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# A TRANSMISSION PROBLEM FOR WAVES UNDER TIME-VARYING DELAY AND TIME-VARYING WEIGHTS*

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**Abstract.** This manuscript focuses on in the transmission problem for one dimensional waves with time-varying weights on the frictional damping and time-varying delay. We prove global existence of solutions using Kato's variable norm technique and we show the exponential stability by the energy method with the construction of a suitable Lyapunov functional.

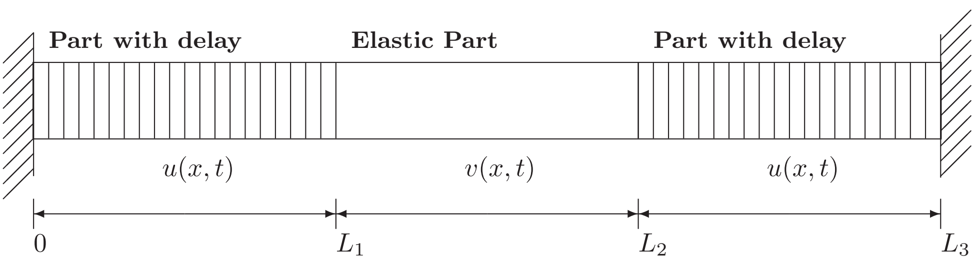
**Key words.** Transmission problem, Time-varying delay, Time-varying weights, Exponential stability.

**Mathematics Subject Classification.** Primary 35L20, 35B40; Secondary 93D15.

**1. Introduction.** In this paper we investigate global existence and decay properties of solutions for a transmission problem for waves with time-varying weights and time-varying delay. We consider the following system

$$\begin{gathered} u_{tt}(x,t) - au_{xx}(x,t) + \mu_1(t)u_t(x,t) + \mu_2(t)u_t(x,t-\tau(t)) = 0 \text{ in } \Omega\times]0,\infty[, \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0 \text{ in } ]L_1, L_2[\times]0,\infty[, \end{gathered} \tag{1.1}$$

where $0 < L_1 < L_2 < L_3$, $\Omega = ]0, L_1[\cup]L_2, L_3[$ and $a$, $b$ are positive constants.




The system (1.1) is subjected to the transmission conditions

$$\begin{aligned} u(L_i,t) &= v(L_i,t), \quad i = 1,2 \\ au_x(L_i,t) &= bv_x(L_i,t), \quad i = 1,2, \end{aligned} \tag{1.2}$$

the boundary conditions

$$u(0,t) = u(L_3,t) = 0 \tag{1.3}$$

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and initial conditions

$$\begin{aligned} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{on } \Omega, \\ u_t(x,t-\tau(0)) = f_0(x,t-\tau(0)) \quad \text{in } \Omega\times]0,\tau(0)[, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x) \quad \text{on } ]L_1,L_2[, \end{aligned} \tag{1.4}$$

where the initial datum $(u_0, u_1, v_0, v_1, f_0)$ belongs to a suitable Sobolev space.

In the first equation of (1.1), $0 < \tau(t)$ is the time-varying delay and $\mu_1(t)$ and $\mu_2(t)$ are time-varying weights acting on the frictional damping. As in [20], we assume that

$$\tau(t) \in W^{2,\infty}([0,T]), \quad \forall T > 0 \tag{1.5}$$

and that there exist positive constants $\tau_0$, $\tau_1$ and $d$ satisfying

$$0 < \tau_0 \le \tau(t) \le \tau_1, \ \ \tau'(t) \le d < 1, \quad \forall t > 0. \tag{1.6}$$

We are interested in proving the exponential stability for the problem (1.1)-(1.4). In order to obtain this, we will assume that

$$\max\{1, \frac{a}{b}\} < \frac{L_1 + L_3 - L_2}{2(L_2 - L_1)}. \tag{1.7}$$

As described in [6], the assumption (1.7) gives the relationship between the boundary regions and the transmission permitted. It can be also seen as a restriction on the wave speeds of the two equations and the damped part of the domain. It is known that for Timoshenko systems [25] and Bresse systems [3] the wave speeds always control the decay rate of the solution. It is an interesting open question to investigate the behavior of the solution when (1.7) is not satisfied.

