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ON THE CONVERGENCE OF A SECOND ORDER NONLINEAR SSP RUNGE-KUTTA METHOD FOR CONVEX SCALAR CONSERVATION LAWS*

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Abstract. A class of strong stability-preserving (SSP) high-order time discretization methods, first developed by Shu [18] and by Shu and Osher [19], has been demonstrated to be very effective in solving time-dependent partial differential equations (PDEs), especially hyperbolic conservation laws. In this paper, we consider an optimal second order SSP Runge-Kutta method, of which the spatial discretization is based on Sweby's flux limiter construction [21] with minmod flux limiter and the E-scheme as the building block. For one-dimensional scalar convex conservation laws, we make minor modification to one of Yang's convergence criteria [24] and then use it to show the entropy convergence of this SSP Runge-Kutta method.

Key words. Conservation law, SSP Runge-Kutta method, Entropy convergence.

Mathematics Subject Classification. 65M12 (35L60).

1. Introduction. We consider numerical approximations to the one-dimensional scalar conservation laws

$$\begin{cases} u_t + f(u)_x = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $f \in C^1(\mathbb{R})$ is convex, and $u_0 \in BV(\mathbb{R})$. Here BV stands for the subspace of L^1_{loc} consisting of functions z with bounded total variation

$$TV(z) := \sup_{h \neq 0} \int_{\mathbb{R}} \frac{|z(x+h) - z(x)|}{|h|}\, dx. \tag{1.2}$$

In particular, we are interested in an optimal second-order SSP Runge-Kutta method [6, 19] that is given by

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n), \\ u^{n+1} &= \frac{1}{2}u^n + \frac{1}{2}u^{(1)} + \Delta t \frac{1}{2} L(u^{(1)}), \end{aligned} \tag{1.3}$$

where $L(u)$ is the spatial operator.

In order to use Yang's last convergence criterion [24] to show the entropy convergence of this scheme, we recast this method as a two-level scheme,

$$u^{n+1} = u^n + \Delta t \frac{1}{2}[L(u^n) + L(u^n + \Delta t L(u^n))]. \tag{1.4}$$

Next, let $L(u^n) = \frac{g^n_{k+\frac{1}{2}} - g^n_{k-\frac{1}{2}}}{-h}$, then we have

$$u^{(1)}_k = u^n_k - \lambda(g^n_{k+\frac{1}{2}} - g^n_{k-\frac{1}{2}}), \tag{1.5}$$

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where $\lambda = \frac{\tau}{h}$ and, $h = \Delta x$ and $\tau = \Delta t$ are spatial and temporal step sizes respectively; $u_k^n = u(x_k, t_n)$ are nodal values of the piecewise constant mesh function $u_h(x,t)$ approximating the solution $u(x,t)$. The numerical flux g is given by

$$g_{k+\frac{1}{2}}^n = g_{k+\frac{1}{2}}[u^n], \tag{1.6}$$

where

$$g_{k+\frac{1}{2}}[v] = g(v_{k-p+1}, v_{k-p+2}, \cdots, v_k, \cdots, v_{k+p}), \tag{1.7}$$

for any data $\{v_j\}$. Now, with $g_{k+\frac{1}{2}}^{(1)} = g_{k+\frac{1}{2}}[u^{(1)}]$ and $L(u^{(1)}) = \frac{g_{k+\frac{1}{2}}^{(1)} - g_{k-\frac{1}{2}}^{(1)}}{-h}$, the scheme (1.4) is then in the conservative form

$$u_k^{n+1} = H(u_{k-2p}^n, \cdots, u_{k+2p}^n) = u_k^n - \lambda(g_{k+\frac{1}{2}}^{n(1)} - g_{k-\frac{1}{2}}^{n(1)}). \tag{1.8}$$

Here, $g_{k+\frac{1}{2}}^{n(1)} = \frac{1}{2}[g_{k+\frac{1}{2}}^n + g_{k+\frac{1}{2}}^{(1)}]$, which is Lipschitz continuous with respect to its $4p$ arguments and is *consistent* with the conservation law in the sense that

$$g(u, u, \cdots, u) \equiv f(u). \tag{1.9}$$

The collection of points $\{x_{k-2p}, \cdots, x_{k+2p}\}$ is said to be *the stencil of the scheme at the point* (x_k, t_n). For a sequence of numerical solutions, we assume that the corresponding sequence of step sizes tends to zero.

For hyperbolic problems, the development of high-resolution finite difference schemes can be illustrated by a series papers of Osher and Chakravarthy, see [4, 5, 16, 17] for example and their references therein. These schemes, in spite of the local degeneracy to the first-order methods at smooth maxima and minima, have some very desirable properties: they are at least second-order accuracy in the smooth regions, having a bound on the variation of the approximate solutions, and produce stable and sharp discrete shock solutions. Nevertheless, the time accuracy of these methods is only first-order. To improve the order of the accuracy in time, the earlier contributions were some of the influential results of Shu and Shu and Osher [18, 19]. Later on, these methods were systematically studied in [6, 7], and a class of high-order SSP methods were developed there as well. A fundamental goal in the designing of these high-order SSP Runge-Kutta methods is to maintain the strong stability properties of first-order Euler time stepping methods. Therefore, some of the clear advantages of using SSP methods are: the ability to preserve stability, in any (semi) norm, of Euler first-order time discretization; having higher order accuracy in time; being able to avoid difficulties associate with oscillations in solving time-dependent PDEs, such as hyperbolic PDEs with shocks; and these methods do not increase the computational cost in comparison with traditional ordinary differential equation (ODE) solvers. Furthermore, it is necessary to use a TVD Runge-Kutta method for hyperbolic problems. The reason for this necessity can be found in [6], where a simple numerical example demonstrates that oscillations may occur when using a linearly stable but non-SSP Runge-Kutta high-order method even when the spatial discretization of the scheme, with the first-order Euler time stepping, is second-order TVD.

For the solutions of time dependent PDEs, SSP methods have been proven to be very effective. The broad range of applications of these methods include, for example, weighted L^2 SSP higher order discretization of spectral schemes discussed in [8] for linear problems, ENO and WENO finite difference and finite volume methods studied in [10, 11, 19, 20] for hyperbolic PDEs, Runge-Kutta discontinuous Galerkin finite element methods explored in [2, 3], and L^∞ SSP higher order methods applied for discontinuous Galerkin and central schemes in [1, 15].

The object of this paper is to establish the entropy consistence of a second-order nonlinear SSP Runge-Kutta method, by using one of Yang's convergence criteria [24]. Here, we would like to mention that Yang's original wavewise entropy inequality, or WEI, was developed for two time level finite difference methods. To achieve our goal, we rewrite the method of (1.3) into the form of (1.4). Also we need to slightly modify Yang's definition of ε-rarefying collection and the convergence criterion in section 3 for our application.

The paper is organized as follows. In section 2, we first review the notions of the extremum paths and the schemes with flux limiters. We then establish the extremum traceableness of the general TVD (total variation diminishing) schemes, which is necessary in analyzing the entropy convergence of the scheme. In section 3, we present a modified version of Yang's convergence criterion, a lemma that simplifies the application of this criterion for our scheme, an important estimate, and finally the proof of the main result.

2. Extremum traceableness of the TVD schemes.

2.1. The Extremum Paths. In this subsection, we review the flux limiter methods described by Sweby [21] and the notions of Yang's extremum paths [24]. For the consistency of notations, we closely follow many of them introduced by Sweby and Yang respectively. Denote

$$(\Delta f_{k+\frac{1}{2}})^{+} = f(u^n_{k+1}) - g^E_{k+\frac{1}{2}}(u^n_k, u^n_{k+1}), \tag{2.1}$$

$$(\Delta f_{k+\frac{1}{2}})^{-} = g^E_{k+\frac{1}{2}}(u^n_k, u^n_{k+1}) - f(u^n_k), \tag{2.2}$$

$$(\Delta f_{k+\frac{1}{2}})^{+(1)} = f(u^{(1)}_{k+1}) - g^E_{k+\frac{1}{2}}(u^{(1)}_k, u^{(1)}_{k+1}), \tag{2.3}$$

$$(\Delta f_{k+\frac{1}{2}})^{-(1)} = g^E_{k+\frac{1}{2}}(u^{(1)}_k, u^{(1)}_{k+1}) - f(u^{(1)}_k), \tag{2.4}$$

where $g^E_{k+\frac{1}{2}} = g^E(u^n_k, u^n_{k+1})$ and $g^{E(1)}_{k+\frac{1}{2}} = g^E_{k+\frac{1}{2}}(u^{(1)}_k, u^{(1)}_{k+1})$ are the fluxes of E-schemes [16] that satisfy

$$\operatorname{sgn}(\, u^n_{k+1} - u^n_k \,)[\, g^E_{k+\frac{1}{2}} - f(u) \,] \leq 0, \tag{2.5}$$

and

$$\operatorname{sgn}(\, u^{(1)}_{k+1} - u^{(1)}_k \,)[\, g^{E(1)}_{k+\frac{1}{2}} - f(u^{(1)}) \,] \leq 0, \tag{2.6}$$

for all u in between u^n_k and u^n_{k+1}, and all $u^{(1)}$ in between $u^{(1)}_k$ and $u^{(1)}_{k+1}$ respectively. Here and thereafter, we add superscript $^{(1)}$ to indicate the arguments of the corresponding functions are of $\{u^{(1)}\}$ instead of $\{u^n\}$.

