Methods and Applications of Analysis.
Vol. 29, No. 2, pp. 161–178, June 2022

A NOTE ON FUNCTIONAL INEQUALITIES AND ENTROPIES ESTIMATES FOR SOME HIGHER-ORDER NONLINEAR PDES*

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Abstract. In this short note, we prove by simple arguments some functional inequalities in arbitrary space dimensions. We investigate the applications of our results in establishing some appropriate a priori estimates (entropy estimates) on the approximate solution of some models related to fluid dynamics system. In particular, we derive some entropy inequalities of the solution to the Navier-Stokes-Korteweg system, and to the fourth-order degenerate diffusion equation in higher dimensional spaces. As a by product, we show that the result obtained recently by D. Bresch, A. Vasseur and C. Yu [Global existence of entropy-weak solutions to the compressible Navier-Stokes equations with non-linear density dependent viscosities, *arXiv:1905.02701*] for viscosity coefficients such that $\mu(\rho) = \rho^m$ and $\lambda(\rho) = 2(m-1)\rho^m$ for $\frac{2}{3} < m < 4$ could be generalized to the case $\frac{2}{3} < m < 6$ in the 3-dimensional setting.

Key words. A priori estimates, Functional inequalities, Navier-Stokes-Korteweg equations, Lubrification equations.

Mathematics Subject Classification. 25A23, 35B45, 35Q30.

1. Introduction. The establishing of appropriate a priori estimates of the solution to nonlinear evolution equations constitutes an important ingredient for the proof of the solution's existence result. For some problems, the bounds that we can deduce from the physical energy, which is in general, a conserved or at least a non-increasing quantity with respect to time, are not sufficient to prove the stability of approximate solution. Additional a priori estimates become required towards this goal. Obtainment of such estimates (at least formally) may be achieved by multiplying the equations by suitable quantities and integrating in space and time. Mathematically speaking, let us consider for example the logarithmic fourth-order equation (called also Derrida-Lebowitz-Speer-Spohn' equation)

$$\partial_t u + \frac{1}{2}\partial_{i,j}\big(u\partial_{i,j}^2 \log u\big) = 0 \qquad u(0,x) = u_0(x) \geq 0, \tag{1.1}$$

where we used the summation convention. This equation has been first derived in [8, 9], and it appears in various places in mathematical physics. Notice that since there is no available maximum principle for fourth-order equations, then proving preservation of positivity or non-negativity of solutions to equation (1.1) becomes more delicate. Consequently, one has to rely on suitable regularization techniques, and a priori estimates. In [17], the authors proved the global-in-time existence of weak solutions to equation (1.1). Their result is based on the following entropy production

$$\frac{dE}{dt} + \frac{1}{2}\int_\Omega u|\nabla\nabla \log u|^2\,dx \leq 0, \tag{1.2}$$

obtained after multiplying (1.1) by $\log u$, and integrating by parts. Here E denotes the physical entropy

$$E = \int_\Omega u \log\Big(\frac{u}{\int u\,dx}\Big)\,dx \geq 0,$$

*Received August 24, 2020; accepted for publication June 18, 2021.

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which is also a Lyapunov functional, $\nabla\nabla \log u$ is the Hessian of $\log u$, and $|\cdot|$ is the Euclidean norm. The fact that u has no lower bound prevents us from deducing the L^2 bound on $\nabla\nabla \log u$. Besides, the authors in [17] proved the following inequality

$$\int_\Omega u|\nabla\nabla \log u|^2\, dx \geq c \int_\Omega |\nabla\nabla \sqrt{u}|^2\, dx, \tag{1.3}$$

which allows us with the help of (1.2) to deduce an L^2 bound on $\nabla\nabla\sqrt{u}$. This inequality was the key point in the proof of global-in-time existence of solutions to equation (1.1). We finish this paragraph by saying that, depending on the mathematical structure of the equations, several complicated inequalities could be useful to conclude some estimate, and hence helping us to achieve the well-posedness result.

The aim of this short note is to prove some functional inequalities, and some entropy estimates for solution of some nonlinear PDEs. Our short paper is then organized as follows. Section 2 is devoted to presenting the main inequalities that we proved, mainly Theorem 2.1. We shall also compare our results with some known ones, see for instance subsection 2.1. In section 3, we investigate the importance of our result by presenting some applications: first we derive some a priori estimates of the solution (admitting such solution exists) to the compressible Navier-Stokes-Korteweg system, and of the solution of a class of lubrification models.

We notice that our computations performed throughout this paper are formal, in the sense that existence of smooth solutions have to be assumed in order to justify the calculations. However, as mentioned in the beginning of this section, the analysis of nonlinear equations relies on appropriate a priori estimates of the solutions (energy estimates and entropy). On other words, in the existence proofs, usually an appropriate approximation of the entropy functional has to be employed to overcome the lack of regularity and to ensure positivity of the approximations. Therefore, the formal computations are a necessary first step to identify possible entropies and, even more importantly, they reveal a lot about the structure of the nonlinear equation.

2. Main result. The aim of this section is to state and prove the following theorem.

THEOREM 2.1. *We assume that ρ is a sufficiently smooth function, and Ω is a bounded domain with periodic boundary condition, namely $\Omega = \mathbb{T}^d$, $d > 1$. Therefore, the following inequalities hold:*

• *Inequality I. Assume that $n > -1$. Then there exist a positive constant c_n such that*

$$0 < c_n \leq \frac{(1+n)(d+2)(d(1-n)+2n) - (d-1)^2(2n-1)^2}{(d+2)^2(1+n)},$$

and the following holds

$$\mathcal{A} := \int_\Omega \rho^2|\nabla\nabla \log\rho|^2\, dx + n\int_\Omega \rho^2|\Delta \log\rho|^2\, dx \geq c_n \int_\Omega |2\nabla\sqrt{\rho}|^4\, dx. \tag{2.1}$$

Using a different strategy, one can prove that for $n \geq 0$, the following holds

$$\mathcal{A} \geq \frac{1}{1+2d}\int_\Omega |\nabla\nabla\rho|^2\, dx + \frac{14}{1+2d}\int_\Omega |\nabla\sqrt{\rho}|^4\, dx. \tag{2.2}$$

Nevertheless for $-1/d < n < 0$, we get

$$\mathcal{A} \geq (1+nd)\Big(\frac{1}{1+2d}\int_\Omega |\nabla\nabla\rho|^2\,dx + \frac{14}{1+2d}\int_\Omega |\nabla\sqrt{\rho}|^4\,dx\Big). \tag{2.3}$$

- *Inequality II. Assume m satisfies the following bounds*

$$\frac{1}{2} < m < 2,$$

then we have

$$\mathcal{B} := \int_\Omega \nabla\nabla\rho^m : \nabla\nabla\rho\,dx \gtrsim \int_\Omega |\nabla\nabla\rho^{\frac{m+1}{2}}|^2\,dx + \int_\Omega |\nabla\rho^{\frac{m+1}{4}}|^4\,dx. \tag{2.4}$$