Time delay is the property of a physical system by which the response to an applied force is delayed in its effect, and the central question is that delays source can destabilize a system that is asymptotically stable in the absence of delays, see [7, 8, 9, 26].

Transmission problems are closely related to the design of material components, attracting considerable attention in recent years, e.g., in the analysis of damping mechanisms in the metallurgical industry or smart materials technology, see [4, 24] and the references therein. Studies of fluid structure interaction and the added mass effect, also known as virtual mass effect, hydrodynamic mass, and hydroelastic vibration of structures were initiated by H. Lamb [14] who investigated the vibrations of a thin elastic circular plate in contact with water. Experimental study of impact on composite plates with fluid-structure interaction was investigated in [13].

From the mathematical point of view a transmission problem for wave propagation consists on a hyperbolic equation for which the corresponding elliptic operator has discontinuous coefficients. We consider the wave propagation over bodies consisting of two physically different materials, one purely elastic and another subject to frictional damping. The type of wave propagation generated by mixed materials originates a transmission (or diffraction) problem.

To the best of our knowledge, the first contribution to the literature regarding transmission problems with time delay was given by A. Benseghir in [6]. More precisely, in [6] the transmission problem

$$\begin{aligned} u_{tt} - a u_{xx} + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, \ \text{in } \Omega\times]0,\infty[, \\ v_{tt} - b v_{xx} = 0, \ \text{in } ]L_1, L_2[\times]0,\infty[. \end{aligned} \tag{1.8}$$

with constant weights $\mu_1, \mu_2$ and time delay $\tau > 0$ was studied. Under an appropriate assumption on the weights of the two feedbacks ($\mu_1 < \mu_2$), it was proved the well-posedness of the system and, under condition (1.7), it was established an exponential decay result.

The results in [6] were improved by S. Zitouni et al. [27]. There, the authors considered the problem with time-varying delay $\tau(t)$ of the form

$$\begin{gathered} u_{tt} - au_{xx} + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau(t)) = 0, \text{ in } \Omega\times]0,\infty[, \\ v_{tt} - bv_{xx} = 0, \text{ in } ]L_1, L_2[\times]0,\infty[ \end{gathered} \tag{1.9}$$

and proved the global existence and exponential stability under suitable assumptions on the delay term and assumption (1.7). The systems (1.8), (1.9) without delay have been investigated in [5].

The transmission problem with history and delay was considered by G. Li et al., in [17] where the equations were expressed as

$$\begin{gathered} u_{tt} - au_{xx} + \int_0^\infty g(s) u_{xx}(x, t-s) ds + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) = 0, \text{ in } \Omega\times]0,\infty[, \\ v_{tt} - bv_{xx} = 0, \text{ in } ]L_1, L_2[\times]0,\infty[. \end{gathered} \tag{1.10}$$

Under suitable assumptions on the delay term and on the function $g$, the authors obtained exponential stability result for two cases. In the first case, they considered $\mu_2 < \mu_1$ and for second case, they assumed that $\mu_2 = \mu_1$.

S. Zitouni et al., [29] extended the results in [17] for varying delay. In [29] was proved existence and the uniqueness of the solution by using the semigroup theory and the exponential stability of the solution was obtained by the energy method for the following problem

$$\begin{gathered} u_{tt} - au_{xx} + \int_0^\infty g(s) u_{xx}(x, t-s) ds + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau(t)) = 0, \text{ in } \Omega\times]0,\infty[, \\ v_{tt} - bv_{xx} = 0, \text{ in } ]L_1, L_2[\times]0,\infty[. \end{gathered} \tag{1.11}$$