We use

$$\nu^+_{k+\frac12}=\frac{\lambda(\Delta f_{k+\frac12})^+}{\Delta u^n_{k+\frac12}},\qquad \nu^-_{k+\frac12}=\frac{\lambda(\Delta f_{k+\frac12})^-}{\Delta u^n_{k+\frac12}},\tag{2.7}$$

$$\nu^{+(1)}_{k+\frac12}=\frac{\lambda(\Delta f_{k+\frac12})^{+(1)}}{\Delta u^{(1)}_{k+\frac12}},\qquad \nu^{-(1)}_{k+\frac12}=\frac{\lambda(\Delta f_{k+\frac12})^{-(1)}}{\Delta u^{(1)}_{k+\frac12}},\tag{2.8}$$

to define a series of local CFL numbers, where, by convention, $\Delta u^n_{k+\frac12}=\Delta u^n_k=\Delta_+u^n_k=\Delta_-u^n_{k+1}=u^n_{k+1}-u^n_k$ and $\Delta u^{(1)}_{k+\frac12}=\Delta u^{(1)}_k=\Delta_+u^{(1)}_k=\Delta_-u^{(1)}_{k+1}=u^{(1)}_{k+1}-u^{(1)}_k$. Clearly, we have $\nu^+_{k+\frac12}\ge 0$, $\nu^-_{k+\frac12}\le 0$, $\nu^{+(1)}_{k+\frac12}\ge 0$, and $\nu^{-(1)}_{k+\frac12}\le 0$, by the defining inequalities (2.5) and (2.6) of E-schemes. We also set

$$\alpha^+_{k+\frac12}=\frac12(1-\nu^+_{k+\frac12}),\qquad \alpha^-_{k+\frac12}=\frac12(1+\nu^-_{k+\frac12}),\tag{2.9}$$

$$\alpha^{+(1)}_{k+\frac12}=\frac12(1-\nu^{+(1)}_{k+\frac12}),\qquad \alpha^{-(1)}_{k+\frac12}=\frac12(1+\nu^{-(1)}_{k+\frac12});\tag{2.10}$$

and

$$r^+_k=\frac{\alpha^+_{k-\frac12}(\Delta f_{k-\frac12})^+}{\alpha^+_{k+\frac12}(\Delta f_{k+\frac12})^+},\qquad r^-_k=\frac{\alpha^-_{k+\frac12}(\Delta f_{k+\frac12})^-}{\alpha^-_{k-\frac12}(\Delta f_{k-\frac12})^-},\tag{2.11}$$

$$r^{+(1)}_k=\frac{\alpha^{+(1)}_{k-\frac12}(\Delta f_{k-\frac12})^{+(1)}}{\alpha^{+(1)}_{k+\frac12}(\Delta f_{k+\frac12})^{+(1)}},\qquad r^{-(1)}_k=\frac{\alpha^{-(1)}_{k+\frac12}(\Delta f_{k+\frac12})^{-(1)}}{\alpha^{-(1)}_{k-\frac12}(\Delta f_{k-\frac12})^{-(1)}}.\tag{2.12}$$

Very often, to enhance the readability, we use Sweby's shorthand notations: $u^k\equiv u^{(1)}_k$, $u_k\equiv u^n_k$, where k and n are the spatial and temporal indexes respectively. Let $g^n_{k+\frac12}$ be the Sweby's numerical flux:

$$g^n_{k+\frac12}=g^E_{k+\frac12}+\varphi(r^+_k)\alpha^+_{k+\frac12}(\Delta f_{k+\frac12})^+-\varphi(r^-_{k+1})\alpha^-_{k+\frac12}(\Delta f_{k+\frac12})^-,\tag{2.13}$$

then $u^{(1)}_k=u^n_k+\tau L(u^n_k)$, with $L(u^n_k)=\frac{g^n_{k+\frac12}-g^n_{k-\frac12}}{-h}$, is given by

$$u^k=u_k-\lambda\,(\,g^n_{k+\frac12}-g^n_{k-\frac12}\,),\tag{2.14}$$

and a flux limiter version of (1.4), with $g^{n(1)}_{k+\frac12}=\frac12[g^n_{k+\frac12}+g^{(1)}_{k+\frac12}]$, is in the conservative form

$$u^{n+1}_k=u_k-\lambda\,(\,g^{n(1)}_{k+\frac12}-g^{n(1)}_{k-\frac12}\,)=u_k-\frac{\lambda}{2}(\,g^n_{k+\frac12}-g^n_{k-\frac12}\,)-\frac{\lambda}{2}(\,g^{(1)}_{k+\frac12}-g^{(1)}_{k-\frac12}\,),\tag{2.15}$$

where

$$g^{(1)}_{k+\frac12}=g^{E(1)}_{k+\frac12}+\varphi(r^{+(1)}_k)\alpha^{+(1)}_{k+\frac12}(\Delta f_{k+\frac12})^{+(1)}-\varphi(r^{-(1)}_{k+1})\alpha^{-(1)}_{k+\frac12}(\Delta f_{k+\frac12})^{-(1)},\tag{2.16}$$

and

$$\varphi(r) = \begin{cases} 0 & r \leq 0, \\ r & 0 < r \leq 1, \\ 1 & r \geq 1. \end{cases} \tag{2.17}$$

Here, $\varphi(r)$ is minmod limiter, which is Lipschitz continuous function and its graph lies in the lower boundary of the second order TVD region of (2.14) that derived by Sweby [21]:

$$\{(r, \varphi_\Phi(r)) : \varphi_\Phi(r) = \max(0, \min(\Phi r, 1), \min(r, \Phi)), 1 \leq \Phi \leq 2, r \in \mathbb{R}\}. \tag{2.18}$$

We shall assume for the remainder of the paper that the local CFL numbers satisfy $|\nu^{\pm}_{k+\frac{1}{2}}| \leq 1$, $|\nu^{\pm(1)}_{k+\frac{1}{2}}| \leq 1$ for all $k \in \mathbb{Z}$; the time step is sufficiently small between u^n and $u^{(1)}$, so that if $u^n_k \geq u^n_{k\pm1}$ ($u^n_k \leq u^n_{k\pm1}$), then $u^{(1)}_k \geq u^{(1)}_{k\pm1}$ ($u^{(1)}_k \leq u^{(1)}_{k\pm1}$). Sweby's arguments [21] also show that scheme (2.15)-(2.17) is second-order accuracy in space away from extreme values and is TVD by Shu and Osher [19]. Thus, we have $m \leq u^n_k \leq M$, where $m = \min u_0$ and $M = \max u_0$.

Using Sweby's increment form for the schemes (2.13)-(2.14), we have

$$u^k = u_k - C_{k-\frac{1}{2}} \Delta u_{k-\frac{1}{2}} + D_{k+\frac{1}{2}} \Delta u_{k+\frac{1}{2}}, \tag{2.19}$$

with

$$C_{k+\frac{1}{2}} = \nu^+_{k+\frac{1}{2}} \{ 1 + \alpha^+_{k+\frac{1}{2}} [\frac{\varphi(r^+_{k+1})}{r^+_{k+1}} - \varphi(r^+_k)] \}, \tag{2.20}$$

and

$$D_{k+\frac{1}{2}} = -\nu^-_{k+\frac{1}{2}} \{ 1 + \alpha^-_{k+\frac{1}{2}} [\frac{\varphi(r^-_k)}{r^-_k} - \varphi(r^-_{k+1})] \}. \tag{2.21}$$

Therefore, the scheme (2.15)-(2.17) can be expressed as:

$$\begin{aligned} u^{n+1}_k = u_k &+ \frac{1}{2}[-C_{k-\frac{1}{2}} \Delta u_{k-\frac{1}{2}} + D_{k+\frac{1}{2}} \Delta u_{k+\frac{1}{2}}] \\ &+ \frac{1}{2}[-C^{(1)}_{k-\frac{1}{2}} \Delta u^{(1)}_{k-\frac{1}{2}} + D^{(1)}_{k+\frac{1}{2}} \Delta u^{(1)}_{k+\frac{1}{2}}], \end{aligned} \tag{2.22}$$

where

$$C^{(1)}_{k+\frac{1}{2}} = \nu^{+(1)}_{k+\frac{1}{2}} \{ 1 + \alpha^{+(1)}_{k+\frac{1}{2}} [\frac{\varphi(r^{+(1)}_{k+1})}{r^{+(1)}_{k+1}} - \varphi(r^{+(1)}_k)] \}, \tag{2.23}$$

$$D^{(1)}_{k+\frac{1}{2}} = -\nu^{-(1)}_{k+\frac{1}{2}} \{ 1 + \alpha^{-(1)}_{k+\frac{1}{2}} [\frac{\varphi(r^{-(1)}_k)}{r^{-(1)}_k} - \varphi(r^{-(1)}_{k+1})] \}, \tag{2.24}$$

and $C_{k+\frac{1}{2}}$, $D_{k+\frac{1}{2}}$ are given by (2.20)-(2.21).

Now, with (2.20)-(2.21) and (2.23)-(2.24), the scheme (2.22) can be written as an increment form:

$$u^{n+1}_k = u_k - \overline{C}_{k-\frac{1}{2}} \Delta u_{k-\frac{1}{2}} + \overline{D}_{k+\frac{1}{2}} \Delta u_{k+\frac{1}{2}}, \tag{2.25}$$

where

$$\overline{C}_{k-\frac{1}{2}} = \frac{1}{2}[(1 - C^{(1)}_{k-\frac{1}{2}} - D^{(1)}_{k+\frac{1}{2}})C_{k-\frac{1}{2}} + (1 - D_{k-\frac{1}{2}} + A_{k-\frac{3}{2}})C^{(1)}_{k-\frac{1}{2}}] \tag{2.26}$$

and

$$\overline{D}_{k+\frac{1}{2}} = \frac{1}{2}[(1 - C^{(1)}_{k-\frac{1}{2}} - D^{(1)}_{k+\frac{1}{2}})D_{k+\frac{1}{2}} + (1 - C_{k+\frac{1}{2}} + B_{k+\frac{3}{2}})D^{(1)}_{k+\frac{1}{2}})], \tag{2.27}$$

with

$$A_{k-\frac{3}{2}} = \frac{C_{k-\frac{3}{2}}\Delta u_{k-\frac{3}{2}}}{\Delta u_{k-\frac{1}{2}}}, \quad B_{k+\frac{3}{2}} = \frac{D_{k+\frac{3}{2}}\Delta u_{k+\frac{3}{2}}}{\Delta u_{k+\frac{1}{2}}}, \quad \Delta u_{k\pm\frac{1}{2}} \neq 0. \tag{2.28}$$

The concept of discrete extremum paths was introduced by Yang (see Definition 6.3 [23] and Definition 2.13 [24]). For the convenience of applications, here, we quote the relevant definitions of the fully-discrete case. Consider a numerical solution u defined on the set of grid points $X := \{(x_j, t_n) : j \in \mathbb{Z}, n \in \mathbb{Z}^+\}$. A finite set of successive grid points $\{x_q, \cdots, x_r\}$ with $r \geq q$ is said to be the *stencil of a spatial maximum*, or simply *an* $\overline{E}$*-stencil* of u at the time t_n, provided $u^n_q = \cdots = u^n_r, u^n_{q-1} < u^n_q$ and $u^n_{r+1} < u^n_r$. Notions of $\underline{E}$*-stencils* for minima and E*-stencils* for general extrema are defined similarly. Throughout the paper, we refer to [24] for the definitions, lemmas and theorems that we have quoted.

