

ON THE HEAT EQUATION WITH DRIFT IN L_{d+1} *

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Abstract. In this paper we deal with the heat equation with drift in L_{d+1} . Basically, we prove that, if the free term is in L_q with high enough q , then the equation is uniquely solvable in a rather unusual class of functions such that $\partial_t u, D^2 u \in L_p$ with $p < d + 1$ and $Du \in L_q$.

Key words. The heat equation, singular first-order terms, non-perturbative technique.

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1. Introduction and first main result. Let \mathbb{R}^d be a Euclidean space of points $x = (x^1, \dots, x^d)$, $d \geq 2$. Define $\mathbb{R}^{d+1} = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d\}$ and for $R > 0$, $(t, x) \in \mathbb{R}^{d+1}$ introduce

$$B_R(x) = \{y \in \mathbb{R}^d : |y - x| < R\}, \quad B_R = B_R(0), \quad C_R = [0, R^2) \times B_R,$$

$$C_R(t, x) = C_R + (t, x).$$

Let $b(t, x)$ be Borel \mathbb{R}^d -valued function on \mathbb{R}^{d+1} such that for any $R > 0$, $(t, x) \in \mathbb{R}^{d+1}$

$$\|b\|_{L_{d+1}(C_R(t,x))}^{d+1} \leq \bar{b}_R^{d+1} R, \tag{1.1}$$

where \bar{b}_R , $R > 0$, is a continuous nondecreasing *bounded* function.

For $f \in L_q(\mathbb{R}^{d+1})$ vanishing for $t \geq 1$ we want to investigate the equation

$$\partial_t u + \Delta u + b^i D_i u = f \tag{1.2}$$

in the class of functions $u \in \bigcup_{T>0} W_p^{1,2}((-T, 1) \times \mathbb{R}^d)$ such that $u = 0$ for $t = 1$, where $p < d + 1$, q is large enough, $\partial_t = \frac{\partial}{\partial t}$, $D_i = \frac{\partial}{\partial x^i}$, and by $W_p^{1,2}(Q)$ we mean the set of functions u such that (Sobolev derivatives) $u, \partial_t u, D_i u, D_i D_j u \in L_p(Q)$, $i, j = 1, \dots, d$. The norm in $W_p^{1,2}(Q)$ is introduced in an obvious way.

A somewhat unusual feature of this problem is that $b^i D_i u \notin L_p((0, 1) \times \mathbb{R}^d)$ for arbitrary $u \in W_p^{1,2}((0, 1) \times \mathbb{R}^d)$ even vanishing for $t = 1$. Therefore, if we solve (1.2) and plug the solution into an equation with different b of the same class, we will generally not obtain a function in L_q even locally. The author is aware of only three similar occasions for equation with the drift term this time growing linearly in x , when the solutions are sought for in usual Hölder or Sobolev spaces without weights. These are found in [1], [2], [3]. As there, the phenomenological explanation of why $b^i D_i u$ can be controlled is that, as a *solution*, u admits a probabilistic representation which shows that, if in some direction the drift is very big, the solution along the drift is almost constant, so that the gradient is almost orthogonal to the drift. This argument does not work if u is just any arbitrary function and it shows that $b^i D_i u$ should not be treated as a perturbation but rather as an integral part of the operator $L = \partial_t + \Delta + b^i D_i$. This is the main reason why we concentrate on first estimating Du .

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Here is our first main result. For $T \in (0, \infty)$ set $\mathbb{R}_T^d = (0, T) \times \mathbb{R}^d$.

THEOREM 1.1. *Additionally to (1.1) suppose that*

$$\|b\|_{L_{d+1}(\mathbb{R}_1^d)} < \infty.$$

Let $p \in (1, d + 1)$ and

$$q = q_p := \frac{p(d + 1)}{d + 1 - p}.$$

Let f have support in C_1 and belong to $L_q(C_1)$. Then there exists $\hat{b} = \hat{b}(d, p) > 0$ such that if $\bar{b}_\infty \leq \hat{b}$, then equation (1.2) has a unique solution such that

$$\partial_t u, D^2 u \in L_p(\mathbb{R}_1^d), \quad Du \in L_q(\mathbb{R}_1^d),$$

and $u(1, \cdot) = 0$. Furthermore, there exist constants $N_1 = N_1(d, p)$ and $N_2 = N_1 \|b\|_{L_{d+1}(\mathbb{R}_1^d)}$ such that

$$\begin{aligned} \|\partial_t u, D^2 u\|_{L_p(\mathbb{R}_1^d)} &\leq N_2 \|f\|_{L_q(\mathbb{R}_1^d)} + N_1 \|f\|_{L_p(\mathbb{R}_1^d)}, \\ \|Du\|_{L_q(\mathbb{R}_1^d)} &\leq N_1 \|f\|_{L_q(\mathbb{R}_1^d)}. \end{aligned}$$

REMARK 1.1. 1. In the second part of the paper we relax the condition $\bar{b}_\infty \leq \hat{b}$ to $\bar{b}_{0+} \leq \hat{b}$ and allow f to be any function in $L_q \cap L_p$ but $q > q_p$. The arguments there are based on some results for diffusion processes with measurable coefficients and are better adapted to be generalized for fully nonlinear parabolic equations with VMO main part.

2. From our proofs one can see that one can replace (1.1) with the requirement that b belongs to more general Morrey classes. We prefer (1.1) for only one reason that in the second part of the paper we use some results from [8] which are proved, so far, only for $b \in L_{p,q}$ with $d/p + 1/q = 1$ satisfying a condition which becomes (1.1) if $p = q$.

3. Hongjie Dong kindly showed the author the way to prove the existence part in Theorem 1.1 by using the theory of parabolic Morrey's spaces. This way is probably the one G. Lieberman had in mind writing his Theorem 25 in [10] (without proof). However, as far as the author understands, this theorem does not cover Theorem 1.1 let alone Theorem 3.1 in what concerns the range of parameters.

REMARK 1.2. Once we have (1.1) the smallness can be always achieved by replacing b with λb , where λ is sufficiently small (of course, this will change the equation).

Also note that (1.1) does not imply higher summability of b . For instance, take $\alpha \in (0, d)$, $\beta \in (0, 1)$ such that $\alpha + 2\beta = d + 1$ and also take a continuous bounded function $h(\tau)$, $\tau \geq 0$, with $h(0) = 0$ and consider the function $g(t, x) = |t|^{-\beta} |x|^{-\alpha} h(|x|)$. Observe that

$$\int_{C_\rho(t,x)} g(s, y) dy ds = \rho \int_{C_1(t',x')} |s|^{-\beta} |y|^{-\alpha} h(\rho|y|) dy ds,$$

where $t' = t/\rho^2$, $x' = x/\rho$. It is not hard to see that the last integral is a bounded function of (ρ, t', x') which tends uniformly to zero as $\rho \downarrow 0$. Hence, the function

$b = g^{1/(d+1)}$ satisfies (1.1) and even $\bar{b}_{0+} = 0$. Also clearly for any $p > d + 1$ one can find h, α and β above such that $g^{1/(d+1)} \notin L_{p, \text{loc}}$.