Stability to localized viscoelastic transmission problem was considered by Muñoz Rivera et al., [19] where they considered the system

$$\begin{gathered} \rho\phi_{tt} + \sigma_x = 0, \quad \text{in } ]0, L_0[\cup]L_0, L_1[\cup]L_1, L[\times]0,\infty[, \\ \sigma(x,t) = \alpha(x)\varphi_x - k(x)\varphi_{xt} - \beta(x)\varphi_{xt} = 0, \end{gathered} \tag{1.12}$$

where $\rho$, $\alpha$, $k$ and $\beta$ are discontinuous in $L_0$, $L_1$ and are bounded and $C^1$ in each of the open intervals $]0, L_0[$, $]L_0, L_1[$ and $]L_1, L[$. The authors investigated the effect of the positions of the dissipative mechanisms on a bar with three components $]0, L_0[$, $]L_0, L_1[$, $]L_1, L[$, and showed that the system is exponentially stable if and only if the viscous component is not in the center of the bar. In other case, they showed the lack of exponential stability, and that the solutions still decay but just polynomially to zero.

The case of time-varying delay has already been considered in other works, such as [12, 16, 22, 28]. A wave equation with time-varying delay and nonlinear weights was considered in the recent work of Barros et al., [2] where it was studied the equation given by

$$u_{tt} - u_{xx} + \mu_1(t) u_t + \mu_2(t) u_t(x, t-\tau(t)) = 0, \text{ in } ]0, L[\times]0,+\infty[. \tag{1.13}$$

Under proper conditions on nonlinear weights $\mu_1(t), \mu_2(t)$ and $\tau(t)$ the authors proved global existence of solutions and obtained an estimate for the decay rate of energy.

In the present work we improve the results in [27] where, for constant weights $\mu_1(t) = \mu_1$, $\mu_2(t) = \mu_2$ and under adequate assumptions regarding the weights and time-varying delay it was proved the well posedness and singularity of solutions by using the semigroup theory. There, the authors also showed exponential stability by introducing an appropriate Lyapunov functional.

Here we consider a transmission problem with time-varying weights and time-varying delay, which is the main characteristic of this work. Although there are some works on laminated beam and on Timoshenko system with delay, all of them consider constant weights, i.e., $\mu_1$ and $\mu_2$ are constants. To the best of our knowledge, there is no result for these systems with nonlinear weights. Moreover, since the weights are nonlinear, a difficulty comes in: the operator is non autonomous. This makes hard the use semigroup theory to study well-posedness. To overcome it we use the Kato's variable norm technique together with semigroup theory to show that the system is well-posed.

The remainder of this paper is organized as follows. In section 2 we introduce some notations and prove the dissipative property for the energy of the system. In the section 3, by using Kato's variable norm technique (see [10]) and under some restriction on the non-linear weights and the time-varying delay, the system is shown to be well-posed. In section 4, we present the result of exponential stability by energy methods, and by using suitable sophisticated estimates for multipliers to construct an appropriated Lyapunov functional.

**2. Preliminaries.** We start by setting the following hypothesis:
**(H1)** $\mu_1 : [0, +\infty[\to]0, +\infty[$ is a non-increasing function of class $C^1$ satisfying

$$\left|\frac{\mu_1'(t)}{\mu_1(t)}\right| \leq M_1, \quad 0 < \mu_0 < \mu_1(t), \quad \forall t \geq 0, \tag{2.1}$$

where $M_1$ and $\mu_0$ are positive constants.

**(H2)** $\mu_2 : [0, +\infty[\to \mathbb{R}$ is a function of class $C^1$, which is not necessarily positive or monotone, such that

$$|\mu_2(t)| \leq \beta\mu_1(t), \tag{2.2}$$

$$|\mu_2'(t)| \leq M_2\mu_1(t), \tag{2.3}$$

for some $0 < \beta < \sqrt{1-d}$ and $M_2 > 0$.