Definition 2.1 (see Definition 2.13 [24]). A nonempty subset of X denoted by $\overline{E}_{t_n,t_m}, n \leq m$, is called *a ridge of the numerical solution* u *from* t_n *to* t_m if

(i) for all ν, $n \leq \nu \leq m$, the set

$$P_{\overline{E}}(\nu) := \{x_j : (x_j, t_\nu) \in \overline{E}_{t_n,t_m}\} = \{x_{q^\nu}, \cdots, x_{r^\nu}\}$$

is not empty and is an $\overline{E}$-stencil of u at t_ν;

(ii) for all ν, $n \leq \nu \leq m - 1$,

$$P_{\overline{E}}(\nu) \cup P_{\overline{E}}(\nu + 1) = \{x_j : \min(q^\nu, q^{\nu+1}) \leq j \leq \max(r^\nu, r^{\nu+1})\}.$$

The set $P_{\overline{E}}(\nu)$ is called *the x-projection of* $\overline{E}_{t_n,t_m}$ at t_ν. The value of u along the ridge is denoted by $V_{\overline{E}}(\nu) : V_{\overline{E}}(\nu) = u^\nu_j$ for $q^\nu \leq j \leq r^\nu$.

If, for all ν, $n \leq \nu \leq m$, the $\overline{E}$-stencil in the item (i) of the definition is replaced by an $\underline{E}$-stencil, then the set is called a *trough* of u from t_n to t_m and is denoted by $\underline{E}_{t_n,t_m}$. The related notions $P_{\underline{E}}(\nu)$ and $V_{\underline{E}}(\nu)$ are defined similarly. Ridges and troughs are also called *extremum paths*. When we do not distinguish between ridges and troughs, we use E_{t_n,t_m}, $P_E(\nu)$, and $V_E(\nu)$ for either type. We write

$$E^1_{t_n,t_m} < (\leq) E^2_{t_n,t_m}, \text{ if } \max P_{E^1}(\nu) < (\leq) \max P_{E^2}(\nu) \text{ for } n \leq \nu \leq m.$$

Definition 2.2 (see Definition 2.14 [24]). A scheme is said to be *extremum traceable* if there exists a positive constant $c \geq 1$ such that for each numerical solution u of the scheme and each integer $N > 0$, there exists a finite or infinite collection of extremum paths $\{E^l_{t_0,t_N}\}^{l_2}_{l=l_1}$ with the following properties:

(i) $\{P_{E^l}(N)\}^{l_2}_{l=l_1}$ is precisely the set of E-stencils of u^n_j at the time t_N arranged in ascending spatial coordinates.

(ii) If $E^l_{t_0,t_N}$ is a ridge (trough), then $V_{E_l}(n)$ is a non increasing (non decreasing) function of n.

(iii) Let $P_{E^l}(n) = \{x_{q^l(n)}, \cdots, x_{r^l(n)}\}$ for $1 \le n \le N$. If $P_{E^l}(n) \cap P_{E^l}(n+1) = \emptyset$, then

$$|\,u^n_{q^l(n+1)} - u^n_{r^l(n)}\,| \le c\,|\,V_{E^l}(n+1) - V_{E^l}(n)\,| \quad \text{when} \quad q^l(n+1) > r^l(n),$$

$$|\,u^n_{r^l(n+1)} - u^n_{q^l(n)}\,| \le c\,|\,V_{E^l}(n+1) - V_{E^l}(n)\,| \quad \text{when} \quad q^l(n) > r^l(n+1).$$

(iv) If $l_2 > l_1$, then $E^l_{t_0,t_N} < E^{l+1}_{t_0,t_N}$ for $l_1 \le l \le l_2 - 1$.

2.2. Extremum Traceableness of the TVD Schemes. Although the numerical solutions of (2.25)-(2.28), under the condition (2.29), share desired TVD property with the exact solution of (1.1). In compliance with Yang's WEI convergence criterion, we need stronger condition (the extremum traceableness) for this scheme and we are able to show the following result [12].

THEOREM 2.3 (see Theorem 2.3 [12]). *The sufficient conditions for the schemes* (2.25)-(2.28) *to be extremum traceable are the following inequalities:*

$$0 \le \overline{C}_{k+\frac{1}{2}}, \quad 0 \le \overline{D}_{k+\frac{1}{2}}, \quad 0 \le \overline{C}_{k+\frac{1}{2}} + \overline{D}_{k+\frac{1}{2}} \le 1, \text{ for all } k; \tag{2.29}$$

there is a positive constant μ such that, if u_k is a space extremum, then

$$\max\{\overline{C}_{k\pm\frac{1}{2}}, \overline{C}_{k\pm\frac{3}{2}}, \overline{D}_{k\pm\frac{1}{2}}, \overline{D}_{k+\frac{3}{2}}\} \le \frac{\mu}{4} < \frac{1}{4}, \tag{2.30}$$

where $\overline{C}_{k+\frac{1}{2}}$ and $\overline{D}_{k+\frac{1}{2}}$ are given by (2.26)-(2.28).

In terms of the local CFL numbers, we assume $|A_{k-\frac{3}{2}}| \le 1$ and $|B_{k+\frac{3}{2}}| \le 1$ for all k, where $A_{k-\frac{3}{2}}$ and $B_{k+\frac{3}{2}}$ are given by (2.28), then the following result is an easy consequence of the Theorem 2.3.

COROLLARY 2.4. *The sufficient conditions for the schemes* (2.25)-(2.28) *to be extremum traceable are the following inequalities:*

$$\nu^+_{k+\frac{1}{2}} - \nu^-_{k+\frac{1}{2}} \le \frac{2}{2+\Phi}, \qquad \text{for all } k, \tag{2.31}$$

$$\nu^{+(1)}_{k+\frac{1}{2}} - \nu^{-(1)}_{k+\frac{1}{2}} \le \frac{1}{2+\Phi}, \qquad \text{for all } k, \tag{2.32}$$

where Φ is given by (2.18)*; when u_k is an extremum and $u^{(1)}_k$ is an extremum, there is a constant μ, $0 \le \mu < 1$, such that*

$$\max\{\nu^+_{k\pm\frac{1}{2}}, \nu^+_{k\pm\frac{3}{2}}, -\nu^-_{k+\frac{3}{2}}, -\nu^-_{k\pm\frac{1}{2}}, \nu^{+(1)}_{k\pm\frac{1}{2}}, \nu^{+(1)}_{k\pm\frac{3}{2}}, -\nu^{-(1)}_{k+\frac{3}{2}}, -\nu^{-(1)}_{k\pm\frac{1}{2}}\} \le \frac{\mu}{12}. \tag{2.33}$$

Proof. Indeed, for all k, by Sweby [21], we have $C_{k+\frac{1}{2}} \ge 0$, $D_{k+\frac{1}{2}} \ge 0$, $C^{(1)}_{k+\frac{1}{2}} \ge 0$, $D^{(1)}_{k+\frac{1}{2}} \ge 0$,

$$C_{k+\frac{1}{2}} + D_{k+\frac{1}{2}} \le (\nu^+_{k+\frac{1}{2}} - \nu^-_{k+\frac{1}{2}})(\frac{2+\Phi}{2}),$$

and

$$C^{(1)}_{k+\frac{1}{2}} + D^{(1)}_{k+\frac{1}{2}} \leq (\nu^{+(1)}_{k+\frac{1}{2}} - \nu^{-(1)}_{k+\frac{1}{2}})(\frac{2+\Phi}{2}).$$

These lead to

$$\begin{aligned}\overline{C}_{k+\frac{1}{2}} + \overline{D}_{k+\frac{1}{2}} &\leq \frac{1}{2}[C_{k+\frac{1}{2}} + 2C^{(1)}_{k+\frac{1}{2}}] + \frac{1}{2}[D_{k+\frac{1}{2}} + 2D^{(1)}_{k+\frac{1}{2}}]\\ &\leq \frac{1}{2}[C_{k+\frac{1}{2}} + D_{k+\frac{1}{2}}] + C^{(1)}_{k+\frac{1}{2}} + D^{(1)}_{k+\frac{1}{2}}\\ &\leq \frac{1}{2} + \frac{1}{2} = 1.\end{aligned}$$

