of a global Maxwellian. Since the term $\|[\bar{u}_x, \bar{\theta}_x]\|^2$ is not integrable with respect to the time t, we need to consider the subtraction of $G(t, x, v)$ by $\overline{G}(t, x, v)$ as (1.22) to cancel the slow time decay terms. However, unlike hard potentials, now the inverse of linearized operator, L_M^{-1} defined as (1.16) is an unbounded operator in $L^2(\mathbb{R}^3)$, which leads to considerable difficulties in our analysis. In order to handle the term involving L_M^{-1}, we will apply the Burnett functions and analyze the fast decay properties above the velocity of the Burnett functions. For the Vlasov-Poisson-Landau system in [7], the dissipation anisotropy norm related the linearized operator contains the derivative of v and this can be used to absorb some velocity derivative. Notice that the dissipation norm in (1.32) has no such a good property. In addition, due to that the background is one-dimensional rarefaction wave profile as in [7], the nonlinear collision terms show stronger nonlinear effects. Thus we use a new weight function $w(\beta)$ as (1.29) to overcome these difficulties. Then we follow the strategy in [7] to perform the energy estimate while we are forced to take care of the role of the weight function $w(\beta)$.

The rest of this paper is arranged as follows. In the next section, we will construct and give some properties for the rarefaction waves. In section 3, we will give some basic estimates used in the latter sections. We establish some non-weighted energy estimates and weighted energy estimates in section 4 and Section 5, respectively. In section 6, we shall derive the fast time decay rates of the electric field term and then establish the existence of global solutions.

2. Approximate rarefaction waves. In this section, we will define the nonlinear time asymptotic rarefaction wave profile for the Cauchy problem (1.6) and (1.7) as in [22, 19, 20]. If we take the G, F_2 and ϕ to be zero in (1.15), we obtain the following 1D compressible Euler system

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = 0, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = 0, \quad i = 2, 3, \\ \big\{\rho(e + \frac{|u|^2}{2})\big\}_t + \big\{\rho u_1(e + \frac{|u|^2}{2}) + p u_1\big\}_x = 0, \end{cases} \tag{2.1}$$

with the Riemann initial data

$$(\rho, u, \theta)(t, x) \mid_{t=0} = (\rho_0^r, u_0^r, \theta_0^r)(x) = \begin{cases} (\rho_+, u_+, \theta_+), & x > 0, \\ (\rho_-, u_-, \theta_-), & x < 0, \end{cases} \tag{2.2}$$

where $(\rho_\pm, u_\pm, \theta_\pm)$ are given by (1.5). By using (2.1), (2.2) and the state equation

$$p = \frac{2}{3}\rho\theta = k\rho^{5/3}\exp(S), \quad k = \frac{1}{2\pi e}, \tag{2.3}$$

where S is the macroscopic entropy, we know that the Euler system (2.1) for (ρ, u_1, S) has three distinct eigenvalues

$$\lambda_i(\rho, u_1, S) = u_1 + (-1)^{\frac{i+1}{2}}\sqrt{p_\rho(\rho, S)}, \ i = 1, 3, \quad \lambda_2(\rho, u_1, S) = u_1.$$

The corresponding right eigenvectors are given as

$$r_i(\rho, u_1, S) = ((-1)^{\frac{i+1}{2}}\rho, \sqrt{p_\rho(\rho, S)}, 0)^t, \ i = 1, 3, \quad r_2(\rho, u_1, S) = (p_S, 0, -p_\rho)^t,$$

where $p_\rho(\rho, S) = \frac{5}{3}k\rho^{\frac{2}{3}}e^S > 0$. In terms of the two Riemann invariants of the i-th eigenvalue $\lambda_i(\rho, u_1, S)$, $i = 1$, 3, we define the i-Rarefaction wave ($i = 1$, 3) as follows (cf. [24])

$$\begin{aligned} R_1(\rho_-, u_{1-}, \theta_-) = \{&(\rho, u_1, \theta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \mid S = S_*, \\ &u_1 + \sqrt{15k}e^{\frac{S}{2}}\rho^{\frac{1}{3}} = u_{1-} + \sqrt{15k}e^{\frac{S_*}{2}}\rho_-^{\frac{1}{3}}, \quad \rho < \rho_-, \quad u_1 > u_{1-}\}, \\ R_3(\rho_-, u_{1-}, \theta_-) = \{&(\rho, u_1, \theta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \mid S = S_*, \\ &u_1 - \sqrt{15k}e^{\frac{S}{2}}\rho^{\frac{1}{3}} = u_{1-} - \sqrt{15k}e^{\frac{S_*}{2}}\rho_-^{\frac{1}{3}}, \quad \rho > \rho_-, \quad u_1 > u_{1-}\}. \end{aligned} \tag{2.4}$$

Here and to the end, $S_* = -\frac{2}{3}\ln\rho_- + \ln(\frac{4}{3}\pi\theta_-) + 1$.

Without the loss of generality, we consider only the stability of 3-rarefaction wave to the Euler system (2.1) with (2.2) in the present paper, and the stability of 1-rarefaction wave can be treated similarly. The 3-rarefaction wave to the Euler system (2.1) with (2.2) can be expressed explicitly by Riemann solution to the inviscid Burgers equation

$$\begin{cases} \omega_t + \omega\omega_x = 0, \\ \omega(0, x) = \begin{cases} \omega_-, & x < 0, \\ \omega_+, & x > 0. \end{cases} \end{cases} \tag{2.5}$$

If two constants $\omega_- < \omega_+$, then (2.5) admits a centered rarefaction wave solution $\omega^r(x, t) = \omega^r(\frac{x}{t})$ connecting ω_- and ω_+ in the form of, cf. [22],

$$\omega^r(\frac{x}{t}) = \begin{cases} \omega_-, & \frac{x}{t} \le \omega_-, \\ \frac{x}{t}, & \omega_- < \frac{x}{t} \le \omega_+, \\ \omega_+, & \frac{x}{t} > \omega_+. \end{cases}$$

For $(\rho_+,u_+,\theta_+)\in R_3(\rho_-,u_-,\theta_-)$, the 3-rarefaction wave $(\rho^r,u^r,\theta^r)(\frac{x}{t})$ to the Riemann problem (2.1) with (2.2) can be defined explicitly by

$$\begin{cases}\lambda_3(\rho^r(\frac{x}{t}),u_1^r(\frac{x}{t}),S_*)=\begin{cases}\lambda_3(\rho_-,u_{1-},S_*), & \frac{x}{t}\le\lambda_3(\rho_-,u_{1-},S_*),\\ \frac{x}{t}, & \lambda_3(\rho_-,u_{1-},S_*)<\frac{x}{t}\le\lambda_3(\rho_+,u_{1+},S_*),\\ \lambda_3(\rho_+,u_{1+},S_*), & \frac{x}{t}>\lambda_3(\rho_+,u_{1+},S_*),\end{cases}\\ u_1^r(\frac{x}{t})-\sqrt{15k}e^{\frac{S_*}{2}}(\rho^r)^{\frac13}(\frac{x}{t})=u_{1-}-\sqrt{15k}e^{\frac{S_*}{2}}\rho_-^{\frac13},\\ u_2^r=u_3^r=0,\quad \theta^r(\frac{x}{t})=\frac32 ke^{S_*}(\rho^r)^{\frac23}(\frac{x}{t}).\end{cases}\tag{2.6}$$

Since the above 3-rarefaction wave is only Lipschitz continuous, we shall construct an approximate smooth rarefaction wave to the 3-rarefaction wave defined in (2.6). Motivated by [22], the approximate smooth rarefaction wave can be constructed by the Burgers equation

$$\begin{cases}\overline{\omega}_t+\overline{\omega}\,\overline{\omega}_x=0,\\ \overline{\omega}(0,x)=\overline{\omega}(x)=\frac{\omega_++\omega_-}{2}+\frac{\omega_+-\omega_-}{2}\tanh(x).\end{cases}\tag{2.7}$$

By the method of characteristic curves, the solution $\overline{\omega}(t,x)$ to the problem (2.7) can be given by

$$\overline{\omega}(t,x)=\overline{\omega}(x_0(t,x)),\quad x=x_0(t,x)+\overline{\omega}(x_0(t,x))t.\tag{2.8}$$

Correspondingly, the smooth approximate rarefaction wave $(\bar\rho,\bar u,\bar\theta)(t,x)$ to the 3-rarefaction wave $(\rho^r,u^r,\theta^r)(\frac{x}{t})$ in (2.6) for the problem (2.1)-(2.2) can be defined by

$$\begin{cases}\overline{\omega}(t,x)=\lambda_3(\bar\rho(t,x),\bar u_1(t,x),S_*),\quad \omega_\pm=\lambda_3(\rho_\pm,u_{1\pm},S_*),\\ \bar u_1(t,x)-\sqrt{15k}e^{\frac{S_*}{2}}\bar\rho^{\frac13}(t,x)=u_{1-}-\sqrt{15k}e^{\frac{S_*}{2}}\rho_-^{\frac13},\quad \bar u_2=\bar u_3=0,\\ \lim\limits_{x\to\pm\infty}(\bar\rho,\bar u_1,\bar\theta)(t,x)=(\rho_\pm,u_{1\pm},\theta_\pm),\quad \bar\theta(t,x)=\frac32 ke^{S_*}\bar\rho^{\frac23}(t,x),\end{cases}\tag{2.9}$$

where $\overline{\omega}(t,x)$ is the solution of (2.7). Then the approximate smooth 3-rarefaction wave $(\bar\rho,\bar u,\bar\theta)(t,x)$ satisfies the following Euler system

$$\begin{cases}\bar\rho_t+(\bar\rho\bar u_1)_x=0,\\ (\bar\rho\bar u_1)_t+(\bar\rho\bar u_1^2+\bar p)_x=0,\\ (\bar\rho\bar u_i)_t+(\bar\rho\bar u_1\bar u_i)_x=0,\quad \text{i=2,3},\\ (\bar\rho\bar\theta)_t+(\bar\rho\bar u_1\bar\theta)_x+\bar p\bar u_{1x}=0,\end{cases}\tag{2.10}$$

where $\bar p=R\bar\rho\bar\theta$. The following properties of $(\bar\rho,\bar u,\bar\theta)(t,x)$ can be found in [22, 20].

Lemma 2.1. *Let* $\delta=|(\rho_+-\rho_-,u_+-u_-,\theta_+-\theta_-)|$, *the 3-rarefaction wave* $(\bar\rho,\bar u,\bar\theta)(t,x)$ *satisfying*

(i) $\bar u_2=\bar u_3=0$, $\bar u_{1x}>0$, *and* $\bar\theta_x=\sqrt{\frac25}\bar\theta^{\frac12}\bar u_{1x}$, $\forall\, x\in\mathbb{R},\ t\ge0$.

(ii) For any $t\ge0$ *and* $q\in[1,+\infty]$, *there exists* $C_q>0$ *such that*

$$\begin{cases}\|(\bar\rho,\bar u_1,\bar\theta)_x(t,\cdot)\|_{L^q}\le C_q\min\{\delta,\delta^{\frac1q}(1+t)^{-1+\frac1q}\},\\ \|\partial_x^j(\bar\rho,\bar u_1,\bar\theta)(t,\cdot)\|_{L^q}\le C_q\min\{\delta,(1+t)^{-1}\},\quad j\ge2.\end{cases}\tag{2.11}$$

(iii) The approximation rarefaction wave and the inviscid rarefaction wave satisfy

$$\lim_{t\to+\infty}\|(\bar\rho,\bar u,\bar\theta)(t,x)-(\rho^r,u^r,\theta^r)(\frac{x}{t})\|_{L^\infty}=0.$$

3. Basic estimates. In this section, we give some basic estimates to be used in later energy analysis. Firstly, we recall the following Burnett functions defined in [1, 8, 31]

$$\hat{A}_j(v) = \frac{|v|^2-5}{2}v_j \quad \text{and} \quad \hat{B}_{ij}(v) = v_iv_j - \frac{1}{3}\delta_{ij}|v|^2 \quad \text{for} \quad i,j=1,2,3. \tag{3.1}$$

Noticing that $\hat{A}_j(\frac{v-u}{\sqrt{R\theta}})M$ and $\hat{B}_{ij}(\frac{v-u}{\sqrt{R\theta}})M$ are orthogonal to the null space $\mathrm{Ker}L_M$ of the linearized operator L_M, we can define $A_j(\frac{v-u}{\sqrt{R\theta}})$ and $B_{ij}(\frac{v-u}{\sqrt{R\theta}})$ such that $P_0A_j(\frac{v-u}{\sqrt{R\theta}})=0$, $P_0B_{ij}(\frac{v-u}{\sqrt{R\theta}})=0$, and

$$A_j(\frac{v-u}{\sqrt{R\theta}}) = L_M^{-1}[\hat{A}_j(\frac{v-u}{\sqrt{R\theta}})M] \quad \text{and} \quad B_{ij}(\frac{v-u}{\sqrt{R\theta}}) = L_M^{-1}[\hat{B}_{ij}(\frac{v-u}{\sqrt{R\theta}})M]. \tag{3.2}$$

The following lemma is borrowed from [31, Lemma 2.4].

LEMMA 3.1. *The Burnett functions have the following properties:*
- $-\langle \hat{A}_i, A_i\rangle$ *is positive and independent of* i;
- $\langle \hat{A}_i, A_j\rangle = 0$ *for any* $i\neq j$; $\langle \hat{A}_i, B_{jk}\rangle = 0$ *for any* i, j, k;
- $\langle \hat{B}_{ij}, B_{kj}\rangle = \langle \hat{B}_{kl}, B_{ij}\rangle = \langle \hat{B}_{ji}, B_{kj}\rangle$, *which is independent of* i, j, *for fixed* k, l;
- $-\langle \hat{B}_{ij}, B_{ij}\rangle$ *is positive and independent of* i, j *when* $i\neq j$;
- $\langle \hat{B}_{ii}, B_{jj}\rangle$ *is positive and independent of* i, j *when* $i\neq j$;
- $-\langle \hat{B}_{ii}, B_{ii}\rangle$ *is positive and independent of* i;
- $\langle \hat{B}_{ij}, B_{kl}\rangle = 0$ *unless either* $(i,j)=(k,l)$ *or* (l,k), *or* i=j *and* k=l;
- $\langle \hat{B}_{ii}, B_{ii}\rangle - \langle \hat{B}_{ii}, B_{jj}\rangle = 2\langle \hat{B}_{ij}, B_{ij}\rangle$ *holds for any* $i\neq j$.

In terms of Burnett functions, the viscosity coefficient $\mu(\theta)$ and heat conductivity coefficient $\kappa(\theta)$ can be represented by

$$\begin{aligned} \mu(\theta) &= -R\theta\int_{\mathbb{R}^3}\hat{B}_{ij}(\frac{v-u}{\sqrt{R\theta}})B_{ij}(\frac{v-u}{\sqrt{R\theta}})dv>0, \quad i\neq j,\\ \kappa(\theta) &= -R^2\theta\int_{\mathbb{R}^3}\hat{A}_j(\frac{v-u}{\sqrt{R\theta}})A_j(\frac{v-u}{\sqrt{R\theta}})dv>0. \end{aligned} \tag{3.3}$$

Notice that these coefficients are positive smooth functions depending only on θ.

Next, we give the following fast decay about the velocity v of the Burnett functions which will be used frequently in the later energy estimates. It can be proved by similar arguments as used in [7, Lemma 6.1] and is omitted for brevity.

LEMMA 3.2. *Suppose that* $U(v)$ *is any polynomial of* $\frac{v-\hat{u}}{\sqrt{R\hat{\theta}}}$ *such that* $U(v)\widehat{M}\in(\ker L_{\widehat{M}})^{\perp}$ *for any Maxwellian* $\widehat{M}=M_{[\widehat{\rho},\widehat{u},\widehat{\theta}]}(v)$ *where* $L_{\widehat{M}}$ *is as* (1.16). *For any* $\epsilon\in(0,1)$ *and any multi-index* β, *there exists constant* $C_\beta>0$ *such that*

$$|\partial_\beta L_{\widehat{M}}^{-1}U(v)\widehat{M}| \leq C_\beta(\widehat{\rho},\widehat{u},\widehat{\theta})\widehat{M}^{1-\epsilon}.$$

In particular, if the assumption (1.20) *holds, there exists constant* $C_\beta>0$ *such that*

$$|\partial_\beta A_j(\frac{v-u}{\sqrt{R\theta}})| + |\partial_\beta B_{ij}(\frac{v-u}{\sqrt{R\theta}})| \leq C_\beta M^{1-\epsilon}. \tag{3.4}$$

Now, we turn to summarize some refined estimates for the collision operators $\mathcal{L}_1$, $\mathcal{L}_2$ and Γ defined in (1.24) and (1.27). We first recall the properties of the linearized operators $\mathcal{L}_1$ and $\mathcal{L}_2$, cf. [12, 16].

Lemma 3.3. *For any $g_1 \in \mathcal{N}_1^{\perp}$ and $g_2 \in \mathcal{N}_2^{\perp}$, there exist constants $\sigma_1 > 0$ and $\sigma_2 > 0$ such that*

$$-\langle \mathcal{L}_1 g_1, g_1\rangle \geq \sigma_1 |g_1|_\nu^2, \quad -\langle \mathcal{L}_2 g_2, g_2\rangle \geq \sigma_2 |g_2|_\nu^2. \tag{3.5}$$

We also shall give the weighted estimates for $\mathcal{L}_1$ and $\mathcal{L}_2$. It can be proved by a straightforward modification of the arguments used in [23, Lemma 2] and we thus omit its proof for brevity.

Lemma 3.4. *Let $-3 < \gamma < 0$ and $w = w(\beta)$ in (1.29) with $q_1 \in [0,1)$ and $q_2 \in (0,1)$ small enough and $\widetilde{L}$ is the linearized operator either $\mathcal{L}_1$ or $\mathcal{L}_2$. For any $|\beta| > 0$ and for all $\epsilon > 0$, there exists $C_\epsilon > 0$ such that*

$$-\langle w^2(\beta)\partial_\beta \widetilde{L} g, \partial_\beta g\rangle \geq |\partial_\beta g|^2_{\nu, w(\beta)} - \epsilon \sum_{|\beta_1|\leq|\beta|} |\partial_{\beta_1} g|^2_{\nu, w(\beta_1)} - C_\epsilon |g|_\nu^2. \tag{3.6}$$

If $|\beta| = 0$, for any $\epsilon > 0$ small enough, there exists $C_\epsilon > 0$ such that

$$-\langle w^2(0)\widetilde{L} g, g\rangle \geq \frac{1}{2}|g|^2_{\nu, w(0)} - C_\epsilon |g|_\nu^2. \tag{3.7}$$

In what follows we recall the weighted estimates on the nonlinear collision operator Γ. By the translation invariant of the collision operator Γ, one has

$$\partial^\alpha_\beta \Gamma(g_1, g_2) \equiv \sum C_\alpha^{\alpha_1\alpha_2} C_\beta^{\beta_0\beta_1\beta_2} \Gamma^0(\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2), \tag{3.8}$$

where the summation is over $\beta_0 + \beta_1 + \beta_2 = \beta$ and $\alpha_1 + \alpha_2 = \alpha$, and Γ^0 is given by

$$\begin{aligned}\Gamma^0(\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2) \equiv &\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v - v_*|^\gamma B(\vartheta)\partial_{\beta_0}[\mu^{1/2}(v_*)]\partial^{\alpha_1}_{\beta_1} g_1(v_*')\partial^{\alpha_2}_{\beta_2} g_2(v') d\omega dv_* \\ &- \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v - v_*|^\gamma B(\vartheta)\partial_{\beta_0}[\mu^{1/2}(v_*)]\partial^{\alpha_1}_{\beta_1} g_1(v_*)\partial^{\alpha_2}_{\beta_2} g_2(v) d\omega dv_*.\end{aligned}$$

Lemma 3.5. *Recall (3.8) with $\beta_0 + \beta_1 + \beta_2 = \beta$, $\alpha_1 + \alpha_2 = \alpha$. Let $-3 < \gamma < 0$ and $w = w(\beta)$ in (1.29) with both $q_1 \in [0,1)$ and $q_2 \in (0,1)$ small enough, for any $\epsilon > 0$ small enough, we have*

$$\begin{aligned}&|\langle w^2(\beta)\Gamma^0(\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2), \partial^\alpha_\beta g_3\rangle| \\ &\leq C\sum_{k\leq 2}\{|\nabla_v^k(\mu^\epsilon \partial^{\alpha_1}_{\beta_1} g_1)|_2 + |\partial^{\alpha_1}_{\beta_1} g_1|_{2, w(\beta_1)}\}|\partial^{\alpha_2}_{\beta_2} g_2|_{\nu, w(\beta_2)}|\partial^\alpha_\beta g_3|_{\nu, w(\beta)},\end{aligned} \tag{3.9}$$

or

$$\begin{aligned}&|\langle w^2(\beta)\Gamma^0(\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2), \partial^\alpha_\beta g_3\rangle| \\ &\leq C\sum_{k\leq 2}\{|\nabla_v^k(\mu^\epsilon \partial^{\alpha_2}_{\beta_2} g_2)|_2 + |\partial^{\alpha_2}_{\beta_2} g_2|_{2, w(\beta_2)}\}|\partial^{\alpha_1}_{\beta_1} g_1|_{\nu, w(\beta_1)}|\partial^\alpha_\beta g_3|_{\nu, w(\beta)}.\end{aligned} \tag{3.10}$$

Set $\varsigma(v) = \langle v\rangle^{-\gamma} = \nu(v)^{-1}$ *and* $\ell \geq 0$*, it holds that*

$$|\varsigma^{\ell}\Gamma(g_1, g_2)|_2^2 \leq C \sum_{|\beta|\leq 2} |\varsigma^{\ell-|\beta|}\partial_\beta g_1|_\nu^2 |\varsigma^{\ell} g_2|_\nu^2. \tag{3.11}$$

Furthermore, under the conditions of (1.20)*, for* $|\alpha_1| \geq 1$*, one has*

$$\begin{aligned}
&|\langle \Gamma^0[\partial_{\beta_1}^{\alpha_1}(\frac{M-\mu}{\sqrt{\mu}}), \partial_{\beta_2}^{\alpha_2} g_2], w^2(\beta)\partial_\beta^\alpha g_3\rangle| \\
&\leq C\Big\{|\partial^{\alpha_1}(\rho, u, \theta)| + \sum_{1\leq|\alpha'|\leq|\alpha_1|} |\partial^{\alpha_1-\alpha'}(\rho, u, \theta)||\partial^{\alpha'}(\rho, u, \theta)| \\
&\quad + \cdots + |(\rho_x, u_x, \theta_x)|^{|\alpha_1|}\Big\}|\partial_{\beta_2}^{\alpha_2} g_2|_{\nu, w(\beta_2)}|\partial_\beta^\alpha g_3|_{\nu, w(\beta)},
\end{aligned} \tag{3.12}$$

and

$$|\langle \Gamma^0[\partial_{\beta_1}(\frac{M-\mu}{\sqrt{\mu}}), \partial_{\beta_2}^{\alpha} g_2], w^2(\beta)\partial_\beta^\alpha g_3\rangle| \leq C\eta_0|\partial_{\beta_2}^{\alpha} g_2|_{\nu, w(\beta_2)}|\partial_\beta^\alpha g_3|_{\nu, w(\beta)}. \tag{3.13}$$

Proof. We can obtain (3.9) (3.10) and (3.11) by employing the similar arguments used to yield the estimates stated in [23, Lemma 3], and thus their proofs are omit for brevity. One also refers to [3, Lemma 3.2].

To prove (3.12) and (3.13), we choose a small constant $\epsilon_1 > q_1 + q_2$, for any β and $b > 0$, then

$$|\langle v\rangle^b \partial_\beta(\frac{M-\mu}{\sqrt{\mu}})|_{2,w(\beta)}^2 \leq C\int_{\mathbb{R}^3} \mu^{-\epsilon_1}|\partial_\beta(\frac{M-\mu}{\sqrt{\mu}})|^2 dv.$$

For the constant $\eta_0 > 0$ in (1.20), there exists $R > 0$ large enough such that

$$\int_{|v|\geq R} \mu^{-\epsilon_1}|\partial_\beta(\frac{M-\mu}{\sqrt{\mu}})|^2 dv \leq C\eta_0^2,$$

and

$$\int_{|v|\leq R} \mu^{-\epsilon_1}|\partial_\beta(\frac{M-\mu}{\sqrt{\mu}})|^2 dv \leq C(|\rho - 1| + |u - 0| + |\theta - \frac{3}{2}|)^2 \leq C\eta_0^2.$$

It follows by the above estimates and (1.30) that

$$|\langle v\rangle^b \partial_\beta(\frac{M-\mu}{\sqrt{\mu}})|_{\nu,w(\beta)}^2 + |\langle v\rangle^b \partial_\beta(\frac{M-\mu}{\sqrt{\mu}})|_{2,w(\beta)}^2 \leq C\eta_0^2. \tag{3.14}$$

This together with (3.9) and (1.9) gives (3.12) and (3.13). We thus complete the proof of Lemma 3.5. □

Next, we prove some linear and nonlinear estimates, which are used in the later section. The first estimates involving the linear terms $\Gamma(\frac{M-\mu}{\sqrt{\mu}}, \mathbf{g})$ and $\Gamma(\mathbf{g}, \frac{M-\mu}{\sqrt{\mu}})$.

LEMMA 3.6. *Let* $|\alpha| + |\beta| \leq N$ *with* $N \geq 6$ *and* $w = w(\beta)$ *in* (1.29) *with both* $q_1 \in [0, 1)$ *and* $q_2 \in (0, 1)$ *small enough. Suppose that* (1.20) *holds and* $\mathcal{E}_{N,l,q}(t) < \varepsilon_0$

with $l \geq N$. If we choose $\eta_0 > 0$ in (1.20), $\delta > 0$ in Lemma 2.1 and $\varepsilon_0 > 0$ small enough, for any $\varepsilon > 0$, one has

$$|(\partial^\alpha_\beta \Gamma(\frac{M-\mu}{\sqrt{\mu}}, \mathbf{g}), w^2(\beta)\partial^\alpha_\beta h)| + |(\partial^\alpha_\beta \Gamma(\mathbf{g}, \frac{M-\mu}{\sqrt{\mu}}), w^2(\beta)\partial^\alpha_\beta h)| \\ \leq C\varepsilon \|\partial^\alpha_\beta h\|^2_{\nu,w(\beta)} + C_\varepsilon(\eta_0 + \delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t), \tag{3.15}$$

and

$$|(\partial^\alpha \Gamma(\frac{M-\mu}{\sqrt{\mu}}, \mathbf{g}), \partial^\alpha h)| + |(\partial^\alpha \Gamma(\mathbf{g}, \frac{M-\mu}{\sqrt{\mu}}), \partial^\alpha h)| \\ \leq C\varepsilon \|\partial^\alpha h\|^2_{\nu} + C_\varepsilon(\eta_0 + \delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t). \tag{3.16}$$

Proof. We only consider the first term on the left hand side of (3.15) since the second term of (3.15) can be treated in the same way. First note that

$$|(\partial^\alpha_\beta \Gamma(\frac{M-\mu}{\sqrt{\mu}}, \mathbf{g}), w^2(\beta)\partial^\alpha_\beta h)| \leq C \sum |(\Gamma^0[\partial^{\alpha_1}_{\beta_1}(\frac{M-\mu}{\sqrt{\mu}}), \partial^{\alpha_2}_{\beta_2}\mathbf{g}], w^2(\beta)\partial^\alpha_\beta h)|, \tag{3.17}$$

where the summation is over $|\beta_1| + |\beta_2| \leq |\beta|$ and $|\alpha_1| + |\alpha_2| = |\alpha|$. If $|\alpha_1| \neq 0$ and $|\alpha_1| + |\beta_1| \leq N/2$ in (3.17), we can deduce from (3.12) and Lemma 2.1 that

$$\begin{aligned} &|(\Gamma^0[\partial^{\alpha_1}_{\beta_1}(\frac{M-\mu}{\sqrt{\mu}}), \partial^{\alpha_2}_{\beta_2}\mathbf{g}], w^2(\beta)\partial^\alpha_\beta h)| \\ &\leq C\big(\|\partial^{\alpha_1}(\rho, u, \theta)\|_{L^\infty} + \cdots + \|(\rho_x, u_x, \theta_x)\|^{|\alpha_1|}_{L^\infty}\big)\|\partial^{\alpha_2}_{\beta_2}\mathbf{g}\|_{\nu,w(\beta_2)}\|\partial^\alpha_\beta h\|_{\nu,w(\beta)} \\ &\leq \varepsilon\|\partial^\alpha_\beta h\|^2_{\nu,w(\beta)} + C_\varepsilon(\delta + \sqrt{\mathcal{E}_{N,l,q}(t)})\|\partial^{\alpha_2}_{\beta_2}\mathbf{g}\|^2_{\nu,w(\beta_2)} \\ &\leq \varepsilon\|\partial^\alpha_\beta h\|^2_{\nu,w(\beta)} + C_\varepsilon(\delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t), \end{aligned}$$

where we have used the following one-dimensional imbedding inequality

$$\|g(x)\|_{L^\infty} \leq \sqrt{2}\|g(x)\|^{\frac{1}{2}}\|g_x(x)\|^{\frac{1}{2}}, \quad \text{for} \quad g(x) \in H^1(\mathbb{R}) \subset L^\infty(\mathbb{R}).$$

Similarly, if $|\alpha_1| \neq 0$ and $|\alpha_2| + |\beta_2| \leq N/2$ in (3.17), it holds that

$$|(\Gamma^0[\partial^{\alpha_1}_{\beta_1}(\frac{M-\mu}{\sqrt{\mu}}), \partial^{\alpha_2}_{\beta_2}\mathbf{g}], w^2(\beta)\partial^\alpha_\beta h)| \leq \varepsilon\|\partial^\alpha_\beta h\|^2_{\nu,w(\beta)} + C_\varepsilon(\delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).$$

On the other hand, if $|\alpha_1| = 0$ in (3.17), we use (3.13) and the smallness of η_0 to get

$$|(\Gamma^0[\partial^{\alpha_1}_{\beta_1}(\frac{M-\mu}{\sqrt{\mu}}), \partial^{\alpha_2}_{\beta_2}\mathbf{g}], w^2(\beta)\partial^\alpha_\beta h)| \leq \varepsilon\|\partial^\alpha_\beta h\|^2_{\nu,w(\beta)} + C_\varepsilon \eta_0 \mathcal{D}_{N,l,q}(t).$$

Plugging the above estimates into (3.17), we obtain

$$|(\partial^\alpha_\beta \Gamma(\frac{M-\mu}{\sqrt{\mu}}, \mathbf{g}), w^2(\beta)\partial^\alpha_\beta h)| \leq C\varepsilon\|\partial^\alpha_\beta h\|^2_{\nu,w(\beta)} + C_\varepsilon(\eta_0 + \delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t). \tag{3.18}$$

Similar arguments as (3.18) imply

$$|(\partial^\alpha_\beta \Gamma(\mathbf{g}, \frac{M-\mu}{\sqrt{\mu}}), w^2(\beta)\partial^\alpha_\beta h)| \leq C\varepsilon\|\partial^\alpha_\beta h\|^2_{\nu,w(\beta)} + C_\varepsilon(\eta_0 + \delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).$$

This completes the proof of (3.15). By Lemma 3.5 and the similar arguments as the above we can prove that (3.16) holds and we omit the details for brevity. □

The second estimates are concerned with the nonlinear term $\Gamma(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}})$.