- *Inequality III. Suppose that n and m are two constants satisfy the following conditions*[1]

$$\begin{gathered} n > -1 \qquad n\times m > 0 \qquad 2n+m+1 > 0,\\ \frac{16}{9}(1-\varepsilon)(1+n)\big(1-\frac{\gamma_1+\gamma_2}{8}\big) + \gamma_1\gamma_2(1+n) + 2(\gamma_1+\gamma_2)(c_n - 1 - n) \geq 0,\\ \text{for some} \quad \varepsilon > 0 \end{gathered} \tag{2.5}$$

where

$$\gamma_1 = \frac{4(m+1)}{2n+m+1} \qquad \gamma_2 = \frac{4(2n-m+1)}{2n+m+1}$$

and

$$c_n = \frac{(1+n)(d+2)(d(1-n)+2n) - (d-1)^2(2n-1)^2}{(d+2)^2(1+n)},$$

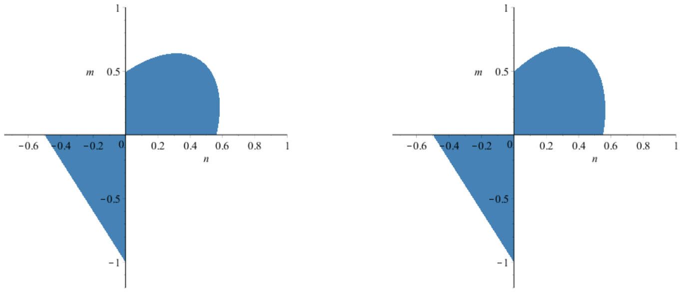
then the following inequality holds

$$\begin{aligned} \mathcal{C} &:= \int_\Omega \rho^{n+1}\nabla\nabla\rho^n : \nabla\nabla\rho^m\,dx + n\int_\Omega \rho^{n+1}\Delta\rho^n\Delta\rho^m\,dx \\ &\gtrsim \int_\Omega \big|\nabla\nabla\rho^{\frac{2n+m+1}{2}}\big|^2\,dx + \int_\Omega |\nabla\rho^{\frac{2n+m+1}{4}}|^4\,dx. \end{aligned} \tag{2.6}$$

Remark 2.1. *The proof of Inequality I is extracted with some minor modifications from [17, 16] and [19]. However, the method used in the proof of Inequalities II and III is new and more simple compared to that used in these papers. We also notice that since it is irrelevant for our proof to get a precise bound in Inequality II and Inequality III, then we did not write their explicit expressions here.*

Remark 2.2. *Unfortunately, it seems complicated to interpret the condition (2.5) algebraically. For that, in Figure 1 below, we show geometrically in which zone Inequality III holds.*

[1]See Remark 2.2 for geometric interpretation.



(a) dimension 2 (b) dimension 3

Fig. 1: Geometric interpretation of condition (2.5) for $\varepsilon = 10^{-5}$.

Remark 2.3. *Notice that the inequalities proved in Theorem 2.1 resemble in some sence the well known logarithmic Sobolev inequality. Indeed, the logarithmic Sobolev inequality reads as follows (see [12])*

$$\int_\Omega |\nabla \varrho|^2\, dx \gtrsim \int_\Omega \left(\varrho^2 \log \varrho - \varrho^2\right) dx, \tag{2.7}$$

for ϱ sufficiently smooth function, and Ω is a bounded smooth domain in $\mathbb{R}^d$. Such inequality constitutes a useful tool in analysis, probability, and measure concentration. It is also equivalent to hypercontractivity for their associated semigroups (cf. [20], [21]).

In order to make the link between (2.7) and (2.6), let us see that if we take for example $n = m = 0$ in (2.6), we get (taking in mind that $\nabla \rho^n \sim \nabla \log \rho$ when $n = 0$)

$$\int_\Omega \rho |\nabla\nabla \log \rho|^2\, dx \gtrsim \int_\Omega |\nabla(\sqrt{\rho}\nabla \log \rho)|^2\, dx + \int_\Omega \rho |\nabla \log \rho|^4\, dx$$

which can be viewed as a higher order of (2.7) (taking $\varrho := \sqrt{\rho}$ in (2.7))

$$\int_\Omega \rho |\nabla \log \rho|^2\, dx \gtrsim \int_\Omega \left(\rho \log \sqrt{\rho} - \rho\right) dx.$$

Proof of Inequality I. The proof of (2.1) is inspired by the extension of the entropy construction method introduced in [17]. To simplify the computations, we keep the same notation as in [17]. We denote by:

$$\theta = \frac{|\nabla \rho|}{\rho} \quad \lambda = \frac{1}{d}\frac{\Delta \rho}{\rho} \quad (\lambda + \xi)\theta^2 = \frac{1}{\rho^3}\nabla\nabla\rho : (\nabla\rho)^2,$$

and $\eta \geq 0$ by

$$|\nabla\nabla\rho|^2 = (d\lambda^2 + \frac{d}{d-1}\mu^2 + \eta^2)\rho^2.$$

We compute $\mathcal{A}$ using the above notations to obtain

$$\mathcal{A} = \int_\Omega \rho^2 \Big((1+nd)d\lambda^2 + \frac{d}{d-1}\xi^2 + \eta^2 - 2\lambda\theta^2(1+nd) - 2\xi\theta^2 + (1+n)\theta^4\Big)\, dx.$$

We need to compare $\mathcal{A}$ to

$$K = 16\int_\Omega |\nabla\sqrt{\rho}|^4\,dx = \int_\Omega \rho^2\theta^4\,dx.$$

We shall rely on the following two dummy integral expressions:

$$F_1 = \int_\Omega \operatorname{div}((\nabla\nabla\rho - \Delta\rho\,\mathbb{I})\cdot\nabla\rho)\,dx \qquad F_2 = \int_\Omega \operatorname{div}(\rho^{-1}|\nabla\rho|^2\nabla\rho)\,dx,$$

where $\mathbb{I}$ is the unit matrix in $\mathbb{R}^d\times\mathbb{R}^d$. Obviously, in view of the boundary conditions, $F_1 = F_2 = 0$. Our purpose now is to find constants c_n, c_1 and c_2 such that

$$\mathcal{A} - c_nK = \mathcal{A} - c_nK + c_1F_1 + c_2F_2 \geq 0.$$

According to the computations performed in [17], one has

$$F_1 = \int_\Omega \rho^2\Big(-d(d-1)\lambda^2 + \frac{d}{d-1}\xi^2 + \eta^2\Big)\,dx \qquad F_2 = \int_\Omega v^{2\gamma}\Big((d+2)\lambda\theta^2 + 2\xi\theta^2 - \theta^4\Big)\,dx.$$

After simple calculation, we obtain that

$$\begin{aligned}
&\mathcal{A} - c_nK + c_1F_1 + c_2F_2\\
&= \int_\Omega \big[((1+nd) - c_1(d-1))d\lambda^2 + \frac{d}{d-1}(1+c_1)\xi^2 + \eta^2(1+c_1)\\
&\quad + \lambda\theta^2(-2(1+nd) + c_2(d+2)) + 2\xi\theta^2(c_2-1) + \theta^4(1+n-c_n-c_2)\big]\,dx.
\end{aligned}$$

We tend to eliminate λ from the above integrand by defining c_1 and c_2 appropriately. The linear system

$$\begin{aligned}
(1+nd) - c_1(d-1) &= 0,\\
-2(1+nd) + c_2(d+2) &= 0,
\end{aligned}$$

has the solution

$$c_1 = \frac{(1+nd)}{d-1} \quad c_2 = 2\,\frac{(1+nd)}{d+2}.$$

Therefore we deduce that

$$\mathcal{A} = \int_\Omega \rho^2(b_1\xi^2 + 2b_2\xi\theta^2 + b_3\theta^4 + b_4\eta^2)\,dx,$$

where we defined b_1, b_2, b_3 and b_4 as follows

$$b_1 = \frac{d^2}{(d-1)^2}(1+n) \quad b_2 = \frac{d}{(d+2)}(2n-1) \quad b_3 = \frac{d-nd+2n}{d+2} - c_n \quad b_4 = \frac{d}{d-1}(1+n).$$

This integral is non negative if the integrand is pointwise non negative. This is the case if and only if

$$b_1 > 0 \;\; b_4 > 0 \quad \text{and} \quad b_1b_3 - b_2^2 \geq 0,$$

which is equivalent to

$$c_n \leq \frac{(1+n)(d+2)(d(1-n)+2n) - (d-1)^2(2n-1)^2}{(d+2)^2(1+n)}.$$

This end the proof of (2.1).