REMARK 1.3. Theorem 1.1 is about the solvability of the terminal value problem with zero terminal data. Concerning nonzero data we refer the reader to Remark 3.2.

2. Auxiliary results. Set $L_0 = \partial_t + \Delta$. If Γ is a measurable subset of \mathbb{R}^{d+1} and f is a function on Γ we denote

$$\int_{\Gamma} f dz = \frac{1}{|\Gamma|} \int_{\Gamma} f dz,$$

where $|\Gamma|$ is the Lebesgue measure of Γ and z stands for (t, x) .

LEMMA 2.1. *Let $v \in W_1^{1,2}(C_R)$ and assume that $L_0 v = 0$ in C_R . Then, for $\kappa \in (0, 1/4]$,*

$$\int_{C_{\kappa R}} \int_{C_{\kappa R}} |Dv(z_1) - Dv(z_2)| dz_1 dz_2 \leq N(d)\kappa \int_{C_R} |Dv(z)| dz.$$

Proof. Since Dv satisfies the same equation, it suffices to prove that

$$\int_{C_{\kappa R}} \int_{C_{\kappa R}} |v(z_1) - v(z_2)| dz_1 dz_2 \leq N\kappa \int_{C_R} |v(z)| dz. \quad (2.1)$$

Self-similar transformations allows us to assume that $R = 1$.

We know (see, for instance, theorem 8.4.4 of [4]) that

$$\int_{C_{\kappa}} \int_{C_{\kappa}} |v(z_1) - v(z_2)| dz_1 dz_2 \leq N\kappa \sup_{C_{\kappa}} (|\partial_t v + |Dv||) \leq N\kappa \sup_{C_{2\kappa}} |v|,$$

where the last supremum is easily estimated through

$$\int_{C_1} |v| dz.$$

The lemma is proved. \square

For $\pi \in (1, d + 2)$ introduce

$$\pi^* = \frac{\pi(d + 2)}{d + 2 - \pi},$$

and observe that, if $\pi < d + 1$,

$$\pi^* < \pi(d + 1)/(d + 1 - \pi) =: q_{\pi},$$

$\|b^i D_i u\|_{L_{\pi}} \leq \|b\|_{L_{d+1}} \|Du\|_{L_{q_{\pi}}}$, whereas by embedding theorems $\partial_t u, D^2 u \in L_{\pi}$ only implies that $Du \in L_{\pi^*}$. This presents the main obstacle on the way of “usual” Sobolev space PDE theory for the operator L when lower-order terms are treated as perturbations.

Define $\partial' C_R = \bar{C}_R \setminus (\{t = 0\} \times B_R)$ and introduce the notation

$$\|g\|_{L_r(C_R)}^r = \int_{C_R} |g|^r dz.$$

LEMMA 2.2. Let $w \in W_{\pi}^{1,2}(C_R)$ and assume that $L_0w = f$ in C_R and $w = 0$ on $\partial' C_R$. Then

$$\|Dw\|_{L_{\pi^*}(C_R)} \leq N(d, \pi) R \|f\|_{L_{\pi}(C_R)}. \tag{2.2}$$

Proof. Rescaling allows us to assume that $R = 1$. In that case the $W_{\pi}^{1,2}(C_1)$ -norm of w is estimate through the $L_{\pi}(C_1)$ -norm of f . After that it only remains to use embedding theorems. The lemma is proved. \square

This result is used below with 1 in place of π^* .

LEMMA 2.3. Let $u \in W_{\pi}^{1,2}(C_R)$. Introduce $L_0u = f$. Then, for $\kappa \in (0, 1/4]$, with $N = N(d, \pi)$,

$$\begin{aligned} & \int_{C_{\kappa R}} \int_{C_{\kappa R}} |Du(z_1) - Du(z_2)| dz_1 dz_2 \\ & \leq N\kappa \int_{C_R} |Du(z)| dz + N\kappa^{-2d-4} R \left(\int_{C_R} |f|^{\pi} dz \right)^{1/\pi}. \end{aligned} \tag{2.3}$$

Proof. Introduce $v \in W_{\pi}^{1,2}(C_R)$ such that $L_0v = 0$ and $v = u$ on $\partial' C_R$ and let $w = u - v$. Then $L_0w = L_0u = f$ and using Lemmas 2.1 and 2.2 we get

$$\int_{C_{\kappa R}} \int_{C_{\kappa R}} |Dv(z_1) - Dv(z_2)| dz_1 dz_2 \leq N\kappa \int_{C_R} |Dv| dz,$$

$$\begin{aligned} \int_{C_R} |Dv| dz & \leq \int_{C_R} |Du| dz + \int_{C_R} |Dw| dz \\ & \leq \int_{C_R} |Du| dz + NR \left(\int_{C_R} |f|^{\pi} dz \right)^{1/\pi}, \end{aligned}$$

$$\int_{C_{\kappa R}} \int_{C_{\kappa R}} |Dw(z_1) - Dw(z_2)| dz_1 dz_2 \leq N\kappa^{-2d-4} R \left(\int_{C_R} |f|^{\pi} dz \right)^{1/\pi}.$$

These computations imply (2.3) and the lemma is proved. \square

THEOREM 2.4. Let $\pi \in (1, d + 1)$ and $u \in W_{\pi}^{1,2}(C_R)$. Set $f = Lu$. Then, for $\kappa \in (0, 1/4]$, with $N = N(d, \pi)$,

$$\begin{aligned} & \int_{C_{\kappa R}} \int_{C_{\kappa R}} |Du(z_1) - Du(z_2)| dz_1 dz_2 \\ & \leq N\kappa \int_{C_R} |Du(z)| dz + N\bar{b}_R \kappa^{-2d-4} \left(\int_{C_R} |Du|^{q_{\pi}} dz \right)^{1/q_{\pi}} \\ & \quad + NR\kappa^{-2d-4} \left(\int_{C_R} |f|^{\pi} dz \right)^{1/\pi}. \end{aligned} \tag{2.4}$$

This result follows from (2.3) and the fact that by Hölder's inequality

$$\left(\int_{C_R} |b|^{\pi} |Du|^{\pi} dz \right)^{1/\pi} \leq \left(\int_{C_R} |b|^{d+1} dz \right)^{1/(d+1)} \left(\int_{C_R} |Du|^{q_{\pi}} dz \right)^{1/q_{\pi}}.$$

The last term in (2.4) presents certain inconvenience which forced us to assume that $f = 0$ outside C_1 . We use the parabolic Hardy-Littlewood maximal function $Mg(z)$ defined as the supremum of

$$\int_C |g| dxdt$$

over all cylinders $C = C_r(s, y)$ containing z .