LEMMA 3.7. *Let $|\alpha|+|\beta|\le N$ with $N\ge 6$ and under the conditions of Lemma 3.6, for any $\varepsilon>0$, one has*

$$
\begin{aligned}
&|(\partial^\alpha_\beta\Gamma(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}),w^2(\beta)\partial^\alpha_\beta h)|\\
&\le C\varepsilon\|\partial^\alpha_\beta h\|^2_{\nu,w(\beta)}+C_\varepsilon\delta^{\frac16}(1+t)^{-\frac76}+C_\varepsilon(\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t),
\end{aligned}\tag{3.19}
$$

and

$$
\begin{aligned}
&|(\partial^\alpha\Gamma(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}),\partial^\alpha h)|\\
&\le C\varepsilon\|\partial^\alpha h\|^2_{\nu}+C_\varepsilon\delta^{\frac16}(1+t)^{-\frac76}+C_\varepsilon(\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).
\end{aligned}\tag{3.20}
$$

Proof. Recall $G=\overline{G}+\sqrt{\mu}\mathbf{g}$, we can see that

$$
\Gamma(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}})=\Gamma(\frac{\overline{G}}{\sqrt{\mu}},\frac{\overline{G}}{\sqrt{\mu}})+\Gamma(\frac{\overline{G}}{\sqrt{\mu}},\mathbf{g})+\Gamma(\mathbf{g},\frac{\overline{G}}{\sqrt{\mu}})+\Gamma(\mathbf{g},\mathbf{g}).\tag{3.21}
$$

By using (3.8), we can obtain

$$
|(\partial^\alpha_\beta\Gamma(\frac{\overline{G}}{\sqrt{\mu}},\frac{\overline{G}}{\sqrt{\mu}}),w^2(\beta)\partial^\alpha_\beta h)|\le C\sum|(\Gamma^0[\partial^{\alpha_1}_{\beta_1}(\frac{\overline{G}}{\sqrt{\mu}}),\partial^{\alpha_2}_{\beta_2}(\frac{\overline{G}}{\sqrt{\mu}})],w^2(\beta)\partial^\alpha_\beta h)|,\tag{3.22}
$$

where the summation is over $|\beta_1|+|\beta_2|\le|\beta|$ and $|\alpha_1|+|\alpha_2|=|\alpha|$. In view of (1.22), (3.1) and (3.2), one can compute that

$$
\overline{G}(t,x,v)=\frac{\sqrt{R}\bar\theta_x}{\sqrt{\theta}}A_1(\frac{v-u}{\sqrt{R\theta}})+\bar u_{1x}B_{11}(\frac{v-u}{\sqrt{R\theta}}),
$$

which implies that for $\beta_1=(1,0,0)$,

$$
\partial_{\beta_1}\overline{G}=\frac{\sqrt{R}\bar\theta_x}{\sqrt{\theta}}\partial_{v_1}A_1(\frac{v-u}{\sqrt{R\theta}})(\frac{1}{\sqrt{R\theta}})+\bar u_{1x}\partial_{v_1}B_{11}(\frac{v-u}{\sqrt{R\theta}})\frac{1}{\sqrt{R\theta}},\tag{3.23}
$$

and

$$
\begin{aligned}
\partial_x\overline{G}=&\frac{\sqrt{R}\bar\theta_{xx}}{\sqrt{\theta}}A_1(\frac{v-u}{\sqrt{R\theta}})-\frac{\sqrt{R}\bar\theta_x\theta_x}{2\sqrt{\theta^3}}A_1(\frac{v-u}{\sqrt{R\theta}})\\
&-\frac{\sqrt{R}\bar\theta_x}{\sqrt{\theta}}\nabla_vA_1(\frac{v-u}{\sqrt{R\theta}})\cdot\frac{u_x}{\sqrt{R\theta}}-\frac{\sqrt{R}\bar\theta_x\theta_x}{\sqrt{\theta}}\nabla_vA_1(\frac{v-u}{\sqrt{R\theta}})\cdot\frac{v-u}{\sqrt{2R\theta^3}}\\
&+\bar u_{1xx}B_{11}(\frac{v-u}{\sqrt{R\theta}})-\frac{\bar u_{1x}u_x}{\sqrt{R\theta}}\cdot\nabla_vB_{11}(\frac{v-u}{\sqrt{R\theta}})-\frac{\bar u_{1x}\theta_x(v-u)}{2\sqrt{R\theta^3}}\cdot\nabla_vB_{11}(\frac{v-u}{\sqrt{R\theta}}).
\end{aligned}\tag{3.24}
$$

And $\partial_t\overline{G}$ has the similar expression as (3.24). By (3.4) and the similar expansion as the above, for any $|\bar\alpha|\ge1$ and $|\bar\beta|\ge0$, we obtain

$$
|\langle v\rangle^b\partial_{\bar\beta}(\frac{\overline{G}}{\sqrt{\mu}})|_{2,w(\bar\beta)}\le C|[\bar u_{1x},\bar\theta_x]|,\tag{3.25}
$$

and

$$|\langle v\rangle^b\partial_{\bar\beta}^{\bar\alpha}(\frac{\overline{G}}{\sqrt{\mu}})|_{2,w(\bar\beta)}\leq C\big\{|\partial^{\bar\alpha}[\bar u_{1x},\bar\theta_x]|+\cdots+|[\bar u_{1x},\bar\theta_x]||\partial^{\bar\alpha}[u,\theta]|\big\}. \tag{3.26}$$

Here we used the fact that $|\langle v\rangle^b w(\bar\beta)\mu^{-\frac{1}{2}}M^{1-\epsilon}|_2\leq C$ by (1.20) for any $b\geq 0$ and $\epsilon>0$ small enough with small positive constants q_1,q_2 in (1.29). Owing to (3.25) and (3.26), one thus get from (3.22), (3.10), the imbedding inequality and Lemma 2.1 as well as $\mathcal{E}_{N,l,q}(t)<\varepsilon_0$ that

$$\begin{aligned}&|(\partial_\beta^\alpha\Gamma(\frac{\overline{G}}{\sqrt{\mu}},\frac{\overline{G}}{\sqrt{\mu}}),w^2(\beta)\partial_\beta^\alpha h)|\\ &\leq C\varepsilon\|\partial_\beta^\alpha h\|_{\nu,w(\beta)}^2+C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C_\varepsilon(\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).\end{aligned} \tag{3.27}$$

For the second term on the right hand side of (3.21), we have from (3.8) and (3.9) that

$$\begin{aligned}&|(\partial_\beta^\alpha\Gamma(\frac{\overline{G}}{\sqrt{\mu}},\mathbf{g}),w^2(\beta)\partial_\beta^\alpha h)|\\ &\leq C\sum|(\Gamma^0(\partial_{\beta_1}^{\alpha_1}(\frac{\overline{G}}{\sqrt{\mu}}),\partial_{\beta_2}^{\alpha_2}\mathbf{g}),w^2(\beta)\partial_\beta^\alpha h)|\\ &\leq C\sum\sum_{k\leq 2}\int_{\mathbb{R}}\{|\nabla_v^k[\mu^\epsilon\partial_{\beta_1}^{\alpha_1}(\frac{\overline{G}}{\sqrt{\mu}})]|_2+|\partial_{\beta_1}^{\alpha_1}(\frac{\overline{G}}{\sqrt{\mu}})|_{2,w(\beta_1)}\}|\partial_{\beta_2}^{\alpha_2}\mathbf{g}|_{\nu,w(\beta_2)}|\partial_\beta^\alpha h|_{\nu,w(\beta)}dx,\end{aligned} \tag{3.28}$$

where the summation is over $|\beta_1|+|\beta_2|\leq|\beta|$ and $|\alpha_1|+|\alpha_2|=|\alpha|$. If $|\alpha_1|=0$ in (3.28), we use (3.25), the imbedding inequality and Lemma 2.1 to get

$$|(\Gamma^0(\partial_{\beta_1}^{\alpha_1}(\frac{\overline{G}}{\sqrt{\mu}}),\partial_{\beta_2}^{\alpha_2}\mathbf{g}),w^2(\beta)\partial_\beta^\alpha h)|\leq C\varepsilon\|\partial_\beta^\alpha h\|_{\nu,w(\beta)}^2+C_\varepsilon\delta^{\frac{1}{6}}\mathcal{D}_{N,l,q}(t).$$

If $|\alpha_1|>0$ and $|\alpha_1|+|\beta_1|\leq N/2$ in (3.28), we have from (3.26) and the imbedding inequality that

$$\begin{aligned}&|(\Gamma^0(\partial_{\beta_1}^{\alpha_1}(\frac{\overline{G}}{\sqrt{\mu}}),\partial_{\beta_2}^{\alpha_2}\mathbf{g}),w^2(\beta)\partial_\beta^\alpha h)|\\ &\leq C\sum_{|\alpha'|\leq 1}\sum_{k\leq 2}\big\{\|\nabla_v^k[\mu^\epsilon\partial_{\beta_1}^{\alpha_1+\alpha'}(\frac{\overline{G}}{\sqrt{\mu}})]\|+\|\partial_{\beta_1}^{\alpha_1+\alpha'}(\frac{\overline{G}}{\sqrt{\mu}})\|_{2,w(\beta_1)}\big\}\|\partial_{\beta_2}^{\alpha_2}\mathbf{g}\|_{\nu,w(\beta_2)}\|\partial_\beta^\alpha h\|_{\nu,w(\beta)}\\ &\leq C\varepsilon\|\partial_\beta^\alpha h\|_{\nu,w(\beta)}^2+C_\varepsilon(\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).\end{aligned}$$

Similarly, if $|\alpha_1|>0$ and $|\alpha_1|+|\beta_1|\geq N/2$ in (3.28), we also have

$$|(\Gamma^0(\partial_{\beta_1}^{\alpha_1}(\frac{\overline{G}}{\sqrt{\mu}}),\partial_{\beta_2}^{\alpha_2}\mathbf{g}),w^2(\beta)\partial_\beta^\alpha h)|\leq C\varepsilon\|\partial_\beta^\alpha h\|_{\nu,w(\beta)}^2+C_\varepsilon(\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).$$

Plugging the above estimates into (3.28), we obtain

$$|(\partial_\beta^\alpha\Gamma(\frac{\overline{G}}{\sqrt{\mu}},\mathbf{g}),w^2(\beta)\partial_\beta^\alpha h)|\leq C\varepsilon\|\partial_\beta^\alpha h\|_{\nu,w(\beta)}^2+C_\varepsilon(\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t). \tag{3.29}$$

By the similar arguments to (3.29), we can prove that the third term on the right hand side of (3.21) has the same estimate.

By using (3.8), (3.9), (3.10) and the imbedding inequality, we deduce that

$$|(\partial_\beta^\alpha \Gamma(\mathbf{g},\mathbf{g}), w^2(\beta)\partial_\beta^\alpha h)| \le \varepsilon \|\partial_\beta^\alpha h\|_{\nu,w(\beta)}^2 + C_\varepsilon \sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t). \tag{3.30}$$

The estimates from (3.27) to (3.30) yields that (3.19). We can deduce (3.20) by using (3.9), (3.10) and the similar arguments as the above estimates. This ends the proof of Lemma 3.7. □

Lastly, we give the basic energy estimates of the terms $\|\partial^\alpha \widetilde{\rho}\|^2$ and $\|\partial^\alpha(\widetilde{\rho}_t, \widetilde{u}_t, \widetilde{\theta}_t)\|^2$.

LEMMA 3.8. *Let $|\alpha| \le N-1$ and $\mathcal{E}_{N,l,q}(t) < \varepsilon_0$ with $\varepsilon_0 > 0$ small enough, for any $\epsilon > 0$ small enough, we have the following results that*

$$\begin{aligned}\|\partial^\alpha \widetilde{\rho}_x\|^2 \le & -C(\partial^\alpha \widetilde{u}_1, \partial^\alpha \widetilde{\rho}_x)_t + C\big\{\|\partial^\alpha[\widetilde{u}_{1x}, \widetilde{\theta}_x]\|^2 + \|\mu^\epsilon \partial^\alpha \mathbf{g}_x\|^2\big\} \\ & + C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t),\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}\|\partial^\alpha(\widetilde{\rho}_t, \widetilde{u}_t, \widetilde{\theta}_t)\|^2 \le & \, C\big\{\|\partial^\alpha(\widetilde{\rho}_x, \widetilde{u}_x, \widetilde{\theta}_x)\|^2 + \|\mu^\epsilon \partial^\alpha \mathbf{g}_x\|^2\big\} \\ & + C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).\end{aligned} \tag{3.32}$$

Proof. Subtracting (2.10) from system (1.15), we obtain

$$\begin{cases} \widetilde{\rho}_t + (\widetilde{\rho}\widetilde{u}_1)_x = -(\bar{\rho}\widetilde{u}_1 + \bar{u}_1\widetilde{\rho})_x, \\ \widetilde{u}_{1t} + \bar{u}_1\widetilde{u}_{1x} + \frac{2}{3}\widetilde{\theta}_x + \frac{2\bar{\theta}}{3\bar{\rho}}\widetilde{\rho}_x + \frac{\partial_x\phi}{\rho}\int_{\mathbb{R}^3} F_2 dv = -J_2 - \int_{\mathbb{R}^3} v_1^2 \frac{G_x}{\rho} dv, \\ \widetilde{u}_{it} + \widetilde{u}_1\widetilde{u}_{ix} + \bar{u}_1\widetilde{u}_{ix} = -\frac{1}{\rho}\int_{\mathbb{R}^3} v_1 v_i G_x dv, \quad i = 2,3, \\ \widetilde{\theta}_t + \frac{2}{3}\bar{\theta}\widetilde{u}_{1x} + \bar{u}_1\widetilde{\theta}_x + \frac{\partial_x\phi}{\rho}(\int_{\mathbb{R}^3} v_1 F_2 dv - u_1\int_{\mathbb{R}^3} F_2 dv) \\ \quad = -J_3 - \frac{1}{\rho}\int_{\mathbb{R}^3} \frac{1}{2} v_1 v\cdot(v-2u) G_x dv, \end{cases} \tag{3.33}$$

where

$$J_2 = \widetilde{u}_1\bar{u}_{1x} + \widetilde{u}_1\widetilde{u}_{1x} + \frac{2}{3}\rho_x\frac{\bar{\rho}\widetilde{\theta} - \widetilde{\rho}\bar{\theta}}{\rho\bar{\rho}}, \quad J_3 = \frac{2}{3}(\widetilde{\theta}\bar{u}_{1x} + \widetilde{\theta}\widetilde{u}_{1x}) + (\widetilde{\theta}_x\widetilde{u}_1 + \bar{\theta}_x\widetilde{u}_1). \tag{3.34}$$

By taking the derivative ∂^α of $(3.33)_2$ with $|\alpha| \le N-1$ and taking the inner product of the resulting equation with $\partial^\alpha \widetilde{\rho}_x$, we get

$$\begin{aligned} & (\frac{2\bar{\theta}}{3\bar{\rho}}\partial^\alpha \widetilde{\rho}_x, \partial^\alpha \widetilde{\rho}_x) \\ = & -\sum_{|\alpha'|<|\alpha|} C_\alpha^{\alpha'}(\partial^{\alpha-\alpha'}(\frac{2\bar{\theta}}{3\bar{\rho}})\partial^{\alpha'}\widetilde{\rho}_x, \partial^\alpha \widetilde{\rho}_x) - (\partial^\alpha \widetilde{u}_{1t}, \partial^\alpha \widetilde{\rho}_x) - (\partial^\alpha(\bar{u}_1\widetilde{u}_{1x} + \frac{2}{3}\widetilde{\theta}_x), \partial^\alpha \widetilde{\rho}_x) \\ & -(\partial^\alpha(\frac{\partial_x\phi}{\rho}\int_{\mathbb{R}^3} F_2 dv), \partial^\alpha \widetilde{\rho}_x) - (\partial^\alpha J_2, \partial^\alpha \widetilde{\rho}_x) - (\int_{\mathbb{R}^3} v_1^2 \partial^\alpha(\frac{G_x}{\rho}) dv, \partial^\alpha \widetilde{\rho}_x). \end{aligned} \tag{3.35}$$

In light of the imbedding inequality and Lemma 2.1 as well as $\mathcal{E}_{N,l,q}(t) < \varepsilon_0$, we get

$$\sum_{|\alpha'|<|\alpha|} C_\alpha^{\alpha'}|(\partial^{\alpha-\alpha'}(\frac{2\bar{\theta}}{3\bar{\rho}})\partial^{\alpha'}\widetilde{\rho}_x, \partial^\alpha \widetilde{\rho}_x)| \le C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).$$

By the integration by parts, $(3.33)_1$ and the Moser-type calculus inequalities, one has

$$
\begin{aligned}
&-(\partial^\alpha\widetilde{u}_{1t},\partial^\alpha\widetilde{\rho}_x)=-(\partial^\alpha\widetilde{u}_1,\partial^\alpha\widetilde{\rho}_x)_t-(\partial^\alpha\widetilde{u}_{1x},\partial^\alpha\widetilde{\rho}_t)\\
=&-(\partial^\alpha\widetilde{u}_1,\partial^\alpha\widetilde{\rho}_x)_t-(\partial^\alpha\widetilde{u}_{1x},\partial^\alpha\{-\bar{\rho}\widetilde{u}_{1x}-\bar{\rho}_x\widetilde{u}_1-(\widetilde{\rho}\widetilde{u}_1)_x-\bar{u}_1\widetilde{\rho}_x-\bar{u}_{1x}\widetilde{\rho}\})\\
\leq&-(\partial^\alpha\widetilde{u}_1,\partial^\alpha\widetilde{\rho}_x)_t+\varepsilon\|\partial^\alpha\widetilde{\rho}_x\|^2+C_\varepsilon\|\partial^\alpha\widetilde{u}_{1x}\|^2+C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).
\end{aligned}
$$

The third term on the right hand side of (3.35) is dominated by

$$
\begin{aligned}
&|(\partial^\alpha(\bar{u}_1\widetilde{u}_{1x}+\frac{2}{3}\widetilde{\theta}_x),\partial^\alpha\widetilde{\rho}_x)|\\
&\leq\varepsilon\|\partial^\alpha\widetilde{\rho}_x\|^2+C_\varepsilon\|\partial^\alpha[\widetilde{u}_{1x},\widetilde{\theta}_x]\|^2+C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C_\varepsilon\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).
\end{aligned}
$$

Recalling $F_2=\sqrt{\mu}\mathbf{f}$ and using the Moser-type calculus inequalities, we arrive at

$$
|(\partial^\alpha(\frac{\partial_x\phi}{\rho}\int_{\mathbb{R}^3}F_2dv),\partial^\alpha\widetilde{\rho}_x)|\leq C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).
$$

The term containing J_2, we have from (3.34) and the elementary inequalities that

$$
|(\partial^\alpha J_2,\partial^\alpha\widetilde{\rho}_x)|\leq C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).
$$

Recalling $G=\overline{G}+\sqrt{\mu}\mathbf{g}$, we have from (3.26) and the Moser-type calculus inequalities that

$$
\begin{aligned}
&|(\int_{\mathbb{R}^3}v_1^2\partial^\alpha(\frac{G_x}{\rho})dv,\partial^\alpha\widetilde{\rho}_x)|\\
&\leq\varepsilon\|\partial^\alpha\widetilde{\rho}_x\|^2+C_\varepsilon\|\mu^\epsilon\partial^\alpha\mathbf{g}_x\|^2+C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C_\varepsilon\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).
\end{aligned}
$$

By combining the above related estimates and choosing $\varepsilon>0$ small enough, we can obtain the estimate (3.31). By using the system (3.33), we readily prove (3.32). This completes the proof of Lemma 3.8. □

4. Non-weighted energy estimates. This section is devoted to deducing the non-weighted energy estimates for the solutions of (1.6)-(1.7). We first derive the lower order energy estimates in the subsection 4.1. Then we establish the high order energy estimates in the subsection 4.2. Lastly, we give the main non-weighted energy estimates.

4.1. Lower order energy estimates. The lower order energy estimates can be stated as follows:

LEMMA 4.1. *Let $(F_1,F_2,\partial_x\phi)$ be a solution to the system* (1.6) *and* (1.7), *and suppose $\mathcal{E}_{N,l,q}(t)<\varepsilon_0$ for some $\varepsilon_0>0$ small enough. Then there exists $\widetilde{C}_1\gg1$ such that*

$$
\begin{aligned}
&\frac{d}{dt}\Big\{\|\mathbf{g}\|^2+\widetilde{C}_1\{\int_{\mathbb{R}}\eta dx-(L_M^{-1}P_1[v_1(\bar{\theta}\ln M)_x-\frac{3}{2}\bar{u}_{1x}v_1^2]M,\frac{\sqrt{\mu}\mathbf{g}}{M})+\lambda\int_{\mathbb{R}}\widetilde{u}_1\widetilde{\rho}_xdx\}\Big\}\\
&+\lambda\big\{\|\sqrt{\bar{u}_{1x}}(\widetilde{\rho},\widetilde{u}_1,\widetilde{\theta})\|^2+\|(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{\theta}_x)\|^2+\|\mathbf{g}\|_\nu^2\big\}\\
\leq&\, C\big\{\|(\widetilde{\rho}_{xx},\widetilde{u}_{xx},\widetilde{\theta}_{xx})\|^2+\|\mathbf{g}_x\|_\nu^2+\|\mathbf{g}_{xx}\|_\nu^2\big\}+C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}\\
&+C(\eta_0+\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t)+C\sum_{|\alpha|\leq1}\|\partial_x\partial^\alpha\phi\|\mathcal{F}_{N,l,q}(t). \qquad (4.1)
\end{aligned}
$$

Here the function η *is defined by* (4.7) *and* $\mathcal{F}_{N,l,q}(t)$ *is given by*

$$\mathcal{F}_{N,l,q}(t) = \sum_{|\alpha|+|\beta|\leq N} \|\langle v\rangle\partial^\alpha_\beta \mathbf{g}(t)\|^2_{2,w(\beta)} + \sum_{|\alpha|+|\beta|\leq N} \|\langle v\rangle\partial^\alpha_\beta \mathbf{f}(t)\|^2_{2,w(\beta)}. \tag{4.2}$$

Proof. In the following, we shall assume that $\mathcal{E}_{N,l,q}(t) < \varepsilon_0$ for some $\varepsilon_0 > 0$ small enough and we have from this and (1.20) that (ρ, u, θ) and $(\bar\rho, \bar u, \bar\theta)$ are close enough to the state $(1, 0, \frac{3}{2})$. As in [17, 19], the following macroscopic entropy S will be estimated for the lower order energy estimates. Set

$$-\frac{3}{2}\rho S = \int_{\mathbb{R}^3} M \ln M\, dv.$$

Multiplying $(1.6)_1$ by $\ln M$ and then integrating over v, we have by a direct calculation that

$$\begin{aligned}(-\frac{3}{2}\rho S)_t + (-\frac{3}{2}\rho u_1 S)_x + (\int_{\mathbb{R}^3} v_1 G \ln M dv)_x \\ - \int_{\mathbb{R}^3} v_1 G(\ln M)_x dv - \int_{\mathbb{R}^3} \partial_x\phi\partial_{v_1} F_2 \ln M dv = 0,\end{aligned} \tag{4.3}$$

where $S = -\frac{2}{3}\ln\rho + \ln(\frac{4\pi}{3}\theta) + 1$ as in (2.3). We denote

$$\begin{cases} m = (m_0, m_1, m_2, m_3, m_4)^t = (\rho, \rho u_1, \rho u_2, \rho u_3, \rho(\theta + \frac{|u|^2}{2}))^t, \\ n = (n_0, n_1, n_2, n_3, n_4)^t = (\rho u_1, \rho u_1^2 + p, \rho u_1 u_2, \rho u_1 u_3, \rho u_1(\theta + \frac{|u|^2}{2}) + p u_1)^t. \end{cases}$$

It follows that

$$(\rho S)_{m_0} = S + \frac{|u|^2}{2\theta} - \frac{5}{3}, \quad (\rho S)_{m_i} = -\frac{u_i}{\theta}, i = 1, 2, 3, \quad (\rho S)_{m_4} = \frac{1}{\theta}. \tag{4.4}$$

Rewrite the conservation laws (1.15) by

$$m_t + n_x = \begin{pmatrix} 0 \\ -\int_{\mathbb{R}^3} v_1^2 G_x dv - \partial_x\phi\int_{\mathbb{R}^3} F_2 dv \\ -\int_{\mathbb{R}^3} v_1 v_2 G_x dv \\ -\int_{\mathbb{R}^3} v_1 v_3 G_x dv \\ -\frac{1}{2}\int_{\mathbb{R}^3} v_1|v|^2 G_x dv - \partial_x\phi\int_{\mathbb{R}^3} v_1 F_2 dv \end{pmatrix}. \tag{4.5}$$

We define an entropy-entropy flux pair (η, q) around a Maxwellian $\overline{M} = M_{[\bar\rho,\bar u,\bar S]}$ $(\bar u_i = 0, i = 2, 3)$ as

$$\begin{cases} \eta = \bar\theta\{-\frac{3}{2}\rho S + \frac{3}{2}\bar\rho\bar S + \frac{3}{2}\nabla_m(\rho S)|_{m=\bar m}\cdot(m - \bar m)\}, \\ q = \bar\theta\{-\frac{3}{2}\rho u_1 S + \frac{3}{2}\bar\rho\bar u_1\bar S + \frac{3}{2}\nabla_m(\rho S)|_{m=\bar m}\cdot(n - \bar n)\}, \end{cases} \tag{4.6}$$

which further implies that

$$\begin{cases} \eta = \rho\bar\theta\Phi(\frac{\bar\rho}{\rho}) + \frac{3}{2}\rho\bar\theta\Phi(\frac{\theta}{\bar\theta}) + \frac{3}{4}\rho|u - \bar u|^2, \\ q = u_1\eta + (u_1 - \bar u_1)(\rho\theta - \bar\rho\bar\theta), \end{cases} \tag{4.7}$$

where the convex function $\Phi(s) = s - \ln s - 1$. There exists a constant $c_1 > 1$ such that

$$c_1^{-1}|(\widetilde\rho, \widetilde u, \widetilde\theta)|^2 \leq \eta \leq c_1|(\widetilde\rho, \widetilde u, \widetilde\theta)|^2. \tag{4.8}$$