With much respect, the proof of (2.2) and (2.3) is extracted with minor modification from article [19]. Indeed, we have

$$\rho\nabla\nabla\log\rho = \nabla\nabla\rho - 4\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}. \tag{2.8}$$

This implies

$$\int_\Omega \rho^2|\nabla\nabla\log\rho|^2\,dx = \int_\Omega |\nabla\nabla\rho|^2\,dx + \int_\Omega |2\nabla\sqrt{\rho}|^4\,dx - 8\int_\Omega \nabla\nabla\rho : \nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}\,dx. \tag{2.9}$$

On the other hand, we can write

$$\begin{aligned}
&\int_\Omega \nabla\nabla\rho : \nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}\,dx\\
&= -\int_\Omega \nabla\rho\cdot\operatorname{div}(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho})\,dx\\
&= -\frac{1}{2}\int_\Omega \nabla\rho\cdot\operatorname{div}(\nabla\log\sqrt{\rho}\otimes\nabla\rho)\,dx\\
&= -\frac{1}{2}\int_\Omega (\nabla\rho)^2\,\Delta\log\sqrt{\rho}\,dx - \frac{1}{2}\int_\Omega \nabla\nabla\rho : \nabla\rho\otimes\nabla\log\sqrt{\rho}\,dx\\
&= -\int_\Omega (\nabla\sqrt{\rho})^2\,\rho\,\Delta\log\rho\,dx - \int_\Omega \nabla\nabla\rho : \nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}\,dx.
\end{aligned} \tag{2.10}$$

Therefore using Cauchy-Schwarz and Young inequalities, we get

$$\begin{aligned}
2\int_\Omega \nabla\nabla\rho : \nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}\,dx &= -\int_\Omega (\nabla\sqrt{\rho})^2\,\rho\,\Delta\log\rho\,dx\\
&\le \Big(\int_\Omega \big(\nabla\sqrt{\rho}\big)^4\,dx\Big)^{1/2}\Big(\int_\Omega \rho^2|\Delta\log\rho|^2\,dx\Big)^{1/2}\\
&\le \frac{1}{2}\int_\Omega \big(\nabla\sqrt{\rho}\big)^4\,dx + \frac{d}{2}\int_\Omega \rho^2|\nabla\nabla\log\rho|^2\,dx.
\end{aligned} \tag{2.11}$$

Thus combining equation (2.9) and estimate (2.11), we infer with

$$\int_\Omega \rho^2|\nabla\nabla\log\rho|^2\,dx \ge \frac{1}{1+2d}\int_\Omega |\nabla\nabla\rho|^2\,dx + \frac{14}{1+2d}\int_\Omega |\nabla\sqrt{\rho}|^4\,dx. \tag{2.12}$$

Hence, if $n\ge 0$, inequality (2.2) is a straightforward application of (2.12). However, if $n<0$, we just have to use the fact that $(\mathrm{D}(\cdot)$ is the symmetric part of the gradient $\nabla(\cdot))$

$$|\operatorname{div} g|^2 \le d|\,\mathrm{D}(g)|^2$$

for all vector g sufficiently smooth, to deduce that

$$n\int_\Omega \rho^2|\Delta\log\rho|^2\,dx \ge nd\int_\Omega \rho^2|\nabla\nabla\log\rho|^2\,dx,$$

and thus

$$\int_\Omega \rho^2|\nabla\nabla\log\rho|^2\,dx + n\int_\Omega \rho^2|\Delta\log\rho|^2\,dx \ge (1+nd)\int_\Omega \rho^2|\nabla\nabla\log\rho|^2\,dx. \tag{2.13}$$

Combining (2.13) and (2.12), we deduce that if $n > -1/d$, then the desired inequality (2.3) holds.

Proof of Inequality II. The idea behind the proof of Inequality II and Inequality III is the same, and different from the one used in proving Inequality I. Indeed, for θ a parameter to be chosen later, one can write

$$\begin{aligned}
\mathcal{B} &= \int_\Omega \nabla\nabla\rho^m : \nabla\nabla\rho \, dx \\
&= \int_\Omega \nabla\Big(\frac{m}{\theta}\rho^{m-\theta}\nabla\rho^\theta\Big) : \nabla\Big(\frac{1}{\theta}\rho^{1-\theta}\nabla\rho^\theta\Big)\, dx \\
&= \frac{m}{\theta^2}\Big[\int_\Omega \Big(\rho^{m-\theta}\nabla\nabla\rho^\theta + (m-\theta)\rho^{m-\theta-1}\nabla\rho\otimes\nabla\rho^\theta\Big) : \\
&\qquad \Big(\rho^{1-\theta}\nabla\nabla\rho^\theta + (1-\theta)\rho^{-\theta}\nabla\rho\otimes\nabla\rho^\theta\Big)\, dx\Big] \\
&= \frac{m}{\theta^2}\Big[\int_\Omega \rho^{m-2\theta+1}|\nabla\nabla\rho^\theta|^2\, dx + (m-2\theta+1)\int_\Omega \rho^{m-2\theta}\nabla\nabla\rho^\theta : \nabla\rho\otimes\nabla\rho^\theta\, dx \\
&\qquad + (m-\theta)(1-\theta)\int_\Omega \rho^{m-2\theta-1}|\nabla\rho\otimes\nabla\rho^\theta|^2\, dx\Big] \\
&= \frac{m}{\theta^2}\Big[\int_\Omega \rho^{m-2\theta+1}|\nabla\nabla\rho^\theta|^2\, dx + (m-2\theta+1)\int_\Omega \rho^{m-2\theta+1}\nabla\nabla\rho^\theta : \nabla\log\rho\otimes\nabla\rho^\theta\, dx \\
&\qquad + (m-\theta)(1-\theta)\int_\Omega \rho^{m-2\theta+1}|\nabla\log\rho\otimes\nabla\rho^\theta|^2\, dx\Big].
\end{aligned}$$

Or by virtue of the following identity

$$\begin{aligned}
\nabla\log\rho\otimes\nabla\rho^\theta &= \theta\frac{\nabla\rho}{\rho}\otimes\rho^{\theta-1}\nabla\rho \\
&= \Big(\theta\times\frac{2^2}{\theta^2}\Big)\frac{\theta}{2}\rho^{\frac{\theta}{2}-1}\nabla\rho\otimes\frac{\theta}{2}\rho^{\frac{\theta}{2}-1}\nabla\rho = \frac{2^2}{\theta}\nabla\rho^{\frac{\theta}{2}}\otimes\nabla\rho^{\frac{\theta}{2}},
\end{aligned}$$

the integral $\mathcal{B}$ becomes

$$\begin{aligned}
\mathcal{B} = \frac{m}{\theta^2}\Big[&\int_\Omega \rho^{m-2\theta+1}|\nabla\nabla\rho^\theta|^2\, dx + (m-2\theta+1)\frac{2^2}{\theta^2}\int_\Omega \rho^{m-2\theta+1}\nabla\nabla\rho^\theta : \nabla\rho^{\frac{\theta}{2}}\otimes\nabla\rho^{\frac{\theta}{2}}\, dx \\
&+ (m-\theta)(1-\theta)\frac{2^4}{\theta^4}\int_\Omega \rho^{m-2\theta+1}|\nabla\rho^{\frac{\theta}{2}}|^4\, dx\Big].
\end{aligned}$$