LEMMA 2.5. *Let $g \geq 0$ have support in C_1 and be integrable. Let $z \in \mathbb{R}^{d+1}$, $\kappa \in (0, 1/4]$. Then for any $R > 0$ and $z_0 \in C_{\kappa R}(z)$*

$$R^\pi \int_{C_R(z)} g dxdt \leq NMg(z_0) + N(|z_0| + 1)^{\pi-d-2} \int_{C_1} g dxdt,$$

where $|z_0| = \sqrt{|t_0|} + |x_0|$ and $N = N(d, \pi)$.

Proof. Introduce $\hat{C} := C_2(-1, 0)$ which is a cylinder strictly containing C_1 and consider a few cases.

Case $z_0 \in \hat{C}$. If $R \leq 1$, then by definition

$$R^\pi \int_{C_R(z)} g dxdt \leq Mg(z_0).$$

However, if $R > 1$, then

$$R^\pi \int_{C_R(z)} g dxdt \leq NR^{\pi-d-2} \int_{C_1} g dxdt \leq N \int_{C_1} g dxdt \leq NMg(z_0).$$

Case $z_0 \notin \hat{C}, t_0 \leq -1$. In this case in order for the intersection of $C_R(z)$ and C_1 be nonempty we have to have $t_0 + R^2 > 0$ and $|x_0| - 2R < 1$, that is $R \geq \max(\sqrt{|t_0|}, (1/2)(|x_0| - 1))$. By taking into account that $|t_0| \geq 1$ it is not hard to see that

$$\max(\sqrt{|t_0|}, (1/2)(|x_0| - 1)) \geq \nu(\sqrt{|t_0|} + |x_0| + 1),$$

where $\nu > 0$ is an absolute constant. In that case

$$R^\pi \int_{C_R(z)} g dxdt \leq NR^{\pi-d-2} \int_{C_1} g dxdt \leq N \frac{1}{(1 + |z_0|)^{d+2-\pi}} \int_{C_1} g dxdt. \quad (2.5)$$

Case $z_0 \notin \hat{C}, t_0 \geq 3$. This time $C_R(z) \cap C_1 \neq \emptyset$ only if $1 + R^2 > t_0$ and $|x_0| - 2R < 1$, that is $R \geq \max(\sqrt{t_0 - 1}, (1/2)(|x_0| - 1))$, which leads to (2.5) again.

Case $z_0 \notin \hat{C}, t_0 \in [-1, 3]$. Here $|x_0| \geq 2$ and $C_R(z) \cap C_1 \neq \emptyset$ only if $|x_0| - 2R < 1$, that is $R \geq (1/2)(|x_0| - 1) \geq (1/8)(|x_0| + 1)$, which leads to (2.5) again. The lemma is proved. \square

Here is the main a priori estimate. Recall that $p \in (1, d+1)$ and $q = p(d+1)/(d+1-p)$.

LEMMA 2.6. *Let $u \in \bigcup_{T>0} W_p^{1,2}((-T, 1) \times \mathbb{R}^d)$ and $Du \in L_q((-\infty, 1) \times \mathbb{R}^d)$. Assume that $u(1, \cdot) = 0$, $f := Lu \in L_q((-\infty, 1) \times \mathbb{R}^d)$, and f has support in C_1 . Then there exists a constant $\hat{b} = \hat{b}(d, p) > 0$ such that, if $\bar{b}_\infty \leq \hat{b}$, then*

$$\|Du\|_{L_q((-\infty, 1) \times \mathbb{R}^d)} \leq N\|f\|_{L_q(C_1)}, \quad (2.6)$$

where $N = N(d, p)$.

Proof. We extend u and f as zero for $t > 1$. Let \mathbb{C} be the collection of $C_R(t, x)$, $R > 0$, $(t, x) \in \mathbb{R}^{d+1}$. For functions $h = h(z)$ on \mathbb{R}^{d+1} for which it makes sense introduce

$$h^\sharp(z) = \sup_{\substack{C \in \mathbb{C}, \\ C \ni z}} \int_C \int_C |h(z_1) - h(z_2)| dz_1 dz_2.$$

Observe that if $z \in \mathbb{R}^{d+1}$ and $z \in C \in \mathbb{C}$, then owing to Theorem 2.4 and Lemma 2.5 with $\pi = (1 + p)/2$

$$\begin{aligned} & \int_C \int_C |Du(z_1) - Du(z_2)| dz_1 dz_2 \\ & \leq N\kappa M |Du|(z) + N\bar{b}_\infty \kappa^{-2d-4} \left(M(|Du|^{q_\pi})(z) \right)^{1/q_\pi} \\ & \quad + N\kappa^{-2d-4} \left(M(|f|^\pi)(z) \right)^{1/\pi} + N\kappa^{-2d-4} \|f\|_{L_\pi(C_1)} h(z), \end{aligned}$$

where $h(z) = (|z| + 1)^{1-(d+2)/\pi}$. Due to the arbitrariness of $C \ni z$ one can replace here the left-hand side with $(Du)^\sharp(z)$. Observe that, $\nu := q_\pi((d+2)/\pi - 1) = (d+2-\pi)(d+1)/(d+1-\pi) > d+2$ and

$$\int_{\mathbb{R}^{d+1}} (|z| + 1)^{q_\pi(1-(d+2)/\pi)} dz = N \int_{\mathbb{R}^d} (|x| + 1)^{2-\nu} dx < \infty.$$

Then by the Fefferman-Stein theorem and by the Hardy-Littlewood maximal function theorem (observe that $q > q_\pi$) we get

$$\|Du\|_{L_q((-\infty, 1) \times \mathbb{R}^d)} \leq N_1(\kappa + \bar{b}_\infty \kappa^{-2d-4}) \|Du\|_{L_q((-\infty, 1) \times \mathbb{R}^d)} + N\kappa^{-2d-4} \|f\|_{L_q(C_1)}.$$

To obtain (2.6) now it only remains to choose first small κ and then \hat{b} so that $N_1(\kappa + \hat{b}\kappa^{-2d-4}) \leq 1/2$. The lemma is proved. \square

Proof of uniqueness in Theorem 1.1. Let $f = 0$, our goal is to show that the only solution u with the specified properties is zero. Since $L_0 u = -b^i D_i u \in L_p(\mathbb{R}_1^d)$, we have that $u \in W_p^{1,2}(\mathbb{R}_1^d)$.