By the definition of (4.6) and a direct but tedious computation, we arrive at

$$
\begin{aligned}
&\eta_t+q_x-\nabla_{[\bar\rho,\bar u,\bar S]}\eta\cdot(\bar\rho,\bar u,\bar S)_t-\nabla_{[\bar\rho,\bar u,\bar S]}q\cdot(\bar\rho,\bar u,\bar S)_x\\
&=\bar\theta\{(-\frac{3}{2}\rho S)_t+(-\frac{3}{2}\rho u_1S)_x\}+\frac{3}{2}\bar\theta\nabla_m(\rho S)|_{m=\bar m}(m_t+n_x)\\
&=\Big(\int_{\mathbb{R}^3}(-v_1\bar\theta\ln M+\frac{3}{2}\bar u_1v_1^2-\frac{3}{4}v_1|v|^2)Gdv\Big)_x+\int_{\mathbb{R}^3}(v_1(\bar\theta\ln M)_x-\frac{3}{2}\bar u_{1x}v_1^2)Gdv\\
&\quad+\bar\theta\partial_x\phi\int_{\mathbb{R}^3}\partial_{v_1}F_2\ln Mdv+\frac{3}{2}\bar u_1\partial_x\phi\int_{\mathbb{R}^3}F_2dv-\frac{3}{2}\partial_x\phi\int_{\mathbb{R}^3}v_1F_2dv. \qquad (4.9)
\end{aligned}
$$

It follows from the similar arguments as [20, 32] that there exists $c_2>0$ such that

$$
\begin{aligned}
&-\{\nabla_{[\bar\rho,\bar u,\bar S]}\eta\cdot(\bar\rho,\bar u,\bar S)_t+\nabla_{[\bar\rho,\bar u,\bar S]}q\cdot(\bar\rho,\bar u,\bar S)_x\}\\
&=\frac{3}{2}\rho\bar u_{1x}(u_1-\bar u_1)^2+\frac{2}{3}\rho\bar\theta\bar u_{1x}\Phi(\frac{\bar\rho}{\rho})+\rho\bar\theta\bar u_{1x}\Phi(\frac{\theta}{\bar\theta})+\frac{3}{2}\rho\bar\theta_x(u_1-\bar u_1)(\frac{2}{3}\ln\frac{\bar\rho}{\rho}+\ln\frac{\theta}{\bar\theta})\\
&\geq c_2\bar u_{1x}(\widetilde\rho^2+\widetilde u_1^2+\widetilde\theta^2). \qquad (4.10)
\end{aligned}
$$

It holds that

$$
\begin{aligned}
&\int_{\mathbb{R}}\Big\{\bar\theta\partial_x\phi\int_{\mathbb{R}^3}\partial_{v_1}F_2\ln Mdv+\frac{3}{2}\bar u_1\partial_x\phi\int_{\mathbb{R}^3}F_2dv-\frac{3}{2}\partial_x\phi\int_{\mathbb{R}^3}v_1F_2dv\Big\}dx\\
&=\int_{\mathbb{R}}\int_{\mathbb{R}^3}\Big\{-\frac{3}{2}\frac{\widetilde\theta}{\theta}\partial_x\phi v_1F_2-\frac{3}{2}\frac{\bar\theta}{\theta}\widetilde u_1\partial_x\phi F_2+\frac{3}{2}\frac{\widetilde\theta}{\theta}\bar u_1\partial_x\phi F_2dv\Big\}dvdx.
\end{aligned}
$$

By using this, (4.9) and (4.10), we arrive at

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}}\eta dx+c_2\|\sqrt{\bar u_{1x}}(\widetilde\rho,\widetilde u_1,\widetilde\theta)\|^2\leq&\int_{\mathbb{R}}\int_{\mathbb{R}^3}(v_1(\bar\theta\ln M)_x-\frac{3}{2}\bar u_{1x}v_1^2)G\,dvdx\\
&+\int_{\mathbb{R}}\int_{\mathbb{R}^3}\Big\{-\frac{3}{2}\frac{\widetilde\theta}{\theta}\partial_x\phi v_1F_2-\frac{3}{2}\frac{\bar\theta}{\theta}\widetilde u_1\partial_x\phi F_2+\frac{3}{2}\frac{\widetilde\theta}{\theta}\bar u_1\partial_x\phi F_2dv\Big\}dvdx. \qquad (4.11)
\end{aligned}
$$

We first estimate the term involving $\partial_x\phi$ in (4.11). Recall $F_2=\sqrt{\mu}\mathbf{f}$, we have that the second line of (4.11) is dominated by

$$
C\|\partial_x\phi\|_{L^\infty}(\|\widetilde u_1\|+\|\widetilde\theta\|)\|\mathbf{f}\|_\nu\leq C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t). \qquad (4.12)
$$

Now we estimate the first term on the right hand side of (4.11). By using (1.17) and the self-adjoint property of L_M^{-1}, one has

$$
\begin{aligned}
&\Big([v_1(\bar\theta\ln M)_x-\frac{3}{2}\bar u_{1x}v_1^2]M,\frac{G}{M}\Big)\\
&=\Big(L_M^{-1}P_1[v_1(\bar\theta\ln M)_x-\frac{3}{2}\bar u_{1x}v_1^2]M,\frac{P_1(v_1M_x)+\Theta}{M}\Big). \qquad (4.13)
\end{aligned}
$$

By using (3.1), (1.10) and (1.11), we have

$$
P_1(v_1M_x)=\frac{\sqrt{R}\theta_x}{\sqrt{\theta}}\hat A_1(\frac{v-u}{\sqrt{R\theta}})M+\sum_{j=1}^3\frac{\partial u_j}{\partial x}\hat B_{1j}(\frac{v-u}{\sqrt{R\theta}})M. \qquad (4.14)
$$

By this, (3.1) and a direct but tedious calculation, we arrive at

$$\begin{aligned}
&P_1 v_1(\bar\theta \ln M)_x M - \frac{3}{2} P_1 \bar u_{1x} v_1^2 M \\
&= \sqrt{R\theta}\widetilde\theta_x \hat A_1(\frac{v-u}{\sqrt{R\theta}})M + \theta\sum_{j=1}^3 \frac{\partial \widetilde u_j}{\partial x}\hat B_{1j}(\frac{v-u}{\sqrt{R\theta}})M \\
&\quad -\frac{\sqrt{R}\widetilde\theta\theta_x}{\sqrt\theta}\hat A_1(\frac{v-u}{\sqrt{R\theta}})M - \widetilde\theta\sum_{j=1}^3 \frac{\partial u_j}{\partial x}\hat B_{1j}(\frac{v-u}{\sqrt{R\theta}})M.
\end{aligned} \tag{4.15}$$

By (3.2) and (4.15), one has

$$\begin{aligned}
&L_M^{-1} P_1[v_1(\bar\theta \ln M)_x - \frac{3}{2}\bar u_{1x} v_1^2]M \\
&= \sqrt{R\theta}\widetilde\theta_x A_1(\frac{v-u}{\sqrt{R\theta}}) + \theta\sum_{j=1}^3 \frac{\partial \widetilde u_j}{\partial x} B_{1j}(\frac{v-u}{\sqrt{R\theta}}) \\
&\quad -\frac{\sqrt{R}\widetilde\theta\theta_x}{\sqrt\theta} A_1(\frac{v-u}{\sqrt{R\theta}}) - \widetilde\theta\sum_{j=1}^3 \frac{\partial u_j}{\partial x} B_{1j}(\frac{v-u}{\sqrt{R\theta}}).
\end{aligned} \tag{4.16}$$

On the other hand, we have from (3.3) and Lemma 3.1 that

$$\langle A_1, \hat A_1\rangle = -\frac{\kappa(\theta)}{R^2\theta}, \quad \langle A_1, \hat B_{jk}\rangle = 0, \quad \langle B_{jk}, \hat A_1\rangle = 0, \quad \text{for any } j,k=1,2,3,$$

and

$$\begin{aligned}
&\langle B_{11}, \hat B_{11}\rangle = -\frac{4}{3}\frac{\mu(\theta)}{R\theta}, \quad \langle B_{12}, \hat B_{12}\rangle = \langle B_{13}, \hat B_{13}\rangle = -\frac{\mu(\theta)}{R\theta}, \\
&\langle B_{1i}, \hat B_{1j}\rangle = 0, \quad \text{for any } i \neq j.
\end{aligned}$$

By using (4.14), (4.16), (3.3) and these relations, we have

$$\begin{aligned}
&\langle L_M^{-1} P_1[v_1(\bar\theta \ln M)_x - \frac{3}{2}\bar u_{1x} v_1^2]M, \frac{P_1(v_1 M_x)}{M}\rangle \\
&= -\frac{\kappa(\theta)}{R\theta}\theta_x\widetilde\theta_x + \frac{\kappa(\theta)}{R\theta^2}\widetilde\theta\theta_x^2 - \frac{4}{3}\frac{\mu(\theta)}{R}u_{1x}\widetilde u_{1x} \\
&\quad -\frac{\mu(\theta)}{R}(u_{2x}\widetilde u_{2x} + u_{3x}\widetilde u_{3x}) + \frac{4}{3}\frac{\mu(\theta)}{R\theta}\widetilde\theta u_{1x}^2 + \frac{\mu(\theta)}{R\theta}\widetilde\theta(u_{2x}\widetilde u_{2x} + u_{3x}\widetilde u_{3x}).
\end{aligned} \tag{4.17}$$

Since both $\mu(\theta)$ and $\kappa(\theta)$ are smooth functions of θ, there exists $c_3 > 1$ such that $\mu(\theta), \kappa(\theta) \in [c_3^{-1}, c_3]$. We use this, the imbedding inequality and the integration by parts to obtain

$$\begin{aligned}
&\int_{\mathbb R}(-\frac{\kappa(\theta)}{R\theta}\theta_x\widetilde\theta_x + \frac{\kappa(\theta)}{R\theta^2}\widetilde\theta\theta_x^2)dx \\
&= \int_{\mathbb R}\Big(-\frac{\bar\theta\kappa(\theta)}{R\theta^2}\widetilde\theta_x^2 + (\frac{\bar\theta\kappa(\theta)}{R\theta^2}\bar\theta_x)_x\widetilde\theta + \frac{\kappa(\theta)}{R\theta^2}(\bar\theta_x + \widetilde\theta_x)\bar\theta_x\widetilde\theta\Big)dx \\
&\le -\lambda\int_{\mathbb R}|\widetilde\theta_x|^2 dx + C\int_{\mathbb R}|\widetilde\theta|(|\bar\theta_{xx}| + |\bar\theta_x||\theta_x| + |\bar\theta_x||\widetilde\theta_x| + |\bar\theta_x|^2)dx \\
&\le -\lambda\|\widetilde\theta_x\|^2 + C\|\widetilde\theta\|_{L^\infty}(\|\bar\theta_{xx}\|_{L^1} + \|\bar\theta_x\|^2 + \|\widetilde\theta_x\|^2) \\
&\le -\lambda\|\widetilde\theta_x\|^2 + C\delta^{\frac16}(1+t)^{-\frac76} + C\sqrt{\varepsilon_0}\mathcal D_{N,l,q}(t).
\end{aligned}$$

And the other terms of (4.17) can be treated in the similar way as the above. It follows that

$$\begin{aligned}&(L_M^{-1}P_1[v_1(\bar\theta\ln M)_x-\frac{3}{2}\bar u_{1x}v_1^2]M,\frac{P_1(v_1M_x)}{M})\\ \leq&-\lambda(\|\widetilde u_x\|^2+\|\widetilde\theta_x\|^2)+C\delta^{\frac16}(1+t)^{-\frac76}+C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).\end{aligned}\tag{4.18}$$

We still deal with the term involving Θ in (4.13). Recalling that

$$\Theta=G_t+P_1(v_1G_x)-P_1(\partial_x\phi\partial_{v_1}F_2)-2Q(G,G).\tag{4.19}$$

We first consider the first term in (4.19). It follows from the fast decay of the Burnett functions (3.4) and the similar arguments as (3.24) that

$$\int_{\mathbb R}\int_{\mathbb R^3}|\frac{\partial_t\overline G}{\sqrt\mu}|^2dvdx\leq C\|[\bar u_{1xt},\bar\theta_{xt}]\|^2+C\|[\bar u_{1x},\bar\theta_x]\cdot[u_t,\theta_t]\|^2.\tag{4.20}$$

For any multi-index β and any $b\geq0$, we have from (4.16), Lemma 3.2 and Lemma 2.1 that

$$\begin{aligned}&\int_{\mathbb R}\int_{\mathbb R^3}\frac{|\langle v\rangle^b\sqrt\mu\partial_\beta L_M^{-1}P_1[v_1(\bar\theta\ln M)_x-\frac32\bar u_{1x}v_1^2]M|^2}{M^2}dvdx\\ \leq&\ C\{\|[\widetilde u_x,\widetilde\theta_x]\|^2+\|\widetilde\theta\cdot[u_x,\theta_x]\|^2\}\\ \leq&\ C\|[\widetilde u_x,\widetilde\theta_x]\|^2+C\delta^{\frac16}(1+t)^{-\frac76}+C(\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).\end{aligned}\tag{4.21}$$

Here we used the fact that, for any $0<\delta_2<\delta_1$ and $b>0$, $|\langle v\rangle^b\mu^{\delta_1}M^{-\delta_2}|_2<C$ by (1.20).

It follows from (4.20), (4.21), Lemma 2.1 and Lemma 3.8 that

$$\begin{aligned}&\Big(L_M^{-1}P_1[v_1(\bar\theta\ln M)_x-\frac32\bar u_{1x}v_1^2]M,\frac{\partial_t\overline G}{M}\Big)\\ =&\Big(\frac{\sqrt\mu L_M^{-1}P_1[v_1(\bar\theta\ln M)_x-\frac32\bar u_{1x}v_1^2]M}{M},\frac{\partial_t\overline G}{\sqrt\mu}\Big)\\ \leq&\ C\varepsilon\|(\widetilde\rho_x,\widetilde u_x,\widetilde\theta_x)\|^2+C_\varepsilon\delta^{\frac16}(1+t)^{-\frac76}+C_\varepsilon(\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).\end{aligned}\tag{4.22}$$

By using the integration by parts about t, we get

$$\begin{aligned}&\Big(L_M^{-1}P_1[v_1(\bar\theta\ln M)_x-\frac32\bar u_{1x}v_1^2]M,\frac{\sqrt\mu\partial_t\mathbf g}{M}\Big)\\ =&\frac{d}{dt}\Big(L_M^{-1}P_1[v_1(\bar\theta\ln M)_x-\frac32\bar u_{1x}v_1^2]M,\frac{\sqrt\mu\mathbf g}{M}\Big)\\ &-\Big(\partial_t[\frac{L_M^{-1}P_1[v_1(\bar\theta\ln M)_x-\frac32\bar u_{1x}v_1^2]M}{M}],\sqrt\mu\mathbf g\Big).\end{aligned}\tag{4.23}$$

Notice that the last line of (4.23) is dominated by

$$\begin{aligned}&\Big\|\mu^{\frac12-\epsilon}\partial_t[\frac{L_M^{-1}P_1[v_1(\bar\theta\ln M)_x-\frac32\bar u_{1x}v_1^2]M}{M}]\Big\|\|\mu^\epsilon\mathbf g\|\\ \leq&\ \varepsilon\|\mathbf g\|_\nu^2+C_\varepsilon(\|(\widetilde\rho_{xx},\widetilde u_{xx},\widetilde\theta_{xx})\|^2+\|\mathbf g_{xx}\|_\nu^2)\\ &+C_\varepsilon\delta^{\frac16}(1+t)^{-\frac76}+C_\varepsilon(\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).\end{aligned}\tag{4.24}$$

Recall $G=\overline{G}+\sqrt{\mu}\mathbf{g}$, we thus have by (4.22), (4.23) and (4.24) that

$$\begin{aligned}&\Big(L_M^{-1}P_1[v_1(\bar{\theta}\ln M)_x-\frac{3}{2}\bar{u}_{1x}v_1^2]M,\frac{\partial_t G}{M}\Big)\\ \leq&\ \frac{d}{dt}\Big(L_M^{-1}P_1[v_1(\bar{\theta}\ln M)_x-\frac{3}{2}\bar{u}_{1x}v_1^2]M,\frac{\sqrt{\mu}\mathbf{g}}{M}\Big)+C\varepsilon(\|(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{\theta}_x)\|^2+\|\mathbf{g}\|_\nu^2)\\ &+C_\varepsilon(\|(\widetilde{\rho}_{xx},\widetilde{u}_{xx},\widetilde{\theta}_{xx})\|^2+\|\mathbf{g}_{xx}\|_\nu^2)+C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C_\varepsilon(\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).\end{aligned}\tag{4.25}$$

For the second term in (4.19), it holds that

$$\begin{aligned}&|(L_M^{-1}P_1[v_1(\bar{\theta}\ln M)_x-\frac{3}{2}\bar{u}_{1x}v_1^2]M,\frac{P_1v_1G_x}{M})|\\ \leq&\ C\varepsilon\|[\widetilde{u}_x,\widetilde{\theta}_x]\|^2+C_\varepsilon\|\mathbf{g}_x\|_\nu^2+C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C_\varepsilon(\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).\end{aligned}\tag{4.26}$$

For the third term in (4.19), recalling that $F_2=\sqrt{\mu}\mathbf{f}$, we have from the integration by parts that

$$\begin{aligned}&|(L_M^{-1}P_1[v_1(\bar{\theta}\ln M)_x-\frac{3}{2}\bar{u}_{1x}v_1^2]M,\frac{P_1\partial_x\phi\partial_{v_1}F_2}{M})|\\ =&\ |(\mu^{\frac{1}{2}-\epsilon}\partial_{v_1}(\frac{L_M^{-1}P_1[v_1(\bar{\theta}\ln M)_x-\frac{3}{2}\bar{u}_{1x}v_1^2]M}{M}),\partial_x\phi\mu^\epsilon\mathbf{f})|.\end{aligned}\tag{4.27}$$

This can be controlled by $C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t)$ by using (3.23), (4.16), Lemma 3.2 and Lemma 2.1.

For the last term in (4.19), in view of (1.24), (4.21) and (3.20), we get

$$\begin{aligned}&|(L_M^{-1}P_1[v_1(\bar{\theta}\ln M)_x-\frac{3}{2}\bar{u}_{1x}v_1^2]M,\frac{Q(G,G)}{M})|\\ =&\ |(\frac{\sqrt{\mu}L_M^{-1}P_1[v_1(\bar{\theta}\ln M)_x-\frac{3}{2}\bar{u}_{1x}v_1^2]M}{M},\Gamma(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}))|\\ \leq&\ C\varepsilon\|[\widetilde{u}_x,\widetilde{\theta}_x]\|^2+C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C_\varepsilon(\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).\end{aligned}\tag{4.28}$$

Hence, combining the estimates (4.25), (4.26), (4.27) and (4.28), we have from (4.19) that

$$\begin{aligned}&\Big(L_M^{-1}P_1[v_1(\bar{\theta}\ln M)_x-\frac{3}{2}\bar{u}_{1x}v_1^2]M,\frac{\Theta}{M}\Big)\\ \leq&\ \frac{d}{dt}\Big(L_M^{-1}P_1[v_1(\bar{\theta}\ln M)_x-\frac{3}{2}\bar{u}_{1x}v_1^2]M,\frac{\sqrt{\mu}\mathbf{g}}{M}\Big)\\ &+C_\varepsilon(\|(\widetilde{\rho}_{xx},\widetilde{u}_{xx},\widetilde{\theta}_{xx})\|^2+\|\mathbf{g}_x\|_\nu^2+\|\mathbf{g}_{xx}\|_\nu^2)\\ &+C\varepsilon(\|(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{\theta}_x)\|^2+\|\mathbf{g}\|_\nu^2)+C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C_\varepsilon(\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).\end{aligned}\tag{4.29}$$

Substituting (4.29) and (4.18) into (4.13), for any $\varepsilon>0$ small enough, we obtain

$$\begin{aligned}&\Big([v_1(\bar{\theta}\ln M)_x-\frac{3}{2}\bar{u}_{1x}v_1^2]M,\frac{G}{M}\Big)+\lambda(\|\widetilde{u}_x\|^2+\|\widetilde{\theta}_x\|^2)\\ \leq&\ \frac{d}{dt}\Big(L_M^{-1}P_1[v_1(\bar{\theta}\ln M)_x-\frac{3}{2}\bar{u}_{1x}v_1^2]M,\frac{\sqrt{\mu}\mathbf{g}}{M}\Big)\\ &+C_\varepsilon(\|(\widetilde{\rho}_{xx},\widetilde{u}_{xx},\widetilde{\theta}_{xx})\|^2+\|\mathbf{g}_x\|_\nu^2+\|\mathbf{g}_{xx}\|_\nu^2)\\ &+C\varepsilon(\|(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{\theta}_x)\|^2+\|\mathbf{g}\|_\nu^2)+C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C_\varepsilon(\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).\end{aligned}\tag{4.30}$$

As a consequence, substituting (4.12) and (4.30) into (4.11) and taking suitably small $\varepsilon > 0$, we have from a suitable linear combination of the resulting equation and Lemma 3.8 that

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_{\mathbb{R}}\eta dx - (L_M^{-1}P_1[v_1(\bar\theta\ln M)_x - \frac{3}{2}\bar u_{1x}v_1^2]M, \frac{\sqrt{\mu}\mathbf{g}}{M}) + \lambda\int_{\mathbb{R}}\widetilde{u}_1\widetilde{\rho}_x dx\Big)\\
&+c_2\|\sqrt{\bar u_{1x}}(\widetilde{\rho},\widetilde{u}_1,\widetilde{\theta})\|^2 + \lambda\|(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{\theta}_x)\|^2\\
\le\ & C\varepsilon\|\mathbf{g}\|_\nu^2 + C_\varepsilon(\|(\widetilde{\rho}_{xx},\widetilde{u}_{xx},\widetilde{\theta}_{xx})\|^2 + \|\mathbf{g}_x\|_\nu^2 + \|\mathbf{g}_{xx}\|_\nu^2)\\
&+C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C_\varepsilon(\delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).
\end{aligned}\tag{4.31}$$

This completes the proof of lower order energy estimates for the macroscopic component $(\widetilde{\rho},\widetilde{u},\widetilde{\theta})$.

Next, we turn to deduce the lower order energy estimates for the microscopic component $\mathbf{g}$. Taking the inner product of (1.23) with $\mathbf{g}$ over $\mathbb{R}\times\mathbb{R}^3$, one has

$$\begin{aligned}
&(\partial_t\mathbf{g} + v_1\partial_x\mathbf{g} - \mathcal{L}_1\mathbf{g},\mathbf{g})\\
=\ & (\Gamma(\mathbf{g},\frac{M-\mu}{\sqrt{\mu}}) + \Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{g}),\mathbf{g}) + (\Gamma(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}),\mathbf{g})\\
&+(\frac{P_0(v_1\sqrt{\mu}\partial_x\mathbf{g})}{\sqrt{\mu}},\mathbf{g}) - (\frac{1}{\sqrt{\mu}}P_1v_1M\{\frac{|v-u|^2\widetilde{\theta}_x}{2R\theta^2} + \frac{(v-u)\cdot\widetilde{u}_x}{R\theta}\},\mathbf{g})\\
&-(\frac{P_1(v_1\partial_x\overline{G})}{\sqrt{\mu}} + \frac{\partial_t\overline{G}}{\sqrt{\mu}},\mathbf{g}) + (\frac{P_1(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}},\mathbf{g}).
\end{aligned}\tag{4.32}$$

We will estimate each term for (4.32). First of all, we have from (3.5) that

$$(\partial_t\mathbf{g} + v_1\partial_x\mathbf{g} - \mathcal{L}_1\mathbf{g},\mathbf{g}) \ge \frac{1}{2}\frac{d}{dt}\|\mathbf{g}\|^2 + \sigma_1\|\mathbf{g}\|_\nu^2.$$

From (3.16) and (3.20), we get

$$\begin{aligned}
&|(\Gamma(\mathbf{g},\frac{M-\mu}{\sqrt{\mu}}),\mathbf{g})| + |(\Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{g}),\mathbf{g})| + |(\Gamma(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}),\mathbf{g})|\\
\le\ & C\varepsilon\|\mathbf{g}\|_\nu^2 + C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C_\varepsilon(\eta_0 + \delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).
\end{aligned}$$

In view of the properties of P_0 in (1.11) and using (1.20), one has

$$|(\frac{P_0(v_1\sqrt{\mu}\partial_x\mathbf{g})}{\sqrt{\mu}},\mathbf{g})| \le C\varepsilon\|\mathbf{g}\|_\nu^2 + C_\varepsilon\|\mathbf{g}_x\|_\nu^2.$$

By the similar arguments as (4.14), one has

$$\begin{aligned}
&P_1v_1M\{\frac{|v-u|^2\widetilde{\theta}_x}{2R\theta^2} + \frac{(v-u)\cdot\widetilde{u}_x}{R\theta}\}\\
=\ & \frac{\sqrt{R}\widetilde{\theta}_x}{\sqrt{\theta}}\hat{A}_1(\frac{v-u}{\sqrt{R\theta}})M + \sum_{j=1}^{3}\widetilde{u}_{jx}\hat{B}_{1j}(\frac{v-u}{\sqrt{R\theta}})M,
\end{aligned}\tag{4.33}$$

which implies that

$$\big|(\frac{1}{\sqrt{\mu}}P_1v_1M\{\frac{|v-u|^2\widetilde{\theta}_x}{2R\theta^2} + \frac{(v-u)\cdot\widetilde{u}_x}{R\theta}\},\mathbf{g})\big| \le C\varepsilon\|\mathbf{g}\|_\nu^2 + C_\varepsilon\|[\widetilde{u}_x,\widetilde{\theta}_x]\|^2.$$

For the term involving $\overline{G}$ in (4.32), we arrive at

$$\big|(\frac{P_1(v_1\partial_x\overline{G})}{\sqrt{\mu}}+\frac{\partial_t\overline{G}}{\sqrt{\mu}},\mathbf{g})\big|\leq C\varepsilon\|\mathbf{g}\|_\nu^2+C_\varepsilon\delta^{\frac16}(1+t)^{-\frac76}+C_\varepsilon(\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).$$

Finally, we handle the complicated term containing $\partial_x\phi$. We use $F_2=\sqrt{\mu}\mathbf{f}$ and (1.11) to get

$$\frac{P_1(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}}=\partial_x\phi\partial_{v_1}\mathbf{f}-\partial_x\phi\frac{v_1}{2}\mathbf{f}-\frac{P_0[\partial_x\phi\partial_{v_1}(\sqrt{\mu}\mathbf{f})]}{\sqrt{\mu}}. \tag{4.34}$$

By using (4.2), we have

$$|(\partial_x\phi\partial_{v_1}\mathbf{f}-\partial_x\phi\frac{v_1}{2}\mathbf{f},\mathbf{g})|\leq C\|\partial_x\phi\|_{L^\infty}(\|\partial_{v_1}\mathbf{f}\|+\|\langle v\rangle\mathbf{f}\|)\|\mathbf{g}\|\leq C\sum_{|\alpha|\leq1}\|\partial_x\partial^\alpha\phi\|\mathcal{F}_{N,l,q}(t).$$

By using (1.11) and the integration by parts, one has

$$\begin{aligned}\big|(\frac{P_0[\partial_x\phi\partial_{v_1}(\sqrt{\mu}\mathbf{f})]}{\sqrt{\mu}},\mathbf{g})\big|&=\Big|\sum_{j=0}^{4}(\langle v\rangle^{-\frac{\gamma}{2}}\frac{1}{\sqrt{\mu}}\langle(\partial_x\phi\partial_{v_1}(\sqrt{\mu}\mathbf{f})),\frac{\chi_j}{M}\rangle\chi_j,\langle v\rangle^{\frac{\gamma}{2}}\mathbf{g})\Big|\\&\leq C\|\partial_x\phi\|_{L^\infty}\|\mathbf{f}\|_\nu\|\mathbf{g}\|_\nu\leq C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).\end{aligned}$$

It follows from the above three estimates that

$$\big|(\frac{P_1(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}},\mathbf{g})\big|\leq C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t)+C\sum_{|\alpha|\leq1}\|\partial_x\partial^\alpha\phi\|\mathcal{F}_{N,l,q}(t).$$

Substituting the above estimates into (4.32) and taking $\varepsilon>0$ small enough, we obtain

$$\begin{aligned}\frac12\frac{d}{dt}\|\mathbf{g}\|^2+\lambda\|\mathbf{g}\|_\nu^2&\leq C(\|[\widetilde{u}_x,\widetilde{\theta}_x]\|^2+\|\mathbf{g}_x\|_\nu^2)+C\delta^{\frac16}(1+t)^{-\frac76}\\&\quad+C(\eta_0+\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t)+C\sum_{|\alpha|\leq1}\|\partial_x\partial^\alpha\phi\|\mathcal{F}_{N,l,q}(t).\end{aligned} \tag{4.35}$$

In summary, for some suitably large constant $\widetilde{C}_1>0$ and any $\varepsilon>0$ small enough, the summation of (4.31) $\times\,\widetilde{C}_1$ and (4.35) gives (4.1). We thus complete the proof of Lemma 4.1. □

4.2. High order energy estimates. In this subsection, we first deduce the high order spatial energy estimates of the fluid variables $(\widetilde{\rho},\widetilde{u},\widetilde{\theta})$ and then we deduce the high order spatial energy estimates of the microscopic component $\mathbf{g}$. Lastly, we give the spatial energy estimates of the function $\mathbf{f}$ in $(1.6)_2$. The main high order energy estimates can be stated as follows:

LEMMA 4.2. *Under the conditions of Lemma 4.1, for any $\varepsilon>0$ and $\eta_0>0$ small enough, there exist suitably large $\widetilde{C}_2>0$ and suitably small $\kappa_1>0$ such that*

$$\begin{aligned}&\frac{d}{dt}\Big\{\sum_{1\leq|\alpha|\leq N-1}H_\alpha(t)+\widetilde{C}_2\{\sum_{1\leq|\alpha|\leq N}\|\frac{\partial^\alpha F_1}{\sqrt{\mu}}\|^2+\sum_{|\alpha|\leq N}(\|\partial^\alpha\mathbf{f}\|^2+\frac12\|\partial^\alpha\partial_x\phi\|^2)+\kappa_1H(t)\}\Big\}\\&+\lambda\Big\{\sum_{1\leq|\alpha|\leq N}\|\partial^\alpha\mathbf{g}\|_\nu^2+\sum_{|\alpha|\leq N}(\|\partial^\alpha\mathbf{f}\|_\nu^2+\|\partial^\alpha\partial_x\phi\|^2)+\sum_{2\leq|\alpha|\leq N}\|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2\Big\}\\&\leq C(\eta_0+\varepsilon)\|\mathbf{g}\|_\nu^2+C_\varepsilon\delta^{\frac16}(1+t)^{-\frac76}+C_\varepsilon(\eta_0+\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t)\\&\quad+C\sum_{|\alpha|\leq N}\|\partial^\alpha\partial_x\phi\|\mathcal{F}_{N,l,q}(t).\end{aligned} \tag{4.36}$$

Here $H_\alpha(t)$, $H(t)$ and $\mathcal{F}_{N,l,q}(t)$ are given by (4.41), (4.73) *and* (4.2), *respectively.*

Proof. We first derive the high order spatial energy estimates of the macroscopic components $(\widetilde{\rho},\widetilde{u},\widetilde{\theta})$. By (1.23) and (4.33), one has

$$\begin{aligned}&\frac{1}{\sqrt{\mu}}\Big(\frac{\sqrt{R}}{\sqrt{\theta}}\hat{A}_1(\frac{v-u}{\sqrt{R\theta}})M\Big)\widetilde{\theta}_x+\frac{1}{\sqrt{\mu}}\sum_{j=1}^{3}\Big(\hat{B}_{1j}(\frac{v-u}{\sqrt{R\theta}})M\Big)\widetilde{u}_{jx}\\&=-\partial_t\mathbf{g}-v_1\partial_x\mathbf{g}+\mathcal{L}_1\mathbf{g}+\Gamma(\mathbf{g},\frac{M-\mu}{\sqrt{\mu}})+\Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{g})+\Gamma(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}})\\&\quad+\frac{P_0(v_1\sqrt{\mu}\partial_x\mathbf{g})}{\sqrt{\mu}}-\frac{P_1(v_1\partial_x\overline{G})}{\sqrt{\mu}}-\frac{\partial_t\overline{G}}{\sqrt{\mu}}+\frac{P_1(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}}.\end{aligned}\tag{4.37}$$

In view of the linear independence of the functions $\hat{A}_1(\frac{v-u}{\sqrt{R\theta}})$ and $\hat{B}_{1j}(\frac{v-u}{\sqrt{R\theta}})$ with $j=1,2,3$, we get

$$\widetilde{\theta}_x=R_1,\quad \widetilde{u}_{jx}=R_{j+1},\quad j=1,2,,3.$$

Here, by the elementary linear algebra, the terms R_j are all of the form

$$\begin{aligned}&\langle-\partial_t\mathbf{g},\sqrt{\mu}\zeta(v)\rangle+\langle-v_1\partial_x\mathbf{g}+\mathcal{L}_1\mathbf{g},\sqrt{\mu}\zeta(v)\rangle+\Gamma(\mathbf{g},\frac{M-\mu}{\sqrt{\mu}})\\&+\langle\Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{g})+\Gamma(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}),\sqrt{\mu}\zeta(v)\rangle\\&+\langle\frac{P_0(v_1\sqrt{\mu}\partial_x\mathbf{g})}{\sqrt{\mu}}-\frac{P_1(v_1\partial_x\overline{G})}{\sqrt{\mu}}-\frac{\partial_t\overline{G}}{\sqrt{\mu}}+\frac{P_1(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}},\sqrt{\mu}\zeta(v)\rangle,\end{aligned}$$

where $\zeta(v)$ is a different linear combination of the functions $\hat{A}_1(\frac{v-u}{\sqrt{R\theta}})$ and $\hat{B}_{1j}(\frac{v-u}{\sqrt{R\theta}})$ with $j=1,2,3$.

For any $1\le|\alpha|\le N-1$ and some function $\zeta_\theta(v)$, by the integration by parts about t, we get

$$\begin{aligned}&(\partial^\alpha\langle-\partial_t\mathbf{g},\sqrt{\mu}\zeta_\theta(v)\rangle,\partial^\alpha\widetilde{\theta}_x)=-\frac{d}{dt}(\partial^\alpha\langle\mathbf{g},\sqrt{\mu}\zeta_\theta(v)\rangle,\partial^\alpha\widetilde{\theta}_x)\\&+(\partial^\alpha\langle\mathbf{g},\sqrt{\mu}\partial_t\zeta_\theta(v)\rangle,\partial^\alpha\widetilde{\theta}_x)-(\partial^\alpha\partial_x\langle\mathbf{g},\sqrt{\mu}\zeta_\theta(v)\rangle,\partial^\alpha\partial_t\widetilde{\theta})\\&\le-\frac{d}{dt}(\partial^\alpha\langle\mathbf{g},\sqrt{\mu}\zeta_\theta(v)\rangle,\partial^\alpha\widetilde{\theta}_x)+C\varepsilon\|\partial^\alpha(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{\theta}_x)\|^2\\&+C_\varepsilon\|\partial^\alpha\mathbf{g}_x\|_\nu^2+C_\varepsilon\delta^{\frac16}(1+t)^{-\frac76}+C_\varepsilon(\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).\end{aligned}$$

Here we used Lemma 3.8, Lemma 2.1 and the elementary inequalities.

Recall $\mathcal{L}_1 g=\Gamma(\sqrt{\mu},g)+\Gamma(g,\sqrt{\mu})$, we deduce from (3.9) and (3.10) that

$$\begin{aligned}(\partial^\alpha\langle-v_1\partial_x\mathbf{g}+\mathcal{L}_1\mathbf{g},\sqrt{\mu}\zeta_\theta(v)\rangle,\partial^\alpha\widetilde{\theta}_x)&\le C\varepsilon\|\partial^\alpha\widetilde{\theta}_x\|^2+C_\varepsilon\|\partial^\alpha\mathbf{g}_x\|_\nu^2+C_\varepsilon\|\partial^\alpha\mathbf{g}\|_\nu^2\\&\quad+C_\varepsilon\delta^{\frac16}(1+t)^{-\frac76}+C_\varepsilon(\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).\end{aligned}$$

With (3.16) and (3.20) in hand, we see

$$\begin{aligned}&(\partial^\alpha\langle\Gamma(\mathbf{g},\frac{M-\mu}{\sqrt{\mu}})+\Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{g})+\Gamma(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}),\sqrt{\mu}\zeta_\theta(v)\rangle,\partial^\alpha\widetilde{\theta}_x)\\&\le C\varepsilon\|\partial^\alpha\widetilde{\theta}_x\|^2+C_\varepsilon\delta^{\frac16}(1+t)^{-\frac76}+C_\varepsilon(\eta_0+\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).\end{aligned}$$

Recalling that $F_2=\sqrt{\mu}\mathbf{f}$ and using (1.11), (3.24), (3.26), Lemma 3.2, Lemma 3.8 and Lemma 2.1, one has

$$
\begin{aligned}
&(\partial^\alpha\langle\frac{P_0(v_1\sqrt{\mu}\partial_x\mathbf{g})}{\sqrt{\mu}}-\frac{P_1(v_1\partial_x\overline{G})}{\sqrt{\mu}}-\frac{\partial_t\overline{G}}{\sqrt{\mu}}+\frac{P_1(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}},\sqrt{\mu}\zeta(v)\rangle,\partial^\alpha\widetilde{\theta}_x)\\
&\leq C\varepsilon\|\partial^\alpha\widetilde{\theta}_x\|^2+C_\varepsilon\|\partial^\alpha\mathbf{g}_x\|_\nu^2+C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}++C_\varepsilon(\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).
\end{aligned}
$$

It follows from the above estimates and any $\varepsilon>0$ small enough that

$$
\begin{aligned}
\|\partial^\alpha\widetilde{\theta}_x\|^2\leq&-\frac{d}{dt}(\partial^\alpha\langle\mathbf{g},\sqrt{\mu}\zeta_\theta(v)\rangle,\partial^\alpha\widetilde{\theta}_x)+C\varepsilon\|\partial^\alpha(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{\theta}_x)\|^2+C_\varepsilon\|\partial^\alpha\mathbf{g}_x\|_\nu^2\\
&+C_\varepsilon\|\partial^\alpha\mathbf{g}\|_\nu^2+C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C_\varepsilon(\eta_0+\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).
\end{aligned}\tag{4.38}
$$

For some function $\zeta_u(v)$, similar arguments as (4.38) imply

$$
\begin{aligned}
\|\partial^\alpha\widetilde{u}_x\|^2\leq&-\frac{d}{dt}(\partial^\alpha\langle\mathbf{g},\sqrt{\mu}\zeta_u(v)\rangle,\partial^\alpha\widetilde{u}_x)+C\varepsilon\|\partial^\alpha(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{u}_x)\|^2+C_\varepsilon\|\partial^\alpha\mathbf{g}_x\|_\nu^2\\
&+C_\varepsilon\|\partial^\alpha\mathbf{g}\|_\nu^2+C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C_\varepsilon(\eta_0+\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).
\end{aligned}\tag{4.39}
$$

Combining (4.38), (4.39) and (3.31) and choosing $\varepsilon>0$ small enough, we arrive at

$$
\begin{aligned}
&\frac{d}{dt}\sum_{1\leq|\alpha|\leq N-1}H_\alpha(t)+\lambda\sum_{2\leq|\alpha|\leq N}\|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2\\
&\leq C\sum_{1\leq|\alpha|\leq N}\|\partial^\alpha\mathbf{g}\|_\nu^2+C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C(\eta_0+\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).
\end{aligned}\tag{4.40}
$$

Here, for $1\leq|\alpha|\leq N-1$ and $\overline{C}_\alpha\gg C_\alpha>0$, the function $H_\alpha(t)$ is defined by

$$
H_\alpha(t)=\int_{\mathbb{R}}\{\overline{C}_\alpha\partial^\alpha\langle\mathbf{g},\sqrt{\mu}\zeta_\theta(v)\rangle\partial^\alpha\widetilde{\theta}_x+\overline{C}_\alpha\partial^\alpha\langle\mathbf{g},\sqrt{\mu}\zeta_u(v)\rangle\partial^\alpha\widetilde{u}_x+C_\alpha\partial^\alpha\widetilde{u}_1\partial^\alpha\widetilde{\rho}_x\}\,dx.\tag{4.41}
$$

Next we will deduce the derivative estimates for the microscopic component $\mathbf{g}$. In terms of $(1.6)_1$, (1.24) and (1.25), one has

$$
\begin{aligned}
&\partial_t(\frac{F_1}{\sqrt{\mu}})+v_1\partial_x(\frac{F_1}{\sqrt{\mu}})-\mathcal{L}_1\mathbf{g}\\
&=\Gamma(\mathbf{g},\frac{M-\mu}{\sqrt{\mu}})+\Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{g})+\Gamma(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}})\\
&\quad+\frac{1}{\sqrt{\mu}}P_1v_1M\{\frac{|v-u|^2\overline{\theta}_x}{2R\theta^2}+\frac{(v-u)\cdot\bar{u}_x}{R\theta}\}+\frac{\partial_x\phi\partial_{v_1}F_2}{\sqrt{\mu}}.
\end{aligned}\tag{4.42}
$$

Taking the derivative ∂^α of (4.42) with $1\leq|\alpha|\leq N$ and taking the inner product with $\frac{\partial^\alpha F_1}{\sqrt{\mu}}$. Then we estimate this inner product term by term. We easily see

$$
\Big(\partial_t(\frac{\partial^\alpha F_1}{\sqrt{\mu}})+v_1\partial_x(\frac{\partial^\alpha F_1}{\sqrt{\mu}}),\frac{\partial^\alpha F_1}{\sqrt{\mu}}\Big)=\frac{1}{2}\frac{d}{dt}\|\frac{\partial^\alpha F_1}{\sqrt{\mu}}\|^2.\tag{4.43}
$$

For the third term on the left hand side of (4.42), recalling that $F_1=M+\overline{G}+\sqrt{\mu}\mathbf{g}$, we have from (3.5) that

$$
-(\mathcal{L}_1\partial^\alpha\mathbf{g},\partial^\alpha\mathbf{g})\geq\sigma_1\|\partial^\alpha\mathbf{g}\|_\nu^2.
$$

Notice that $\mathcal{L}_1 g = \Gamma(\sqrt{\mu}, g) + \Gamma(g, \sqrt{\mu})$, we have from (3.9), (3.10), (3.26) and Lemma 2.1 that

$$|(\mathcal{L}_1\partial^\alpha \mathbf{g}, \frac{\partial^\alpha \overline{G}}{\sqrt{\mu}})| \leq C\|\partial^\alpha \mathbf{g}\|_\nu \|\frac{\partial^\alpha \overline{G}}{\sqrt{\mu}}\|_\nu \leq \varepsilon\|\partial^\alpha \mathbf{g}\|_\nu^2 + C_\varepsilon \delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).$$

We now deal with the complicated term involving M. For $|\bar{\alpha}| \geq 2$, one has

$$\begin{aligned}
\partial^{\bar{\alpha}} M &= M\Big(\frac{\partial^{\bar{\alpha}}\rho}{\rho} - \frac{3\partial^{\bar{\alpha}}\theta}{2\theta} + \frac{(v-u)^2\partial^{\bar{\alpha}}\theta}{2R\theta^2} + \sum_{i=1}^3 \frac{\partial^{\bar{\alpha}} u_i (v_i - u_i)}{R\theta}\Big) + \cdots \\
&= \Big(\mu + (M-\mu)\Big)\Big(\frac{\partial^{\bar{\alpha}}\rho}{\rho} - \frac{3\partial^{\bar{\alpha}}\theta}{2\theta} + \frac{(v-u)^2\partial^{\bar{\alpha}}\theta}{2R\theta^2} + \sum_{i=1}^3 \frac{\partial^{\bar{\alpha}} u_i (v_i - u_i)}{R\theta}\Big) + \cdots \\
&= J_1^{\bar{\alpha}} + J_2^{\bar{\alpha}} + J_3^{\bar{\alpha}}.
\end{aligned} \tag{4.44}$$

Here the terms J_1 and J_2 are the higher order derivatives of (ρ, u, θ) with μ and $M-\mu$ and J_3 is the low order derivatives with M. For $|\alpha| = 1$, we use the integration by parts to get

$$(\mathcal{L}_1\partial^\alpha \mathbf{g}, \frac{\partial^\alpha M}{\sqrt{\mu}}) = -(\mathcal{L}_1 \mathbf{g}, \frac{\partial_{xx} M}{\sqrt{\mu}}) = -(\mathcal{L}_1 \mathbf{g}, \frac{J_1^2}{\sqrt{\mu}}) - (\mathcal{L}_1 \mathbf{g}, \frac{J_2^2}{\sqrt{\mu}}) - (\mathcal{L}_1 \mathbf{g}, \frac{J_3^2}{\sqrt{\mu}}).$$

Since $\frac{J_1^2}{\sqrt{\mu}} \in \ker \mathcal{L}_1$, it follows that $(\mathcal{L}_1 \mathbf{g}, \frac{J_1^2}{\sqrt{\mu}}) = 0$. For the term $\frac{J_2^2}{\sqrt{\mu}}$, we have from (3.9), (3.10), (3.13) and Lemma 2.1 that

$$|(\mathcal{L}_1 \mathbf{g}, \frac{J_2^2}{\sqrt{\mu}})| \leq C\|\mathbf{g}\|_\nu \|\frac{J_2^2}{\sqrt{\mu}}\|_\nu \leq C\eta_0(\|\mathbf{g}\|_\nu^2 + \|\partial_{xx}(\widetilde{\rho}, \widetilde{u}, \widetilde{\theta})\|^2) + C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}.$$

Similarly, it holds that

$$|(\mathcal{L}_1 \mathbf{g}, \frac{J_3^2}{\sqrt{\mu}})| \leq C\|\mathbf{g}\|_\nu \|\frac{J_3^2}{\sqrt{\mu}}\|_\nu \leq C\varepsilon\|\mathbf{g}\|_\nu^2 + C_\varepsilon \delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C_\varepsilon\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).$$

Hence, for $|\alpha| = 1$, we can deduce from the above related estimates that

$$\begin{aligned}
&|(\mathcal{L}_1\partial_x \mathbf{g}, \frac{\partial_x M}{\sqrt{\mu}})| \\
&\leq C(\eta_0+\varepsilon)\|\mathbf{g}\|_\nu^2 + C\eta_0\|\partial_{xx}(\widetilde{\rho}, \widetilde{u}, \widetilde{\theta})\|^2 + C_\varepsilon \delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C_\varepsilon\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).
\end{aligned}$$

For $2 \leq |\alpha| \leq N$, carrying out the similar calculations as the above, we can obtain

$$\begin{aligned}
&(\mathcal{L}_1\partial^\alpha \mathbf{g}, \frac{\partial^\alpha M}{\sqrt{\mu}}) \\
&= -(\mathcal{L}_1\partial^\alpha \mathbf{g}, \frac{J_1^\alpha}{\sqrt{\mu}}) - (\mathcal{L}_1\partial^\alpha \mathbf{g}, \frac{J_2^\alpha}{\sqrt{\mu}}) - (\mathcal{L}_1\partial^\alpha \mathbf{g}, \frac{J_3^\alpha}{\sqrt{\mu}}) \\
&\leq C(\eta_0+\varepsilon)\|\partial^\alpha \mathbf{g}\|_\nu^2 + C\eta_0\|\partial^\alpha(\widetilde{\rho}, \widetilde{u}, \widetilde{\theta})\|^2 + C_\varepsilon \delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C_\varepsilon\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).
\end{aligned}$$

As a consequence, for the third term on the left hand side of (4.42), if $|\alpha| = 1$, we get

$$\begin{aligned}
(\mathcal{L}_1\partial^\alpha \mathbf{g}, \frac{\partial^\alpha F_1}{\sqrt{\mu}}) \leq &- \sigma_1\|\partial^\alpha \mathbf{g}\|_\nu^2 + C(\eta_0+\varepsilon)(\|\mathbf{g}\|_\nu^2 + \|\partial_{xx}(\widetilde{\rho}, \widetilde{u}, \widetilde{\theta})\|^2) + C\varepsilon\|\partial^\alpha \mathbf{g}\|_\nu^2 \\
&+ C_\varepsilon \delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C_\varepsilon\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).
\end{aligned} \tag{4.45}$$

If $2 \le |\alpha| \le N$, we have

$$(\mathcal{L}_1\partial^\alpha \mathbf{g}, \frac{\partial^\alpha F_1}{\sqrt{\mu}}) \le -\sigma_1\|\partial^\alpha \mathbf{g}\|_\nu^2 + C(\eta_0+\varepsilon)(\|\partial^\alpha \mathbf{g}\|_\nu^2 + \|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2) + C_\varepsilon \delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C_\varepsilon\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t). \quad (4.46)$$

For $2 \le |\bar{\alpha}| \le N$, we have from (3.26) and (4.44) that

$$\begin{aligned}\|\frac{\partial^{\bar{\alpha}} F_1}{\sqrt{\mu}}\|_\nu^2 &\le C\|\frac{\partial^{\bar{\alpha}}\sqrt{\mu}\mathbf{g}}{\sqrt{\mu}}\|_\nu^2 + \|\frac{\partial^{\bar{\alpha}}\overline{G}}{\sqrt{\mu}}\|_\nu^2 + \|\frac{\partial^{\bar{\alpha}} M}{\sqrt{\mu}}\|_\nu^2 \\ &\le C(\|\partial^{\bar{\alpha}}\mathbf{g}\|_\nu^2 + \|\partial^{\bar{\alpha}}(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2) + C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t). \quad (4.47)\end{aligned}$$

For the first term on the right hand side of (4.42), if $|\alpha| = 1$, we use the integration by parts, (3.16) and (4.47) to obtain

$$\begin{aligned}\big(\partial^\alpha\Gamma(\mathbf{g}, \frac{M-\mu}{\sqrt{\mu}}), \frac{\partial^\alpha F_1}{\sqrt{\mu}}\big) &= -\big(\Gamma(\mathbf{g}, \frac{M-\mu}{\sqrt{\mu}}), \frac{\partial_{xx} F_1}{\sqrt{\mu}}\big) \\ &\le C\varepsilon(\|\partial_{xx}\mathbf{g}\|_\nu^2 + \|\partial_{xx}(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2) + C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} \\ &\quad + C_\varepsilon(\eta_0 + \delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t). \quad (4.48)\end{aligned}$$

If $2 \le |\alpha| \le N$, we get from (3.16) and (4.47) that

$$\big(\partial^\alpha\Gamma(\mathbf{g}, \frac{M-\mu}{\sqrt{\mu}}), \frac{\partial^\alpha F_1}{\sqrt{\mu}}\big) \le C\varepsilon(\|\partial^\alpha\mathbf{g}\|_\nu^2 + \|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2) + C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C_\varepsilon(\eta_0 + \delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t). \quad (4.49)$$

The second term on the right hand side of (4.42) can be handled in the same way. For the third term on the right hand side of (4.42), if $|\alpha| = 1$, we can deduce from the integration by parts, (3.20) and (4.47) that

$$\begin{aligned}&\big(\partial^\alpha\Gamma(\frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}}), \frac{\partial^\alpha F_1}{\sqrt{\mu}}\big) \\ =\ &\big(\Gamma(\frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}}), \frac{\partial_{xx} F_1}{\sqrt{\mu}}\big) \\ \le\ &C\varepsilon\|\frac{\partial_{xx}F_1}{\sqrt{\mu}}\|_\nu^2 + C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C_\varepsilon(\delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t) \\ \le\ &C\varepsilon(\|\partial_{xx}\mathbf{g}\|_\nu^2 + \|\partial_{xx}(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2) + C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C_\varepsilon(\delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t). \quad (4.50)\end{aligned}$$

If $2 \le |\alpha| \le N$, by (3.20) and (4.47), one has

$$\big(\partial^\alpha\Gamma(\frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}}), \frac{\partial^\alpha F_1}{\sqrt{\mu}}\big) \le C\varepsilon(\|\partial^\alpha\mathbf{g}\|_\nu^2 + \|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2) + C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C_\varepsilon(\delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t). \quad (4.51)$$

For the fourth term on the right hand side of (4.42), if $|\alpha| = 1$, we can deduce from (4.33), the imbedding inequality, Lemma 2.1 that

$$\big(\frac{1}{\sqrt{\mu}}\partial^\alpha P_1 v_1 M\big\{\frac{|v-u|^2\overline{\theta}_x}{2R\theta^2} + \frac{(v-u)\cdot\bar{u}_x}{R\theta}\big\}, \frac{\partial^\alpha M}{\sqrt{\mu}}\big) \le C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).$$

Here we used the fact that $|\langle v\rangle^b\mu^{-\frac12}M|_2\le C$ for any $b\ge 0$ by (1.20). Moreover, it also holds that

$$\begin{aligned}&\Big(\frac{1}{\sqrt{\mu}}\partial^\alpha P_1 v_1 M\big\{\frac{|v-u|^2\overline{\theta}_x}{2R\theta^2}+\frac{(v-u)\cdot\bar{u}_x}{R\theta}\big\},\frac{\partial^\alpha(\overline{G}+\sqrt{\mu}\mathbf{g})}{\sqrt{\mu}}\Big)\\&\le C\varepsilon\|\partial^\alpha\mathbf{g}\|_\nu^2+C_\varepsilon\delta^{\frac16}(1+t)^{-\frac76}+C_\varepsilon\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).\end{aligned}$$

Therefore, for $|\alpha|=1$, we have from the above two estimates and $F_1=M+\overline{G}+\sqrt{\mu}\mathbf{g}$ that

$$\begin{aligned}&\Big(\frac{1}{\sqrt{\mu}}\partial^\alpha P_1 v_1 M\big\{\frac{|v-u|^2\overline{\theta}_x}{2R\theta^2}+\frac{(v-u)\cdot\bar{u}_x}{R\theta}\big\},\frac{\partial^\alpha F_1}{\sqrt{\mu}}\Big)\\&\le C\varepsilon\|\partial^\alpha\mathbf{g}\|_\nu^2+C_\varepsilon\delta^{\frac16}(1+t)^{-\frac76}+C_\varepsilon\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).\end{aligned}\tag{4.52}$$

For $2\le|\alpha|\le N$, we use (4.47) and the similar arguments as (4.52) to get

$$\begin{aligned}&\Big(\frac{1}{\sqrt{\mu}}\partial^\alpha P_1 v_1 M\big\{\frac{|v-u|^2\overline{\theta}_x}{2R\theta^2}+\frac{(v-u)\cdot\bar{u}_x}{R\theta}\big\},\frac{\partial^\alpha F_1}{\sqrt{\mu}}\Big)\\&\le C\varepsilon(\|\partial^\alpha\mathbf{g}\|_\nu^2+\|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2)+C_\varepsilon\delta^{\frac16}(1+t)^{-\frac76}+C_\varepsilon\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).\end{aligned}\tag{4.53}$$

Finally, we handle the term containing $\partial_x\phi$ in (4.42). Recall $F_2=\sqrt{\mu}\mathbf{f}$, one has

$$\begin{aligned}\frac{\partial^\alpha(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}}=&\,\partial_x\phi\partial_{v_1}\partial^\alpha\mathbf{f}+\sum_{1\le|\alpha_1|\le|\alpha|}C_\alpha^{\alpha_1}\partial^{\alpha_1}\partial_x\phi\partial_{v_1}\partial^{\alpha-\alpha_1}\mathbf{f}\\&-\sum_{|\alpha_1|\le|\alpha|}C_\alpha^{\alpha_1}\partial^{\alpha_1}\partial_x\phi\frac{v_1}{2}\partial^{\alpha-\alpha_1}\mathbf{f}.\end{aligned}\tag{4.54}$$

If $|\alpha_1|\le\frac{|\alpha|}{2}$, by using (4.2) and the imbedding inequality, we have

$$\begin{aligned}|(\partial^{\alpha_1}\partial_x\phi\frac{v_1}{2}\partial^{\alpha-\alpha_1}\mathbf{f},\partial^\alpha\mathbf{g})|&\le C\|\partial^{\alpha_1}\partial_x\phi\|_{L^\infty}\|\langle v\rangle\partial^{\alpha-\alpha_1}\mathbf{f}\|\|\langle v\rangle\partial^\alpha\mathbf{g}\|\\&\le C\sum_{|\alpha'|\le N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t).\end{aligned}\tag{4.55}$$

Similarly, if $\frac{|\alpha|}{2}<|\alpha_1|\le|\alpha|$, it holds that

$$|(\partial^{\alpha_1}\partial_x\phi\frac{v_1}{2}\partial^{\alpha-\alpha_1}\mathbf{f},\partial^\alpha\mathbf{g})|\le C\sum_{|\alpha'|\le N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t).\tag{4.56}$$

Similarly, we also have

$$\sum_{1\le|\alpha_1|\le|\alpha|}C_\alpha^{\alpha_1}|(\partial^{\alpha_1}\partial_x\phi\partial_{v_1}\partial^{\alpha-\alpha_1}\mathbf{f},\partial^\alpha\mathbf{g})|\le C\sum_{|\alpha'|\le N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t).\tag{4.57}$$

By using (4.55), (4.56) and (4.57), we have from (4.54) that

$$\Big(\frac{\partial^\alpha(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}},\partial^\alpha\mathbf{g}\Big)\le(\partial_x\phi\partial_{v_1}\partial^\alpha\mathbf{f},\partial^\alpha\mathbf{g})+C\sum_{|\alpha'|\le N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t).\tag{4.58}$$

On the other hand, from the integration by parts, it holds that

$$(\frac{\partial^\alpha(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}},\frac{\partial^\alpha(M+\overline{G})}{\sqrt{\mu}})\leq C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C(\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t). \tag{4.59}$$

Hence, for $1\leq|\alpha|\leq N$, we have from $F_1=M+\overline{G}+\sqrt{\mu}\mathbf{g}$ and the estimates (4.58) and (4.59) that

$$\begin{aligned}&(\frac{\partial^\alpha(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}},\frac{\partial^\alpha F_1}{\sqrt{\mu}})-(\partial_x\phi\partial_{v_1}\partial^\alpha\mathbf{f},\partial^\alpha\mathbf{g})\\ \leq\ & C\sum_{|\alpha'|\leq N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t)+C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C(\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t). \end{aligned}\tag{4.60}$$