When u_k is an extremum and $u^{(1)}_k$ is an extremum, we have the following equalities:

$$C_{k-\frac{1}{2}} = \nu^{+}_{k-\frac{1}{2}}[\,1 - \alpha^{+}_{k-\frac{1}{2}}\varphi(r^{+}_{k-1})\,], \quad D_{k+\frac{1}{2}} = -\nu^{-}_{k+\frac{1}{2}}[\,1 - \alpha^{-}_{k+\frac{1}{2}}\varphi(r^{-}_{k+1})\,],$$

$$C_{k+\frac{1}{2}} = \nu^{+}_{k+\frac{1}{2}}[\,1 + \alpha^{+}_{k+\frac{1}{2}}\frac{\varphi(r^{+}_{k+1})}{r^{+}_{k+1}}\,], \quad D_{k-\frac{1}{2}} = -\nu^{-}_{k-\frac{1}{2}}[\,1 + \alpha^{-}_{k-\frac{1}{2}}\frac{\varphi(r^{-}_{k-1})}{r^{-}_{k-1}}\,],$$

$$C^{(1)}_{k-\frac{1}{2}} = \nu^{+(1)}_{k-\frac{1}{2}}[\,1 - \alpha^{+(1)}_{k-\frac{1}{2}}\varphi(r^{+(1)}_{k-1})\,], \quad D^{(1)}_{k+\frac{1}{2}} = -\nu^{-(1)}_{k+\frac{1}{2}}[\,1 - \alpha^{-(1)}_{k+\frac{1}{2}}\varphi(r^{-(1)}_{k+1})\,],$$

$$C^{(1)}_{k+\frac{1}{2}} = \nu^{+(1)}_{k+\frac{1}{2}}[\,1 + \alpha^{+(1)}_{k+\frac{1}{2}}\frac{\varphi(r^{+(1)}_{k+1})}{r^{+(1)}_{k+1}}\,], \quad D^{(1)}_{k-\frac{1}{2}} = -\nu^{-(1)}_{k-\frac{1}{2}}[\,1 + \alpha^{-(1)}_{k-\frac{1}{2}}\frac{\varphi(r^{-(1)}_{k-1})}{r^{-(1)}_{k-1}}\,],$$

and then one can easily verify that indeed we have the desired estimate

$$\max\{\overline{C}_{k\pm\frac{1}{2}}, \overline{C}_{k\pm\frac{3}{2}}, \overline{D}_{k\pm\frac{1}{2}}, \overline{D}_{k+\frac{3}{2}}\} \leq \frac{\mu}{4} < \frac{1}{4}.$$

□

Notice that the inequalities of (2.29) are the Harten's sufficient TVD conditions [9] for the scheme (2.15)-(2.17). Also by Lax-Wendroff theorem [13] and Helly's theorem, the numerical solutions of the scheme (2.15)-(2.17) converge to a weak solution of (1.1). We will show the entropy convergence of the scheme (2.15)-(2.17) in the next section, where we will need the following lemmas.

LEMMA 2.5 (see Lemma 2.15 [24]). *An extremum traceable scheme is TVD.*

LEMMA 2.6 (see Theorem 15.3 [14]). *Any TVD method is monotonicity preserving.*

3. the convergence with minmod flux limiter. In view of the consistency relation (1.9), The following separation property at the spatial extrema is a generic property of the TVD numerical fluxes. Similar conditions can been found in [22], where these types of properties have been used to characterize the convenient TVD conditions by E. Tadmor.

ASSUMPTION 3.1. *The numerical fluxes* $g^{n(1)}_{k+\frac{1}{2}}$, $k \in \mathbb{Z}$, *satisfy*

$$g^{n(1)}_{k+\frac{1}{2}} \geq f(u^n_k) \geq g^{n(1)}_{k-\frac{1}{2}}, \quad if\ u^n_k \geq u^n_{k\pm1},$$

and

$$g^{n(1)}_{k+\frac{1}{2}} \leq f(u^n_k) \leq g^{n(1)}_{k-\frac{1}{2}}, \quad if\ u^n_k \leq u^n_{k\pm1}.$$

LEMMA 3.2. *The schemes of the form* (2.15)-(2.17) *satisfy the Assumption* 3.1.

Proof. If $u^n_k \geq u^n_{k\pm1}$, then

$$\begin{aligned} g^n_{k+\frac{1}{2}} &= g^E_{k+\frac{1}{2}} - \varphi(r^-_{k+1})\alpha^-_{k+\frac{1}{2}}(\Delta f_{k+\frac{1}{2}})^- \\ &> g^E_{k+\frac{1}{2}} - (g^E_{k+\frac{1}{2}} - f(u^n_k)) = f(u^n_k), \end{aligned}$$

and

$$\begin{aligned} g^n_{k-\frac{1}{2}} &= g^E_{k-\frac{1}{2}} + \varphi(r^+_{k-1})\alpha^+_{k-\frac{1}{2}}(\Delta f_{k-\frac{1}{2}})^+ \\ &< g^E_{k-\frac{1}{2}} + (\Delta f_{k-\frac{1}{2}})^+ = f(u^n_k). \end{aligned}$$

Similarly, we can show that if $u^n_k \leq u^n_{k\pm1}$, then $g^n_{k+\frac{1}{2}} < f(u^n_k) < g^n_{k-\frac{1}{2}}$. Likewise, following the same arguments, we have that if $u^{(1)}_k \geq u^{(1)}_{k\pm1}$, then $g^{(1)}_{k+\frac{1}{2}} > f(u^{(1)}_k) > g^{(1)}_{k-\frac{1}{2}}$; and if $u^{(1)}_k \leq u^{(1)}_{k\pm1}$, then $g^{(1)}_{k+\frac{1}{2}} < f(u^{(1)}_k) < g^{(1)}_{k-\frac{1}{2}}$. Also, for sufficiently small $\mu = |\sup_\xi \lambda f'(\xi)|$, we can make $|f(u^{(1)}_k) - f(u^n_k)| = |f'(\xi)(u^{(1)}_k - u^n_k)| = |\lambda f'(\xi)(g^n_{k+\frac{1}{2}} - g^n_{k-\frac{1}{2}})|$ small enough such that the desired inequalities of Assumption 3.1 are hold for scheme (2.15)-(2.17). □

Denote $\tilde{v}_j = H(v_{j-2p}, \cdots, v_{j+2p})$ (see (1.8)), $\bar{v}_j = \frac{v_j + \tilde{v}_j}{2}$ for any collection of data $\{v_j\}$, and $f[w; L, R]$ be the linear function interpolating $f(w)$ at $w = L$ and $w = R$. Recall that we have assumed $f''(w) \geq 0$.

DEFINITION 3.3 (see Definition 2.20 [24]). We call an ordered pair of numbers $\{L, R\}$ a rarefying pair if $L < R$ and $f[w; L, R] > f(w)$ when $L < w < R$. We call a collection of data $\Gamma = \{v_j\}^{J+p}_{j=I-p}$ an ε-rarefying collection of the scheme to the rarefying pair $\{L, R\}$ if, for $\varepsilon > 0$,

(i) $L = v_I \leq v_{I+1} \leq \cdots \leq v_J = R$;
(ii) $\tilde{v}_I \leq \tilde{v}_{I+1} \leq \cdots \leq \tilde{v}_J$, $|L - \tilde{v}_I| < \varepsilon$, $|R - \tilde{v}_J| < \varepsilon$;
(iii) either $v_{I-1} \geq v_I$ or $v_I = v_{I+1}$; and either $v_{J+1} \leq v_J$ or $v_{J-1} = v_J$.

The conditions (i) and (ii) imply that $\bar{v}_I \leq \bar{v}_{I+1} \leq \cdots \leq \bar{v}_J$, $|L - \bar{v}_I| < \frac{\varepsilon}{2}$, and $|R - \bar{v}_J| < \frac{\varepsilon}{2}$.

A 0-rarefying collection $\Gamma = \{v_j\}_{j=I-2}^{J+2}$ of the scheme to the pair $\{L, R\}$ that satisfies

$$L = v_{I-2} = v_{I-1} = v_I = v_{I+1} \leq \cdots \leq v_{J-1} = v_J = v_{J+1} = v_{J+2} = R \tag{3.1}$$

is called *a normal collection*.

Definition 3.4 (modified Definition 2.20 [24]). We call an ordered pair of numbers $\{L, R\}$ a rarefying pair if $L < R$ and $f[w; L, R] > f(w)$ when $L < w < R$. We call a collection of data $\Gamma = \{v_j\}_{j=I-2p}^{J+2p}$ an m-ε-rarefying collection of the scheme to the rarefying pair $\{L, R\}$ if, for $\varepsilon > 0$,

(i) $L = v_I \leq v_{I+1} \leq \cdots \leq v_J = R$;

(ii) $\tilde{v}_I \leq \tilde{v}_{I+1} \leq \cdots \leq \tilde{v}_J$, $|L - \tilde{v}_I| < \varepsilon$, $|R - \tilde{v}_J| < \varepsilon$;

(iii) either $v_{I-1} \geq v_I$ and $|v_{I\pm 1} - L| < \varepsilon$ or $v_I = v_{I+1}$ and $|v_{I-1} - L| < \varepsilon$, also in both cases, $|v_{I-4} - v_{I-3}| < \varepsilon$, $|v_{I-3} - v_{I-2}| < \varepsilon$, $|v_{I-2} - v_{I-1}| < \varepsilon$; and either $v_{J+1} \leq v_J$ and $|v_{J\pm 1} - R| < \varepsilon$ or $v_{J-1} = v_J$ and $|v_{J+1} - R| < \varepsilon$, also in both cases, $|v_{J+4} - v_{J+3}| < \varepsilon$, $|v_{J+3} - v_{J+2}| < \varepsilon$, $|v_{J+2} - v_{J+1}| < \varepsilon$.

The conditions (i) and (ii) imply that $\bar{v}_I \leq \bar{v}_{I+1} \leq \cdots \leq \bar{v}_J$, $|L - \bar{v}_I| < \frac{\varepsilon}{2}$, and $|R - \bar{v}_J| < \frac{\varepsilon}{2}$. On the interval $(\bar{v}_I, \bar{v}_J)$, we define the piecewise constant function g_Γ associated with the m-ε-rarefying collection Γ as follows:

$$g_\Gamma(w) = g_{j+\frac{1}{2}}^{n(1)}[v] \quad \text{for } w \in (\bar{v}_j, \bar{v}_{j+1}), \quad I \leq j \leq J-1.$$

For the extremum traceable schemes of the form (2.15)-(2.17), the condition (i) of the Definition 3.4, Lemma 2.5 and Lemma 2.6 also ensure that we have $v_I^{(1)} \leq v_{I+1}^{(1)} \leq \cdots \leq v_J^{(1)}$ as well.