Now fix a $t_0 > 0$ close to zero, such that $u(t_0, \cdot) \in W_p^2(\mathbb{R}^d)$ and for $t \leq t_0$ define w as a solution given by means of the heat semigroup of the equation $L_0 w = 0$, $t \leq t_0$, with terminal data $w(t_0, \cdot) = u(t_0, \cdot)$. For $t \in [t_0, 1]$ set $w = u$. Then w is of class $\bigcup_{T>0} W_p^{1,2}((-T, 1) \times \mathbb{R}^d)$ and satisfies $L_0 w + I_{t>t_0} b^i D_i w = 0$ in $(-\infty, 1) \times \mathbb{R}^d$ with zero terminal condition. By using the explicit representation of w for $t \leq t_0$ and the fact that by assumption $Du \in L_q(\mathbb{R}_1^d)$, one easily shows that $Dw \in L_q((-\infty, 1) \times \mathbb{R}^d)$. But then owing to (2.6), $Dw = 0$ and $L_0 w = 0$ in $(-\infty, 1) \times \mathbb{R}^d$ and $L_0 u = 0$ in $(t_0, 1) \times \mathbb{R}^d$. It follows that $u = 0$ for $t \in [t_0, 1]$ and since t_0 can be chosen arbitrarily close to 0, $u = 0$ in \mathbb{R}_1^d , and the uniqueness of solutions is established. \square

Now comes the last step needed to prove the existence part in Theorem 1.1.

LEMMA 2.7. *Let $f \in C_0^\infty(C_1)$, $f = 0$ outside C_1 , and $b \in C_0^\infty(\mathbb{R}^{d+1})$. Define u as the classical solution of $Lu = f$ for $t \leq 1$ with terminal condition $u(1, \cdot) = 0$. Assume that $\bar{b}_\infty \leq \hat{b}$. Then*

$$\|Du\|_{L_q(\mathbb{R}_1^d)} \leq N \|f\|_{L_q(\mathbb{R}_1^d)}, \tag{2.7}$$

where $N = N(d, p)$. Furthermore,

$$\|\partial_t u, D^2 u\|_{L_p(\mathbb{R}_1^d)} \leq N_1 \|f\|_{L_q(\mathbb{R}_1^d)} + N_2 \|f\|_{L_p(\mathbb{R}_1^d)}, \quad (2.8)$$

where $N_1 = N(d, p) \|b\|_{L_{d+1}(\mathbb{R}_1^d)}$, $N_2 = N_2(d, p)$.

Proof. The existence of smooth bounded u is a classical result. For $t \leq 0$, define $u(t, x)$ as the solution of $L_0 u = 0$ with terminal data $u(0, \cdot)$. For $t < 0$, $u(t, x)$ is just a caloric function and it is represented by means of the fundamental solution of the heat equation. Furthermore, we have $q > (d + 2)/(d + 1)$ so that simple estimates show that $Du \in L_q((-\infty, 1) \times \mathbb{R}^d)$. Now (2.7) follows from Lemma 2.6.

Estimate (2.7) and Hölder's inequality show that

$$\|b^i D_i u\|_{L_p(\mathbb{R}_1^d)} \leq \|b\|_{L_{d+1}(\mathbb{R}_1^d)} \|Du\|_{L_q(\mathbb{R}_1^d)} < \infty,$$

which implies that $f - b^i D_i u \in L_p(\mathbb{R}_1^d)$, so that (2.8) is a classical result. The lemma is proved. \square

Proof of Theorem 1.1. The uniqueness part is taken care of above. To prove the existence, take $f_n \in C_0^\infty(C_1)$ converging to $f \in L_q(C_1)$ and $b_n \in C_0^\infty(\mathbb{R}^{d+1})$ converging to b in $L_{d+1}(\mathbb{R}_1^d)$ and having \bar{b}_R the same for all n (just use mollifiers and cut-off's). Then by Lemma 2.7 we have solutions u_n of $L_0 u_n + b_n^i D_i u_n = f_n$ admitting estimates (2.7) and (2.8) with u_n and f_n in place of u and f and with the constants independent of n . Now to prove the theorem it only remains to check that, if $Du_n \rightarrow Du$ weakly in $L_q(\mathbb{R}_1^d)$, then

$$b^i D_i u_n \rightarrow b^i D_i u$$

weakly in $L_p(\mathbb{R}_1^d)$. As we have seen a few times the sequence $b^i D_i u_n$ is bounded in $L_p(\mathbb{R}_1^d)$, so we need

$$\int_{\mathbb{R}_1^d} \phi b^i D_i u_n dz \rightarrow \int_{\mathbb{R}_1^d} \phi b^i D_i u dz$$

for any $\phi \in L_{p/(p-1)}(\mathbb{R}_1^d)$. The latter holds indeed, since by Hölder's inequality $\phi b \in L_{q/(q-1)}(\mathbb{R}_1^d)$. The theorem is proved. \square

3. Case when \bar{b}_{0+} is small. We suppose that assumption (1.1) is satisfied and $\|b\|_{L_{d+1}(\mathbb{R}^{d+1})} < \infty$. For $\delta \in (0, 1)$ take the finite continuous function $\bar{N}(d, d + 1, \delta)$ introduced in Theorem 2.3 of [7] and assume that there exists $\underline{R} \in (0, \infty)$ such that

$$\bar{N}(d, d + 1, 1/2) \bar{b}_{\underline{R}} < 1.$$

Next, let $d_0 = d_0(d, 1/2, \underline{R}) \in (d/2, d)$ be taken from [7].

Below, in Theorem 3.1 (for $\delta = 1/2$) $p \in [d_0 + 1, d + 1)$ and

$$q > q_p = \frac{p(d + 1)}{d + 1 - p}.$$

THEOREM 3.1. *There is a constant $\hat{b} > 0$, depending only on d , \underline{R} , p , q , $\bar{b}_{\underline{R}}$, $\|b\|_{L_{d+1}(\mathbb{R}^{d+1})}$, and the function $\bar{N}(d, d + 1, \cdot)$, such that if*

$$\bar{b}_{R_0} \leq \hat{b} \quad (3.1)$$

for an $R_0 \in (0, \underline{R}]$, then there exists a constant N_0 , depending only on what \hat{b} depends on and on R_0 and \bar{b}_∞ , such that for any $\lambda > N_0$ and $f \in L_p(\mathbb{R}^{d+1}) \cap L_q(\mathbb{R}^{d+1})$ there exists a unique solution of $Lu - \lambda u = f$ in the class of functions such that

$$\partial_t u, D^2 u \in L_p(\mathbb{R}^{d+1}), \quad Du \in L_q(\mathbb{R}^{d+1}), \quad u \in L_p(\mathbb{R}^{d+1}) \cap L_q(\mathbb{R}^{d+1}). \quad (3.2)$$

We prove Theorem 3.1 after some preparations. For $\gamma \in (0, 1)$ and $\rho > 0$ introduce the restricted sharp function of h by the formula

$$h_{\gamma, \rho}^\sharp(z) = \sup \{ I_r(h, z_0) : z_0 \in (0, \infty) \times \mathbb{R}^d, r \in (0, \rho], C_r(z_0) \ni z \}, \quad (3.3)$$

where

$$I_r(h, z) = \left(\int_{C_r(z)} \int_{C_r(z)} |h(z_1) - h(z_2)|^\gamma dz_1 dz_2 \right)^{1/\gamma}.$$

Here is Theorem C.2.4 of [6].