In summary, for any $\varepsilon>0$ small enough, by the estimates from (4.43) to (4.60), we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{1\leq|\alpha|\leq N}\|\frac{\partial^\alpha F_1}{\sqrt{\mu}}\|^2+\sigma_1\sum_{1\leq|\alpha|\leq N}\|\partial^\alpha\mathbf{g}\|_\nu^2-\sum_{1\leq|\alpha|\leq N}(\partial_x\phi\partial_{v_1}\partial^\alpha\mathbf{f},\partial^\alpha\mathbf{g})\\ \leq\ & C(\eta_0+\varepsilon)\{\sum_{|\alpha|\leq N}\|\partial^\alpha\mathbf{g}\|_\nu^2+\sum_{2\leq|\alpha|\leq N}\|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2\}\\ &+C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C_\varepsilon(\eta_0+\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t)\\ &+C\sum_{|\alpha|\leq N}\|\partial^\alpha\partial_x\phi\|\mathcal{F}_{N,l,q}(t). \end{aligned}\tag{4.61}$$

Now we consider the energy estimates of the function $\mathbf{f}$. Since $\ker\mathcal{L}_2$ is one dimensional and the estimates of the macroscopic component of $\mathbf{f}$ can be compensated through the Poisson equation (1.28), we will not separate $\mathbf{f}$ to be the macroscopic and microscopic components. Taking ∂^α derivative of equation (1.26) with $|\alpha|\leq N$ and taking the inner product of the resulting equation with $\partial^\alpha\mathbf{f}$, we obtain

$$\begin{aligned}&(\partial^\alpha\partial_t\mathbf{f}+v_1\partial^\alpha\partial_x\mathbf{f}+\partial^\alpha\partial_x\phi v_1\sqrt{\mu},\partial^\alpha\mathbf{f})-(\partial^\alpha\mathcal{L}_2\mathbf{f},\partial^\alpha\mathbf{f})\\ =\ &(\partial^\alpha\Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{f}),\partial^\alpha\mathbf{f})+(\partial^\alpha\Gamma(\frac{\overline{G}}{\sqrt{\mu}},\mathbf{f}),\partial^\alpha\mathbf{f})+(\partial^\alpha\Gamma(\mathbf{g},\mathbf{f}),\partial^\alpha\mathbf{f})\\ &+\big(\partial^\alpha(\frac{\partial_x\phi\partial_{v_1}(M-\mu)}{\sqrt{\mu}})+\partial^\alpha(\frac{\partial_x\phi\partial_{v_1}\overline{G}}{\sqrt{\mu}}),\partial^\alpha\mathbf{f}\big)+\big(\partial^\alpha(\frac{\partial_x\phi\partial_{v_1}(\sqrt{\mu}\mathbf{g})}{\sqrt{\mu}}),\partial^\alpha\mathbf{f}\big). \end{aligned}\tag{4.62}$$

We will estimate each term for (4.62). Recalling that $F_2=\sqrt{\mu}\mathbf{f}$, one has from $(1.6)_2$ that

$$\langle\partial_t\mathbf{f},\sqrt{\mu}\rangle+\langle v_1\partial_x\mathbf{f},\sqrt{\mu}\rangle=0. \tag{4.63}$$

From (4.63), (1.28) and the integration by parts, we get

$$\begin{aligned}(\partial^\alpha\partial_x\phi v_1\sqrt{\mu},\partial^\alpha\mathbf{f})&=-(\partial^\alpha\phi v_1\sqrt{\mu},\partial_x\partial^\alpha\mathbf{f})\\ &=\frac{1}{2}(\partial^\alpha\phi,\partial_t\partial^\alpha(-\partial_{xx}\phi))=\frac{1}{4}\frac{d}{dt}\|\partial^\alpha\partial_x\phi\|^2. \end{aligned}\tag{4.64}$$

By the integration by parts, one has

$$(\partial^\alpha\partial_t\mathbf{f}+v_1\partial^\alpha\partial_x\mathbf{f}+\partial^\alpha\partial_x\phi v_1\sqrt{\mu},\partial^\alpha\mathbf{f})=\frac{1}{2}\frac{d}{dt}(\|\partial^\alpha\mathbf{f}\|^2+\frac{1}{2}\|\partial^\alpha\partial_x\phi\|^2).$$

Note that $\mathbf{f}=P_2\mathbf{f}+(\mathbf{I}-P_2)\mathbf{f}$, we have from (3.5) that

$$-(\partial^\alpha\mathcal{L}_2\mathbf{f},\partial^\alpha\mathbf{f})\geq\sigma_2\|\partial^\alpha(\mathbf{I}-P_2)\mathbf{f}\|_\nu^2.$$

By using (3.16), (3.29) and (3.30), we obtain

$$\begin{aligned}&(\partial^\alpha\Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{f})+\partial^\alpha\Gamma(\frac{\overline{G}}{\sqrt{\mu}},\mathbf{f}),\partial^\alpha\mathbf{f})+(\partial^\alpha\Gamma(\mathbf{g},\mathbf{f}),\partial^\alpha\mathbf{f})\\&\leq C\varepsilon\|\partial^\alpha\mathbf{f}\|_\nu^2+C_\varepsilon(\eta_0+\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).\end{aligned}$$

By using (1.20), (3.25), (3.26) and the imbedding inequality, we get

$$(\partial^\alpha(\frac{\partial_x\phi\partial_{v_1}(M-\mu)}{\sqrt{\mu}})+\partial^\alpha(\frac{\partial_x\phi\partial_{v_1}\overline{G}}{\sqrt{\mu}}),\partial^\alpha\mathbf{f})\leq C(\eta_0+\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).$$

As for the last term of (4.62), we perform the similar calculations as (4.55)-(4.57) to get

$$(\frac{\partial^\alpha(\partial_x\phi\partial_{v_1}(\sqrt{\mu}\mathbf{g}))}{\sqrt{\mu}},\partial^\alpha\mathbf{f})-(\partial_x\phi\partial_{v_1}\partial^\alpha\mathbf{g},\partial^\alpha\mathbf{f})\leq C\sum_{|\alpha'|\leq N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t).$$

Combining the above estimates, for any $\varepsilon>0$ small enough, we arrive at

$$\begin{aligned}&\frac12\frac{d}{dt}\sum_{|\alpha|\leq N}(\|\partial^\alpha\mathbf{f}\|^2+\frac12\|\partial^\alpha\partial_x\phi\|^2)+\sigma_2\sum_{|\alpha|\leq N}\|\partial^\alpha(\mathbf{I}-P_2)\mathbf{f}\|_\nu^2-\sum_{|\alpha|\leq N}(\partial_x\phi\partial_{v_1}\partial^\alpha\mathbf{g},\partial^\alpha\mathbf{f})\\&\leq C\varepsilon\sum_{|\alpha|\leq N}\|\partial^\alpha\mathbf{f}\|_\nu^2+C_\varepsilon(\eta_0+\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t)+C\sum_{|\alpha|\leq N}\|\partial^\alpha\partial_x\phi\|\mathcal{F}_{N,l,q}(t).\end{aligned}\tag{4.65}$$

By the integration by parts, one has

$$-\sum_{1\leq|\alpha|\leq N}(\partial_x\phi\partial_{v_1}\partial^\alpha\mathbf{f},\partial^\alpha\mathbf{g})-\sum_{|\alpha|\leq N}(\partial_x\phi\partial_{v_1}\partial^\alpha\mathbf{g},\partial^\alpha\mathbf{f})=-(\partial_x\phi\partial_{v_1}\mathbf{g},\mathbf{f}),\tag{4.66}$$

which is dominated by $C\|\partial_x\phi\|\mathcal{F}_{N,l,q}(t)$. By this, (4.61) and (4.65), we arrive at

$$\begin{aligned}&\frac{d}{dt}\Big\{\sum_{1\leq|\alpha|\leq N}\|\frac{\partial^\alpha F_1}{\sqrt{\mu}}\|^2+\sum_{|\alpha|\leq N}(\|\partial^\alpha\mathbf{f}\|^2+\frac12\|\partial^\alpha\partial_x\phi\|^2)\Big\}\\&+\lambda\sum_{1\leq|\alpha|\leq N}\|\partial^\alpha\mathbf{g}\|_\nu^2+\lambda\sum_{|\alpha|\leq N}\|\partial^\alpha(\mathbf{I}-P_2)\mathbf{f}\|^2\\&\leq C(\eta_0+\varepsilon)\big\{\sum_{|\alpha|\leq N}\|\partial^\alpha\mathbf{g}\|_\nu^2+\sum_{2\leq|\alpha|\leq N}\|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2+\sum_{|\alpha|\leq N}\|\partial^\alpha\mathbf{f}\|_\nu^2\big\}\\&+C_\varepsilon\delta^{\frac16}(1+t)^{-\frac76}+C_\varepsilon(\eta_0+\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t)+C\sum_{|\alpha|\leq N}\|\partial^\alpha\partial_x\phi\|\mathcal{F}_{N,l,q}(t).\end{aligned}\tag{4.67}$$

Notice that $P_2\mathbf{f}$ and the electric field $\partial_x\phi$ are not included in the dissipation of (4.67). Next we shall derive these estimates. Letting $P_2\mathbf{f}=\mathbf{a}(t,x)\sqrt{\mu}$, we have from (1.26) that

$$\begin{aligned}\partial_x\mathbf{a}+\partial_x\phi=&\langle-\partial_t\mathbf{f},v_1\sqrt{\mu}\rangle+\langle-v_1\partial_x(\mathbf{I}-P_2)\mathbf{f},v_1\sqrt{\mu}\rangle+\langle\mathcal{L}_2(\mathbf{I}-P_2)\mathbf{f},v_1\sqrt{\mu}\rangle\\&+\big\langle\Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{f}),v_1\sqrt{\mu}\big\rangle+\big\langle\Gamma(\frac{\overline{G}}{\sqrt{\mu}},\mathbf{f}),v_1\sqrt{\mu}\big\rangle+\big\langle\Gamma(\mathbf{g},\mathbf{f}),v_1\sqrt{\mu}\big\rangle\\&+\big\langle\frac{\partial_x\phi\partial_{v_1}(M-\mu)}{\sqrt{\mu}}+\frac{\partial_x\phi\partial_{v_1}\overline{G}}{\sqrt{\mu}}+\frac{\partial_x\phi\partial_{v_1}(\sqrt{\mu}\mathbf{g})}{\sqrt{\mu}},v_1\sqrt{\mu}\big\rangle.\end{aligned}\tag{4.68}$$

Taking ∂^α derivative of equation (4.68) with $|\alpha| \le N-1$ and we shall estimate each term. By using (4.63) and the fact that $P_2\mathbf{f} = \mathbf{a}(t,x)\sqrt{\mu}$, one has

$$\partial^\alpha \mathbf{a}_t + \langle \partial_x \partial^\alpha (\mathbf{I} - P_2)\mathbf{f}, v_1\sqrt{\mu}\rangle = 0. \tag{4.69}$$

By using this and the integration by parts, we have

$$(\langle -\partial_t \partial^\alpha \mathbf{f}, v_1\sqrt{\mu}\rangle, \partial_x\partial^\alpha \mathbf{a}) \le -\frac{d}{dt}(\langle \partial^\alpha \mathbf{f}, v_1\sqrt{\mu}\rangle, \partial_x\partial^\alpha \mathbf{a}) + C\|\partial_x\partial^\alpha(\mathbf{I}-P_2)\mathbf{f}\|_\nu^2.$$

Recalling that $\mathcal{L}_2 g = \Gamma(\sqrt{\mu}, g)$ in (1.27), we have from (3.9) that

$$\begin{aligned}&(\langle -v_1\partial_x\partial^\alpha(\mathbf{I}-P_2)\mathbf{f}, v_1\sqrt{\mu}\rangle + \langle \mathcal{L}_2\partial^\alpha(\mathbf{I}-P_2)\mathbf{f}, v_1\sqrt{\mu}\rangle, \partial_x\partial^\alpha\mathbf{a})\\ &\le C\varepsilon\|\partial_x\partial^\alpha\mathbf{a}\|^2 + C_\varepsilon\|\partial_x\partial^\alpha(\mathbf{I}-P_2)\mathbf{f}\|_\nu^2 + C_\varepsilon\|\partial^\alpha(\mathbf{I}-P_2)\mathbf{f}\|_\nu^2.\end{aligned}$$

By the similar arguments as (3.18), (3.29), (3.30) and Lemma 3.5, we have

$$\begin{aligned}&(\langle \partial^\alpha\Gamma(\frac{M-\mu}{\sqrt{\mu}}, \mathbf{f}) + \partial^\alpha\Gamma(\frac{\overline{G}}{\sqrt{\mu}}, \mathbf{f}) + \partial^\alpha\Gamma(\mathbf{g},\mathbf{f}), v_1\sqrt{\mu}\rangle, \partial_x\partial^\alpha\mathbf{a})\\ &\le C\varepsilon\|\partial_x\partial^\alpha\mathbf{a}\|^2 + C_\varepsilon(\eta_0 + \delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\sum_{|\alpha'|\le|\alpha|}\|\partial^{\alpha'}\mathbf{f}\|_\nu^2.\end{aligned}$$

For the last line of (4.68), owing to the fact that $\overline{G}$ and $\sqrt{\mu}\mathbf{g}$ satisfy (1.12), one has

$$\Big\langle \frac{\partial_x\phi\partial_{v_1}(M-\mu)}{\sqrt{\mu}} + \frac{\partial_x\phi\partial_{v_1}\overline{G}}{\sqrt{\mu}} + \frac{\partial_x\phi\partial_{v_1}(\sqrt{\mu}\mathbf{g})}{\sqrt{\mu}}, v_1\sqrt{\mu}\Big\rangle = -\partial_x\phi(\rho-1).$$

By using the fact that $|\rho - 1| < \eta_0$, we have

$$(-\partial^\alpha(\partial_x\phi(\rho-1)), \partial_x\partial^\alpha\mathbf{a}) \le \varepsilon\|\partial_x\partial^\alpha\mathbf{a}\|^2 + C_\varepsilon(\eta_0 + \delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\sum_{|\alpha'|\le N}\|\partial^{\alpha'}\partial_x\phi\|^2.$$

By using the Poisson equation (1.28) and the fact that $P_2\mathbf{f} = \mathbf{a}(t,x)\sqrt{\mu}$, one has

$$(\partial^\alpha\partial_x\phi, \partial_x\partial^\alpha\mathbf{a}) = -(\partial^\alpha\partial_{xx}\phi, \partial^\alpha\mathbf{a}) = 2\|\partial^\alpha\mathbf{a}\|^2.$$

Hence, for $|\alpha| \le N-1$ and any $\varepsilon > 0$ small enough, by using the above estimates, we can deduce

$$\begin{aligned}&\|\partial_x\partial^\alpha\mathbf{a}\|^2 + \|\partial^\alpha\mathbf{a}\|^2\\ &\le -C\frac{d}{dt}(\langle\partial^\alpha\mathbf{f}, v_1\sqrt{\mu}\rangle, \partial_x\partial^\alpha\mathbf{a}) + C\|\partial_x\partial^\alpha(\mathbf{I}-P_2)\mathbf{f}\|_\nu^2 + C\|\partial^\alpha(\mathbf{I}-P_2)\mathbf{f}\|_\nu^2\\ &\quad + C(\eta_0 + \delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\sum_{|\alpha'|\le N}(\|\partial^{\alpha'}\mathbf{f}\|_\nu^2 + \|\partial^{\alpha'}\partial_x\phi\|^2).\end{aligned} \tag{4.70}$$

On the other hand, by taking the inner product of (4.68) with $\partial_x\phi$ and then the similar arguments as (4.70), we arrive at

$$\begin{aligned}\|\mathbf{a}\|^2 + \|\partial_x\phi\|^2 \le& -C\frac{d}{dt}(\langle\mathbf{f}, v_1\sqrt{\mu}\rangle, \partial_x\phi) + C\|\partial_x(\mathbf{I}-P_2)\mathbf{f}\|_\nu^2\\ &+ C\|(\mathbf{I}-P_2)\mathbf{f}\|_\nu^2 + C(\eta_0 + \delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})(\|\mathbf{f}\|_\nu^2 + \|\partial_x\phi\|^2).\end{aligned} \tag{4.71}$$

By using (4.70), (4.71) and the fact that $\|\partial^\alpha P_2\mathbf{f}\| \approx \|\partial^\alpha \mathbf{a}(t,x)\| \approx \|\partial^\alpha\partial_{xx}\phi\|$, we have

$$\sum_{|\alpha|\le N}(\|\partial^\alpha P_2\mathbf{f}\|^2+\|\partial^\alpha\partial_x\phi\|^2) \le -\frac{d}{dt}H(t)+C\sum_{|\alpha|\le N}\|\partial^\alpha(\mathbf{I}-P_2)\mathbf{f}\|_\nu^2$$
$$+C(\eta_0+\delta^{\frac16}+\sqrt{\varepsilon_0})\sum_{|\alpha|\le N}(\|\partial^\alpha\mathbf{f}\|_\nu^2+\|\partial^\alpha\partial_x\phi\|^2). \quad (4.72)$$

Here the function $H(t)$ is defined as

$$\begin{aligned} H(t) &= C\sum_{|\alpha|\le N-1}((\langle\partial^\alpha\mathbf{f}, v_1\sqrt{\mu}\rangle, \partial_x\partial^\alpha\mathbf{a})+(\langle\mathbf{f}, v_1\sqrt{\mu}\rangle, \partial_x\phi)\\ &\le C\sum_{|\alpha|\le N}(\|\partial^\alpha\mathbf{f}\|^2+\|\partial^\alpha\partial_x\phi\|^2). \end{aligned} \quad (4.73)$$

As a consequence, we get from (4.67) and (4.72)$\times\kappa_1$ with a small constant $\kappa_1>0$ that

$$\begin{aligned} &\frac{d}{dt}\Big\{\sum_{1\le|\alpha|\le N}\|\frac{\partial^\alpha F_1}{\sqrt{\mu}}\|^2+\sum_{|\alpha|\le N}(\|\partial^\alpha\mathbf{f}\|^2+\frac12\|\partial^\alpha\partial_x\phi\|^2)+\kappa_1 H(t)\Big\}\\ &+\lambda\sum_{1\le|\alpha|\le N}\|\partial^\alpha\mathbf{g}\|_\nu^2+\lambda\sum_{|\alpha|\le N}(\|\partial^\alpha\mathbf{f}\|_\nu^2+\|\partial^\alpha\partial_x\phi\|^2)\\ &\le C(\eta_0+\varepsilon)(\sum_{2\le|\alpha|\le N}\|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2+\|\mathbf{g}\|_\nu^2)+C_\varepsilon\delta^{\frac16}(1+t)^{-\frac76}\\ &+C_\varepsilon(\eta_0+\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t)+C\sum_{|\alpha|\le N}\|\partial^\alpha\partial_x\phi\|\mathcal{F}_{N,l,q}(t). \end{aligned} \quad (4.74)$$

Here we have used the smallness of $\varepsilon>0$, $\delta>0$ and $\eta_0>0$ and

$$\|\partial^\alpha\mathbf{f}\|_\nu^2 \le C\|\partial^\alpha(\mathbf{I}-P_2)\mathbf{f}\|_\nu^2+C\|\partial^\alpha P_2\mathbf{f}\|_\nu^2. \quad (4.75)$$

In summary, for suitably large constant $\widetilde{C}_2>0$ and any $\varepsilon>0$ small enough, we can obtain (4.36) by (4.74)$\times\widetilde{C}_2$ and (4.40). This completes the proof of Lemma 4.2. □

4.3. First main energy estimates. We will give the estimates for the pure spatial derivatives of the solution without the velocity weight by Lemma 4.1 and Lemma 4.2.

LEMMA 4.3. *Under the conditions of Lemma 4.1, one has*

$$\begin{aligned} &\frac{d}{dt}\widetilde{\mathcal{E}_N}(t)+\lambda\|\sqrt{\bar{u}_{1x}}(\widetilde{\rho},\widetilde{u}_1,\widetilde{\theta})\|^2\\ &+\lambda\Big\{\sum_{|\alpha|\le N}(\|\partial^\alpha\mathbf{g}\|_\nu^2+\|\partial^\alpha\mathbf{f}\|_\nu^2+\|\partial^\alpha\partial_x\phi\|^2)+\sum_{1\le|\alpha|\le N}\|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2\Big\}\\ &\le C\delta^{\frac16}(1+t)^{-\frac76}+C(\eta_0+\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t)\\ &+C\sum_{|\alpha|\le N}\|\partial^\alpha\partial_x\phi\|\mathcal{F}_{N,l,q}(t). \end{aligned} \quad (4.76)$$

Here $\mathcal{F}_{N,l,q}(t)$ *is given by* (4.2) *and* $\widetilde{\mathcal{E}_N}(t)$ *is given by*

$$\begin{aligned}\widetilde{\mathcal{E}_N}(t)=&\Big\{\|\mathbf{g}\|^2+\widetilde{C}_1\{\int_{\mathbb{R}}\eta dx-(L_M^{-1}P_1[v_1(\bar{\theta}\ln M)_x\\&-\frac{3}{2}\bar{u}_{1x}v_1^2]M,\frac{\sqrt{\mu}\mathbf{g}}{M})+\lambda\int_{\mathbb{R}}\widetilde{u}_1\widetilde{\rho}_x dx\}\Big\}\\&+\widetilde{C}_3\Big\{\sum_{1\le|\alpha|\le N-1}H_\alpha(t)+\widetilde{C}_2\{\sum_{1\le|\alpha|\le N}\|\frac{\partial^\alpha F_1}{\sqrt{\mu}}\|^2\\&+\sum_{|\alpha|\le N}(\|\partial^\alpha\mathbf{f}\|^2+\frac{1}{2}\|\partial^\alpha\partial_x\phi\|^2)+\kappa_1H(t)\}\Big\},\end{aligned}\tag{4.77}$$

for constants $\widetilde{C}_3\gg\widetilde{C}_1\gg1$, *where* $H_\alpha(t)$ *and* $H(t)$ *are given by* (4.41) *and* (4.73), *respectively.*

Proof. Letting $\widetilde{C}_3\gg\widetilde{C}_1\gg1$, we see the estimate (4.76) follows from (4.36)$\times\widetilde{C}_3$ and (4.1) by further taking $\varepsilon>0$, $\eta_0>0$ small enough. This finishes the proof of Lemma 4.3. □

5. Weighted energy estimates. This section is devoted to deducing the weighted energy estimates for the solution of (1.6) and (1.7) in order to close the energy estimates.

5.1. Pure spatial derivative estimates. The pure spatial derivative estimates of the solution with the weighted function can be stated as follows.

LEMMA 5.1. *Under the conditions of Lemma 4.1, it holds that*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big\{\sum_{|\alpha|\le1}\|\partial^\alpha\mathbf{g}\|_{2,w}^2+\sum_{2\le|\alpha|\le N}\|\frac{\partial^\alpha F_1}{\sqrt{\mu}}\|_{2,w}^2+\sum_{|\alpha|\le N}\|\partial^\alpha\mathbf{f}\|_{2,w}^2\Big\}\\&+\lambda\sum_{|\alpha|\le N}\frac{q_2q_3}{(1+t)^{1+q_3}}\big(\|\langle v\rangle\partial^\alpha\mathbf{g}\|_{2,w}^2+\|\langle v\rangle\partial^\alpha\mathbf{f}\|_{2,w}^2\big)+\lambda\sum_{|\alpha|\le N}(\|\partial^\alpha\mathbf{g}\|_{\nu,w}^2+\|\partial^\alpha\mathbf{f}\|_{\nu,w}^2)\\\le{}&C\sum_{1\le|\alpha|\le N}\|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2+C\sum_{|\alpha|\le N}\big(\|\partial^\alpha\mathbf{g}\|_\nu^2+\|\partial^\alpha\mathbf{f}\|_\nu^2+\|\partial^\alpha\partial_x\phi\|^2\big)\\&+C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C(\eta_0+\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t)+C\sum_{|\alpha|\le N}\|\partial^\alpha\partial_x\phi\|\mathcal{F}_{N,l,q}(t).\end{aligned}\tag{5.1}$$

Here $\mathcal{F}_{N,l,q}(t)$ *and* $w=w(0)$ *are defined by* (4.2) *and* (1.29), *respectively.*

Proof. We take the derivative ∂^α of (1.23) with $|\alpha|\le1$ and then take the inner product with $w^2(0)\partial^\alpha\mathbf{g}$ over $\mathbb{R}\times\mathbb{R}^3$ to get

$$\begin{aligned}&(\partial_t\partial^\alpha\mathbf{g},w^2(0)\partial^\alpha\mathbf{g})+(v_1\partial_x\partial^\alpha\mathbf{g},w^2(0)\partial^\alpha\mathbf{g})-(\mathcal{L}_1\partial^\alpha\mathbf{g},w^2(0)\partial^\alpha\mathbf{g})\\={}&(\partial^\alpha\Gamma(\mathbf{g},\frac{M-\mu}{\sqrt{\mu}})+\partial^\alpha\Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{g}),w^2(0)\partial^\alpha\mathbf{g})+(\partial^\alpha\Gamma(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}),w^2(0)\partial^\alpha\mathbf{g})\\&+(\partial^\alpha\frac{P_0(v_1\sqrt{\mu}\partial_x\mathbf{g})}{\sqrt{\mu}}-\frac{1}{\sqrt{\mu}}\partial^\alpha P_1v_1M\{\frac{|v-u|^2\widetilde{\theta}_x}{2R\theta^2}+\frac{(v-u)\cdot\widetilde{u}_x}{R\theta}\},w^2(0)\partial^\alpha\mathbf{g})\\&-(\frac{\partial^\alpha P_1(v_1\partial_x\overline{G})}{\sqrt{\mu}}+\frac{\partial^\alpha\partial_t\overline{G}}{\sqrt{\mu}},w^2(0)\partial^\alpha\mathbf{g})+(\frac{\partial^\alpha P_1(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}},w^2(0)\partial^\alpha\mathbf{g}).\end{aligned}\tag{5.2}$$

Now we will estimate (5.2) term by term. Recalling the weight function $w = w(0)$ in (1.29), one has

$$(\partial_t\partial^\alpha \mathbf{g}, w^2(0)\partial^\alpha \mathbf{g}) = \frac{1}{2}\frac{d}{dt}\|\partial^\alpha \mathbf{g}\|_{2,w}^2 + \frac{q_2 q_3}{2(1+t)^{1+q_3}}\|\langle v\rangle\partial^\alpha \mathbf{g}\|_{2,w}^2.$$

And the second term on the left hand side of (5.2) vanishes by integration by parts. By (3.7), one has

$$-(\mathcal{L}_1\partial^\alpha \mathbf{g}, w^2(0)\partial^\alpha \mathbf{g}) \geq \frac{1}{2}\|\partial^\alpha \mathbf{g}\|_{\nu,w}^2 - C\|\partial^\alpha \mathbf{g}\|_\nu^2.$$

By (3.15) and (3.19), we see

$$\begin{aligned}&|(\partial^\alpha\Gamma(\mathbf{g}, \frac{M-\mu}{\sqrt{\mu}}) + \partial^\alpha\Gamma(\frac{M-\mu}{\sqrt{\mu}}, \mathbf{g}), w^2(0)\partial^\alpha \mathbf{g})| + |(\partial^\alpha\Gamma(\frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}}), w^2(0)\partial^\alpha \mathbf{g})|\\ &\leq C\varepsilon\|\partial^\alpha \mathbf{g}\|_{\nu,w}^2 + C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C_\varepsilon(\eta_0 + \delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).\end{aligned}$$

By (1.11), the imbedding inequality and Lemma 2.1, one has

$$\begin{aligned}&|(\frac{\partial^\alpha P_0(v_1\sqrt{\mu}\partial_x \mathbf{g})}{\sqrt{\mu}}, w^2(0)\partial^\alpha \mathbf{g})|\\ &\leq C\varepsilon\|\partial^\alpha \mathbf{g}\|_{\nu,w}^2 + C_\varepsilon\|\partial^\alpha \mathbf{g}_x\|_\nu^2 + C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C_\varepsilon\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t),\end{aligned}$$

where we have used the fact that $|\langle v\rangle^b\mu^{-\frac{1}{2}}M^{1-\epsilon}|_{2,w} \leq C$ for any $b \geq 0$ and some positive constants ϵ, q_1, q_2 small enough by (1.20).