An m-0-rarefying collection $\Gamma = \{v_j\}_{j=I-4}^{J+4}$ of the scheme to the pair $\{L, R\}$ (hence $\bar{v}_I = L$, and $\bar{v}_J = R$ for an m-0-rarefying collection) that satisfies

$$\begin{aligned} L &= v_{I-4} = v_{I-3} = v_{I-2} = v_{I-1} = v_I = v_{I+1} = v_{I+2} \\ &\leq \cdots \leq v_{J-2} = v_{J-1} = v_J = v_{J+1} = v_{J+2} = v_{J+3} = v_{J+4} = R \end{aligned} \tag{3.2}$$

is called *an m-normal collection.*

Clearly, we have that $\{m$-normal collection$\}\subseteq\{$normal collection$\}$.

Theorem 3.5 (modified Theorem 2.21[24]). *An extremum traceable scheme that satisfies Assumption* 3.1 *converges for convex conservation laws if, for every rarefying pair* $\{L, R\}$ *and m-ε-rarefying collection to the pair,*

$$\int_L^R f[w; L, R]\, dw - \int_{\bar{v}_I}^{\bar{v}_J} g_\Gamma(w)\, dw > \delta \tag{3.3}$$

for some constant $\delta > 0$ *depending only on the exact flux* f*, the numerical flux function* g*, and the two numbers* L *and* R*, provided that* ε *is sufficiently small.*

Remark. Although $\{m$-ε-rarefying collection$\}$ is a subset of $\{\varepsilon$-rarefying collection$\}$, Yang's original proof for his Theorem 2.21[24] still works for Theorem 3.5 with minor modifications:

(1). Now the inequality (33) in [24] would refer to every m-ε-rarefying collection.

(2). To verify the estimation of (32) in [24], we now want (32) holds for some m-ε-rarefying collection $\Gamma = \{v_j\}_{j=I-2p}^{J+2p}$ of the scheme to the pair $\{L, R\}$. To achieve this, in the Step 1 of his construction of $\Gamma = \{v_j\}$ we modify (39) in [24] and set

$$w_j = \begin{cases} L \ \text{ if } \hat{I} \le j \le l, \\ R \ \text{ if } r \le j \le \hat{J}, \\ |w_j - R| < \varepsilon_k \ \text{ if } j = \hat{J} \pm 1, \\ |w_j - L| < \varepsilon_k \ \text{ if } j = \hat{I} \pm 1, \\ u_j \ \text{ otherwise}; \end{cases} \tag{3.4}$$

also for $j \in ([I-4, I-2] \cap \mathbb{Z}) \cup ([J+2, J+4] \cap \mathbb{Z})$, we set the values of w_j such that

$$|w_{I-i} - w_{I-(i-1)}| < \varepsilon_k, \ |w_{J+i} - w_{J+(i-1)}| < \varepsilon_k, \ \text{ for } i \in [2, 4] \cap \mathbb{Z}.$$

Then we follow his Step 2 to modify $\Omega = \{w_j\}$. Finally we set $v_j = w_j$ for $j \in [I-4, J+4] \cap \mathbb{Z}$, all of which do not affect the rest of his proof to establish the inequality (32).

For the class of flux limiter schemes concerned, Theorem 3.5 can be simplified by the following Lemma.

Lemma 3.6. *An extremum traceable scheme of the form* (2.15)-(2.17) *converges for convex conservation laws, provided that for each rarefying pair* $\{L, R\}$ *there is a constant* $\delta > 0$ *such that the inequality* (3.3) *holds for all* m*-normal collections of the scheme to the pair* $\{L, R\}$*.*

Proof. To simplify the notations, throughout the proof we denote $j \in [A, B] \cap \mathbb{Z}$ by $j \in [A, B]$ and c be a generic constant. Let $\Lambda = \{\kappa_{P-4}, \cdots, \kappa_{Q+4}\}$ be an arbitrary m-ε-rarefying collection of the scheme to the pair $\{L, R\}$. Let

$$S' = \int_{\bar{\kappa}_P}^{\bar{\kappa}_Q} g_\Lambda(w)\, dw = \sum_{j=P}^{Q-1} (\bar{\kappa}_{j+1} - \bar{\kappa}_j)\, g_{j+\frac{1}{2}}^{n(1)}[\kappa]. \tag{3.5}$$

By (i) and (iii) of Definition 3.4, κ_P is either a minimum or a maximum. In either case, Assumption 3.1 and the condition (ii) of Definition 3.4 imply that

$$\varepsilon > |L - \tilde{\kappa}_P| = |\tilde{\kappa}_P - \kappa_P| = \lambda |g_{P+\frac{1}{2}}^{n(1)}[\kappa] - g_{P-\frac{1}{2}}^{n(1)}[\kappa]| \ge \lambda |g_{P\pm\frac{1}{2}}^{n(1)}[\kappa] - f(L)|. \tag{3.6}$$

Similarly, we have

$$\varepsilon > |R - \tilde{\kappa}_Q| \ge \lambda |g_{Q\pm\frac{1}{2}}^{n(1)}[\kappa] - f(R)|. \tag{3.7}$$

Next, we construct an m-normal collection $\Gamma = \{v_j\}_{j=I-4}^{J+4}$ as follows. First, let $I = P-1\,(P = I+1)$ and $J = Q+1\,(Q = J-1)$. We also set $v_{I-4} = v_{I-3} = v_{I-2} = v_{I-1} = v_I = v_{I+1} = v_{I+2} = L$, $v_{J-2} = v_{J-1} = v_J = v_{J+1} = v_{J+2} = v_{J+3} = v_{J+4} = R$, and $v_j = \kappa_j$ for $j \in [I+3, J-3]$. Then, we have

$$\begin{aligned} & f(L) = g_{I-\frac{5}{2}}^n[v] = g_{I-\frac{3}{2}}^n[v] = g_{I\pm\frac{1}{2}}^n[v] = g_{I+\frac{3}{2}}^n[v], \\ & f(R) = g_{J-\frac{3}{2}}^n[v] = g_{J\pm\frac{1}{2}}^n[v] = g_{J+\frac{3}{2}}^n[v] = g_{J+\frac{5}{2}}^n[v], \\ & g_{j+\frac{1}{2}}^n[v] = g_{j+\frac{1}{2}}^n[\kappa], \qquad \text{for } j \in [I+4, J-5]; \end{aligned} \tag{3.8}$$

and by (3.2) and (iii) of Definition 3.4, we have $|\,g^n_{j+\frac12}[v]-g^n_{j+\frac12}[\kappa]\,|<c\varepsilon$, for $j\in[I+1,I+3]\cup[J-4,J-2]$, which imply that,

$$\begin{aligned}
&L=v^{(1)}_{I-2}=v^{(1)}_{I-1}=v^{(1)}_{I}=v^{(1)}_{I+1}\le v^{(1)}_{I+2},\quad |\,\kappa^{(1)}_{P-1}-L\,|<c\varepsilon,\\
&v^{(1)}_{J-2}\le R=v^{(1)}_{J-1}=v^{(1)}_{J}=v^{(1)}_{J+1}=v^{(1)}_{J+2},\quad |\,\kappa^{(1)}_{Q+1}-R\,|<c\varepsilon,\\
&v^{(1)}_{j}=\kappa^{(1)}_{j}\quad \text{for } j\in[I+5,J-5],\\
&|\,v^{(1)}_{I+j}-\kappa^{(1)}_{(P-1)+j}\,|<c\varepsilon\quad \text{for } j\in[1,4],\\
&|\,v^{(1)}_{J-j}-\kappa^{(1)}_{(Q+1)-j}\,|<c\varepsilon\quad \text{for } j\in[1,4].
\end{aligned}\tag{3.9}$$

Then we have

$$\begin{aligned}
&g^{(1)}_{I\pm\frac12}[v]=f(L),\quad g^{(1)}_{J\pm\frac12}[v]=f(R),\\
&g^{(1)}_{j+\frac12}[v]=g^{(1)}_{j+\frac12}[\kappa]\quad \text{for } j\in[I+6,J-7]=[P+5,Q-6],\\
&|\,g^{(1)}_{j+\frac12}[v]-g^{(1)}_{j+\frac12}[\kappa]\,|<c\varepsilon\quad \text{for } j\in[I+1,I+5]\cup[J-6,J-1].
\end{aligned}\tag{3.10}$$

Using the relationships of Λ and Γ; $\{v^{(1)}_j\}$ and $\{\kappa^{(1)}_j\}$; $\{g^n_{j+\frac12}[v]\}$ and $\{g^n_{j+\frac12}[\kappa]\}$; $\{g^{(1)}_{j+\frac12}[v]\}$ and $\{g^{(1)}_{j+\frac12}[\kappa]\}$, we obtain the following

$$\begin{aligned}
&g^{n(1)}_{I\pm\frac12}[v]=f(L),\quad g^{n(1)}_{J\pm\frac12}[v]=f(R),\\
&g^{n(1)}_{j+\frac12}[v]=g^{n(1)}_{j+\frac12}[\kappa]\quad \text{for } j\in[I+6,J-7]=[P+5,Q-6],\\
&|\,g^{n(1)}_{j+\frac12}[v]-g^{n(1)}_{j+\frac12}[\kappa]\,|<c\varepsilon\quad \text{for } j\in[I+1,I+5]\cup[J-6,J-1].
\end{aligned}\tag{3.11}$$

Thus

$$\bar v_I=\tilde v_I=v_I=L,\qquad \bar v_J=\tilde v_J=v_J=R,\tag{3.12}$$

and $\Gamma=\{v_j\}_{j=I-4}^{J+4}$ indeed is an m-normal collection.