THEOREM 3.2. *Let $q \in (1, \infty)$, $\kappa \in (0, 1]$, $R \in (0, \infty)$, and $h \in L_q(C_{R(1+2\kappa)})$. Then*

$$\|h\|_{L_q(C_R)} \leq N \|h_{\gamma, \kappa R}^\sharp\|_{L_q(C_R)} + N\kappa^{-\chi} \|h\|_{L_\gamma(C_R)}, \quad (3.4)$$

where $\chi = (d + 2)/\gamma$ and the constants N depend only on d, γ , and q .

We also need a very particular case of Theorem 5.3 of [8].

THEOREM 3.3. *There is $\gamma \in (0, 1)$ depending only on d, \underline{R} such that for any $R \in (0, \underline{R}]$, $u \in W_{d_0+1}^{1,2}(C_R)$*

$$\|Du\|_{L_\gamma(C_R)} \leq NR \|f\|_{L_{d_0+1}(C_R)} + NR^{-1} \operatorname{osc}_{\partial' C_R} u,$$

where $f = Lu$ and the constants N depend only on $d, d_0, \underline{R}, \bar{b}_\underline{R}$ and the function $\bar{N}(d, d + 1, \cdot)$.

By combining this with embedding theorems (see, for instance, Lemma 2.3.3 in [9]) and taking into account that $d_0 + 1 > d/2 + 1$ we come to the following.

LEMMA 3.4. *For γ from Theorem 3.3 and the same type of constants N , for any $R \in (0, \underline{R}]$, $\varepsilon \in (0, 1]$, and $u \in W_{d_0+1}^{1,2}(C_R)$ we have*

$$\begin{aligned} \|Du\|_{L_\gamma(C_R)} &\leq NR \|f\|_{L_{d_0+1}(C_R)} + N\varepsilon^{(2d_0-d)/(d_0+1)} R \|\partial_t u, D^2 u\|_{L_{d_0+1}(C_R)} \\ &\quad + N\varepsilon^{-(d+2)/(d_0+1)} R^{-1} \|u\|_{L_{d_0+1}(C_R)}, \end{aligned}$$

or taking $\varepsilon = R^\alpha$, where $\alpha = (d + 2)/(2d_0 - d)$,

$$\|Du\|_{L_\gamma(C_R)} \leq NR \|f\|_{L_{d_0+1}(C_R)} + NR^\xi \|\partial_t u, D^2 u\|_{L_{d_0+1}(C_R)} + NR^{-\eta} \|u\|_{L_{d_0+1}(C_R)},$$

where $f = Lu$, $\xi = 1 + (d + 2)/(d_0 + 1)$, and $\eta = (d + 2)^2(d_0 + 1)^{-1}(2d_0 - d)^{-1} + 1$.

REMARK 3.1. Below we use the fact that by Hölder's inequality if $q \geq d_0 + 1$ and $\kappa \in (0, 1]$ that

$$\|f\|_{L_{d_0+1}(C_R)} \leq \|f\|_{L_q(C_R)} \leq N(d) \|f\|_{L_q(C_{R(1+2\kappa)})}.$$

LEMMA 3.5. *Let $p \in [d_0 + 1, d + 1)$ and*

$$q > q_p.$$

Take $\kappa \in (0, 1]$, $R \in (0, \underline{R}]$, and $u \in W_p^{1,2}(C_{R+2\kappa R})$. Set $f = Lu$. Then

$$\begin{aligned} \#Du\|_{L_q(C_R)} &\leq N(\kappa + \bar{b}_R \kappa^{-2d-4}) \#Du\|_{L_q(C_{R+2\kappa R})} + NR\kappa^{-\hat{\chi}} \|f\|_{L_q(C_{R+2\kappa R})} \\ &\quad + N\kappa^{-\chi} R^\xi \#\partial_t u, D^2u\|_{L_{d_0+1}(C_R)} + N\kappa^{-\chi} R^{-\eta} \|u\|_{L_{d_0+1}(C_R)}, \end{aligned} \quad (3.5)$$

where $\hat{\chi} = \chi \vee (2d + 4)$ and the constants N depend only on $d, d_0, \underline{R}, p, q, \bar{b}_\underline{R}$ and the function $\bar{N}(d, d + 1, \cdot)$.

Proof. Let $h = Du$. Then for $z \in C_R$, $r \leq R$, $z_0 \in (0, \infty) \times \mathbb{R}^d$, and $C_{\kappa r}(z_0) \ni z$ we have $C_r(z_0) \subset C_{R+2\kappa R}$. It follows from Theorem 2.4 that

$$\begin{aligned} I_{\kappa r}(h, z_0) &\leq N(\kappa + \bar{b}_r \kappa^{-2d-4}) \left(\int_{C_r(z_0)} |h|^{q_p} dx dt \right)^{1/q_p} \\ &\quad + NR\kappa^{-2d-4} \left(\int_{C_r(z_0)} |f|^p dx dt \right)^{1/p}. \end{aligned}$$

Hence, on C_R

$$\begin{aligned} h_{\gamma, \kappa R}^\#(z) &\leq N(\kappa + \bar{b}_R \kappa^{-2d-4}) \left(\int_{C_r(z_0)} I_{C_{R+2\kappa R}} |h|^{q_p} dx dt \right)^{1/q_p} \\ &\quad + NR\kappa^{-2d-4} \left(\int_{C_r(z_0)} I_{C_{R+2\kappa R}} |f|^p dx dt \right)^{1/p} \\ &\leq N(\kappa + \bar{b}_R \kappa^{-2d-4}) \left(M \left(I_{C_{R+2\kappa R}} |h|^{q_p} \right) (z) \right)^{1/q_p} \\ &\quad + NR\kappa^{-2d-4} \left(M \left(I_{C_{R+2\kappa R}} |f|^p \right) (z) \right)^{1/p}. \end{aligned}$$

For $q > q_p$ by Hardy-Littlewood

$$\|h_{\gamma, \kappa R}^\#\|_{L_q(C_R)} \leq N(\kappa + \bar{b}_R \kappa^{-2d-4}) \|h\|_{L_q(C_{R+2\kappa R})} + NR\kappa^{-2d-4} \|f\|_{L_q(C_{R+2\kappa R})}$$

and this along with Theorem 3.2, Lemma 3.4, and Remark 3.1 yields the desired result.