It follows from (4.33), the imbedding inequality and Lemma 3.2 that

$$\begin{aligned}&|(\frac{1}{\sqrt{\mu}}\partial^\alpha P_1 v_1 M\{\frac{|v-u|^2\widetilde{\theta}_x}{2R\theta^2} + \frac{(v-u)\cdot\widetilde{u}_x}{R\theta}\}, w^2(0)\partial^\alpha \mathbf{g})|\\ &\leq C\varepsilon\|\partial^\alpha \mathbf{g}\|_{\nu,w}^2 + C_\varepsilon\|[\partial^\alpha\widetilde{u}_x, \partial^\alpha\widetilde{\theta}_x]\|^2 + C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C_\varepsilon\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).\end{aligned}$$

By (3.24), (3.26) and Lemma 3.8, we have

$$\begin{aligned}&|(\frac{\partial^\alpha P_1(v_1\partial_x\overline{G})}{\sqrt{\mu}} + \frac{\partial_t\partial^\alpha\overline{G}}{\sqrt{\mu}}, w^2(0)\partial^\alpha \mathbf{g})|\\ &\leq C\varepsilon\|\partial^\alpha \mathbf{g}\|_{\nu,w}^2 + C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C_\varepsilon(\delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).\end{aligned}$$

Now we deal with the complicated term containing $\partial_x\phi$. First note that

$$\frac{\partial^\alpha P_1(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}} = \partial^\alpha(\partial_x\phi\partial_{v_1}\mathbf{f}) - \partial^\alpha(\partial_x\phi\frac{v_1}{2}\mathbf{f}) - \frac{\partial^\alpha P_0[\partial_x\phi\partial_{v_1}(\sqrt{\mu}\mathbf{f})]}{\sqrt{\mu}}. \tag{5.3}$$

For $|\alpha| \leq 1$, we get from the imbedding inequality and (4.2) that

$$\begin{aligned}&\big|(\partial^\alpha(\partial_x\phi\partial_{v_1}\mathbf{f}), w^2(0)\partial^\alpha \mathbf{g})\big|\\ &\leq C\sum_{|\alpha_1|\leq|\alpha|}\|\partial^{\alpha_1}\partial_x\phi\|_{L^\infty}\|\langle v\rangle^{\frac{|\gamma|}{2}}\langle v\rangle^{-|\gamma|}w(0)\partial_{v_1}\partial^{\alpha-\alpha_1}\mathbf{f}\|\|\langle v\rangle^{\frac{|\gamma|}{2}}w(0)\partial^\alpha \mathbf{g}\|\\ &\leq C\sum_{|\alpha'|\leq N}\|\partial_x\partial^{\alpha'}\phi\|\|\langle v\rangle\partial_{v_1}\partial^{\alpha-\alpha_1}\mathbf{f}\|_{2,w(1)}\|\langle v\rangle\partial^\alpha \mathbf{g}\|_{2,w(0)}\\ &\leq C\sum_{|\alpha'|\leq N}\|\partial_x\partial^{\alpha'}\phi\|\mathcal{F}_{N,l,q}(t),\end{aligned} \tag{5.4}$$

where we used the fact that $\langle v\rangle^{\frac{|\gamma|}{2}}\langle v\rangle^{-|\gamma|}w(0)=\langle v\rangle^{\frac{|\gamma|}{2}}w(1)\leq\langle v\rangle w(1)$ for $|\gamma|\leq 2$ by (1.29).

Similarly we also have

$$\big|(\partial^\alpha(\partial_x\phi\frac{v_1}{2}\mathbf{f}),w^2(0)\partial^\alpha\mathbf{g})\big|\leq C\sum_{|\alpha'|\leq N}\|\partial_x\partial^{\alpha'}\phi\|\mathcal{F}_{N,l,q}(t). \tag{5.5}$$

It follows from (1.10), (1.11) and the elementary inequalities that

$$\big|(-\frac{\partial^\alpha P_0[\partial_x\phi\partial_{v_1}(\sqrt{\mu}\mathbf{f})]}{\sqrt{\mu}},w^2(0)\partial^\alpha\mathbf{g})\big|\leq C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t). \tag{5.6}$$

Hence, by the estimates from (5.3) to (5.6), we get

$$\begin{aligned}&\big|(\frac{\partial^\alpha P_1(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}},w^2(0)\partial^\alpha\mathbf{g})\big|\\ &\leq C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t)+C\sum_{|\alpha'|\leq N}\|\partial_x\partial^{\alpha'}\phi\|\mathcal{F}_{N,l,q}(t).\end{aligned}$$

Collecting the above related estimates and taking $\varepsilon>0$ small enough, we arrive at

$$\begin{aligned}&\sum_{|\alpha|\leq 1}\Big\{\frac{1}{2}\frac{d}{dt}\|\partial^\alpha\mathbf{g}\|^2_{2,w}+\frac{q_2q_3}{2(1+t)^{1+q_3}}\|\langle v\rangle\partial^\alpha\mathbf{g}\|^2_{2,w}+\lambda\|\partial^\alpha\mathbf{g}\|^2_{\nu,w}\Big\}\\ &\leq C\sum_{|\alpha|\leq 1}\big\{\|[\partial^\alpha\widetilde{u}_x,\partial^\alpha\widetilde{\theta}_x]\|^2+\|\partial^\alpha\mathbf{g}\|^2_\nu+\|\partial^\alpha\mathbf{g}_x\|^2_\nu\big\}+C\sum_{|\alpha|\leq N}\|\partial_x\partial^\alpha\phi\|\mathcal{F}_{N,l,q}(t)\\ &\quad+C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C(\eta_0+\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).\end{aligned} \tag{5.7}$$

Next, we consider the higher order weighted spatial derivative estimates for the microscopic component $\mathbf{g}$. We take the derivative ∂^α of (4.42) with $2\leq|\alpha|\leq N$ and then take the inner product with $w^2(0)\frac{\partial^\alpha F_1}{\sqrt{\mu}}$ to get

$$\begin{aligned}&(\partial_t(\frac{\partial^\alpha F_1}{\sqrt{\mu}}),w^2(0)\frac{\partial^\alpha F_1}{\sqrt{\mu}})+(v_1\partial_x(\frac{\partial^\alpha F_1}{\sqrt{\mu}}),w^2(0)\frac{\partial^\alpha F_1}{\sqrt{\mu}})-(\mathcal{L}_1\partial^\alpha\mathbf{g},w^2(0)\frac{\partial^\alpha F_1}{\sqrt{\mu}})\\ =&(\partial^\alpha\Gamma(\mathbf{g},\frac{M-\mu}{\sqrt{\mu}})+\partial^\alpha\Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{g}),w^2(0)\frac{\partial^\alpha F_1}{\sqrt{\mu}})+(\partial^\alpha\Gamma(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}),w^2(0)\frac{\partial^\alpha F_1}{\sqrt{\mu}})\\ &+(\frac{1}{\sqrt{\mu}}\partial^\alpha P_1v_1M\big\{\frac{|v-u|^2\overline{\theta}_x}{2R\theta^2}+\frac{(v-u)\cdot\bar{u}_x}{R\theta}\big\},w^2(0)\frac{\partial^\alpha F_1}{\sqrt{\mu}})\\ &+(\partial^\alpha(\frac{\partial_x\phi\partial_{v_1}F_2}{\sqrt{\mu}}),w^2(0)\frac{\partial^\alpha F_1}{\sqrt{\mu}}).\end{aligned} \tag{5.8}$$

We will estimate (5.8) term by term. First note that

$$(\partial_t(\frac{\partial^\alpha F_1}{\sqrt{\mu}}),w^2(0)\frac{\partial^\alpha F_1}{\sqrt{\mu}})=\frac{1}{2}\frac{d}{dt}\|\frac{\partial^\alpha F_1}{\sqrt{\mu}}\|^2_{2,w}+\frac{q_2q_3}{2(1+t)^{1+q_3}}\|\langle v\rangle\frac{\partial^\alpha F_1}{\sqrt{\mu}}\|^2_{2,w}. \tag{5.9}$$

Next we estimate the last term in (5.9). Recalling that $F_1=M+\overline{G}+\sqrt{\mu}\mathbf{g}$, we have from (4.44), (3.26), the imbedding inequality and Lemma 2.1 that

$$\begin{aligned}
&\|\langle v\rangle\frac{\partial^\alpha F_1}{\sqrt{\mu}}\|_{2,w}^2\\
&\geq \|\langle v\rangle\partial^\alpha\mathbf{g}\|_{2,w}^2-\|\langle v\rangle\frac{\partial^\alpha M}{\sqrt{\mu}}\|_{2,w}^2-\|\langle v\rangle\frac{\partial^\alpha\overline{G}}{\sqrt{\mu}}\|_{2,w}^2\\
&\geq \|\langle v\rangle\partial^\alpha\mathbf{g}\|_{2,w}^2-C\|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2-C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}-C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).
\end{aligned}\tag{5.10}$$

Here we have used the fact that $|\langle v\rangle^b\mu^{-\frac{1}{2}}M^{1-\epsilon}|_{2,w}\leq C$ by (1.20) for any $b\geq 0$ and some positive constants ϵ, q_1, q_2 small enough.

The second term on the left hand side of (5.8) vanishes by the integration by parts. For the third term, recalling that $\mathcal{L}_1\mathbf{g}=\Gamma(\sqrt{\mu},\mathbf{g})+\Gamma(\mathbf{g},\sqrt{\mu})$ and using (3.9), (3.10), (3.26), (4.44) and Lemma 2.1, we have

$$\begin{aligned}
&|(\mathcal{L}_1\partial^\alpha\mathbf{g},w^2(0)\frac{\partial^\alpha(M+\overline{G})}{\sqrt{\mu}})|\\
&\leq C\|\partial^\alpha\mathbf{g}\|_\nu^2+C\|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2+C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).
\end{aligned}$$

From (3.7), we obtain

$$-(\mathcal{L}_1\partial^\alpha\mathbf{g},w^2(0)\partial^\alpha\mathbf{g})\geq\frac{1}{2}\|\partial^\alpha\mathbf{g}\|_{\nu,w}^2-C\|\partial^\alpha\mathbf{g}\|_\nu^2.$$

It follows from the above two estimates that

$$\begin{aligned}
(\mathcal{L}_1\partial^\alpha\mathbf{g},w^2(0)\frac{\partial^\alpha F_1}{\sqrt{\mu}})\leq&-\frac{1}{2}\|\partial^\alpha\mathbf{g}\|_{\nu,w}^2+C\|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2\\
&+C\|\partial^\alpha\mathbf{g}\|_\nu^2+C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).
\end{aligned}\tag{5.11}$$

For $2\leq|\alpha|\leq N$, by the fact that $F_1=M+\overline{G}+\sqrt{\mu}\mathbf{g}$, we have from the similar arguments as (4.47) that

$$\begin{aligned}
\|w(0)\frac{\partial^\alpha F_1}{\sqrt{\mu}}\|_\nu^2\leq& C(\|\partial^\alpha\mathbf{g}\|_{\nu,w}^2+\|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2)\\
&+C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).
\end{aligned}\tag{5.12}$$

By this, (3.15) and (3.19), one has

$$\begin{aligned}
&\big|(\partial^\alpha\Gamma(\mathbf{g},\frac{M-\mu}{\sqrt{\mu}})+\partial^\alpha\Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{g}),w^2(0)\frac{\partial^\alpha F_1}{\sqrt{\mu}})\big|+\big|(\partial^\alpha\Gamma(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}),w^2(0)\frac{\partial^\alpha F_1}{\sqrt{\mu}})\big|\\
&\leq C\varepsilon(\|\partial^\alpha\mathbf{g}\|_{\nu,w}^2+\|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2)+C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C_\varepsilon(\eta_0+\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).
\end{aligned}$$

We now estimate the last line of (5.8). With the help of (4.33) and (5.12), we arrive at

$$\begin{aligned}
&\big|(\frac{1}{\sqrt{\mu}}\partial^\alpha P_1v_1M\big\{\frac{|v-u|^2\overline{\theta}_x}{2R\theta^2}+\frac{(v-u)\cdot\bar{u}_x}{R\theta}\big\},w^2(0)\frac{\partial^\alpha F_1}{\sqrt{\mu}})\big|\\
&\leq C\varepsilon(\|\partial^\alpha\mathbf{g}\|_{\nu,w}^2+\|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2)+C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C_\varepsilon\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t).
\end{aligned}$$

Then we deal with the complicated term containing $\partial_x\phi$. Recalling $F_2 = \sqrt{\mu}\mathbf{f}$, one has

$$\frac{\partial^\alpha(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}} = \partial_x\phi\partial_{v_1}\partial^\alpha\mathbf{f} + \sum_{1\le|\alpha_1|\le|\alpha|} C_\alpha^{\alpha_1}\partial^{\alpha_1}\partial_x\phi\partial_{v_1}\partial^{\alpha-\alpha_1}\mathbf{f} - \sum_{|\alpha_1|\le|\alpha|} C_\alpha^{\alpha_1}\partial^{\alpha_1}\partial_x\phi\frac{v_1}{2}\partial^{\alpha-\alpha_1}\mathbf{f}.$$

For $1 \le |\alpha_1| \le \frac{|\alpha|}{2}$ with $2 \le |\alpha| \le N$, similar arguments as (5.4) imply

$$\begin{aligned}
&\big|(\partial^{\alpha_1}\partial_x\phi\partial_{v_1}\partial^{\alpha-\alpha_1}\mathbf{f}, w^2(0)\partial^\alpha\mathbf{g})\big| \\
&\le C\|\partial^{\alpha_1}\partial_x\phi\|_{L^\infty}\|\langle v\rangle^{\frac{|\gamma|}{2}}\langle v\rangle^{-|\gamma|}w(0)\partial_{v_1}\partial^{\alpha-\alpha_1}\mathbf{f}\|\|\langle v\rangle^{\frac{|\gamma|}{2}}w(0)\partial^\alpha\mathbf{g}\| \\
&\le C\sum_{|\alpha'|\le N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t).
\end{aligned} \tag{5.13}$$

Here we used the fact that $\langle v\rangle^{\frac{|\gamma|}{2}}\langle v\rangle^{-|\gamma|}w(0) = \langle v\rangle^{\frac{|\gamma|}{2}}w(1) \le \langle v\rangle w(1)$ for $|\gamma| \le 2$ by (1.29).

For $\frac{|\alpha|}{2} \le |\alpha_1| \le |\alpha|$ with $2 \le |\alpha| \le N$, we have from the similar arguments as (5.13) that

$$\big|(\partial^{\alpha_1}\partial_x\phi\partial_{v_1}\partial^{\alpha-\alpha_1}\mathbf{f}, w^2(0)\partial^\alpha\mathbf{g})\big| \le C\sum_{|\alpha'|\le N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t). \tag{5.14}$$

By (5.13) and (5.14), one has

$$\sum_{1\le|\alpha_1|\le|\alpha|} C_\alpha^{\alpha_1}\big|(\partial^{\alpha_1}\partial_x\phi\partial_{v_1}\partial^{\alpha-\alpha_1}\mathbf{f}, w^2(0)\partial^\alpha\mathbf{g})\big| \le C\sum_{|\alpha'|\le N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t).$$

Similarly, it holds that

$$\sum_{|\alpha_1|\le|\alpha|}|(\partial^{\alpha_1}\partial_x\phi\frac{v_1}{2}\partial^{\alpha-\alpha_1}\mathbf{f}, w^2(0)\partial^\alpha\mathbf{g})| \le C\sum_{|\alpha'|\le N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t).$$

For $2 \le |\alpha| \le N$, we have from the above estimates that

$$\begin{aligned}
&(\frac{\partial^\alpha(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}}, w^2(0)\partial^\alpha\mathbf{g}) - (\partial_x\phi\partial_{v_1}\partial^\alpha\mathbf{f}, w^2(0)\partial^\alpha\mathbf{g}) \\
&\le C\sum_{|\alpha'|\le N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t).
\end{aligned} \tag{5.15}$$

For $2 \le |\alpha| \le N$, by the integration by parts and the fact that $F_2 = \sqrt{\mu}\mathbf{f}$, one has

$$\begin{aligned}
&|(\frac{\partial^\alpha(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}}, w^2(0)\frac{\partial^\alpha(M+\overline{G})}{\sqrt{\mu}})| \\
&= |(\partial^\alpha(\partial_x\phi\mathbf{f}\sqrt{\mu}), \partial_{v_1}(\frac{w^2(0)\partial^\alpha(M+\overline{G})}{\mu}))| \\
&\le C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C(\delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).
\end{aligned} \tag{5.16}$$

By the estimates (5.15), (5.16) and the fact that $F_1 = M + \overline{G} + \sqrt{\mu}\mathbf{g}$, we can obtain

$$\begin{aligned}&(\frac{\partial^\alpha(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}}, w^2(0)\frac{\partial^\alpha F_1}{\sqrt{\mu}}) - (\partial_x\phi\partial_{v_1}\partial^\alpha\mathbf{f}, w^2(0)\partial^\alpha\mathbf{g})\\ &\le C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C(\delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t) + C\sum_{|\alpha'|\le N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t). \end{aligned}\tag{5.17}$$

Hence, we have from those above estimates and small $\varepsilon > 0$ that

$$\begin{aligned}&\sum_{2\le|\alpha|\le N}\Big\{\frac{1}{2}\frac{d}{dt}\|\frac{\partial^\alpha F_1}{\sqrt{\mu}}\|^2_{2,w} + \frac{q_2q_3}{2(1+t)^{1+q_3}}\|\langle v\rangle\partial^\alpha\mathbf{g}\|^2_{2,w}\\ &+\lambda\|\partial^\alpha\mathbf{g}\|^2_{\nu,w} - (\partial_x\phi\partial_{v_1}\partial^\alpha\mathbf{f}, w^2(0)\partial^\alpha\mathbf{g})\Big\}\\ &\le C\sum_{2\le|\alpha|\le N}(\|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2 + \|\partial^\alpha\mathbf{g}\|^2_\nu) + C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}\\ &+C(\eta_0 + \delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t) + C\sum_{|\alpha|\le N}\|\partial^\alpha\partial_x\phi\|\mathcal{F}_{N,l,q}(t).\end{aligned}\tag{5.18}$$

This ends the proof of the high order weighted spatial energy estimates of $\mathbf{g}$.

Now, we turn to deduce the weighted estimates of the function $\mathbf{f}$. Taking ∂^α derivative of equation (1.26) with $|\alpha| \le N$ and taking the inner product with $w^2(0)\partial^\alpha\mathbf{f}$, we obtain

$$\begin{aligned}&(\partial^\alpha\partial_t\mathbf{f}, w^2(0)\partial^\alpha\mathbf{f}) + (v_1\partial^\alpha\partial_x\mathbf{f}, w^2(0)\partial^\alpha\mathbf{f}) + (\partial^\alpha\partial_x\phi v_1\sqrt{\mu}, w^2(0)\partial^\alpha\mathbf{f})\\ &= (\partial^\alpha\mathcal{L}_2\mathbf{f}, w^2(0)\partial^\alpha\mathbf{f}) + (\partial^\alpha\Gamma(\frac{M-\mu}{\sqrt{\mu}}, \mathbf{f}) + \partial^\alpha\Gamma(\frac{\overline{G}}{\sqrt{\mu}}, \mathbf{f}) + \partial^\alpha\Gamma(\mathbf{g}, \mathbf{f}), w^2(0)\partial^\alpha\mathbf{f})\\ &+(\partial^\alpha(\frac{\partial_x\phi\partial_{v_1}(M-\mu)}{\sqrt{\mu}}) + \partial^\alpha(\frac{\partial_x\phi\partial_{v_1}\overline{G}}{\sqrt{\mu}}), w^2(0)\partial^\alpha\mathbf{f})\\ &+(\partial^\alpha(\frac{\partial_x\phi\partial_{v_1}(\sqrt{\mu}\mathbf{g})}{\sqrt{\mu}}), w^2(0)\partial^\alpha\mathbf{f}).\end{aligned}\tag{5.19}$$

We will estimate (5.19) term by term. Recalling the weight function $w = w(0)$ in (1.29), one has

$$(\partial_t\partial^\alpha\mathbf{f}, w^2(0)\partial^\alpha\mathbf{f}) = \frac{1}{2}\frac{d}{dt}\|\partial^\alpha\mathbf{f}\|^2_{2,w} + \frac{q_2q_3}{2(1+t)^{1+q_3}}\|\langle v\rangle\partial^\alpha\mathbf{f}\|^2_{2,w}.$$

And the second term on the left hand side of (5.19) vanishes by the integration by parts. It holds that

$$|(\partial^\alpha\partial_x\phi v_1\sqrt{\mu}, w^2(0)\partial^\alpha\mathbf{f})| \le C\|\partial^\alpha\partial_x\phi\|^2 + C\|\partial^\alpha\mathbf{f}\|^2_\nu,$$

where we used the fact that $|\langle v\rangle^b\mu^{\frac{1}{2}}w^2(0)|_2 \le C$ by (1.29) for any $b \ge 0$ and some positive constants q_1, q_2 small enough. It follows from (3.7) that

$$-(\mathcal{L}_2\partial^\alpha\mathbf{f}, w^2(0)\partial^\alpha\mathbf{f}) \ge \frac{1}{2}\|\partial^\alpha\mathbf{f}\|^2_{\nu,w} - C\|\partial^\alpha\mathbf{f}\|^2_\nu.$$

By the similar arguments as (3.15), (3.29) and (3.30), we obtain

$$\begin{aligned}&|(\partial^\alpha\Gamma(\frac{M-\mu}{\sqrt{\mu}}, \mathbf{f}) + \partial^\alpha\Gamma(\frac{\overline{G}}{\sqrt{\mu}}, \mathbf{f}) + \partial^\alpha\Gamma(\mathbf{g}, \mathbf{f}), w^2(0)\partial^\alpha\mathbf{f})|\\ &\le C\varepsilon\|\partial^\alpha\mathbf{f}\|^2_{\nu,w} + C_\varepsilon(\eta_0 + \delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).\end{aligned}$$

In terms of (1.20), (3.14), (4.44), Lemma 2.1 and the imbedding inequality, we arrive at

$$
\begin{aligned}
&|(\partial^\alpha(\frac{\partial_x\phi\partial_{v_1}(M-\mu)}{\sqrt{\mu}})+\partial^\alpha(\frac{\partial_x\phi\partial_{v_1}\overline{G}}{\sqrt{\mu}}),w^2(0)\partial^\alpha\mathbf{f})|\\
&\leq C(\eta_0+\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t). \qquad (5.20)
\end{aligned}
$$

For the last term of (5.19), we have from the similar arguments as (5.15) that

$$
\begin{aligned}
&(\frac{\partial^\alpha(\partial_x\phi\partial_{v_1}(\sqrt{\mu}\mathbf{g}))}{\sqrt{\mu}},w^2(0)\partial^\alpha\mathbf{f})\\
&\leq \big(\partial_x\phi\partial_{v_1}\partial^\alpha\mathbf{g},w^2(0)\partial^\alpha\mathbf{f}\big)+C\sum_{|\alpha'|\leq N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t).
\end{aligned}
$$

By taking $\varepsilon>0$ small enough, we have from the above estimates that

$$
\begin{aligned}
&\sum_{|\alpha|\leq N}\Big\{\frac{1}{2}\frac{d}{dt}\|\partial^\alpha\mathbf{f}\|_{2,w}^2+\frac{q_2q_3}{2(1+t)^{1+q_3}}\|\langle v\rangle\partial^\alpha\mathbf{f}\|_{2,w}^2\\
&+\lambda\|\partial^\alpha\mathbf{f}\|_{\nu,w}^2-(\partial_x\phi\partial_{v_1}\partial^\alpha\mathbf{g},w^2(0)\partial^\alpha\mathbf{f})\Big\}\\
&\leq C\sum_{|\alpha|\leq N}(\|\partial^\alpha\partial_x\phi\|^2+\|\partial^\alpha\mathbf{f}\|_\nu^2)+C(\eta_0+\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t)\\
&+C\sum_{|\alpha|\leq N}\|\partial^\alpha\partial_x\phi\|\mathcal{F}_{N,l,q}(t). \qquad (5.21)
\end{aligned}
$$

By the integration by parts and the direct calculations, we have

$$
\begin{aligned}
&-\sum_{2\leq|\alpha|\leq N}(\partial_x\phi\partial_{v_1}\partial^\alpha\mathbf{f},w^2(0)\partial^\alpha\mathbf{g})\\
&-\sum_{|\alpha|\leq N}(\partial_x\phi\partial_{v_1}\partial^\alpha\mathbf{g},w^2(0)\partial^\alpha\mathbf{f})\leq C\sum_{|\alpha|\leq 1}\|\partial_x\partial^\alpha\phi\|\mathcal{F}_{N,l,q}(t). \qquad (5.22)
\end{aligned}
$$

By this, (5.7), (5.18) and (5.21), we have that (5.1) holds. This completes the proof of the weighted spatial energy estimates of the functions $\mathbf{g}$ and $\mathbf{f}$. □

5.2. Mixed spatial-velocity derivative estimates. The mixed spatial-velocity spatial derivative estimates of the solution with the weighted function can be stated as follows.

Lemma 5.2. *Under the conditions of Lemma 4.1, we have*

$$
\begin{aligned}
&\sum_{|\alpha|=0}^{N-1}C_{|\alpha|}\sum_{j=1}^{N-|\alpha|}C_j\sum_{|\beta|=j}\frac{d}{dt}\{\|\partial_\beta^\alpha\mathbf{g}\|_{2,w}^2+\|\partial_\beta^\alpha\mathbf{f}\|_{2,w}^2\}\\
&+\sum_{|\alpha|+|\beta|\leq N,|\beta|\geq1}\Big\{\frac{\lambda q_2q_3}{2(1+t)^{1+q_3}}(\|\langle v\rangle\partial_\beta^\alpha\mathbf{g}\|_{2,w}^2+\|\langle v\rangle\partial_\beta^\alpha\mathbf{f}\|_{2,w}^2)+\lambda(\|\partial_\beta^\alpha\mathbf{g}\|_{\nu,w}^2+\|\partial_\beta^\alpha\mathbf{f}\|_{\nu,w}^2)\Big\}\\
&\leq C\sum_{|\alpha|\leq N-1}\Big\{\|\partial^\alpha\mathbf{g}\|_\nu^2+\|\partial^\alpha\mathbf{f}\|_\nu^2+\|\partial_x\partial^\alpha\mathbf{g}\|_{\nu,w}^2+\|\partial_x\partial^\alpha\mathbf{f}\|_{\nu,w}^2+\|\partial^\alpha\partial_x\phi\|^2+\|\partial^\alpha[\widetilde{u}_x,\widetilde{\theta}_x]\|^2\Big\}\\
&+C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}+C(\eta_0+\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t)+C\sum_{|\alpha|\leq N}\|\partial^\alpha\partial_x\phi\|\mathcal{F}_{N,l,q}(t). \qquad (5.23)
\end{aligned}
$$

Here $\mathcal{F}_{N,l,q}(t)$ and $w=w(\beta)$ are defined by (4.2) *and* (1.29), *respectively.*

Proof. We take the derivative ∂^α_β of (1.23) with $|\alpha|+|\beta|\le N$ and $|\beta|\ge 1$ and then take the inner product of the resulting equation with $w^2(\beta)\partial^\alpha_\beta\mathbf{g}$ over $\mathbb{R}\times\mathbb{R}^3$ to get

$$\begin{aligned}
&(\partial_t\partial^\alpha_\beta\mathbf{g},w^2(\beta)\partial^\alpha_\beta\mathbf{g})+(v_1\partial_x\partial^\alpha_\beta\mathbf{g},w^2(\beta)\partial^\alpha_\beta\mathbf{g})\\
&+(C^{\beta-e_1}_\beta\partial_x\partial^\alpha_{\beta-e_1}\mathbf{g},w^2(\beta)\partial^\alpha_\beta\mathbf{g})-(\partial^\alpha_\beta\mathcal{L}_1\mathbf{g},w^2(\beta)\partial^\alpha_\beta\mathbf{g})\\
=&(\partial^\alpha_\beta\Gamma(\mathbf{g},\frac{M-\mu}{\sqrt{\mu}})+\partial^\alpha_\beta\Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{g}),w^2(\beta)\partial^\alpha_\beta\mathbf{g})+(\partial^\alpha_\beta\Gamma(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}),w^2(\beta)\partial^\alpha_\beta\mathbf{g})\\
&+(\partial^\alpha_\beta(\frac{P_0(v_1\sqrt{\mu}\partial_x\mathbf{g})}{\sqrt{\mu}})-\partial^\alpha_\beta(\frac{1}{\sqrt{\mu}}P_1v_1M\{\frac{|v-u|^2\widetilde{\theta}_x}{2R\theta^2}+\frac{(v-u)\cdot\widetilde{u}_x}{R\theta}\}),w^2(\beta)\partial^\alpha_\beta\mathbf{g})\\
&-(\partial^\alpha_\beta(\frac{P_1(v_1\partial_x\overline{G})}{\sqrt{\mu}})-\partial^\alpha_\beta(\frac{\partial_t\overline{G}}{\sqrt{\mu}}),w^2(\beta)\partial^\alpha_\beta\mathbf{g})+(\partial^\alpha_\beta(\frac{P_1(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}}),w^2(\beta)\partial^\alpha_\beta\mathbf{g}). \quad (5.24)
\end{aligned}$$

Here $e_1=(1,0,0)$. We shall estimate (5.24) term by term. First note that

$$(\partial_t\partial^\alpha_\beta\mathbf{g},w^2(\beta)\partial^\alpha_\beta\mathbf{g})=\frac{1}{2}\frac{d}{dt}\|\partial^\alpha_\beta\mathbf{g}\|^2_{2,w}+\frac{q_2q_3}{2(1+t)^{1+q_3}}\|\langle v\rangle\partial^\alpha_\beta\mathbf{g}\|^2_{2,w}.$$

And the second term on the left hand side of (5.24) vanishes by the integration by parts. By (1.29), one has

$$\begin{aligned}
|(\partial_x\partial^\alpha_{\beta-e_1}\mathbf{g},w^2(\beta)\partial^\alpha_\beta\mathbf{g})|&\le C\|\langle v\rangle^{-\frac{\gamma}{2}}w(\beta)\partial_x\partial^\alpha_{\beta-e_1}\mathbf{g}\|\|\langle v\rangle^{\frac{\gamma}{2}}w(\beta)\partial^\alpha_\beta\mathbf{g}\|\\
&=C\|\langle v\rangle^{-\frac{\gamma}{2}}\langle v\rangle^{\gamma}w(\beta-e_1)\partial_x\partial^\alpha_{\beta-e_1}\mathbf{g}\|\|\langle v\rangle^{\frac{\gamma}{2}}w(\beta)\partial^\alpha_\beta\mathbf{g}\|\\
&\le\varepsilon\|\partial^\alpha_\beta\mathbf{g}\|^2_{\nu,w}+C_\varepsilon\|\partial_x\partial^\alpha_{\beta-e_1}\mathbf{g}\|^2_{\nu,w(\beta-e_1)}. \quad (5.25)
\end{aligned}$$

Here we have used that $w(\beta)=\langle v\rangle^\gamma w(\beta-e_1)$ for $|\beta-e_1|=|\beta|-1$ by (1.29).