Secondly, let G be the Lipschitz constant of the numerical flux g, and $K=\max\{|f(L)|,|f(R)|\}+G(R-L)$. Denote

$$S=\int_L^R g_\Gamma(w)\,dw=\sum_{j=I}^{J-1}(\bar v_{j+1}-\bar v_j)g^{n(1)}_{j+\frac12}[v],\tag{3.13}$$

then a-priori estimate $|S-S'|\le cK\varepsilon$ holds. Here, again c is a generic constant. Let δ' be a constant such that for all m-normal collections of the scheme to the pair $\{L,R\}$ the inequality (3.3) holds for $\delta=\delta'$. Thus, for $\delta=\delta'$, the inequality (3.3) also holds for the m-normal collection $\Gamma=\{v_j\}_{j=I-4}^{J+4}$. Therefore, for $\delta=\frac{\delta'}{2}$, the inequality (3.3) holds for all m-ε-collection of the scheme to the pair $\{L,R\}$ provided that $\varepsilon\le\frac{\delta}{cK}$. It remains to show the a-priori estimate. Notice that $g^{n(1)}_{j+\frac12}[\kappa]=g^{n(1)}_{j+\frac12}[v]$, for $j\in[P+5,Q-6]$, thus, $\bar\kappa_j=\bar v_j$ for $j\in[P+6,Q-6]$. For some terms on the right hand side of (3.13), we introduce the following shorthand notations:

$$
\begin{aligned}
& I - term[v] = (\bar{v}_{I+1} - \bar{v}_I)\, g^{n(1)}_{I+\frac{1}{2}}[v], \\
& I + j - term[v] = (\bar{v}_{(I+1)+j} - \bar{v}_{I+j})\, g^{n(1)}_{(I+j)+\frac{1}{2}}[v], \quad \text{for } j \in [1,5], \\
& I + 6 - term[v] = (\bar{v}_{I+7} - \bar{v}_{I+6})\, g^{n(1)}_{I+\frac{13}{2}}[v], \\
& J - 1 - term[v] = (\bar{v}_J - \bar{v}_{J-1})\, g^{n(1)}_{J-\frac{1}{2}}[v], \\
& (J-1) - j - term[v] = (\bar{v}_{J-j} - \bar{v}_{(J-1)-j})\, g^{n(1)}_{(J-1)-j+\frac{1}{2}}[v], \quad \text{for } j \in [1,5] \\
& J - 7 - term[v] = (\bar{v}_{J-6} - \bar{v}_{J-7})\, g^{n(1)}_{J-\frac{13}{2}}[v],
\end{aligned} \tag{3.14}
$$

and use the same notations for some terms in the sum of (3.5):

$$
\begin{aligned}
& (P-1) + j - term[\kappa] = (\bar{\kappa}_{P+j} - \bar{\kappa}_{(P-1)+j})\, g^{n(1)}_{(P-1)+j+\frac{1}{2}}[\kappa], \ \text{for } j \in [1,5], \\
& P + 5 - term[\kappa] = (\bar{\kappa}_{P+6} - \bar{\kappa}_{P+5})\, g^{n(1)}_{P+\frac{11}{2}}[\kappa], \\
& Q - j - term[\kappa] = (\bar{\kappa}_{(Q+1)-j} - \bar{\kappa}_{Q-j})\, g^{n(1)}_{(Q-j)+\frac{1}{2}}[\kappa], \ \text{for } j \in [1,5], \\
& Q - 6 - term[\kappa] = (\bar{\kappa}_{Q-5} - \bar{\kappa}_{Q-6})\, g^{n(1)}_{Q-\frac{11}{2}}[\kappa].
\end{aligned} \tag{3.15}
$$

Then, it is easy to verify that

$$
(|I + j - term[v]) - ((P-1) + j - term[\kappa])| < c\varepsilon, \quad \text{for } j \in [1,5],
$$

$$
(|(J-1) - j - term[v]) - (Q - j - term[\kappa])| < c\varepsilon, \quad \text{for } j \in [1,5],
$$

which help us to obtain

$$
\begin{aligned}
|S - S'| \le\ & |\bar{v}_{I+1} - \bar{v}_I||g^{n(1)}_{I+\frac{1}{2}}[v]| + |\bar{v}_{I+6} - \bar{\kappa}_{P+5}||g^{n(1)}_{I+\frac{13}{2}}[v]| + \\
& |\bar{v}_J - \bar{v}_{J-1}||g^{n(1)}_{J-\frac{1}{2}}[v]| + |\bar{v}_{J-6} - \bar{\kappa}_{Q-5}||g^{n(1)}_{J-\frac{13}{2}}[v]| + c\varepsilon.
\end{aligned} \tag{3.16}
$$

The relationship of Λ and Γ and the inequalities (3.6)-(3.12) yield:

$$
|\bar{v}_{I+1} - \bar{\kappa}_{I+1}| < c\varepsilon, \quad |\bar{v}_{J-1} - \bar{\kappa}_{J-1}| < c\varepsilon, \tag{3.17}
$$

$$
|\bar{v}_{I+1} - \bar{v}_I| = |\bar{v}_{I+1} - L| \le |\bar{v}_{I+1} - \bar{\kappa}_{I+1}| + |\bar{\kappa}_P - L| < c\varepsilon, \tag{3.18}
$$

and

$$
|\bar{v}_J - \bar{v}_{J-1}| = |\bar{v}_{J-1} - R| \le |\bar{v}_{J-1} - \bar{\kappa}_{J-1}| + |\bar{\kappa}_Q - R| < c\varepsilon. \tag{3.19}
$$

Also, one can verify that

$$
|\bar{v}_{I+6} - \bar{\kappa}_{P+5}| < c\varepsilon, \ \text{and } |\bar{v}_{J-6} - \bar{\kappa}_{Q-5}| < c\varepsilon. \tag{3.20}
$$

Finally, $|S - S'| < cK\varepsilon$ follows from the inequalities (3.16)-(3.20). □

For a normal collection $\Gamma = \{v_j\}_{j=I-4}^{J+4}$, we denote the vertex $(v_j, f(v_j))$ by V_j and the area of convex polygon $V_{j_1} V_{j_2} \cdots V_{j_r}$ by $S_{j_1,\ldots,j_r}$.

If $\{v_j^{(1)}\}_{j=I-4}^{J+4}$ happens to satisfy (3.1), we use $V_j^{(1)}$ to denote the vertex $(v_j^{(1)}, f(v_j^{(1)}))$, and use $S^{(1)}_{j_1,\dots,j_r}$ to denote the area of convex polygon $V^{(1)}_{j_1}V^{(1)}_{j_2}\cdots V^{(1)}_{j_r}$.

Let

$$\sigma_\Gamma = \max_{I-4\le j\le J+4}\{\,|\nu^{\pm}_{j\pm\frac12}|,\ |\nu^{\pm(1)}_{j\pm\frac12}|,\ \lambda|f'(\xi_j)|\},$$

for ξ_j in between v_j and $v_j^{(1)}$, or v_j and $\bar v_j$, or $\bar v_j$ and $v_j^{(1)}$, and let

$$\alpha_j = \begin{cases} 0.5 & \text{if } \Delta v_{j-2} = \Delta v_{j+1} = 0, \text{ or if } \Delta v^{(1)}_{j-2} = \Delta v^{(1)}_{j+1} = 0,\\ 1 & \text{otherwise.}\end{cases}$$

The proof of Theorem 3.9 depends on Lemma 3.6, as well as the following two Lemmas. The proof of the first one, a very important estimate, will be given at the end of this section. The second one is Yang's original result.

LEMMA 3.7. *Let* $\Gamma = \{v_j\}_{j=I-4}^{J+4}$ *be a normal collection to a rarefying pair* $\{L, R\}$. *Then the numerical solutions of the scheme of the form* (2.15)-(2.17) *for convex conservation laws satisfy, for a sufficiently small* σ_Γ, *the following inequality*

$$\int_L^R (f[w;L,R]-g_\Gamma)dw \ge S_{I,I+1,\dots,J} - \sum_{j=I+1}^{J-1}\alpha_j S_{j-1,j,j+1}. \tag{3.21}$$

If $\{v_j^{(1)}\}_{j=I-4}^{J+4}$ *satisfies (3.1) to the rarefying pair* $\{L, R\}$, *then we also have the inequality:*

$$\int_L^R (f[w;L,R]-g_\Gamma)dw \ge S^{(1)}_{I,I+1,\dots,J} - \sum_{j=I+1}^{J-1}\alpha_j S^{(1)}_{j-1,j,j+1}. \tag{3.22}$$

LEMMA 3.8 (see Lemma 3.7 [24]). *We have*

$$S_{I,I+1,\dots,J} - \sum_{j=I+1}^{J-1} S_{j-1,j,j+1} \ge S_{I,i,i+1,J} - (S_{I,i,i+1} + S_{i,i+1,J}),$$

and

$$S^{(1)}_{I,I+1,\dots,J} - \sum_{j=I+1}^{J-1} S^{(1)}_{j-1,j,j+1} \ge S^{(1)}_{I,i,i+1,J} - (S^{(1)}_{I,i,i+1} + S^{(1)}_{i,i+1,J}),$$

for $I < i < J-1$.

Let $\sigma = \lambda\max_{w,w^{(1)},\xi}\{|f'(w)|, |f'(w^{(1)})|, |f'(\xi)|\}$, for ξ in between w and $w^{(1)}$, or w and $\bar w$, or $\bar w$ and $w^{(1)}$. We can show the following desired entropy convergence result. The proof is similar to the one given by Yang [24] for the MUSCL schemes.

THEOREM 3.9. *The numerical solutions of the scheme* (2.15)-(2.17), *for strictly convex problem* (1.1), *converge provided that* φ *is minmod flux limiter,* $g^E(\cdot,\cdot)$ *is an E-flux of Godunov or Engquist-Osher scheme, and* σ *is sufficiently small.*

Proof. For each m-normal collection $\Gamma = \{v_i\}_{i=I-4}^{J+4}$ to a rarefying pair $\{L, R\}$, we set

$$d_1(\Gamma) = \max_{I \le i \le J} \min(v_i - L, R - v_i).$$

Since $J - I$ is finite, $d_1(\Gamma) = \min(v_j - L, R - v_j)$ for some j between I and J. We then let

$$d_2(\Gamma) = \max_{I \le i \le J, i \ne j} \min(v_i - L, R - v_i).$$

We also have $d_2(\Gamma) = \min(v_k - L, R - v_k)$ for some $k \ne j$ between I and J. Clearly, we can choose j and k so that $|j - k| = 1$.