Now, if we choose the parameter θ such that $\theta = \frac{m+1}{2}$, we conclude that

$$\mathcal{B} = \frac{m^2}{\theta^2}\Big[\int_\Omega |\nabla\nabla\rho^\theta|^2\, dx - \frac{2^4(m-1)^2}{(m+1)^2}\int_\Omega |\nabla\rho^{\frac{\theta}{2}}|^4\, dx\Big]. \tag{2.14}$$

Since the coefficient in front of the second integral on the right hand side is negative, we need to control it by the help of the good sign term. So the idea is to find a relation between the two terms written on the right hand side of equation (2.14). Indeed, according to equation (2.10), we have

$$\begin{aligned}
\int_\Omega \nabla\nabla\rho^\theta : \nabla\rho^{\frac{\theta}{2}}\otimes\nabla\rho^{\frac{\theta}{2}}\, dx &= -\frac{1}{2}\int_\Omega (\nabla\rho^{\frac{\theta}{2}})^2\,\rho^\theta\,\Delta\log\rho^\theta\, dx \\
&= -\frac{1}{2}\int_\Omega (\nabla\rho^{\frac{\theta}{2}})^2\,\Delta\rho^\theta\, dx + 2\int_\Omega (\nabla\rho^{\frac{\theta}{2}})^4\, dx.
\end{aligned}$$

Therefore, using Cauchy-Schwarz inequality, one immediately obtains

$$2\int_\Omega (\nabla\rho^{\frac{\theta}{2}})^4\,dx \le \frac{3}{2}\Big(\int_\Omega (\nabla\rho^{\frac{\theta}{2}})^4\,dx\Big)^{1/2}\Big(\int_\Omega (\nabla\nabla\rho^{\theta})^2\,dx\Big)^{1/2}.$$

Thus, we deduce that

$$\int_\Omega (\nabla\rho^{\frac{\theta}{2}})^4\,dx \le \frac{9}{16}\int_\Omega (\nabla\nabla\rho^{\theta})^2\,dx. \tag{2.15}$$

Consequently, using estimate (3.16), the integral $\mathcal{B}$ becomes

$$\mathcal{B} \ge \frac{4m}{(m+1)^2}\Big[\varepsilon\int_\Omega |\nabla\nabla\rho^\theta|^2\,dx + \Big(\frac{16}{9}(1-\varepsilon) - \frac{2^4(m-1)^2}{(m+1)^2}\Big)\int_\Omega |\nabla\rho^{\frac{\theta}{2}}|^4\,dx\Big], \tag{2.16}$$

where ε is a positive parameter to be chosen sufficiently small. Thus, in order to deduce Inequality II from (2.16), we then have to assume the following condition on m

$$\frac{16}{9}(1-\varepsilon) - \frac{16(m-1)^2}{(m+1)^2} \ge 0.$$

We can check that it is sufficient to choose m such that

$$\frac{1}{2} < m < 2.$$

We mention here that inequality (3.16) is a multi-dimensional version of an inequality of Bernis which holds in space dimension $d = 1$ (see Theorem 1 in [1]).

Proof of Inequality III. First, we denote by

$$\mathcal{C}_1 := \int_\Omega \rho^{n+1}\nabla\nabla\rho^n : \nabla\nabla\rho^m\,dx \qquad \mathcal{C}_2 := \int_\Omega \rho^{n+1}\Delta\rho^n\,\Delta\rho^m\,dx.$$

Below, we shall perform some computations on $\mathcal{C}_1$ and $\mathcal{C}_2$. Indeed, we have

$$\begin{aligned}
\mathcal{C}_1 &= \int_\Omega \rho^{n+1}\nabla\nabla\rho^n : \nabla\nabla\rho^m\,dx\\
&= \int_\Omega \rho^{n+1}\nabla\Big(\frac{n}{\theta}\,\rho^{n-\theta}\nabla\rho^\theta\Big) : \nabla\Big(\frac{m}{\theta}\,\rho^{m-\theta}\nabla\rho^\theta\Big)\,dx\\
&= \frac{n\,m}{\theta^2}\Big[\int_\Omega \rho^{2n+m+1-2\theta}|\nabla\nabla\rho^\theta|^2\,dx + \int_\Omega \rho^{2n+1-\theta}\nabla\nabla\rho^\theta : \nabla\rho^{m-\theta}\otimes\nabla\rho^\theta\,dx\\
&\quad + \int_\Omega \rho^{n+1+m-\theta}\nabla\nabla\rho^\theta : \nabla\rho^{n-\theta}\otimes\nabla\rho^\theta\,dx\\
&\quad + \int_\Omega \rho^{n+1}\nabla\rho^{n-\theta}\otimes\nabla\rho^\theta : \nabla\rho^{m-\theta}\otimes\nabla\rho^\theta\,dx\Big].
\end{aligned}$$

Notice that the following identities hold

$$\int_\Omega \rho^{2n+1-\theta}\nabla\nabla\rho^\theta : \nabla\rho^{m-\theta}\otimes\nabla\rho^\theta\,dx = \frac{4(m-\theta)}{\theta}\int_\Omega \rho^{2n+m+1-2\theta}\nabla\nabla\rho^\theta : \nabla\rho^{\theta/2}\otimes\nabla\rho^{\theta/2}\,dx$$

$$\int_\Omega \rho^{n+1+m-\theta}\nabla\nabla\rho^\theta : \nabla\rho^{n-\theta}\otimes\nabla\rho^\theta\,dx = \frac{4(n-\theta)}{\theta}\int_\Omega \rho^{2n+m+1-2\theta}\nabla\nabla\rho^\theta : \nabla\rho^{\theta/2}\otimes\nabla\rho^{\theta/2}\,dx$$

$$\int_\Omega \rho^{n+1}\nabla\rho^{n-\theta}\otimes\nabla\rho^\theta : \nabla\rho^{m-\theta}\otimes\nabla\rho^\theta\,dx = \frac{4(n-\theta)}{\theta}\,\frac{4(m-\theta)}{\theta}\int_\Omega \rho^{2n+m+1-2\theta}(\nabla\rho^{\theta/2})^4\,dx.$$

Therefore the integral $\mathcal{C}_1$ becomes

$$\begin{aligned}\mathcal{C}_1 = \frac{n\,m}{\theta^2}\Big[&\int_\Omega \rho^{2n+m+1-2\theta}|\nabla\nabla\rho^\theta|^2\,dx - \gamma_1\int_\Omega \rho^{2n+m+1-2\theta}\nabla\nabla\rho^\theta : \nabla\rho^{\theta/2}\otimes\nabla\rho^{\theta/2}\,dx\\ &- \gamma_2\int_\Omega \rho^{2n+m+1-2\theta}\nabla\nabla\rho^\theta : \nabla\rho^{\theta/2}\otimes\nabla\rho^{\theta/2}\,dx\\ &+ \gamma_1\,\gamma_2\int_\Omega \rho^{2n+m+1-2\theta}(\nabla\rho^{\theta/2})^4\,dx\Big],\end{aligned}$$

where γ_1 and γ_2 are two constants defined by

$$\gamma_1 = \frac{4(\theta-n)}{\theta} \qquad \gamma_2 = \frac{4(\theta-m)}{\theta}.$$