Now we are going to replace C with $C(z)$ in (3.5) thus obtaining an inequality between two functions on \mathbb{R}^{d+1} and then take the L_q -norms of both sides as a functions on \mathbb{R}^{d+1} . We need a lemma.

LEMMA 3.6. *Let h be a nonnegative function on \mathbb{R}^{d+1} and let $\infty > q \geq t \geq 1$, $r, s \in [1, \infty)$ be such that*

$$1 + \frac{t}{q} = \frac{1}{r} + \frac{1}{s}, \quad (3.6)$$

Then for any $R \in (0, \infty)$

$$\left(\int_{\mathbb{R}^{d+1}} \#h\|_{L_t(C_R(z))}^q dz \right)^{1/q} \leq N(d) R^{-(d+2)(1-1/s)/t} \|h^t\|_{L_r(\mathbb{R}^{d+1})}^{1/t} \quad (3.7)$$

Proof. Observe that

$$\|h\|_{L_t(C_R(z))}^t = NR^{-d-2}h^t * I_{C_R}(z).$$

Therefore, the left-hand side of (3.7) is

$$NR^{-(d+2)/t}\|h^t * I_{C_R}\|_{L_{q/t}(\mathbb{R}^{d+1})}^{1/t}.$$

By Young's inequality the $L_{q/t}$ -norm of the above convolution is dominated by

$$\|h^t\|_{L_r(\mathbb{R}^{d+1})}\|I_{C_R}\|_{L_s(\mathbb{R}^{d+1})} = NR^{(d+2)/s}\|h^t\|_{L_r(\mathbb{R}^{d+1})}$$

and the result follows.

Under the conditions of Lemma 3.5 we see that (3.5) with $C(z)$ in place of C yields

$$\|Du\|_{L_q(\mathbb{R}^{d+1})} \leq N(\kappa + \bar{b}_R\kappa^{-2d-4})\|Du\|_{L_q(\mathbb{R}^{d+1})} + NR\kappa^{-\hat{\chi}}\|f\|_{L_q(\mathbb{R}^{d+1})} + I,$$

where I is the sum of the $L_q(\mathbb{R}^{d+1})$ -norms of the last two terms in (3.5) with $C(z)$ in place of C . To estimate these we use Lemma 3.6 by taking $t = d_0 + 1$, $r = p/t$, and $s > 1$ defined from (3.6). Then we see that

$$\left(\int_{\mathbb{R}^{d+1}} \|\partial_t u, D^2 u\|_{L_{d_0+1}(C_R(z))}^q dz\right)^{1/q} \leq NR^{-(d+2)(1/p-1/q)}\|\partial_t u, D^2 u\|_{L_p(\mathbb{R}^{d+1})},$$

where $(d+2)(1/p-1/q) \leq (d+2)/(d_0+1) = \xi - 1$. Similarly we treat the last term in (3.5) and conclude that

$$\begin{aligned} \|Du\|_{L_q(\mathbb{R}^{d+1})} &\leq N_1(\kappa + \bar{b}_R\kappa^{-2d-4})\|Du\|_{L_q(\mathbb{R}^{d+1})} + NR\kappa^{-\hat{\chi}}\|f\|_{L_q(\mathbb{R}^{d+1})} \\ &\quad + N\kappa^{-\chi}R\|\partial_t u, D^2 u\|_{L_p(\mathbb{R}^{d+1})} \\ &\quad + N\kappa^{-\chi}R^{-(d+2)(1/p-1/q)-\eta}\|u\|_{L_p(\mathbb{R}^{d+1})}. \end{aligned} \quad (3.8)$$

We fix κ so that $N_1\kappa \leq 1/4$, observe that N_1 depends only on $d, d_0, \underline{R}, p, q, \bar{b}_R$, and the function $\bar{N}(d, d+1, \cdot)$, and in the future will only concentrate on R such that

$$N_2\bar{b}_R \leq 1/4,$$

where $N_2 = N_1\kappa^{-2d-4}$. In that case provided that the left-hand side of (3.8) is finite we get

$$\begin{aligned} \|Du\|_{L_q(\mathbb{R}^{d+1})} &\leq NR\|f\|_{L_q(\mathbb{R}^{d+1})} + NR\|\partial_t u, D^2 u\|_{L_p(\mathbb{R}^{d+1})} \\ &\quad + NR^{-(d+2)(1/p-1/q)-\eta}\|u\|_{L_p(\mathbb{R}^{d+1})}. \end{aligned} \quad (3.9)$$

THEOREM 3.7. *Under the conditions of Lemma 3.5 there exists $\hat{b} > 0$, depending only on $d, d_0, \underline{R}, p, q, \bar{b}_R, \|b\|_{L_{d+1}(\mathbb{R}^{d+1})}$, and the function $\bar{N}(d, d+1, \cdot)$, such that, if $\bar{b}_{R_0} \leq \hat{b}$ is satisfied for an $R_0 \in (0, \underline{R}]$, then for any $u \in C_0^\infty(\mathbb{R}^{d+1})$ and $\lambda \geq 0$,*

$$\begin{aligned} &\|\partial_t u, D^2 u\|_{L_p(\mathbb{R}^{d+1})} + \|Du\|_{L_q(\mathbb{R}^{d+1})} + (\lambda - N)\|u\|_{L_p(\mathbb{R}^{d+1})} + (\lambda - N)\|u\|_{L_q(\mathbb{R}^{d+1})} \\ &\leq N\|Lu - \lambda u\|_{L_q(\mathbb{R}^{d+1})} + N\|Lu - \lambda u\|_{L_p(\mathbb{R}^{d+1})}, \end{aligned} \quad (3.10)$$

where N depend only on $d, d_0, \underline{R}, p, q, R_0, \bar{b}_R, \bar{b}_\infty, \|b\|_{L_{d+1}(\mathbb{R}^{d+1})}$, and the function $\bar{N}(d, d+1, \cdot)$

Proof. By Theorem 5.2 of [7] for $\lambda \geq 1$

$$\lambda \|u\|_{L_p(\mathbb{R}^{d+1})} \leq N \|Lu - \lambda u\|_{L_p(\mathbb{R}^{d+1})}, \quad \lambda \|u\|_{L_q(\mathbb{R}^{d+1})} \leq N \|Lu - \lambda u\|_{L_q(\mathbb{R}^{d+1})}.$$

It follows that it suffices to prove (3.10) for $\lambda = 0$.