It follows from (3.6) that

$$-(\partial^\alpha_\beta\mathcal{L}_1\mathbf{g},w^2(\beta)\partial^\alpha_\beta\mathbf{g})\ge\|\partial^\alpha_\beta\mathbf{g}\|^2_{\nu,w}-\varepsilon\sum_{|\beta_1|\le|\beta|}\|\partial^\alpha_{\beta_1}\mathbf{g}\|^2_{\nu,w(\beta_1)}-C_\varepsilon\|\partial^\alpha\mathbf{g}\|^2_\nu.$$

By (3.15) and (3.19), we get

$$\begin{aligned}
&|(\partial^\alpha_\beta\Gamma(\mathbf{g},\frac{M-\mu}{\sqrt{\mu}})+\partial^\alpha_\beta\Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{g}),w^2(\beta)\partial^\alpha_\beta\mathbf{g})|+|(\partial^\alpha_\beta\Gamma(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}),w^2(\beta)\partial^\alpha_\beta\mathbf{g})|\\
\le&\,C\varepsilon\|\partial^\alpha_\beta\mathbf{g}\|^2_{\nu,w}+C_\varepsilon\delta^{\frac16}(1+t)^{-\frac76}+C_\varepsilon(\eta_0+\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).
\end{aligned}$$

By (1.11), (4.33) and the imbedding inequality as well as Lemma 2.1, we arrive at

$$\begin{aligned}
&|(\partial^\alpha_\beta(\frac{P_0(v_1\sqrt{\mu}\partial_x\mathbf{g})}{\sqrt{\mu}})-\partial^\alpha_\beta(\frac{1}{\sqrt{\mu}}P_1v_1M\{\frac{|v-u|^2\widetilde{\theta}_x}{2R\theta^2}+\frac{(v-u)\cdot\widetilde{u}_x}{R\theta}\}),w^2(\beta)\partial^\alpha_\beta\mathbf{g})|\\
\le&\,C\varepsilon\|\partial^\alpha_\beta\mathbf{g}\|^2_{\nu,w}+C_\varepsilon\|\partial^\alpha\mathbf{g}_x\|^2_\nu+C_\varepsilon\|\partial^\alpha[\widetilde{u}_x,\widetilde{\theta}_x]\|^2+C_\varepsilon\delta^{\frac16}(1+t)^{-\frac76}\\
&+C_\varepsilon(\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).
\end{aligned}$$

For the term containing $\overline{G}$, we have from (1.11), (3.25), (3.26), Lemma 3.8 and Lemma 2.1 that

$$\begin{aligned}
&\big|(\partial^\alpha_\beta(\frac{P_1(v_1\partial_x\overline{G})}{\sqrt{\mu}})-\partial^\alpha_\beta(\frac{\partial_t\overline{G}}{\sqrt{\mu}}),w^2(\beta)\partial^\alpha_\beta\mathbf{g})\big|\\
\le&\,C\varepsilon\|\partial^\alpha_\beta\mathbf{g}\|^2_{\nu,w}+C_\varepsilon\delta^{\frac16}(1+t)^{-\frac76}+C_\varepsilon(\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).
\end{aligned}$$

For the last term of (5.24), recalling $F_2 = \sqrt{\mu}\mathbf{f}$ and using (1.11), one has

$$\partial^\alpha_\beta(\frac{P_1(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}}) = \partial^\alpha_\beta(\partial_x\phi\partial_{v_1}\mathbf{f}) - \partial^\alpha_\beta(\partial_x\phi\frac{v_1}{2}\mathbf{f}) - \partial^\alpha_\beta(\frac{P_0[\partial_x\phi\partial_{v_1}(\sqrt{\mu}\mathbf{f})]}{\sqrt{\mu}}). \quad (5.26)$$

We now expand the first term on the right hand side of (5.26) as

$$\partial^\alpha_\beta(\partial_x\phi\partial_{v_1}\mathbf{f}) = \partial_x\phi\partial^\alpha_\beta\partial_{v_1}\mathbf{f} + \sum_{|\alpha_1|<|\alpha|} C^{\alpha_1}_\alpha \partial^{\alpha-\alpha_1}\partial_x\phi\partial^{\alpha_1}_\beta\partial_{v_1}\mathbf{f}. \quad (5.27)$$

Notice that the last term of (5.27) vanishes when $|\alpha| = 0$. For $1 \le |\alpha - \alpha_1| \le \frac{N}{2}$ with $|\alpha| \neq 0$, we have

$$\begin{aligned}
&\big|(\partial^{\alpha-\alpha_1}\partial_x\phi\partial^{\alpha_1}_\beta\partial_{v_1}\mathbf{f}, w^2(\beta)\partial^\alpha_\beta\mathbf{g})\big| \\
&\le C\|\partial^{\alpha-\alpha_1}\partial_x\phi\|_{L^\infty}\|\langle v\rangle^{\frac{|\gamma|}{2}}\langle v\rangle^{-|\gamma|}w(\beta)\partial_{v_1}\partial^{\alpha_1}_\beta\mathbf{f}\|\|\langle v\rangle^{\frac{|\gamma|}{2}}w(\beta)\partial^\alpha_\beta\mathbf{g}\| \\
&\le C\|\partial^{\alpha-\alpha_1}\partial_x\phi\|_{L^\infty}\|\langle v\rangle\partial_{v_1}\partial^{\alpha_1}_\beta\mathbf{f}\|_{2,w(\beta+e_1)}\|\langle v\rangle\partial^\alpha_\beta\mathbf{g}\|_{2,w} \\
&\le C\sum_{|\alpha'|\le N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t). \quad (5.28)
\end{aligned}$$

Here we have used that $\langle v\rangle^{-|\gamma|}w(\beta) = w(\beta+e_1)$ and $\langle v\rangle^{\frac{|\gamma|}{2}} \le \langle v\rangle$ for $|\gamma| \le 2$ by (1.29) and $e_1 = (1,0,0)$.

For $\frac{N}{2} \le |\alpha - \alpha_1| \le N$ with $|\alpha| \neq 0$, the similar arguments as (5.28) imply

$$\big|(\partial^{\alpha-\alpha_1}\partial_x\phi\partial_{v_1}\partial^{\alpha_1}_\beta\mathbf{f}, w^2(\beta)\partial^\alpha_\beta\mathbf{g})\big| \le C\sum_{|\alpha'|\le N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t). \quad (5.29)$$

With the help of (5.28) and (5.29), we have from (5.27) that

$$\begin{aligned}
\big(\partial^\alpha_\beta(\partial_x\phi\partial_{v_1}\mathbf{f}), w^2(\beta)\partial^\alpha_\beta\mathbf{g}\big) \le{}& \big(\partial_x\phi\partial^\alpha_\beta\partial_{v_1}\mathbf{f}, w^2(\beta)\partial^\alpha_\beta\mathbf{g}\big) \\
&+C\sum_{|\alpha'|\le N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t). \quad (5.30)
\end{aligned}$$

Next we expand the second term on the right hand side of (5.26) as

$$\partial^\alpha_\beta(\partial_x\phi\frac{v_1}{2}\mathbf{f}) = \sum_{|\alpha_1|\le|\alpha|} C^{\alpha_1}_\alpha(\partial^{\alpha_1}\partial_x\phi\frac{v_1}{2}\partial^{\alpha-\alpha_1}_\beta\mathbf{f} + \frac{1}{2}C^{\beta-e_1}_\beta\partial^{\alpha_1}\partial_x\phi\partial^{\alpha-\alpha_1}_{\beta-e_1}\mathbf{f}), \quad (5.31)$$

where $e_1 = (1,0,0)$. If $|\alpha_1| \le \frac{|\alpha|}{2}$, we have

$$\begin{aligned}
&\big(\partial^{\alpha_1}\partial_x\phi\frac{v_1}{2}\partial^{\alpha-\alpha_1}_\beta\mathbf{f} + \frac{1}{2}C^{\beta-e_1}_\beta\partial^{\alpha_1}\partial_x\phi\partial^{\alpha-\alpha_1}_{\beta-e_1}\mathbf{f}, w^2(\beta)\partial^\alpha_\beta\mathbf{g}\big) \\
&\le C\|\partial^{\alpha_1}\partial_x\phi\|_{L^\infty}(\|\langle v\rangle\partial^{\alpha-\alpha_1}_\beta\mathbf{f}\|_{2,w}\|\langle v\rangle\partial^\alpha_\beta\mathbf{g}\|_{2,w} + \|\partial^{\alpha-\alpha_1}_{\beta-e_1}\mathbf{f}\|_{2,w(\beta-e_1)}\|\partial^\alpha_\beta\mathbf{g}\|_{2,w}) \\
&\le C\sum_{|\alpha'|\le N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t). \quad (5.32)
\end{aligned}$$

Here we have used the fact that $w(\beta) \le w(\beta - e_1)$ by (1.29). If $\frac{|\alpha|}{2} \le |\alpha_1| \le |\alpha|$, we also have

$$\begin{aligned}
&\big(\partial^{\alpha_1}\partial_x\phi\frac{v_1}{2}\partial^{\alpha-\alpha_1}_\beta\mathbf{f} + \frac{1}{2}C^{\beta-e_1}_\beta\partial^{\alpha_1}\partial_x\phi\partial^{\alpha-\alpha_1}_{\beta-e_1}\mathbf{f}, w^2(\beta)\partial^\alpha_\beta\mathbf{g}\big) \\
&\le C\sum_{|\alpha'|\le N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t). \quad (5.33)
\end{aligned}$$

With the help of (5.32) and (5.33), we have from (5.31) that

$$\big(\partial^\alpha_\beta(\partial_x\phi\frac{v_1}{2}\mathbf{f}), w^2(\beta)\partial^\alpha_\beta\mathbf{g}\big) \leq C\sum_{|\alpha'|\leq N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t). \tag{5.34}$$

The last term on the right hand side of (5.26) can be dominated by

$$|\big(\partial^\alpha_\beta(\frac{P_0[\partial_x\phi\partial_{v_1}(\sqrt{\mu}\mathbf{f})]}{\sqrt{\mu}}), w^2(\beta)\partial^\alpha_\beta\mathbf{g}\big)| \leq C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t). \tag{5.35}$$

By (5.30), (5.34) and (5.35), we have from (5.26) that

$$\begin{aligned}&(\partial^\alpha_\beta(\frac{P_1(\partial_x\phi\partial_{v_1}F_2)}{\sqrt{\mu}}), w^2(\beta)\partial^\alpha_\beta\mathbf{g}) - \big(\partial_x\phi\partial^\alpha_\beta\partial_{v_1}\mathbf{f}, w^2(\beta)\partial^\alpha_\beta\mathbf{g}\big)\\ &\leq C\sqrt{\varepsilon_0}\mathcal{D}_{N,l,q}(t) + C\sum_{|\alpha'|\leq N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t).\end{aligned} \tag{5.36}$$

Hence, for $|\alpha|+|\beta|\leq N$ with $|\beta|\geq 1$ and any $\varepsilon>0$ small enough, we have from the above estimates that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha_\beta\mathbf{g}\|^2_{2,w} + \frac{q_2q_3}{2(1+t)^{1+q_3}}\|\langle v\rangle\partial^\alpha_\beta\mathbf{g}\|^2_{2,w} + \lambda\|\partial^\alpha_\beta\mathbf{g}\|^2_{\nu,w} - \big(\partial_x\phi\partial^\alpha_\beta\partial_{v_1}\mathbf{f}, w^2(\beta)\partial^\alpha_\beta\mathbf{g}\big)\\ &\leq C\varepsilon\sum_{|\beta_1|\leq|\beta|}\|\partial^\alpha_{\beta_1}\mathbf{g}\|^2_{\nu,w(\beta_1)} + C_\varepsilon\big(\|\partial_x\partial^\alpha_{\beta-e_1}\mathbf{g}\|^2_{\nu,w(\beta-e_1)} + \|\partial^\alpha\mathbf{g}\|^2_\nu + \|\partial^\alpha\partial_x\mathbf{g}\|^2_\nu\\ &\quad+\|\partial^\alpha[\widetilde{u}_x,\widetilde{\theta}_x]\|^2\big) + C_\varepsilon\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C_\varepsilon(\eta_0+\delta^{\frac{1}{6}}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t)\\ &\quad+C\sum_{|\alpha'|\leq N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t).\end{aligned} \tag{5.37}$$

This completes the proof of the weighted mixed derivative estimates of the function $\mathbf{g}$.

In the following, we deduce the weighted mixed derivative estimates of the function $\mathbf{f}$. Taking the derivative ∂^α_β to (1.26) with $|\alpha|+|\beta|\leq N$ and $|\beta|\geq 1$, for $e_1=(1,0,0)$, one has

$$\begin{aligned}&\partial_t\partial^\alpha_\beta\mathbf{f} + v_1\partial_x\partial^\alpha_\beta\mathbf{f} + C^{\beta-e_1}_\beta\partial_x\partial^\alpha_{\beta-e_1}\mathbf{f} + \partial^\alpha_\beta(\partial_x\phi v_1\sqrt{\mu}) - \partial^\alpha_\beta\mathcal{L}_2\mathbf{f}\\ &= \partial^\alpha_\beta\Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{f}) + \partial^\alpha_\beta\Gamma(\frac{\overline{G}}{\sqrt{\mu}},\mathbf{f}) + \partial^\alpha_\beta\Gamma(\mathbf{g},\mathbf{f}) + \partial^\alpha_\beta(\frac{\partial_x\phi\partial_{v_1}(M-\mu)}{\sqrt{\mu}})\\ &\quad+\partial^\alpha_\beta(\frac{\partial_x\phi\partial_{v_1}\overline{G}}{\sqrt{\mu}}) + \partial^\alpha_\beta(\frac{\partial_x\phi\partial_{v_1}(\sqrt{\mu}\mathbf{g})}{\sqrt{\mu}}).\end{aligned} \tag{5.38}$$

We take the inner product of (5.38) with $w^2(\beta)\partial^\alpha_\beta\mathbf{f}$ over $\mathbb{R}\times\mathbb{R}^3$ and then estimate term by term. First of all, one has

$$(\partial_t\partial^\alpha_\beta\mathbf{f}, w^2(\beta)\partial^\alpha_\beta\mathbf{f}) = \frac{1}{2}\frac{d}{dt}\|\partial^\alpha_\beta\mathbf{f}\|^2_{2,w} + \frac{q_2q_3}{2(1+t)^{1+q_3}}\|\langle v\rangle\partial^\alpha_\beta\mathbf{f}\|^2_{2,w}.$$

And the inner product of the second term vanishes by the integration by parts. By the similar arguments as (5.25), one has

$$|(\partial_x\partial^\alpha_{\beta-e_1}\mathbf{f}, w^2(\beta)\partial^\alpha_\beta\mathbf{f})| \leq \varepsilon\|\partial^\alpha_\beta\mathbf{f}\|^2_{\nu,w} + C_\varepsilon\|\partial_x\partial^\alpha_{\beta-e_1}\mathbf{f}\|^2_{\nu,w(\beta-e_1)}.$$

It follows that

$$|(\partial^\alpha_\beta(\partial_x\phi v_1\sqrt{\mu}),w^2(\beta)\partial^\alpha_\beta\mathbf{f})|\leq\varepsilon\|\partial^\alpha_\beta\mathbf{f}\|^2_{\nu,w}+C_\varepsilon\|\partial^\alpha\partial_x\phi\|^2.$$

By (3.6), one has

$$-(\partial^\alpha_\beta\mathcal{L}_2\mathbf{f},w^2(\beta)\partial^\alpha_\beta\mathbf{f})\geq\|\partial^\alpha_\beta\mathbf{f}\|^2_{\nu,w}-\varepsilon\sum_{|\beta_1|\leq|\beta|}\|\partial^\alpha_{\beta_1}\mathbf{f}\|^2_{\nu,w(\beta_1)}-C_\varepsilon\|\partial^\alpha\mathbf{f}\|^2_\nu.$$

By the similar arguments as (3.15), (3.29) and (3.30), we have

$$\begin{aligned}&|(\partial^\alpha_\beta\Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{f})+\partial^\alpha_\beta\Gamma(\frac{\overline{G}}{\sqrt{\mu}},\mathbf{f})+\partial^\alpha_\beta\Gamma(\mathbf{g},\mathbf{f}),w^2(\beta)\partial^\alpha_\beta\mathbf{f})|\\&\leq C\varepsilon\|\partial^\alpha_\beta\mathbf{f}\|^2_{\nu,w(\beta)}+C_\varepsilon(\eta_0+\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).\end{aligned}$$

Following the similar methods used as (5.20), we can obtain

$$(\partial^\alpha_\beta(\frac{\partial_x\phi\partial_{v_1}(M-\mu)}{\sqrt{\mu}})+\partial^\alpha_\beta(\frac{\partial_x\phi\partial_{v_1}\overline{G}}{\sqrt{\mu}}),w^2(\beta)\partial^\alpha_\beta\mathbf{f})|\leq C(\eta_0+\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t).$$

By using the similar method used as (5.30) and (5.34), we arrive at

$$\begin{aligned}&\big(\partial^\alpha_\beta(\frac{\partial_x\phi\partial_{v_1}(\sqrt{\mu}\mathbf{g})}{\sqrt{\mu}}),w^2(\beta)\partial^\alpha_\beta\mathbf{f}\big)\\&\leq\big(\partial_x\phi\partial_{v_1}\partial^\alpha_\beta\mathbf{g},w^2(\beta)\partial^\alpha_\beta\mathbf{f}\big)+C\sum_{|\alpha'|\leq N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t).\end{aligned}\tag{5.39}$$

Hence, for $|\alpha|+|\beta|\leq N$ with $|\beta|\geq1$ and any $\varepsilon>0$ small enough, we arrive at

$$\begin{aligned}&\frac12\frac{d}{dt}\|\partial^\alpha_\beta\mathbf{f}\|^2_{2,w}+\frac{q_2q_3}{2(1+t)^{1+q_3}}\|\langle v\rangle\partial^\alpha_\beta\mathbf{f}\|^2_{2,w}+\lambda\|\partial^\alpha_\beta\mathbf{f}\|^2_{\nu,w}-\big(\partial_x\phi\partial_{v_1}\partial^\alpha_\beta\mathbf{g},w^2(\beta)\partial^\alpha_\beta\mathbf{f}\big)\\&\leq C\varepsilon\sum_{|\beta_1|\leq|\beta|}\|\partial^\alpha_{\beta_1}\mathbf{f}\|^2_{\nu,w(\beta_1)}+C_\varepsilon\big(\|\partial_x\partial^\alpha_{\beta-e_1}\mathbf{f}\|^2_{\nu,w(\beta-e_1)}+\|\partial^\alpha\mathbf{f}\|^2_\nu+\|\partial^\alpha\partial_x\phi\|^2\big)\\&\quad+C_\varepsilon(\eta_0+\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t)+C\sum_{|\alpha'|\leq N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t).\end{aligned}\tag{5.40}$$

This completes the proof of the weighted mixed derivative estimates of the function $\mathbf{f}$.

In summary, for $|\alpha|+|\beta|\leq N$ with $|\beta|\geq1$ and any $\varepsilon>0$ small enough, we have from (5.37) and (5.40) that

$$\begin{aligned}&\frac12\frac{d}{dt}(\|\partial^\alpha_\beta\mathbf{g}\|^2_{2,w}+\|\partial^\alpha_\beta\mathbf{f}\|^2_{2,w})+\frac{q_2q_3}{2(1+t)^{1+q_3}}(\|\langle v\rangle\partial^\alpha_\beta\mathbf{g}\|^2_{2,w}+\|\langle v\rangle\partial^\alpha_\beta\mathbf{f}\|^2_{2,w})\\&\quad+\lambda(\|\partial^\alpha_\beta\mathbf{g}\|^2_{\nu,w}+\|\partial^\alpha_\beta\mathbf{f}\|^2_{\nu,w})\\&\leq C\varepsilon\sum_{|\beta_1|\leq|\beta|}\big\{\|\partial^\alpha_{\beta_1}\mathbf{g}\|^2_{\nu,w(\beta_1)}+\|\partial^\alpha_{\beta_1}\mathbf{f}\|^2_{\nu,w(\beta_1)}\big\}\\&\quad+C_\varepsilon\big\{\|\partial_x\partial^\alpha_{\beta-e_1}\mathbf{g}\|^2_{\nu,w(\beta-e_1)}+\|\partial_x\partial^\alpha_{\beta-e_1}\mathbf{f}\|^2_{\nu,w(\beta-e_1)}\big\}\\&\quad+C_\varepsilon\big\{\|\partial^\alpha\mathbf{g}\|^2_\nu+\|\partial^\alpha\mathbf{f}\|^2_\nu+\|\partial^\alpha\partial_x\phi\|^2+\|\partial^\alpha[\widetilde{u}_x,\widetilde{\theta}_x]\|^2\big\}+C_\varepsilon\delta^{\frac16}(1+t)^{-\frac76}\\&\quad+C_\varepsilon(\eta_0+\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t)+C\sum_{|\alpha'|\leq N}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t).\end{aligned}\tag{5.41}$$

Here we have used the fact that

$$
\begin{aligned}
&-(\partial_x\phi\partial^\alpha_\beta\partial_{v_1}\mathbf{f},w^2(\beta)\partial^\alpha_\beta\mathbf{g})-(\partial_x\phi\partial_{v_1}\partial^\alpha_\beta\mathbf{g},w^2(\beta)\partial^\alpha_\beta\mathbf{f})\\
&\le C\sum_{|\alpha'|\le 1}\|\partial^{\alpha'}\partial_x\phi\|\mathcal{F}_{N,l,q}(t).
\end{aligned}\tag{5.42}
$$

Notice that the coefficients in the second term of the second line in (5.41) is large. We will use the induction in $|\beta|$ and then suitably choose $\varepsilon>0$ small enough to this term. By the suitable linear combinations, we can obtain (5.23). This ends the proof of Lemma 5.2. □

5.3. Second main energy estimates. In this subsection, we will give the weighted energy estimates to the solution in terms of Lemma 5.1 and Lemma 5.2.

LEMMA 5.3. *Under the conditions of Lemma 4.1, we have*

$$
\begin{aligned}
&\frac{d}{dt}\widehat{\mathcal{E}}_N(t)+\frac{\lambda q_2q_3}{(1+t)^{1+q_3}}\mathcal{F}_{N,l,q}+\lambda\sum_{|\alpha|+|\beta|\le N}\{\|\partial^\alpha_\beta\mathbf{g}\|^2_{\nu,w}+\|\partial^\alpha_\beta\mathbf{f}\|^2_{\nu,w}\}\\
&\le C\sum_{1\le|\alpha|\le N}\|\partial^\alpha(\widetilde{\rho},\widetilde{u},\widetilde{\theta})\|^2+C\sum_{|\alpha|\le N}\{\|\partial^\alpha\mathbf{g}\|^2_\nu+\|\partial^\alpha\mathbf{f}\|^2_\nu+\|\partial^\alpha\partial_x\phi\|^2\}\\
&\quad+C\delta^{\frac16}(1+t)^{-\frac76}+C(\eta_0+\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t)+C\sum_{|\alpha|\le N}\|\partial^\alpha\partial_x\phi\|\mathcal{F}_{N,l,q}(t).
\end{aligned}\tag{5.43}
$$

Here the function $\widehat{\mathcal{E}}_N(t)$ *is given by*

$$
\begin{aligned}
\widehat{\mathcal{E}}_N(t)=&\widetilde{C}_4\Big\{\sum_{|\alpha|\le 1}\|\partial^\alpha\mathbf{g}\|^2_{2,w}+\sum_{2\le|\alpha|\le N}\|\frac{\partial^\alpha F_1}{\sqrt{\mu}}\|^2_{2,w}+\sum_{|\alpha|\le N}\|\partial^\alpha\mathbf{f}\|^2_{2,w}\Big\}\\
&+\sum_{|\alpha|=0}^{N-1}C_{|\alpha|}\sum_{j=1}^{N-|\alpha|}C_j\sum_{|\beta|=j}\{\|\partial^\alpha_\beta\mathbf{g}\|^2_{2,w}+\|\partial^\alpha_\beta\mathbf{f}\|^2_{2,w}\}
\end{aligned}\tag{5.44}
$$

for some large constant $\widetilde{C}_4>0$ *and* $\mathcal{F}_{N,l,q}(t)$ *is given by* (4.2).

Proof. For some suitably large constant $\widetilde{C}_4>0$, the estimate (5.43) follows from (5.1)$\times\widetilde{C}_4$ and (5.23). This completes the proof of Lemma 5.3. □

6. The Proof of Main Result. In this section, we shall prove our main theorem. To this end, we first need to establish the crucial time decay rates of the electric field term. Then we prove the a priori estimate is closed and establish global solution for the system (1.6) and (1.7). Finally, we prove the time asymptotic stability of rarefaction waves for the solution of the system (1.6) and (1.7).

6.1. Time decay rates of the electric field. In this subsection, we follow the methods in [11] and make use of the smallness of the wave strength and the instant energy functional to establish the following lemma.