To complete the proof, we argue by contradiction. Hence, we assume that for certain convex f, the scheme of the form (2.15)-(2.17) does not converge. By Lemma 3.6, there is a rarefying pair $\{L, R\}$ such that for each $\delta > 0$, there is a m-normal collection $\Gamma = \{v_j\}_{j=I-4}^{J+4}$ of the scheme to the pair that satisfies

$$\int_L^R \{f[w; L, R] - g_\Gamma(w)\} dw \le \delta.$$

It follows that there is a sequence of m-normal collections $\{\Gamma_\nu\}_{\nu=1}^{\infty}$, where $\Gamma_\nu = \{v_j^\nu\}_{j=I^\nu-4}^{J^\nu+4}$ such that

$$\lim_{\nu \to \infty} \int_L^R \{f[w; L, R] - g_{\Gamma_\nu}(w)\} \le 0. \tag{3.23}$$

The following three cases exhaust all possibilities.

Case 1. $\limsup_{\nu\to\infty} d_2(\Gamma_\nu) > 0$. Set $\rho = \frac{1}{2} \limsup_{\nu\to\infty} d_2(\Gamma_\nu)$. Then, there is a subsequence of the m-normal collections, still denoted by $\{\Gamma_\nu\}_{\nu=1}^{\infty}$, and a corresponding sequence of integers $\{i^\nu)\}_{\nu=1}^{\infty}$ such that

$$L + \rho \le v_{i^\nu}^\nu \le v_{i^\nu+1}^\nu \le R - \rho,$$

and $\sup_\nu \sigma_{\Gamma_\nu} \le \sigma$. For simplicity, we fix a ν and drop it from the notation. Set $\gamma = f[\frac{L+R}{2}; L, R] - f(\frac{L+R}{2})$. It is a positive constant since $\{L, R\}$ is a rarefying pair. Applying the first inequalities of Lemmas 3.7 and 3.8, we have

$$\begin{aligned} &\int_L^R \{f[w; L, R] - g_{\Gamma_\nu}(w)\} dw \ge S_{I,i,i+1,J} - (S_{I,i,i+1} + S_{i,i+1,J}) \\ &= \frac{1}{2}\{(v_i - v_I)(f[v_{i+1}; L, R] - f(v_{i+1})) + (v_J - v_{i+1})(f[v_i; L, R] - f(v_i))\} \\ &> \eta, \end{aligned} \tag{3.24}$$

if $\eta = 2\rho^2\gamma/(R - L)$. This contradicts (3.23).

Case 2. $\limsup_{\nu\to\infty} d_1(\Gamma_\nu) > \limsup_{\nu\to\infty} d_2(\Gamma_\nu) = 0$. Set $\rho = \frac{1}{2} \limsup_{\nu\to\infty} d_1(\Gamma_\nu)$. Then, there is a subsequence of the m-normal collections, still denoted by $\{\Gamma_\nu\}_{\nu=1}^{\infty}$, and a corresponding sequence of integers $\{i^\nu\}_{\nu=1}^{\infty}$ such that $\lim_{\nu\to\infty} v_{i^\nu-1}^\nu = L, \lim_{\nu\to\infty} v_{i^\nu+1}^\nu = R$, and $\lim_{\nu\to\infty} v_{i^\nu}^\nu = v \in [L + \rho, R - \rho]$. We then have

$$\int_L^R (f[w; L, R] - g_{\Gamma_\nu}(w)) dw \to \int_L^R (f[w; L, R] - g_\Gamma(w)) dw,$$

where Γ is the following m-normal collection: $I=-1, J=5, v_{-5}=v_{-4}=v_{-3}=v_{-2}=v_{-1}=v_0=v_1=L, v_2=v$, and $v_3=v_4=v_5=v_6=v_7=v_8=v_9=R$. By Lemma 3.7, we have

$$\int_L^R (f[w;L,R]-g_\Gamma(w))dw \geq S_{1,2,3}-\alpha_2 S_{1,2,3}=\frac{1}{2}S_{1,2,3}>0$$

for $\alpha_2=\frac{1}{2}$ since $\Delta v_0=\Delta v_3=0$. This contradicts (3.23).

Case 3. $\limsup_{\nu\to\infty} d_1(\Gamma_\nu)=0$. Then, there exists a sequence of integers $\{i^\nu\}$ with $I^\nu+1\leq i^\nu<J^\nu-1$ such that $\lim_{\nu\to\infty} v^\nu_{i^\nu}=L, \lim_{\nu\to\infty} v^\nu_{i^\nu+1}=R$. We then have

$$\int_L^R (f[w;L,R]-g_{\Gamma_\nu}(w))dw \to \int_L^R (f[w;L,R]-g_\Gamma(w))dw,$$

where Γ is the following m-normal collection: $I=0, J=5, v_{-4}=v_{-3}=v_{-2}=v_{-1}=v_0=v_1=v_2=L, v_3=v_4=v_5=v_6=v_7=v_8=v_9=R$. Then one can verify that

$$\begin{aligned} L&=v^{(1)}_{I-3}=v^{(1)}_{I-2}=v^{(1)}_{I-1}=v^{(1)}_{I}=v^{(1)}_{I+1}\\ &\leq v^{(1)}_{I+2}\leq v^{(1)}_{J-2}\leq v^{(1)}_{J-1}=v^{(1)}_{J}=v^{(1)}_{J+1}=v^{(1)}_{J+2}=v^{(1)}_{J+3}=R. \end{aligned} \tag{3.25}$$

Notice that

$$0<v^{(1)}_{I+2}-v_{I+2}=\lambda[g^n_{I+\frac{3}{2}}-g^n_{I+\frac{5}{2}}]=\lambda[g^E_{0+\frac{3}{2}}(L,L)-g^E_{0+\frac{5}{2}}(L,R)] \overset{denote}{=} A,$$

and

$$0>v^{(1)}_{J-2}-v_{J-2}=-\lambda[g^n_{J-\frac{3}{2}}-g^n_{J-\frac{5}{2}}]=-\lambda[g^E_{5-\frac{3}{2}}(R,R)-g^E_{5-\frac{5}{2}}(L,R)] \overset{denote}{=} -B.$$

Let $\rho=\min\frac{1}{2}\{A,B\}$. Then we have

$$L+\rho\leq v^{(1)}_{I+2}\leq v^{(1)}_{J-2}\leq R-\rho.$$

Using the second inequalities of Lemmas 3.7 and 3.8, similar to (3.24), we have

$$\begin{aligned} &\int_L^R (f[w;L,R]-g_\Gamma(w))dw \geq S^{(1)}_{I+1,i,i+1,J-1}-(S^{(1)}_{I+1,i,i+1}+S^{(1)}_{i,i+1,J-1})\\ &=\frac{1}{2}\{(v^{(1)}_i-v^{(1)}_{I+1})(f[v^{(1)}_{i+1};L,R]-f(v^{(1)}_{i+1}))+(v^{(1)}_{J-1}-v_{i+1})(f[v^{(1)}_i;L,R]-f(v^{(1)}_i))\}\\ &>\eta, \end{aligned} \tag{3.26}$$

if $\eta=2\rho^2\gamma/(R-L)$. This contradicts (3.23). We have thus completed the proof of Theorem 3.9. □

Finally, we finish this section by presenting the proof of Lemma 3.7.

Proof of Lemma 3.7. In the proof, we keep the same notations $(\Delta f_{j+\frac{1}{2}})^\pm$ and $r^\pm_j$ for $\{v_j\}$ instead of $\{u_j\}$. To justify the inequality (3.21) and (3.22), it suffices to show the following inequalities:

$$\int_L^R g_\Gamma(w)dw-\sum_{j=I}^{J-1}\int_{v_j}^{v_{j+1}} f[w;v_j,v_{j+1}]dw\leq \sum_{j=I+1}^{J-1}\alpha_j S_{j-1,j,j+1}, \tag{3.27}$$

and

$$\int_L^R g_\Gamma(w)dw - \sum_{j=I}^{J-1}\int_{v_j^{(1)}}^{v_{j+1}^{(1)}} f[w; v_j^{(1)}, v_{j+1}^{(1)}]dw \le \sum_{j=I+1}^{J-1} \alpha_j S_{j-1,j,j+1}^{(1)}, \tag{3.28}$$

respectively.