Now, if we choose $\theta = \frac{2n+m+1}{2}$, we conclude that

$$\begin{aligned}\mathcal{C}_1 = \frac{n\,m}{\theta^2}\Big[&\int_\Omega |\nabla\nabla\rho^\theta|^2\,dx - (\gamma_1+\gamma_2)\int_\Omega \nabla\nabla\rho^\theta : \nabla\rho^{\theta/2}\otimes\nabla\rho^{\theta/2}\,dx\\ &+ \gamma_1\gamma_2\int_\Omega (\nabla\rho^{\theta/2})^4\,dx\Big].\end{aligned}$$

A similar computation on $\mathcal{C}_2$ yields to

$$\mathcal{C}_2 = \frac{n\,m}{\theta^2}\Big[\int_\Omega |\Delta\rho^\theta|^2\,dx - (\gamma_1+\gamma_2)\int_\Omega \Delta\rho^\theta\,(\nabla\rho^{\theta/2})^2\,dx + \gamma_1\gamma_2\int_\Omega (\nabla\rho^{\theta/2})^4\,dx\Big].$$

Gathering $\mathcal{C}_1$ and $\mathcal{C}_2$ together and taking in mind for periodic boundary condition, we infer that

$$\begin{aligned}\mathcal{C} &= \mathcal{C}_1 + n\mathcal{C}_2\\ &= \frac{n\,m}{\theta^2}\Big[(1+n)\int_\Omega |\nabla\nabla\rho^\theta|^2\,dx + \gamma_1\gamma_2(1+n)\int_\Omega (\nabla\rho^{\theta/2})^4\,dx\\ &\quad - (\gamma_1+\gamma_2)\Big(\int_\Omega \nabla\nabla\rho^\theta : \nabla\rho^{\theta/2}\otimes\nabla\rho^{\theta/2}\,dx + n\int_\Omega \Delta\rho^\theta\,(\nabla\rho^{\theta/2})^2\,dx\Big)\Big].\end{aligned} \tag{2.17}$$

In the sequel, we want to establish an estimate of

$$-(\gamma_1+\gamma_2)\Big(\int_\Omega \nabla\nabla\rho^\theta : \nabla\rho^{\theta/2}\otimes\nabla\rho^{\theta/2}\,dx + n\int_\Omega \Delta\rho^\theta\,(\nabla\rho^{\theta/2})^2\,dx\Big).$$

For this purpose, let us in the first place observe that the two following identities hold

$$\varrho\nabla\nabla\log\varrho = \nabla\nabla\varrho - 4\nabla\sqrt{\varrho}\otimes\nabla\sqrt{\varrho}, \qquad \varrho\Delta\log\varrho = \Delta\varrho - 4(\nabla\sqrt{\varrho})^2.$$

This implies that

$$\begin{aligned}\int_\Omega \varrho^2|\nabla\nabla\log\varrho|^2\,dx &= \int_\Omega |\nabla\nabla\varrho|^2\,dx + \int_\Omega |2\nabla\sqrt{\varrho}|^4\,dx - 8\int_\Omega \nabla\nabla\varrho : \nabla\sqrt{\varrho}\otimes\nabla\sqrt{\varrho}\,dx\\ \int_\Omega \varrho^2|\Delta\log\varrho|^2\,dx &= \int_\Omega |\Delta\varrho|^2\,dx + \int_\Omega |2\nabla\sqrt{\varrho}|^4\,dx - 8\int_\Omega \Delta\varrho\,(\nabla\sqrt{\varrho})^2\,dx.\end{aligned}$$

Therefore using (2.1)

$$\begin{aligned} &-8\Big(\int_\Omega \nabla\nabla\varrho : \nabla\sqrt{\varrho}\otimes\nabla\sqrt{\varrho}\,dx + n\int_\Omega \Delta\varrho\,(\nabla\sqrt{\varrho})^2\,dx\Big)\\ &=\int_\Omega \varrho^2|\nabla\nabla\log\varrho|^2\,dx + n\int_\Omega \varrho^2|\Delta\log\varrho|^2\,dx - (1+n)\int_\Omega|\nabla\nabla\varrho|^2\,dx\\ &\quad-(1+n)\int_\Omega|2\nabla\sqrt{\varrho}|^4\,dx\\ &\geq (c_n-1-n)\int_\Omega|2\nabla\sqrt{\varrho}|^4\,dx-(1+n)\int_\Omega|\nabla\nabla\varrho|^2\,dx \end{aligned} \tag{2.18}$$

where c_n is given by

$$c_n=\frac{(1+n)(d+2)(d(1-n)+2n)-(d-1)^2(2n-1)^2}{(d+2)^2(1+n)}.$$

Taking $\varrho=\rho^\theta$ in inequality (2.18), and substituting the result into equation (2.17), we obtain after assuming that $\gamma_1+\gamma_2>0$

$$\begin{aligned} \mathcal{C}\geq \frac{nm}{\theta^2}\Big[(1+n)\big(1-\frac{\gamma_1+\gamma_2}{8}\big)\int_\Omega|\nabla\nabla\rho^\theta|^2\,dx+\big(\gamma_1\gamma_2(1+n)\\ +2(\gamma_1+\gamma_2)(c_n-1-n)\big)\int_\Omega|\nabla\rho^{\theta/2}|^4\,dx\Big]. \end{aligned}$$

Arguing as in (2.16) by introducing a small parameter ε, we conclude that if the following constraints on the coefficients hold

$$\begin{aligned} &n\times m>0\quad \gamma_1+\gamma_2>0\\ &\frac{16}{9}(1-\varepsilon)(1+n)\big(1-\frac{\gamma_1+\gamma_2}{8}\big)+\gamma_1\gamma_2(1+n)+2(\gamma_1+\gamma_2)(c_n-1-n)\geq 0, \end{aligned}$$

then the proof of inequality (2.6) is finished.

2.1. Comments on our results. In this paragraph, we show how our result could be seen as a generalization of two known inequalities used to prove the well posedness result of two important models in fluids dynamics system: the Derrida-Lebowitz-Speer-Spohn equation ([17]), and the compressible Navier-Stokes equations [7]. Indeed, in [[17], Lemma 4], A. Jüngel and D. Mattews proved the following inequality

$$\frac{1}{m}\int_\Omega \rho^2\nabla\nabla\log\rho:\nabla\nabla\rho^m\,dx\geq k_m\int_\Omega|\Delta\rho^{\frac{m+2}{2}}|^2\,dx\quad\text{for}\quad -2<m<\frac{2d}{d+2}. \tag{2.19}$$

for ρ sufficiently smooth positive function. k_m is a given constant which is positive if and only if $m\in]\frac{-2-4\sqrt{d}}{d+2},\frac{-2+4\sqrt{d}}{d+2}[$. Now, we aim to show how the above inequality can be seen as a particular case of our inequality (2.6). Indeed, using the fact that $\nabla\log\rho^\alpha=\alpha\nabla\log\rho$, then we can write ($r=\rho^2$)

$$\int_\Omega\rho^2\nabla\nabla\log\rho:\nabla\nabla\rho^m\,dx=\frac{1}{2}\int_\Omega r\nabla\nabla\log r:\nabla\nabla r^{\frac{m}{2}}\,dx. \tag{2.20}$$

Arguing as in the proof of Inequality III, we can check that

$$\int_\Omega r\nabla\nabla\log r : \nabla\nabla r^{\frac{m}{2}}\,dx$$
$$= \frac{m}{2\theta^2}\Big[\int_\Omega r^{\frac{m}{2}+1-2\theta}|\nabla\nabla r^\theta|^2\,dx - (\gamma_1+\gamma_2)\int_\Omega r^{\frac{m}{2}+1-2\theta}\nabla\nabla r^\theta : \nabla r^{\frac{\theta}{2}}\otimes\nabla r^{\frac{\theta}{2}}\,dx$$
$$+\gamma_1\gamma_2\int_\Omega r^{\frac{m}{2}+1-2\theta}|\nabla r^{\frac{\theta}{2}}|^4\,dx\Big],$$

where θ,γ_1 and γ_2 represent

$$\theta = \frac{m+2}{4} \qquad \gamma_1 = 4-\frac{2m}{\theta} \qquad \gamma_2 = 4.$$