LEMMA 6.1. *Suppose* $\mathcal{E}_{N,l,q}(t)<\varepsilon_0$ *for* $N\ge 6$ *and* $l\ge\max\{N,\frac12+\frac{1}{|\gamma|}\}$ *with* $-2\le\gamma<0$. *For any* $q_3\ge 0$, *any* $q_1\in[0,1)$ *and* $q_2\in[0,1)$ *small enough in* (1.29). *If we choose* $\eta_0>0$ *in* (1.20), $\delta>0$ *in Lemma 2.1 and* $\varepsilon_0>0$ *small enough, one has*

$$
\sum_{|\alpha|\le N-1}\|\partial^\alpha\mathbf{f}\|^2+\sum_{|\alpha|\le N}\|\partial^\alpha\partial_x\phi\|^2\le C\varepsilon_0(1+t)^{-(2l-4)},\quad q_1\ge 0,\tag{6.1}
$$

and there exists $c>0$ such that that

$$\sum_{|\alpha|\leq N-1}\|\partial^\alpha \mathbf{f}\|^2+\sum_{|\alpha|\leq N}\|\partial^\alpha\partial_x\phi\|^2\leq C\varepsilon_0 e^{-c(1+t)^{\frac{2}{2+|\gamma|}}},\quad q_1>0. \tag{6.2}$$

Proof. We have from (4.62) with $|\alpha|\leq N-1$, (4.63), (4.64) and (3.5) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\partial^\alpha\mathbf{f}\|^2+\frac{1}{2}\|\partial^\alpha\partial_x\phi\|^2)+\sigma_2\|\partial^\alpha(\mathbf{I}-P_2)\mathbf{f}\|_\nu^2\\ &\leq(\partial^\alpha\Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{f})+\partial^\alpha\Gamma(\frac{\overline{G}}{\sqrt{\mu}},\mathbf{f}),\partial^\alpha\mathbf{f})+(\partial^\alpha\Gamma(\mathbf{g},\mathbf{f}),\partial^\alpha\mathbf{f})\\ &\quad+(\partial^\alpha(\frac{\partial_x\phi\partial_{v_1}\overline{G}}{\sqrt{\mu}})+\partial^\alpha(\frac{\partial_x\phi\partial_{v_1}(M-\mu)}{\sqrt{\mu}}),\partial^\alpha\mathbf{f})+(\partial^\alpha(\frac{\partial_x\phi\partial_{v_1}(\sqrt{\mu}\mathbf{g})}{\sqrt{\mu}}),\partial^\alpha\mathbf{f}).\end{aligned} \tag{6.3}$$

By (3.8), (3.12), (3.13), (3.26), the imbedding inequality and Lemma 2.1, one has

$$\begin{aligned}&|(\partial^\alpha\Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{f})+\partial^\alpha\Gamma(\frac{\overline{G}}{\sqrt{\mu}},\mathbf{f}),\partial^\alpha\mathbf{f})|+|(\partial^\alpha\Gamma(\mathbf{g},\mathbf{f}),\partial^\alpha\mathbf{f})|\\ &\leq C(\eta_0+\delta^{\frac{1}{6}})\|\partial^\alpha\mathbf{f}\|_\nu^2+C(\delta^{\frac{1}{6}}+\sqrt{\mathcal{E}_{N,0,0}(t)})\sum_{|\alpha'|\leq N-1}\|\partial^{\alpha'}\mathbf{f}\|_\nu^2.\end{aligned} \tag{6.4}$$

On the other hand, we also have

$$\begin{aligned}&|(\partial^\alpha(\frac{\partial_x\phi\partial_{v_1}\overline{G}}{\sqrt{\mu}})+\partial^\alpha(\frac{\partial_x\phi\partial_{v_1}(M-\mu)}{\sqrt{\mu}}),\partial^\alpha\mathbf{f})|\\ &\leq C(\eta_0+\delta^{\frac{1}{6}}+\sqrt{\mathcal{E}_{N,0,0}(t)})\sum_{|\alpha'|\leq N-1}(\|\partial^{\alpha'}\mathbf{f}\|_\nu^2+\|\partial^{\alpha'}\partial_x\phi\|^2).\end{aligned} \tag{6.5}$$

For the last term of (6.3), we have

$$\frac{\partial^\alpha(\partial_x\phi\partial_{v_1}(\sqrt{\mu}\mathbf{g}))}{\sqrt{\mu}}=\sum_{|\alpha_1|\leq|\alpha|}C_\alpha^{\alpha_1}(\partial^{\alpha_1}\partial_x\phi\partial_{v_1}\partial^{\alpha-\alpha_1}\mathbf{g}-\partial^{\alpha_1}\partial_x\phi\frac{v_1}{2}\partial^{\alpha-\alpha_1}\mathbf{g}).$$

If $|\alpha_1|\leq|\alpha|/2$, we have from the imbedding inequality that

$$\begin{aligned}&|(\partial^{\alpha_1}\partial_x\phi\partial_{v_1}\partial^{\alpha-\alpha_1}\mathbf{g}-\partial^{\alpha_1}\partial_x\phi\frac{v_1}{2}\partial^{\alpha-\alpha_1}\mathbf{g},\partial^\alpha\mathbf{f})|\\ &\leq C\big\{\|\langle v\rangle^{-\frac{\gamma}{2}}\partial_{v_1}\partial^{\alpha-\alpha_1}\mathbf{g}\|+\|\langle v\rangle^{-\frac{\gamma}{2}+1}\partial^{\alpha-\alpha_1}\mathbf{g}\|\big\}\|\langle v\rangle^{\frac{\gamma}{2}}\partial^\alpha\mathbf{f}\|\|\partial^{\alpha_1}\partial_x\phi\|_{L^\infty}\\ &\leq C\sqrt{\mathcal{E}_{N,l,0}(t)}\sum_{|\alpha'|\leq N-1}(\|\partial^{\alpha'}\mathbf{f}\|_\nu^2+\|\partial^{\alpha'}\partial_x\phi\|^2).\end{aligned} \tag{6.6}$$

Here we have used the fact that, for $-2\leq\gamma<0$,

$$\begin{aligned}&\langle v\rangle^{-\frac{\gamma}{2}}\leq\langle v\rangle^{|\gamma|(l-1)}\quad\text{and}\quad\langle v\rangle^{-\frac{\gamma}{2}+1}\leq\langle v\rangle^{|\gamma|l},\\ &\text{that is}\ \ l\geq\max\{\frac{3}{2},\frac{1}{2}+\frac{1}{|\gamma|}\}.\end{aligned} \tag{6.7}$$

Similarly, if $|\alpha_1|/2 < |\alpha_1| \le |\alpha|$, one has

$$(\partial^{\alpha_1}\partial_x\phi\partial_{v_1}\partial^{\alpha-\alpha_1}\mathbf{g} - \partial^{\alpha_1}\partial_x\phi\frac{v_1}{2}\partial^{\alpha-\alpha_1}\mathbf{g}, \partial^\alpha\mathbf{f}) \le C\sqrt{\mathcal{E}_{N,l,0}(t)}\sum_{|\alpha'|\le N-1}\big\{\|\partial^{\alpha'}\mathbf{f}\|_\nu^2 + \|\partial^{\alpha'}\partial_x\phi\|^2\big\}. \tag{6.8}$$

By using the above estimates and $\mathcal{E}_{N,l,0}(t) < \varepsilon_0$ with $l \ge \max\{N, \frac{1}{2}+\frac{1}{|\gamma|}\}$, we arrive at

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\le N-1}\big\{\|\partial^\alpha\mathbf{f}\|^2 + \frac{1}{2}\|\partial^\alpha\partial_x\phi\|^2\big\} + \sigma_2\sum_{|\alpha|\le N}\|\partial^\alpha(\mathbf{I}-P_2)\mathbf{f}\|_\nu^2 \le C(\eta_0 + \delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\sum_{|\alpha|\le N-1}\big\{\|\partial^\alpha\mathbf{f}\|_\nu^2 + \|\partial^\alpha\partial_x\phi\|^2\big\}. \tag{6.9}$$

By the similar arguments as (4.72), we can obtain

$$\sum_{|\alpha|\le N-1}(\|\partial^\alpha P_2\mathbf{f}\|^2 + \|\partial^\alpha\partial_x\phi\|^2) \le -\frac{d}{dt}H_1(t) + C\sum_{|\alpha|\le N-1}\|\partial^\alpha(\mathbf{I}-P_2)\mathbf{f}\|_\nu^2 + C(\eta_0 + \delta^{\frac{1}{6}} + \sqrt{\varepsilon_0})\sum_{|\alpha|\le N-1}(\|\partial^\alpha\mathbf{f}\|_\nu^2 + \|\partial^\alpha\partial_x\phi\|^2). \tag{6.10}$$

Here the function $H_1(t)$ is defined as

$$H_1(t) = C\sum_{|\alpha|\le N-2}(\langle\partial^\alpha\mathbf{f}, v_1\sqrt{\mu}\rangle, \partial_x\partial^\alpha\mathbf{a}) + C(\langle\mathbf{f}, v_1\sqrt{\mu}\rangle, \partial_x\phi) \le C\sum_{|\alpha|\le N-1}(\|\partial^\alpha\mathbf{f}\|^2 + \|\partial^\alpha\partial_x\phi\|^2). \tag{6.11}$$

For some positive constants η_0, δ, ε_0 and κ_2 small enough, we have from (6.9) and (6.10) that

$$\frac{d}{dt}G(t) + \lambda\sum_{|\alpha|\le N-1}\big\{\|\partial^\alpha\mathbf{f}\|_\nu^2 + \|\partial^\alpha\partial_x\phi\|^2\big\} \le 0. \tag{6.12}$$

Here the function $G(t)$ is given by

$$G(t) = \sum_{|\alpha|\le N-1}\big\{\|\partial^\alpha\mathbf{f}\|^2 + \frac{1}{2}\|\partial^\alpha\partial_x\phi\|^2 + \kappa_2 H_1(t)\big\} \approx \sum_{|\alpha|\le N-1}\big\{\|\partial^\alpha\mathbf{f}\|^2 + \|\partial^\alpha\partial_x\phi\|^2\big\}. \tag{6.13}$$

In what follows we establish the polynomial time decay by the interpolation methods developed in [11]. For $|\alpha| \le N-1$ and $l > 2$, we have

$$\|\partial^\alpha\mathbf{f}\|^2 = \int_{\mathbb{R}}\int_{\mathbb{R}^3}\langle v\rangle^{\gamma\frac{2l-4}{2l-3}}\langle v\rangle^{-\gamma\frac{2l-4}{2l-3}}|\partial^\alpha\mathbf{f}|^{2(\frac{2l-4}{2l-3}+\frac{1}{2l-3})}dvdx \le C\Big(\int_{\mathbb{R}}\int_{\mathbb{R}^3}\langle v\rangle^\gamma|\partial^\alpha\mathbf{f}|^2dvdx\Big)^{\frac{2l-4}{2l-3}}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}^3}\langle v\rangle^{|\gamma|(2l-4)}|\partial^\alpha\mathbf{f}|^2dvdx\Big)^{\frac{1}{2l-3}}, \tag{6.14}$$

which further implies that

$$\begin{aligned}\|\partial^\alpha \mathbf{f}\|_\nu^2 &\geq \lambda \|\partial^\alpha \mathbf{f}\|^{\frac{2(2l-3)}{2l-4}} \|\langle v\rangle^{|\gamma|(l-2)}\partial^\alpha \mathbf{f}\|^{-\frac{2}{2l-4}} \\ &\geq \lambda \|\partial^\alpha \mathbf{f}\|^{\frac{2(2l-3)}{2l-4}} \{\sup_{0\leq s\leq T} \mathcal{E}_{N-1,l,0}(s)\}^{-\frac{1}{2l-4}}. \end{aligned} \tag{6.15}$$

Similarly, we can obtain

$$\begin{aligned}\|\partial^\alpha \partial_x \phi\|^2 &= \|\partial^\alpha \partial_x \phi\|^{\frac{2(2l-3)}{2l-4}} \|\partial^\alpha \partial_x \phi\|^{-\frac{2}{2l-4}} \\ &\geq \lambda \|\partial^\alpha \partial_x \phi\|^{\frac{2(2l-3)}{2l-4}} \{\sup_{0\leq s\leq T} \mathcal{E}_{N-1,l,0}(s)\}^{-\frac{1}{2l-4}}. \end{aligned} \tag{6.16}$$

It follows from (6.12), (6.13), (6.15) and (6.16) that

$$\frac{d}{dt}G(t) + \lambda G(t)^{\frac{2l-3}{2l-4}} \{\sup_{0\leq s\leq T} \mathcal{E}_{N-1,l,0}(s)\}^{-\frac{1}{2l-4}} \leq 0. \tag{6.17}$$

By the fact that $\sup_{0\leq s\leq T} \mathcal{E}_{N-1,l,0}(s) < \varepsilon_0$, we solve this inequality to get

$$G(t) \leq C \sup_{0\leq s\leq T} \mathcal{E}_{N-1,l,0}(s)(1+t)^{-(2l-4)}. \tag{6.18}$$

By the Poisson equation (1.28) and the fact that $P_2\mathbf{f} = \mathbf{a}(t,x)\sqrt{\mu}$, we have

$$-\partial_{xx}\phi = 2\mathbf{a}(t,x) \quad \text{and} \quad \|\partial^\alpha P_2 \mathbf{f}\| \approx \|\partial^\alpha \mathbf{a}(t,x)\|. \tag{6.19}$$

This implies that

$$\sum_{1\leq|\alpha|\leq N} \|\partial^\alpha \partial_x \phi\|^2 = \sum_{|\alpha|\leq N-1} \|\partial^\alpha \partial_{xx}\phi\|^2 \leq C \sum_{|\alpha|\leq N-1} \|\partial^\alpha P_2 \mathbf{f}\|^2 \leq C \sum_{|\alpha|\leq N-1} \|\partial^\alpha \mathbf{f}\|^2. \tag{6.20}$$

The estimate (6.1) follows from this, (6.18) and (6.13).

We now turn to prove the stretched exponential time decay by the splitting methods developed in [11]. For any $\varrho > 0$, $\epsilon_2 > 0$ and $\varrho(1+t)^{\epsilon_2} \geq 1$, one has by $-2 \leq \gamma < 0$ that

$$\begin{aligned}\|\partial^\alpha \mathbf{f}\|_\nu^2 &= \int_{\mathbb{R}}\int_{\langle v\rangle \leq \varrho(1+t)^{\epsilon_2}} \langle v\rangle^\gamma |\partial^\alpha \mathbf{f}|^2\, dvdx + \int_{\mathbb{R}}\int_{\langle v\rangle > \varrho(1+t)^{\epsilon_2}} \langle v\rangle^\gamma |\partial^\alpha \mathbf{f}|^2\, dvdx \\ &\geq \{\varrho(1+t)^{\epsilon_2}\}^{-|\gamma|} \int_{\mathbb{R}}\int_{\langle v\rangle \leq \varrho(1+t)^{\epsilon_2}} |\partial^\alpha \mathbf{f}|^2\, dvdx \\ &= \{\varrho(1+t)^{\epsilon_2}\}^{-|\gamma|} \|\partial^\alpha \mathbf{f}\|^2 - \{\varrho(1+t)^{\epsilon_2}\}^{-|\gamma|} \int_{\mathbb{R}}\int_{\langle v\rangle > \varrho(1+t)^{\epsilon_2}} |\partial^\alpha \mathbf{f}|^2\, dvdx. \end{aligned}$$

For $q_1 > 0$ and $t \in [0,T]$, the last term on the above inequality is dominated by

$$\begin{aligned} &\{\varrho(1+t)^{\epsilon_2}\}^{-|\gamma|} \int_{\mathbb{R}}\int_{\langle v\rangle > \varrho(1+t)^{\epsilon_2}} e^{\frac{q_1}{2}\langle v\rangle^2 - \frac{q_1}{2}\varrho^2(1+t)^{2\epsilon_2}} |\partial^\alpha \mathbf{f}|^2 dvdx \\ &\leq C\{\varrho(1+t)^{\epsilon_2}\}^{-|\gamma|} e^{-\frac{q_1}{2}\varrho^2(1+t)^{2\epsilon_2}} \sup_{0\leq s\leq T} \mathcal{E}_{N-1,l,q}(s). \end{aligned}$$

It follows from the above estimates that

$$\|\partial^\alpha \mathbf{f}\|_\nu^2 \geq \{\varrho(1+t)^{\epsilon_2}\}^{-|\gamma|}\|\partial^\alpha \mathbf{f}\|^2 - C\{\varrho(1+t)^{\epsilon_2}\}^{-|\gamma|}e^{-\frac{q_1}{2}\varrho^2(1+t)^{2\epsilon_2}}\sup_{0\leq s\leq T}\mathcal{E}_{N-1,l,q}(s).$$

For $\varrho(1+t)^{\epsilon_2}\geq 1$ and $-2\leq\gamma<0$, we see

$$\|\partial^\alpha\partial_x\phi\|^2 \geq \{\varrho(1+t)^{\epsilon_2}\}^{-|\gamma|}\|\partial^\alpha\partial_x\phi\|^2.$$

With the help of the above two inequalities, we have from (6.12) that

$$\frac{d}{dt}G(t)+\lambda\{\varrho(1+t)^{\epsilon_2}\}^{-|\gamma|}G(t)\leq C\{\varrho(1+t)^{\epsilon_2}\}^{-|\gamma|}e^{-\frac{q_1}{2}\varrho^2(1+t)^{2\epsilon_2}}\sup_{0\leq s\leq T}\mathcal{E}_{N-1,l,q}(s),$$

which further implies that

$$\frac{d}{dt}\big(e^{\epsilon(t)}G(t)\big)\leq Ce^{\epsilon(t)}\{\varrho(1+t)^{\epsilon_2}\}^{-|\gamma|}e^{-\frac{q_1}{2}\varrho^2(1+t)^{2\epsilon_2}}\sup_{0\leq s\leq T}\mathcal{E}_{N-1,l,q}(s). \tag{6.21}$$

Here $\epsilon(t)$ is defined as

$$\epsilon(t)=\frac{\lambda(1+t)^{1-\epsilon_2|\gamma|}}{(1-\epsilon_2|\gamma|)\varrho^{|\gamma|}}. \tag{6.22}$$

Noticing that $G(0)\leq C\sup_{0\leq s\leq T}\mathcal{E}_{N-1,l,q}(s)$ for any $T>0$ by (6.13). It follows from (6.21) that

$$\begin{aligned} G(t)&\leq Ce^{-\epsilon(t)}(1+\int_0^t e^{\epsilon(s)}\{\varrho(1+s)^{\epsilon_2}\}^{-|\gamma|}e^{-\frac{q_1}{2}\varrho^2(1+s)^{2\epsilon_2}}ds)\sup_{0\leq s\leq T}\mathcal{E}_{N-1,l,q}(s)\\ &\leq Ce^{-\epsilon(t)}\sup_{0\leq s\leq T}\mathcal{E}_{N-1,l,q}(s). \end{aligned}\tag{6.23}$$

Here we have chosen $\varrho>0$ large enough and $\epsilon_2=\frac{1}{2+|\gamma|}$ and $q_1>0$ such that

$$\int_0^t e^{\epsilon(s)}(\varrho(1+s)^{\epsilon_2})^{-|\gamma|}e^{-\frac{q_1}{2}\varrho^2(1+s)^{2\epsilon_2}}ds<\infty.$$

Hence, combining (6.23) with (6.13) and taking $\epsilon_2=\frac{1}{2+|\gamma|}$ in (6.22), we have

$$\sum_{|\alpha|\leq N-1}(\|\partial^\alpha\mathbf{f}\|^2+\|\partial^\alpha\partial_x\phi\|^2)\leq C\varepsilon_0 e^{-c(1+t)^{\frac{2}{2+|\gamma|}}}.$$

The estimate (6.2) follows from this and (6.20). We thus complete the proof of Lemma 6.1. □

6.2. Global existence. In this subsection, we are now in a position to complete the proof of Theorem 1.1. By a suitable linear combination of (4.76) and (5.43), we can obtain

$$\begin{aligned} &\frac{d}{dt}\overline{\mathcal{E}}_{N,l,q}(t)+\frac{\lambda q_2q_3}{(1+t)^{1+q_3}}\mathcal{F}_{N,l,q}(t)+\lambda\|\sqrt{\bar{u}_{1x}}(\widetilde{\rho},\widetilde{u}_1,\widetilde{\theta})\|^2+\lambda\mathcal{D}_{N,l,q}(t)\\ &\leq C\delta^{\frac16}(1+t)^{-\frac76}+C(\eta_0+\delta^{\frac16}+\sqrt{\varepsilon_0})\mathcal{D}_{N,l,q}(t)+C\sum_{|\alpha|\leq N}\|\partial^\alpha\partial_x\phi\|\mathcal{F}_{N,l,q}(t). \end{aligned}\tag{6.24}$$

Here $\mathcal{D}_{N,l,q}(t)$ and $\mathcal{F}_{N,l,q}(t)$ are defined by (1.32) and (4.2), respectively, and $\overline{\mathcal{E}}_{N,l,q}(t)$ is given by

$$\overline{\mathcal{E}}_{N,l,q}(t) = \widetilde{C}_5\widetilde{\mathcal{E}}_N(t) + \widehat{\mathcal{E}}_N(t), \tag{6.25}$$

for some suitably large constant $\widetilde{C}_5 > 0$, where $\widetilde{\mathcal{E}}_N(t)$ and $\widehat{\mathcal{E}}_N(t)$ are defined by (4.77) and (5.44), respectively.

Assuming $\mathcal{E}_{N,l,q}(t) < \varepsilon_0$ for $N \geq 6$ and $l \geq \max\{N, \frac{1}{2} + \frac{1}{|\gamma|}\}$ with $-2 \leq \gamma < 0$ and $q = (q_1, q_2, q_3)$ with both $q_1 \geq 0$ and $q_2 > 0$ small enough and $q_3 \in (0, l-3)$ in (1.29), we have from (6.1) and (6.24) that

$$\begin{aligned}&\frac{d}{dt}\overline{\mathcal{E}}_{N,l,q}(t) + \frac{\lambda q_2 q_3}{(1+t)^{1+q_3}}\mathcal{F}_{N,l,q}(t) + \lambda\|\sqrt{\bar{u}_{1x}}(\widetilde{\rho}, \widetilde{u}_1, \widetilde{\theta})\|^2 + \lambda\mathcal{D}_{N,l,q}(t)\\ &\leq C_1\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}.\end{aligned} \tag{6.26}$$

Here we have further used the smallness of η_0, δ and ε_0. In view of (6.25), (4.77), (5.44), (1.31) and Lemma 2.1, there exists a constant $\bar{C} > 1$ such that

$$\bar{C}^{-1}(\mathcal{E}_{N,l,q}(t) - \delta^{\frac{1}{6}}) \leq \overline{\mathcal{E}}_{N,l,q}(t) \leq \bar{C}(\mathcal{E}_{N,l,q}(t) + \delta^{\frac{1}{6}}). \tag{6.27}$$

Assume that $\mathcal{E}_{N,l,q}(0) \leq \varepsilon_1$ and for some $T > 0$, we make a priori estimate as follows

$$\sup_{0\leq t\leq T} \mathcal{E}_{N,l,q}(t) < C_0(\varepsilon_1 + \delta^{\frac{1}{6}}) = \varepsilon_0, \tag{6.28}$$

with $C_0 = 3(\bar{C}^2 + \bar{C}C_1 + \bar{C})$, where $\eta_0 > 0$, $\epsilon_0 > 0$ and $\delta > 0$ are small enough. It follows from (6.26) that

$$\sup_{0\leq t\leq T} \overline{\mathcal{E}}_{N,l,q}(t) < \overline{\mathcal{E}}_{N,l,q}(0) + C_1\delta^{\frac{1}{6}}.$$

This together with (6.27) and (6.28) implies that

$$\begin{aligned}\sup_{0\leq t\leq T} \mathcal{E}_{N,l,q}(t) &< \bar{C}\sup_{0\leq t\leq T} \overline{\mathcal{E}}_{N,l,q}(t) + \bar{C}\delta^{\frac{1}{6}} < \bar{C}(\overline{\mathcal{E}}_{N,l,q}(0) + C_1\delta^{\frac{1}{6}}) + \bar{C}\delta^{\frac{1}{6}}\\ &\leq \bar{C}^2\mathcal{E}_{N,l,q}(0) + (\bar{C}^2 + \bar{C}C_1 + \bar{C})\delta^{\frac{1}{6}} < C_0(\varepsilon_1 + \delta^{\frac{1}{6}}) = \varepsilon_0.\end{aligned} \tag{6.29}$$

Thus the a priori estimate (6.28) is closed.

The local existence of the solutions for the VPB system (1.1) near a global Maxwellian was proved in [9, 10]. By a straightforward modification of the argument there, we can obtain the local existence of the solutions for the VPB system (1.4) and (1.5) with $F_{\pm}(t, x, v) \geq 0$ under the conditions of Theorem 1.1. For brevity, we omit the proof. By the a priori estimate (6.28) and the local existence of the solutions, the standard continuity argument, we can obtain the global existence and uniqueness of the solutions for the VPB system (1.6) and (1.7).

Next, we need to justify the time asymptotic stability of planar rarefaction waves as (1.33). In fact, for any $t > 0$, we have from (6.26) that

$$\overline{\mathcal{E}}_{N,l,q}(t) + \lambda\int_0^t \mathcal{D}_{N,l,q}(s)ds \leq C\varepsilon_0. \tag{6.30}$$

In terms of (3.26), the expression of $\partial_x M$ and $\partial_x \overline{M} = \partial_x M_{[\bar\rho,\bar u,\bar\theta]}$ in (1.19), we have from the imbedding inequality and Lemma 2.1 that

$$\|\frac{(M-\overline{M})_x}{\sqrt{\mu}}\|^2 + \|\frac{\overline{G}_x}{\sqrt{\mu}}\|^2 \le C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C\mathcal{D}_{N,l,q}(t).$$

For $-2 \le \gamma < 0$, we have by (1.32) and $l \ge N$ with $N \ge 6$ that

$$\|\mathbf{g}_x\|^2 = \|\langle v\rangle^{|\gamma|/2}\mathbf{g}_x\|_\nu^2 \le C\mathcal{D}_{N,l,q}(t).$$

Since $F_1 = M + \overline{G} + \sqrt{\mu}\mathbf{g}$, it follows from the above two estimates that

$$\begin{aligned}\|\frac{(F_1-\overline{M})_x}{\sqrt{\mu}}\|^2 &\le C\big\{\|\frac{(M-\overline{M})_x}{\sqrt{\mu}}\|^2 + \|\frac{\overline{G}_x}{\sqrt{\mu}}\|^2 + \|\mathbf{g}_x\|^2\big\}\\ &\le C\mathcal{D}_{N,l,q}(t) + C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}.\end{aligned} \tag{6.31}$$

From (4.42), one has

$$\begin{aligned}&\partial_t\partial_x(\frac{F_1-\overline{M}}{\sqrt{\mu}})\\ &= -\partial_t\partial_x(\frac{\overline{M}}{\sqrt{\mu}}) - v_1\partial_{xx}(\frac{F_1}{\sqrt{\mu}}) + \partial_x\mathcal{L}_1\mathbf{g} + \partial_x\Gamma(\mathbf{g},\frac{M-\mu}{\sqrt{\mu}}) + \partial_x\Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{g})\\ &\quad+\partial_x\Gamma(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}}) + \frac{1}{\sqrt{\mu}}\partial_x P_1 v_1 M\big\{\frac{|v-u|^2\overline{\theta}_x}{2R\theta^2} + \frac{(v-u)\cdot\bar u_x}{R\theta}\big\} + \partial_x(\frac{\partial_x\phi\partial_{v_1}F_2}{\sqrt{\mu}}).\end{aligned} \tag{6.32}$$

For any $l \ge \frac{1}{2} + \frac{1}{|\gamma|}$ with $-2 \le \gamma < 0$, $\langle v\rangle^{-\frac{\gamma}{2}+1} \le \langle v\rangle^{|\gamma|l}$. Thus we have

$$\|v_1\partial_{xx}\mathbf{g}\|^2 \le C\|\langle v\rangle^{-\frac{\gamma}{2}+1}\partial_{xx}\mathbf{g}\|_\nu^2 \le C\mathcal{D}_{N,l,q}(t).$$

By using this, (1.19), (4.44), (3.26), (1.20) and Lemma 2.1, we arrive at

$$\begin{aligned}&\|\partial_t\partial_x(\frac{\overline{M}}{\sqrt{\mu}})\|^2 + \|v_1\partial_{xx}(\frac{F_1}{\sqrt{\mu}})\|^2 + \|\frac{1}{\sqrt{\mu}}\partial_x P_1 v_1 M\big\{\frac{|v-u|^2\overline{\theta}_x}{2R\theta^2} + \frac{(v-u)\cdot\bar u_x}{R\theta}\big\}\|^2\\ &\le C\|(\bar\rho_{xt},\bar u_{xt},\bar\theta_{xt})\|^2 + \|(\bar\rho_x,\bar u_x,\bar\theta_x)\cdot(\bar\rho_t,\bar u_t,\bar\theta_t)\|^2 + \|v_1\partial_{xx}(\frac{M}{\sqrt{\mu}})\|^2 + \|v_1\partial_{xx}(\frac{\overline{G}}{\sqrt{\mu}})\|^2\\ &\quad+\|v_1\partial_{xx}\mathbf{g}\|^2 + C\|(\bar u_{xx},\bar\theta_{xx})\|^2 + \|(\rho_x,u_x,\theta_x)\cdot(\bar\rho_x,\bar u_x,\bar\theta_x)\|^2\\ &\le C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C\mathcal{D}_{N,l,q}(t).\end{aligned} \tag{6.33}$$

Note that $\mathcal{L}_1 g = \Gamma(\sqrt{\mu},g) + \Gamma(g,\sqrt{\mu})$, we have from (3.11) that

$$\|\partial_x\mathcal{L}_1\mathbf{g}\|^2 + \|\partial_x\Gamma(\mathbf{g},\frac{M-\mu}{\sqrt{\mu}})\|^2 + \|\partial_x\Gamma(\frac{M-\mu}{\sqrt{\mu}},\mathbf{g})\|^2 \le C\mathcal{D}_{N,l,q}(t).$$

By using (3.21), (3.24), (3.26), (3.11) and Lemma 2.1, we arrive at

$$\|\partial_x\Gamma(\frac{G}{\sqrt{\mu}},\frac{G}{\sqrt{\mu}})\|^2 + \|\partial_x(\frac{\partial_x\phi\partial_{v_1}F_2}{\sqrt{\mu}})\|^2 \le C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C\mathcal{D}_{N,l,q}(t).$$

It follows from (6.32) and the above estimates that

$$\|\partial_t\partial_x(\frac{F_1-\overline{M}}{\sqrt{\mu}})\|^2 \le C\delta^{\frac{1}{6}}(1+t)^{-\frac{7}{6}} + C\mathcal{D}_{N,l,q}(t). \tag{6.34}$$

By this, (6.30) and (6.31), one has

$$\int_0^{+\infty} \|\frac{(F_1-\overline{M})_x}{\sqrt{\mu}}\|^2 dt + \int_0^{+\infty} |\frac{d}{dt}\|\frac{(F_1-\overline{M})_x}{\sqrt{\mu}}\|^2| dt < C(\varepsilon_0+\delta^{\frac{1}{6}}),$$

which implies that

$$\lim_{t\to+\infty}\|\frac{(F_1-\overline{M})_x}{\sqrt{\mu}}\|^2 = 0.$$

By this and the imbedding inequality, we get

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}|\frac{(F_1-\overline{M})}{\sqrt{\mu}}|_2^2 = 0. \tag{6.35}$$

By Lemma 2.1, we can obtain

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}|\frac{M_{[\bar\rho,\bar u,\bar\theta]}-M_{[\rho^r,u^r,\theta^r]}}{\sqrt{\mu}}|_2^2 = 0. \tag{6.36}$$

Therefore, the time asymptotic convergence of the solution F_1 to the 3-rarefaction wave $M_{[\rho^r,u^r,\theta^r]}$ can be derived directly from (6.35) and (6.36). Then the proof of Theorem 1.1 is completed.

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