By classical results

$$\|\partial_t u, D^2 u\|_{L_p(\mathbb{R}^{d+1})} \leq N \|L_0 u\|_{L_p(\mathbb{R}^{d+1})} \leq N \|Lu\|_{L_p(\mathbb{R}^{d+1})} + N_2 \|b^i D_i u\|_{L_p(\mathbb{R}^{d+1})},$$

where the last term by Hölder's inequality is dominated by the product $\|b\|_{L_{d+1}(\mathbb{R}^{d+1})} \|Du\|_{L_{q_p}(\mathbb{R}^{d+1})}$ and, for any $\varepsilon > 0$,

$$\begin{aligned} \|Du\|_{L_{q_p}(\mathbb{R}^{d+1})} &\leq \varepsilon \|Du\|_{L_{p^*}(\mathbb{R}^{d+1})} + N(\varepsilon) \|Du\|_{L_q(\mathbb{R}^{d+1})} \\ &\leq N_3 \varepsilon \|\partial_t u, D^2 u\|_{L_p(\mathbb{R}^{d+1})} + N(\varepsilon) \|Du\|_{L_q(\mathbb{R}^{d+1})}, \end{aligned} \quad (3.11)$$

where $p^* = p(d+2)/(d+2-p)$ and the last inequality is a consequence of embedding theorems. Hence,

$$N_2 \|b^i D_i u\|_{L_p(\mathbb{R}^{d+1})} \leq N_3 \varepsilon \|\partial_t u, D^2 u\|_{L_p(\mathbb{R}^{d+1})} + N(\varepsilon) \|Du\|_{L_q(\mathbb{R}^{d+1})}.$$

We also take into account (3.9) and conclude that

$$\begin{aligned} &\|\partial_t u, D^2 u\|_{L_p(\mathbb{R}^{d+1})} \\ &\leq N \|Lu\|_{L_p(\mathbb{R}^{d+1})} + N_3 \varepsilon \|\partial_t u, D^2 u\|_{L_p(\mathbb{R}^{d+1})} \\ &\quad + N(\varepsilon) \left(R \|Lu\|_{L_q(\mathbb{R}^{d+1})} + R \|\partial_t u, D^2 u\|_{L_p(\mathbb{R}^{d+1})} + R^{-\beta} \|u\|_{L_p(\mathbb{R}^{d+1})} \right), \end{aligned}$$

where $\beta > 0$ is defined obviously. We choose and fix $\varepsilon > 0$ so that $N_3 \varepsilon \leq 1/4$ and after that we make the final choice for \hat{b} and R_0 by requiring not only $N_1 \bar{b}_{R_0} \leq 1/4$, but also $N(\varepsilon) R_0 \leq 1/4$. Then we get

$$\|\partial_t u, D^2 u\|_{L_p(\mathbb{R}^{d+1})} \leq N \|Lu\|_{L_p(\mathbb{R}^{d+1})} + N \|Lu\|_{L_q(\mathbb{R}^{d+1})} + N \|u\|_{L_p(\mathbb{R}^{d+1})}.$$

After that it only remains to use (3.9) again. The theorem is proved. \square

Proof of Theorem 3.1. Uniqueness follows from Theorem 5.2 of [7]. To prove the existence, first assume that b is bounded, smooth, and belongs to $L_{d+1}(\mathbb{R}^{d+1})$. Then by classical results (for any $\lambda > 0$) we have a unique solution in $W_p^{1,2}(\mathbb{R}^{d+1}) \cap W_q^{1,2}(\mathbb{R}^{d+1})$. Actually, it is probably hard to find this result in the literature. The reader may like to repeat the proof of the similar result for elliptic equations with variable coefficients given in Section 11.6 of [5]. Alternatively one can treat the first-order terms as perturbations as usual and invoke Theorem 5.2 (for equations with measurable coefficients) of [7] for estimating the zeroth order norm of u . These comments relate to the situation which is far more general than ours and there should be a shorter way to get what we need here. Then take $\zeta \in C_0^\infty(\mathbb{R}^{d+1})$ with unit integral and support in the unit ball, for $n = 1, 2, \dots$ define $\zeta_n(z) = n^{d+1} \zeta(nz)$, and let $b_n = b * \zeta_n$. Observe that the quantities \bar{b}_R remain the same for all b_n . Therefore, for an appropriate N_0 , for the solution u_n of $(L_0 + b_n^i D_i)u_n - \lambda u_n = f$ for $\lambda > N_0$ we obtain uniform estimates of the left-hand sides of (3.10) with u_n in place of u . By finding

u with the properties in (3.2) and a subsequence n' such that $\partial_t u_{n'}, D^2 u_{n'} \rightarrow \partial_t u, D^2 u$ weakly in $L_p(\mathbb{R}^{d+1})$, $Du_{n'} \rightarrow Du$ weakly in $L_q(\mathbb{R}^{d+1})$ and in $L_{q_p}(\mathbb{R}^{d+1})$ (see (3.11)), and $u_{n'} \rightarrow u$ weakly in $L_p(\mathbb{R}^{d+1})$, and observing that then $b_n^i D_i u_{n'} \rightarrow b^i D_i u$ weakly in $L_p(\mathbb{R}^{d+1})$, we easily pass to the limit in $L_0 u_{n'} + b_n^i D_i u_{n'} - \lambda u_{n'} = f$. The theorem is proved. \square

As a corollary of this theorem we obtain the following result about solvability of terminal-value problem with zero data at the final time. This result is obtained just by taking $f(t, x) = 0$ for $t \geq T$ and multiplying functions by $e^{\lambda t}$.

THEOREM 3.8. *Let $T \in (0, \infty)$, $p \in [d_0 + 1, d + 1)$, $q > q_p$, and let $f \in L_p((0, T) \times \mathbb{R}^d) \cap L_q((0, T) \times \mathbb{R}^d)$. Assume that condition (3.1) is satisfied. Then there exists a unique solution of the equation $Lu = f$ in $(0, T) \times \mathbb{R}^d$ with terminal condition $u(T, \cdot) = 0$ in the class of functions such that*

$$\partial_t u, D^2 u \in L_p((0, T) \times \mathbb{R}^d), \quad Du \in L_q((0, T) \times \mathbb{R}^d),$$

$$u \in L_p((0, T) \times \mathbb{R}^d) \cap L_q((0, T) \times \mathbb{R}^d).$$

REMARK 3.2. In Theorem 3.8 the terminal data is zero. One can easily consider more general data, say $g(x)$ such that there exists $g \in (W_p^{1,2} \cap W_q^{1,2})((T, T + 1) \times \mathbb{R}^d)$ such that $g(T, x) = g(x)$ and $g(T + 1, x) = 0$. Indeed, then one would apply Theorem 3.8 with $T + 1$ in place of T to $b(t, x)I_{t < T}$ in place of b and $fI_{t < T} + (\partial_t g + \Delta g)I_{t > T}$ in place of f .