Now, let us assume the following constraint

$$\gamma_1+\gamma_2>0 \qquad \frac{16}{9}(1-\varepsilon)\big(1-\frac{\gamma_1+\gamma_2}{8}\big)+\gamma_1\gamma_2+2(\gamma_1+\gamma_2)(c_0-1)\geq 0,$$

where c_0 is given by

$$c_0 = c_n(n=0) = \frac{4d-1}{(d+2)^2}.$$

This condition turns out to assume

$$-2 < m < \frac{9(4d-1)}{4(d+2)^2},$$

Thus we conclude that if the above condition holds, then we have

$$\frac{1}{m}\int_\Omega \rho^2\nabla\nabla\log\rho : \nabla\nabla\rho^m\,dx = \frac{1}{2m}\int_\Omega r\nabla\nabla\log r : \nabla\nabla r^{\frac{m}{2}}\,dx$$
$$\geq \alpha\int_\Omega \big|\nabla\nabla r^\theta\big|^2\,dx + \beta\int_\Omega |\nabla r^{\frac{\theta}{2}}|^4\,dx,$$
$$\geq \alpha\int_\Omega \big|\nabla\nabla\rho^{\frac{m+2}{2}}\big|^2\,dx + \beta\int_\Omega |\nabla\rho^{\frac{m+2}{4}}|^4\,dx,$$

where α and β denotes two positives constants.

Notice that the interest of inequality (2.19) proved by Jüngel-Mattews is when m and k_m are positive which leads to take $m\in]0,\frac{-2+4\sqrt{d}}{d+2}[$ which is similar to that furnished by our approach $]0,\frac{9(4d-1)}{4(d+2)^2}[$.

Let us switch now to the inequality that was proved by D. Bresch, A. Vasseur and C. Yu in [[7], Lemma 2.1] ; for ρ again sufficiently smooth positive function, we have

$$\int_\Omega \rho^{n+1}|\nabla\nabla\rho^n|^2\,dx \gtrsim \int_\Omega |\nabla\nabla\rho^{\frac{3n+1}{2}}|^2\,dx + \int_\Omega |\nabla\rho^{\frac{3n+1}{4}}|^4\,dx \quad \text{if} \quad -\frac{1}{3}<n<3. \tag{2.21}$$

We remark that the above inequality could be seen also as a particular case of our inequality (2.6). Indeed, proceeding as above, we can write

$$\int_\Omega \rho^{n+1}|\nabla\nabla\rho^n|^2\,dx = \frac{n^2}{\theta^2}\Big[\int_\Omega|\nabla\nabla\rho^\theta|^2\,dx - (\gamma_1+\gamma_2)\int_\Omega\nabla\nabla\rho^\theta : \nabla\rho^{\theta/2}\otimes\nabla\rho^{\theta/2}\,dx$$
$$+\gamma_1\gamma_2\int_\Omega(\nabla\rho^{\theta/2})^4\,dx\Big]$$

where

$$\theta = \frac{3n+1}{2} \qquad \gamma_1 = \gamma_2 = \frac{4(n+1)}{3n+1}.$$

Therefore, we deduce that if

$$\frac{8(n+1)}{(3n+1)} > 0 \qquad \frac{32}{9}(1-\varepsilon)n(3n+1) + 16(n+1)^2 - 16(n+1)(3n+1)\frac{d^2+5}{(d+2)^2} \geq 0,$$

which turns out to assume

$$-\frac{1}{3} < n < 3.96 \quad \text{for} \quad d=2 \qquad -\frac{1}{3} < n < 5.11 \quad \text{for} \quad d=3$$

then the following holds

$$\int_\Omega \rho^{n+1}|\nabla\nabla\rho^n|^2\,dx \gtrsim \int_\Omega |\nabla\nabla\rho^{\frac{3n+1}{2}}|^2\,dx + \int_\Omega |\nabla\rho^{\frac{3n+1}{4}}|^4\,dx.$$

This leads to an improvement on the range of parameter n under which inequality (2.21) holds, and consequently could allows us to extend the result proved in [7] concerning the global existence of weak solutions of the compressible Navier-Stokes equations for viscosity coefficients such that $\mu(\rho) = \rho^m$ and $\lambda(\rho) = 2(m-1)\rho^m$ for $\frac{2}{3} < m < 4$ to the case $\frac{2}{3} < m < 6$ in the 3-dimensional setting. See for instance Remark 3.1 in Section 3.

3. Application to fluid dynamics systems. In this section, we give some applications of our result proved previously. Indeed, we show how our functional inequalities proved in Theorem 2.1 can help us in establishing some important entropy estimates of the solution of some mathematical models coming from fluid dynamics systems. Nevertheless, this section is divided into two subsections. In the first subsection, we consider the Navier-Stokes-Korteweg model while the second one concerns a class of lubrification models.

We point out here that we are not interested in the question of well posedness of such systems since it needs more work (this will be the subject for a forthcoming paper). However, these estimates established here constitute the main ingredient that we need towards this goal.

3.1. Navier-Stokes-Korteweg. The Navier-Stokes-Korteweg system introduced by Korteweg (see for instance [15]) which used to model phase transition phenomena in fluids. The equations are:

$$\begin{gathered} \partial_t\rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho)\,\mathrm{D}(u) + \lambda(\rho)\operatorname{div} u\,\mathbb{I}) + \nabla p = \operatorname{div}(\mathbb{S}) \end{gathered} \tag{3.1}$$

where $\operatorname{div}(\mathbb{S})$ is the capillary tensor which reads as follows

$$\mathbb{S} = \big(\rho\operatorname{div}(K(\rho)\nabla\rho) + \frac{1}{2}(K(\rho) - \rho K'(\rho)|\nabla\rho|^2\big)\,\mathbb{I} - K(\rho)\nabla\rho\otimes\nabla\rho, \tag{3.2}$$

with $K : (0,\infty) \to (0,\infty)$ is a smooth function, u stands for the velocity of fluid, and $\mathrm{D}(u) = \frac{1}{2}(\nabla u + \nabla^t u)$ is the strain tensor. The function $p(\rho)$ is the pressure that we assume here equal to $a\rho^\gamma$ where $a>0$, $\gamma>0$. The viscosity coefficients μ and λ are the Lamé coefficients that should obey the following physical conditions

$$\mu(\rho)>0 \quad 2\mu(\rho)+d\lambda(\rho)>0. \tag{3.3}$$

Notice that, as mentioned in [10], the Korteweg tensor can be written in the form

$$\operatorname{div}(\mathbb{S})=\rho\nabla\big(\sqrt{K(\rho)}\,\Delta\big(\int_0^\rho \sqrt{K(s)}\,ds\big)\big). \tag{3.4}$$

The global-in-time existence of weak solutions to system (3.1) for a general Korteweg tensor is, as far as we know, an open problem. The main difficulty that appears when we look at this question lies in the difficulty of establishing the appropriate a priori estimates, which are necessary to prove the stability of an approximate solution. Despite of having these a priori estimates at hand, proving the stability of an approximate sequence is not obvious. In fact this is due to the strongly non linear third-order differential operator, and the dispersive structure of the momentum equation. For these reasons, several results have been proved for particular choices of viscosity and capillarity coefficients (see for instance [6], [14], [18] and references therein). As for example, in [18] the authors proved the existence of global-in-time of weak solutions to the so called quantum Navier-Stokes system which corresponds to the case when

$$\mu(\rho)=\rho \qquad \lambda(\rho)=0 \qquad K(\rho)=\rho^{-1}. \tag{3.5}$$

By virtue of our inequality (2.6), we are able to prove that under more general choices of capillarity and viscosity coefficients, such for instance

$$\mu(\rho)=\rho^{m+1} \quad \lambda(\rho)=2m\rho^{m+1} \quad K(\rho)=\rho^{2n-1},$$

where n and m should satisfy the constraint (2.5), the solutions of the Navier-Stokes-Korteweg system enjoys two kind of entropy estimates. The first one is the classical energy estimate, while the second one is the so-called B-D entropy estimate. These estimates may allow us to control the degeneracy of the model close to vacuum, and, hence, helping us to study the well posedness question of such system as done in [18]. Indeed, we have the following Lemma.