To estimate the LHS of (3.27), we have, from (2.13), that

$$\begin{aligned} g_{j+\frac{1}{2}}^n &= g_{j+\frac{1}{2}}^E + \alpha_{j+\frac{1}{2}}^+ \varphi(r_j^+)\Delta f_{j+\frac{1}{2}}^+ - \alpha_{j+\frac{1}{2}}^- \varphi(r_{j+1}^-)\Delta f_{j+\frac{1}{2}}^- \\ &\le g_{j+\frac{1}{2}}^E + \frac{1}{2}\Delta f_{j+\frac{1}{2}}^+ - \frac{1}{2}\Delta f_{j+\frac{1}{2}}^- \\ &= \frac{1}{2}(f(v_j) + f(v_{j+1})). \end{aligned}$$

Similarly, we obtain that $g_{j+\frac{1}{2}}^{(1)} \le \frac{1}{2}(f(v_j^{(1)}) + f(v_{j+1}^{(1)}))$. Therefore,

$$\begin{aligned} 2g_{j+\frac{1}{2}}^{n(1)} &= g_{j+\frac{1}{2}}^n + g_{j+\frac{1}{2}}^{(1)} \\ &\le \frac{1}{2}[(f(v_j) + f(v_{j+1})) + (f(v_j^{(1)}) + f(v_{j+1}^{(1)})]. \end{aligned} \tag{3.29}$$

Also from (2.25), we have

$$\Delta\tilde{v}_{j+\frac{1}{2}} = \Delta v_{j+\frac{1}{2}} - (\overline{C}_{j+\frac{1}{2}} + \overline{D}_{j+\frac{1}{2}})\Delta v_{j+\frac{1}{2}} + \overline{D}_{j+\frac{3}{2}}\Delta v_{j+\frac{3}{2}} + \overline{C}_{j-\frac{1}{2}}\Delta v_{j-\frac{1}{2}},$$

which gives that

$$\begin{aligned} &2\Delta\bar{v}_{j+\frac{1}{2}} = \Delta v_{j+\frac{1}{2}} + \Delta\tilde{v}_{j+\frac{1}{2}} \\ &= 2\Delta v_{j+\frac{1}{2}} + [-(\overline{C}_{j+\frac{1}{2}} + \overline{D}_{j+\frac{1}{2}})\Delta v_{j+\frac{1}{2}} + \overline{D}_{j+\frac{3}{2}}\Delta v_{j+\frac{3}{2}} + \overline{C}_{j-\frac{1}{2}}\Delta v_{j-\frac{1}{2}}]. \end{aligned} \tag{3.30}$$

Then,

$$\begin{aligned} \text{LHS of (3.27)} &= \sum_{j=I}^{J-1} g_{j+\frac{1}{2}}^{n(1)}\Delta\bar{v}_{j+\frac{1}{2}} - \sum_{j=I}^{J-1}\int_{v_j}^{v_{j+1}} f[w; v_j, v_{j+1}]dw \\ &= \sum_{j=I}^{J-1} g_{j+\frac{1}{2}}^{n(1)}\frac{2\Delta\bar{v}_{j+\frac{1}{2}}}{2} - \sum_{j=I}^{J-1}\frac{f(v_j) + f(v_{j+1})}{2}\Delta v_{j+\frac{1}{2}} \\ &= \frac{1}{2}\sum_{j=I}^{J-1}[g_{j+\frac{1}{2}}^{n(1)}2\Delta\bar{v}_{j+\frac{1}{2}} - (f(v_j) + f(v_{j+1}))\Delta v_{j+\frac{1}{2}}], \end{aligned} \tag{3.31}$$

and

$$\begin{aligned}
\text{LHS of (3.28)} &= \sum_{j=I}^{J-1} g^{n(1)}_{j+\frac{1}{2}} \Delta\bar{v}_{j+\frac{1}{2}} - \sum_{j=I}^{J-1} \int_{v_j^{(1)}}^{v_{j+1}^{(1)}} f[w; v_j^{(1)}, v_{j+1}^{(1)}] dw \\
&= \sum_{j=I}^{J-1} g^{n(1)}_{j+\frac{1}{2}} \frac{2\Delta\bar{v}_{j+\frac{1}{2}}}{2} - \sum_{j=I}^{J-1} \frac{f(v_j^{(1)}) + f(v_{j+1}^{(1)})}{2} \Delta v^{(1)}_{j+\frac{1}{2}} \\
&= \frac{1}{2} \sum_{j=I}^{J-1} [g^{n(1)}_{j+\frac{1}{2}} 2\Delta\bar{v}_{j+\frac{1}{2}} - (f(v_j^{(1)}) + f(v_{j+1}^{(1)}))\Delta v^{(1)}_{j+\frac{1}{2}}] \\
&= \frac{1}{2} \sum_{j=I}^{J-1} [g^{n(1)}_{j+\frac{1}{2}} 2\Delta\bar{v}_{j+\frac{1}{2}} - (f(v_j^{(1)}) + f(v_{j+1}^{(1)}))\Delta v_{j+\frac{1}{2}}] \\
&- \frac{1}{2} \sum_{j=I}^{J-1} [(f(v_j^{(1)}) + f(v_{j+1}^{(1)}))(\Delta v^{(1)}_{j+\frac{1}{2}} - \Delta v_{j+\frac{1}{2}})]. \qquad (3.32)
\end{aligned}$$

Now,

$$\begin{aligned}
&\text{the } j\text{th-term of the last summation of (3.31)} \\
&= 2g^{n(1)}_{j+\frac{1}{2}} \Delta\bar{v}_{j+\frac{1}{2}} - (f(v_j) + f(v_{j+1}))\Delta v_{j+\frac{1}{2}} \\
&= 2g^{n(1)}_{j+\frac{1}{2}} \Delta v_{j+\frac{1}{2}} - (f(v_j) + f(v_{j+1}))\Delta v_{j+\frac{1}{2}} \\
&+ g^{n(1)}_{j+\frac{1}{2}} [-(\overline{C}_{j+\frac{1}{2}} + \overline{D}_{j+\frac{1}{2}})\Delta v_{j+\frac{1}{2}} + \overline{D}_{j+\frac{3}{2}} \Delta v_{j+\frac{3}{2}} + \overline{C}_{j-\frac{1}{2}} \Delta v_{j-\frac{1}{2}}]. \qquad (3.33)
\end{aligned}$$

With the help of (3.29), we estimate the first term of the last sum of (3.33)

$$\begin{aligned}
&2g^{n(1)}_{j+\frac{1}{2}} \Delta v_{j+\frac{1}{2}} - (f(v_j) + f(v_{j+1}))\Delta v_{j+\frac{1}{2}} \\
&\le \frac{1}{2}[(f(v_j^{(1)}) + f(v_{j+1}^{(1)})) - (f(v_j) + f(v_{j+1}))]\Delta v_{j+\frac{1}{2}} \\
&= \frac{1}{2}[f'(\xi_j)(v_j^{(1)} - v_j) + f'(\xi_{j+1})(v_{j+1}^{(1)} - v_{j+1})]\Delta v_{j+\frac{1}{2}} \\
&= \frac{1}{2}[-\lambda f'(\xi_j)(g^n_{j+\frac{1}{2}} - g^n_{j-\frac{1}{2}}) - \lambda f'(\xi_{j+1})(g^n_{j+\frac{3}{2}} - g^n_{j+\frac{1}{2}})]\Delta v_{j+\frac{1}{2}}, \qquad (3.34)
\end{aligned}$$

where ξ_j is in between $v_j^{(1)}$ and v_j, for $j \in [I, J]$.

Clearly, it is feasible that

$$\text{LHS of (3.27)} \le \frac{1}{2} \sum_{j=I+1}^{J-1} S_{j-1,j,j+1} \le \sum_{j=I+1}^{J-1} \alpha_j S_{j-1,j,j+1},$$

provided that σ_Γ is sufficiently small.

Similarly,

$$\begin{aligned}
&\text{the } j\text{th-term of the first sum in the last summation of (3.32)} \\
&= 2g^{n(1)}_{j+\frac{1}{2}} \Delta\bar{v}_{j+\frac{1}{2}} - (f(v_j^{(1)}) + f(v_{j+1}^{(1)}))\Delta v_{j+\frac{1}{2}} \\
&= 2g^{n(1)}_{j+\frac{1}{2}} \Delta v_{j+\frac{1}{2}} - (f(v_j^{(1)}) + f(v_{j+1}^{(1)}))\Delta v_{j+\frac{1}{2}} \\
&+ g^{n(1)}_{j+\frac{1}{2}} [-(\overline{C}_{j+\frac{1}{2}} + \overline{D}_{j+\frac{1}{2}})\Delta v_{j+\frac{1}{2}} + \overline{D}_{j+\frac{3}{2}} \Delta v_{j+\frac{3}{2}} + \overline{C}_{j-\frac{1}{2}} \Delta v_{j-\frac{1}{2}}], \qquad (3.35)
\end{aligned}$$

and

$$
\begin{aligned}
&2g^{n(1)}_{j+\frac{1}{2}}\Delta v_{j+\frac{1}{2}} - (f(v^{(1)}_j) + f(v^{(1)}_{j+1}))\Delta v_{j+\frac{1}{2}} \\
\le& -\frac{1}{2}[(f(v^{(1)}_j) + f(v^{(1)}_{j+1})) - (f(v_j) + f(v_{j+1}))]\Delta v_{j+\frac{1}{2}} \\
=& -\frac{1}{2}[f'(\xi_j)(v^{(1)}_j - v_j) + f'(\xi_{j+1})(v^{(1)}_{j+1} - v_{j+1})]\Delta v_{j+\frac{1}{2}} \\
=& \frac{1}{2}[\lambda f'(\xi_j)(g^n_{j+\frac{1}{2}} - g^n_{j-\frac{1}{2}}) + \lambda f'(\xi_{j+1})(g^n_{j+\frac{3}{2}} - g^n_{j+\frac{1}{2}})]\Delta v_{j+\frac{1}{2}}, \qquad (3.36)
\end{aligned}
$$

where ξ_j is in between $v^{(1)}_j$ and v_j, for $j \in [I, J]$.

Also,

$$
\begin{aligned}
&\text{the } j\text{th-term of the second sum in the last summation of (3.32)} \\
=& (f(v^{(1)}_j) + f(v^{(1)}_{j+1}))(\Delta v^{(1)}_{j+\frac{1}{2}} - \Delta v_{j+\frac{1}{2}}) \\
=& (f(v^{(1)}_j) + f(v^{(1)}_{j+1}))[-(C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}})\Delta v_{j+\frac{1}{2}} + D_{j+\frac{3}{2}}\Delta v_{j+\frac{3}{2}} + C_{j-\frac{1}{2}}\Delta v_{j-\frac{1}{2}}]. \qquad (3.37)
\end{aligned}
$$

Therefore, it is feasible that

$$
\text{LHS of (3.28)} \le \frac{1}{2}\sum_{j=I+1}^{J-1} S^{(1)}_{j-1,j,j+1} \le \sum_{j=I+1}^{J-1} \alpha_j S^{(1)}_{j-1,j,j+1},
$$

provided that σ_Γ is sufficiently small. Thus, we have completed the proof of Lemma 3.7. □

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