4. Application to Itô's equations. As we know from [7] there are weak solutions of the equation

$$x_t = w_t + \int_0^t b(s, x_s) ds, \tag{4.1}$$

where w_t is a d -dimensional Wiener process.

THEOREM 4.1. *Assume (3.1) and $\|b\|_{L_{d+1}(\mathbb{R}^{d+1})} < \infty$. Then all solutions of (4.1) have the same finite-dimensional distributions.*

Proof. As is quite common we will rely on solutions of the corresponding parabolic equations and use Itô's formula. The only difficulty is that we have to prove that the formula is applicable with our u and b .

Take $T > 0$ and a smooth bounded $f(t, x)$ with compact support and let u be the function from Theorem 3.8. It is convenient to extend $u(t, x)$ for $t > T$ as zero. This was actually the way it was meant to be constructed, by solving the equation with $f(t, x) = 0$ for $t > T$. Introduce $u_n = \zeta_n * u$. By Itô's formula for any stopping time $\tau \leq T$ and $t \leq \tau$

$$u_n(\tau, x_\tau) = u_n(t, x_t) + \int_t^\tau Lu_n(s, x_s) ds + \int_t^\tau Du_n(s, x_s) dw_s.$$

As for any stochastic integral one can choose $\tau_n \uparrow T$ such that

$$E\left(\int_t^{\tau_n} Du_n(s, x_s) dw_s \mid \mathcal{F}_t^x\right) = 0.$$

In that case

$$E(u_n(\tau_n, x_{\tau_n}) \mid \mathcal{F}_t^x) = u_n(t, x_t) + E\left(\int_t^{\tau_n} Lu_n(s, x_s) ds \mid \mathcal{F}_t^x\right). \quad (4.2)$$

According to Theorem 4.9 of [7]

$$\begin{aligned} E \int_t^{\tau_n} |Lu_n - Lu|(s, x_s) ds &\leq E \int_t^T |Lu_n - Lu|(s, x_s) ds \\ &\leq N(1+T)^\chi \|\Phi_{1/T}(Lu_n - Lu)\|_{L_p((0,T)\times\mathbb{R}^d)}, \end{aligned}$$

where $\Phi_\lambda(t, x) = \exp(-\sqrt{\lambda}(|x| + \sqrt{t})\theta)$ and χ and θ are independent of T , u_n , u . Observe that $\partial_t u_n, D^2 u_n \rightarrow \partial_t u, D^2 u$ in $L_p((0, T) \times \mathbb{R}^d)$ and, owing to the fact that $Du_n \rightarrow Du$ in any $L_r((0, T) \times \mathbb{R}^d)$ ($f \in L_r(\mathbb{R}^{d+1})$) for any $r \in (q_p, \infty)$, we also have (by Hölder's inequality) $b^i D_i u_n \rightarrow b^i D_i u$ in $L_p(\Gamma)$, where $\Gamma \subset (0, T) \times \mathbb{R}^d$ is any standard cylinder with base that is unit ball. It turns out that because of that $\Phi_{1/T}(b^i D_i u_n - b^i D_i u) \rightarrow 0$ in $L_p((0, T) \times \mathbb{R}^d)$ and $\Phi_{1/T}(Lu_n - Lu) \rightarrow 0$ in $L_p((0, T) \times \mathbb{R}^d)$.

Indeed, let ζ be the indicator of C_1 . Then as we said above, by Hölder's inequality, for any $r > q_p$,

$$\begin{aligned} I_n(t, x) &:= \|\zeta(t - \cdot, x - \cdot)|b| |D(u_n - u)|\|_{L_p((0,T)\times\mathbb{R}^d)}^p \\ &\leq N \|b\|_{L_{d+1}}^p \|D(u_n - u)\|_{L_r((0,T)\times\mathbb{R}^d)}^p, \end{aligned}$$

which shows that $I_n(t, x) \rightarrow 0$ uniformly on \mathbb{R}^{d+1} . After that it only remains to note that

$$\begin{aligned} &\|\Phi_{1/T}|b| |D(u_n - u)|\|_{L_p((0,T)\times\mathbb{R}^d)}^p \\ &= N \int_0^{T+1} \int_{\mathbb{R}^d} \|\zeta(t - \cdot, x - \cdot)\Phi_{1/T}|b| |D(u_n - u)|\|_{L_p((0,T)\times\mathbb{R}^d)}^p dx dt \\ &\leq N \int_0^{T+1} \int_{\mathbb{R}^d} \Phi_{1/T}(t, x) I_n(t, x) dt dx \rightarrow 0, \end{aligned}$$

where the last inequality holds because $\zeta(t - s, x - y)\Phi_{1/T}(s, y)$ and $\zeta(t - s, x - y)\Phi_{1/T}(t, x)$ are comparable.

By similar reasons

$$E \int_t^T |Lu|(s, x_s) ds \leq N(1+T)^\chi \|\Phi_{1/T}Lu\|_{L_p((0,T)\times\mathbb{R}^d)} < \infty.$$

Also observe that by embedding theorems u in $t \leq T$, and, hence, u_n in $t \leq T$ are uniformly continuous (even Hölder continuous since $p > d/2 + 1$). This allows us to pass to the limit in (4.2) and conclude that

$$-u(t, x_t) = E\left(\int_t^T f(s, x_s) ds \mid \mathcal{F}_t^x\right). \quad (4.3)$$

Now suppose that for some $n = 0, 1, 2, \dots$, any $0 \leq t_0 \leq t_1 \leq \dots \leq t_n$ (no t_1 if $n = 0$) and any continuous $f_0(x), \dots, f_n(x)$ with compact support the quantity

$$Ef_0(x_{t_0}) \cdot \dots \cdot f_n(x_{t_n})$$

is independent of which solution of (4.1) we take. Automatically, of course, the same holds if f_k 's are just bounded and continuous.

This induction hypothesis holds true for $n = 0$, because (4.3) with $t = 0$ implies that

$$\int_0^T Ef(s, x_s) ds \quad \text{and, hence,} \quad Ef(s, x_s)$$

is independent of which solution of (4.1) we take.

To show that the induction works observe that by using (4.3) with $t = t_n$, $T = t_{n+1}$ we get that

$$Ef_0(x_{t_0}) \cdot \dots \cdot f_n(x_{t_n}) \int_{t_n}^{t_{n+1}} f(s, x_s) ds$$

is independent of which solution of (4.1) we take. As above this leads to the conclusion that

$$Ef_0(x_{t_0}) \cdot \dots \cdot f_n(x_{t_n}) f(t_{n+1}, x_{t_{n+1}})$$

is independent of which solution of (4.1) we take. This, obviously, proves the theorem.

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