LEMMA 3.1. *Assume that n and m satisfy condition* (2.5) *in Inequality III. Then the approximate solutions (ρ,u) (admitting such solution exist) of system* (3.1) *satisfy the following estimates*

$$\begin{aligned}
&\frac12\frac{d}{dt}\int_\Omega \rho|u|^2\,dx+2\int_\Omega\rho^{m+1}|\,\mathrm{D}(u)|^2\,dx+2m\int_\Omega\rho^{m+1}|\operatorname{div}u|^2\,dx\\
&\qquad+\frac{d}{dt}\int_\Omega\frac{a}{\gamma-1}\rho^\gamma\,dx+\frac12\frac{d}{dt}\int_\Omega\big|\nabla\big(\int_0^\rho\sqrt{s^{2n-1}}\,ds\big)\big|^2\,dx\le 0\\
&\frac{d}{dt}\int_\Omega\rho\big|u+\frac{2\nabla\rho^{m+1}}{\rho}|\,dx+2\int_\Omega\rho^{m+1}|A(u)|^2\,dx+\frac{d}{dt}\int_\Omega\frac{a}{\gamma-1}\rho^\gamma\,dx\\
&\qquad+2a\gamma(m+1)\int_\Omega\rho^{m+\gamma-2}|\nabla\rho|^2\,dx+\frac12\frac{d}{dt}\int_\Omega\big|\nabla\big(\int_0^\rho\sqrt{s^{2n-1}}\,ds\big)\big|^2\,dx\\
&\qquad+\alpha\int_\Omega\big|\nabla\nabla\rho^{\frac{2n+m+1}{2}}\big|^2\,dx+\beta\int_\Omega|\nabla\rho^{\frac{2n+m+1}{4}}|^4\,dx\le 0
\end{aligned}$$

where α and β are two positive constants and $A(u):=\frac12(\nabla u-\nabla^t u)$.

Before proceeding with the proof of Lemma 3.1, let us prove the following identity which can be seen as a particular case of the generalized Böhm identity proved in [5].

LEMMA 3.2. *For any smooth function $\rho(x)$, we have*

$$\rho\nabla\big(\sqrt{\rho^{2n-1}}\Delta\big(\int_0^\rho \sqrt{s^{2n-1}}ds\big)\big) = \frac{1}{n(n+1)}[\operatorname{div}(\rho^{n+1}\nabla\nabla\rho^n) + n\nabla(\rho^{n+1}\Delta\rho^n)]. \tag{3.6}$$

Proof. By straightforward computation, we can write

$$\begin{aligned}
&\rho\partial_j\big(\rho^{n-1/2}\partial_i^2\big(\int_0^\rho s^{n-1/2}\big)\big)\\
&= \frac{1}{n}\rho\,\partial_j\big(\rho^{n-1/2}\partial_i(\rho^{1/2}\partial_i\rho^n)\big)\\
&= \frac{1}{n}\rho\,\partial_j\big(\rho^n\partial_i^2\rho^n + \frac{1}{2}\rho^{n-1}\partial_i\rho\,\partial_i\rho^n\big)\\
&= \frac{1}{n}\partial_j\big(\rho^{n+1}\partial_i^2\rho^n\big) - \frac{1}{n}\rho^n\partial_j\rho\,\partial_i^2\rho^n + \frac{1}{2\,n^2}\rho\,\partial_j\big((\partial_i\rho^n)^2\big)\\
&= \frac{1}{n}\partial_j\big(\rho^{n+1}\partial_i^2\rho^n\big) - \frac{1}{n(n+1)}\partial_j\rho^{n+1}\,\partial_i^2\rho^n + \frac{1}{n(n+1)}\partial_i\rho^{n+1}\partial_j\partial_i\rho^n\\
&= \frac{1}{n(n+1)}[(n+1)\partial_j(\rho^{n+1}\partial_i^2\rho^n) - \partial_j\rho^{n+1}\partial_i^2\rho^n + \partial_i(\rho^{n+1}\partial_i\partial_j\rho^n) - \rho^{n+1}\partial_j\partial_i^2\rho^n]\\
&= \frac{1}{n(n+1)}[\partial_i(\rho^{n+1}\partial_i\partial_j\rho^n) + n\,\partial_j(\rho^{n+1}\partial_i^2\rho^n)].
\end{aligned}$$

□

Proof of Lemma 3.1. The proof of the first inequality is classical. It sufficient to multiply the equation of conservation of momentum $(3.1)_2$ by u, integrate by parts, and use the mass conservation equation. For the second one, we start by multiplying the mass conservation by $(m+1)\rho^m$ to obtain

$$\partial_t\rho^{m+1} + \operatorname{div}(\rho^{m+1}u) + m\rho^{m+1}\operatorname{div}u = 0.$$

Now, differentiating the above equation with respect to x, we get

$$\partial_t\nabla\rho^{m+1} + \operatorname{div}(u\otimes\nabla\rho^{m+1}) + \operatorname{div}(\rho^{m+1}\nabla^t u) + m\nabla(\rho^{m+1}\operatorname{div}u) = 0.$$

Multiplying the above equation by 2, and adding it to $(3.1)_2$, we obtain

$$\begin{aligned}
&\partial_t\Big(\rho\big(u + \frac{2\nabla\rho^{m+1}}{\rho}\big)\Big) + \operatorname{div}\Big(\rho u\otimes\big(u + \frac{2\nabla\rho^{m+1}}{\rho}\big)\Big)\\
&- 2\operatorname{div}\big(\rho^{m+1}A(u)\big) + a\nabla\rho^\gamma = \rho\nabla\Big(\sqrt{\rho^{2n-1}}\Delta\big(\int_0^\rho\sqrt{s^{2n-1}}\,ds\big)\Big)
\end{aligned} \tag{3.7}$$

Now, multiplying equation (3.7) by $\big(u+\frac{2\nabla\rho^{m+1}}{\rho}\big)$ and integrating by parts, we deduce

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \rho\big|u + \frac{2\nabla\rho^{m+1}}{\rho}\big|^2\,dx + 2\int_\Omega \rho^{m+1}|A(u)|^2\,dx + \frac{d}{dt}\int_\Omega\frac{a\rho^\gamma}{\gamma-1}\,dx\\
&+ 2a\gamma(m+1)\int_\Omega \rho^{m+\gamma-2}|\nabla\rho|^2\,dx = I,
\end{aligned}$$

where we denote

$$I := \int_\Omega \rho \nabla\big(\sqrt{\rho^{2n-1}}\Delta\big(\int_0^\rho \sqrt{s^{2n-1}}ds\big)\big) \cdot \big(u + \frac{2\nabla\rho^{m+1}}{\rho}\big)\, dx.$$

Now, using the mass conservation equation and identity (3.6), we get

$$\begin{aligned} I = &- \frac{1}{2}\frac{d}{dt}\int_\Omega \big|\nabla\big(\int_0^\rho \sqrt{s^{2n-1}}\, ds\big)\big|^2\, dx \\ &- \frac{2(m+1)}{nm(n+1)}\Big[\int_\Omega \rho^{n+1}\nabla\nabla\rho^n : \nabla\nabla\rho^m\, dx + n\int_\Omega \rho^{n+1}\Delta\rho^n\, \Delta\rho^m\, dx\Big]. \end{aligned}$$

By virtue of our inequality (2.6), the proof of Lemma 3.1 is complete.

Remark 3.1. *The proof of well-posedness result of the compressible Navier-Stokes equations (which corresponds to* $\mathrm{div}(\mathbb{S}) = 0$ *in* (3.1)*) proved in [7] is based on the consideration of an approximated system which is, in fact, the Navier-Stokes-Korteweg system with non-linear drags terms. Consequently, as the approximated solution should enjoy the two energy estimates showed in Lemma 3.1, so one can easily remark the interest of establishing a functional inequality like* (2.21) *or more generally like inequality III in such proof.*

3.2. Lubrification model. In this part, we will consider the following fourth order lubrication approximation

$$\partial_t h + \mathrm{div}(f(h)\nabla\Delta h - \nabla h^m) = 0. \tag{3.8}$$

This equation is relevant to surface tension dominated motion of thin viscous films and spreading droplets (see for instance [11, 3]). Here $h(t,x)$ is the thickness of the film. The equation without the second order term is derived from a lubrification approximation. The second-order term in the equation arises as a cut off of van der Waals interactions. It has a regularization effect since it removes the singularity associated with the movement of the "contact line" identified in [3].

In one dimensional space, and taking the following choice of $f(h) = h^n$, the authors in [3] (see also [4] for results obtained by the same authors concerning the same equation without the second-order "porous media" term) present rigorous weak existence theory for the above equation for all $n > 0$ and $m \in (1,2)$. The argumentation in the one-dimensional setting strongly relies on the following dissipative estimate:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_x h|^2\, dx + \int_\Omega h^n|\partial_x^3 h|^2\, dx + m\int_\Omega h^{m-1}|\partial_x^2 h|^2\, dx \\ + \frac{m(m-1)(2-m)}{3}\int_\Omega h^{m-3}|h_x|^4\, dx \le 0 \end{aligned} \tag{3.9}$$

obtained after multiplying the equation (3.8) by $-\partial_x^2 h$, and integrating in space (one needs to perform an integration by parts for the last term). Besides, since there is no available lower bound on h, then it is not clear what bounds can we deduce on the unknown h from the second and the third terms in (3.9). Indeed, the authors in [2] benefit from the so-called Bernis's inequality (see [1])

$$\int_\Omega \big(\partial_x^3 h^{\frac{n+2}{2}}\big)^2\, dx \le C\int_\Omega h^n(\partial_x^3 h)^2\, dx \qquad \text{for} \quad n \in \big(\frac{1}{2}, 3\big) \tag{3.10}$$

and using the fact that this identity holds

$$\int_\Omega (\partial_x^2 h^{\frac{m+1}{2}})^2\,dx = \Big(\frac{m+1}{2}\Big)^2\Big[\int_\Omega h^{m-1}(\partial_x^2 h)^2\,dx + \big((m-1)^2 + \frac{(m-1)(m-2)}{3}\big)\int_\Omega h^{m-3}(\partial_x h)^4\,dx\Big], \tag{3.11}$$

they conclude that under the assumption that $n \in (\frac{1}{2}, 3)$ and $1 < m < 2$, the following inequality holds

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_x h|^2\,dx + \int_\Omega \big(\partial_x^3 h^{\frac{n+2}{2}}\big)^2\,dx + \int_\Omega |\partial_x^2 h^{\frac{n+1}{2}}|^2\,dx + \int_\Omega |\partial_x h^{\frac{m+1}{4}}|^4\,dx \le 0. \tag{3.12}$$

Notice that our above computation is formal since h may vanish and so one has to be careful when dividing by h. However, all this formal computation could be justified through regularization techniques. Moreover, we emphasize that even though estimate (3.12) holds for $n \in (\frac{1}{2}, 3)$, the authors in [3] have obtained an existence theory result for all $n > 0$. This is due to the regularization effect that is coming from the porous media term. We notice that, without the porous media term, we have an existence result for $n \in (0, 3)$ in one dimensional space ([4]). Actually, the techniques used in the parameter range $n \in (0; 2)$ ($n \in (0, \frac{1}{2}]$ included), differ from those applied in the range $n \in [2; 3)$. In the former case, the reasoning is based on the so called α-entropy estimate (see [13, 2] for details).

Our goal at this stage is to derive an analogue dissipative estimate in higher dimension. Indeed, we have the following Lemma.

Lemma 3.3. *Assume that n and m satisfy the following constraints*

$$2 - \sqrt{1 - \frac{d}{8+d}} < n < 3 \qquad \frac{1}{2} < m < 2,$$

then the approximate solution (admitting such solutions exists) of equation (3.8) *satisfies the following dissipative entropy estimate*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla h|^2\,dx + \int_\Omega |\nabla\Delta h^{\frac{n+2}{2}}|^2\,dx + \int_\Omega |\nabla h^{\frac{n+2}{6}}|^6\,dx + \int_\Omega |\nabla\nabla h^{\frac{m+1}{2}}|^2\,dx + \int_\Omega |\nabla h^{\frac{m+1}{4}}|^4\,dx \le 0. \tag{3.13}$$

Proof. We start by multiplying equation (3.8) by $-\Delta h$ and integrating by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla h|^2\,dx + \int_\Omega h^n|\nabla\Delta h|^2\,dx + \int_\Omega \Delta h^m \Delta h\,dx \le 0. \tag{3.14}$$

By integration by parts, we deduce using Inequality II in Theorem 2.1 the following estimate

$$\int_\Omega \Delta h^m \Delta h\,dx = \int_\Omega \nabla\nabla h^m : \nabla\nabla h\,dx \gtrsim \int_\Omega |\nabla\nabla h^{\frac{m+1}{2}}|^2\,dx + \int_\Omega |\nabla h^{\frac{m+1}{4}}|^4\,dx \qquad \frac{1}{2} < m < 2. \tag{3.15}$$

As h may vanish, then one needs a multi-dimensional version of Bernis's inequality (3.10) in order to be able to extract information on h from the fact that the second term in estimate (3.14) is bounded. Indeed, G. Grün in [Theorem 1.1, [13]] proved the following inequality

$$\int_\Omega h^n |\nabla \Delta h|^2\,dx \gtrsim \int_\Omega |\nabla \Delta h^{\frac{n+2}{2}}|^2\,dx + \int_\Omega |\nabla h^{\frac{n+2}{6}}|^6\,dx \qquad 2 - \sqrt{1 - \frac{d}{8+d}} < n < 3. \tag{3.16}$$

Combining estimates (3.14), (3.15) and (3.16), the proof of Lemma 3.3 becomes complete. □

Remark 3.2. *It is still an open problem whether the lower bound on n given in lemma 3.3 is optimal. For the upper bound $n < 3$, however, optimality follows by similar arguments as mentioned in [13]. Concerning the parameter m, our estimate* (3.13) *holds for $\frac{1}{2} < m < 2$ instead of $1 < m < 2$ in* (3.12).

Acknowledgment. The author would like to thank Didier Bresch for insightful discussions. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

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