As in Nicaise and Pignotti [20] we introduce the new variable

$$z(x, \rho, t) = u_t(x, t - \tau(t)\rho), \quad (x, \rho) \in \Omega\times]0, 1[, \ t > 0. \tag{2.4}$$

It is easily verified that the new variable satisfies

$$\tau(t)z_t(x, \rho, t) + (1 - \tau'(t)\rho)z_\rho(x, \rho, t) = 0$$

and the system (1.1) is equivalent to the system

$$\begin{aligned} u_{tt}(x, t) - au_{xx}(x, t) + \mu_1(t)u_t(x, t) + \mu_2(t)z(x, 1, t) = 0 \quad \text{in } \Omega\times]0, \infty[, \\ v_{tt}(x, t) - bv_{xx}(x, t) = 0 \quad \text{in } ]L_1, L_2[\times]0, \infty[, \\ \tau(t)z_t(x, \rho, t) + (1 - \tau'(t)\rho)z_\rho(x, \rho, t) = 0 \quad \text{in } \Omega\times]0, 1[\times]0, \infty[, \end{aligned} \tag{2.5}$$

subject to the transmission conditions

$$\begin{aligned} u(L_i,t) &= v(L_i,t), \quad i=1,2,\\ au_x(L_i,t) &= bv_x(L_i,t), \quad i=1,2, \end{aligned} \tag{2.6}$$

the boundary conditions

$$u(0,t) = u(L_3,t) = 0, \tag{2.7}$$

and the initial conditions

$$\begin{aligned} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{on } \Omega,\\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x) \quad \text{on } \Omega,\\ z(x,\rho,0) = u_t(x,-\tau(0)\rho) = f_0(x,-\tau(0)\rho), \quad (x,\rho) \quad \text{in } \Omega\times]0,1[. \end{aligned} \tag{2.8}$$

For any regular solution $u$ of (2.5), we define its energy at time $t>0$ as

$$\begin{aligned} E_1(t) &= \frac{1}{2}\int_\Omega \left(|u_t(x,t)|^2 + a|u_x(x,t)|^2\right)\,dx,\\ E_2(t) &= \frac{1}{2}\int_{L_1}^{L_2} \left(|v_t(x,t)|^2 + b|v_x(x,t)|^2\right)\,dx. \end{aligned}$$

The total energy is defined by

$$E(t) = E_1(t) + E_2(t) + \frac{\xi(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx, \tag{2.9}$$

where

$$\xi(t) = \bar{\xi}\mu_1(t) \tag{2.10}$$

is a non-increasing function of class $C^1$ in $[0,+\infty[$ and $\bar{\xi}$ be a positive constant such that

$$\frac{\beta}{\sqrt{1-d}} < \bar{\xi} < 2 - \frac{\beta}{\sqrt{1-d}}. \tag{2.11}$$

Our first result states that the energy is a non-increasing function.

Lemma 2.1. *Let $(u,v,z)$ be a solution to the system* (2.5)-(2.8). *Then the energy functional defined by* (2.9) *satisfies*

$$\begin{aligned} E'(t) \le& -\mu_1(t)\left(1-\frac{\bar{\xi}}{2}-\frac{\beta}{2\sqrt{1-d}}\right)\int_\Omega u_t^2\,dx\\ &- \mu_1(t)\left(\frac{\bar{\xi}(1-\tau'(t))}{2}-\frac{\beta\sqrt{1-d}}{2}\right)\int_\Omega z_1^2(x,\rho,t)\,dx\\ \le&\ 0. \end{aligned} \tag{2.12}$$

*Proof.* For any fixed $t>0$ we multiply the first and second equations of (2.5) by $u_t(x,t)$ and $v_t(x,t)$ respectively. Then, integrating by parts on $\Omega$ and $]L_1,L_2[$ respectively, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(u_t^2 + au_x^2\right)\,dx = -\,\mu_1(t)\int_\Omega u_t^2\,dx - \mu_2(t)\int_\Omega z(x,1,t)u_t\,dx + a\left[u_xu_t\right]_{\partial\Omega}, \tag{2.13}$$

$$\frac{1}{2}\frac{d}{dt}\int_{L_1}^{L_2}\left(v_t^2+bv_x^2\right)\,dx=b\left[v_xv_t\right]_{L_1}^{L_2}. \tag{2.14}$$

Now multiplying the third equation of (2.5) by $\xi(t)z(x,\rho,t)$ and integrating on $\Omega\times]0,1[$, we obtain

$$\tau(t)\xi(t)\int_\Omega\int_0^1 z_t(x,\rho,t)z(x,\rho,t)\,d\rho\,dx=-\frac{\xi(t)}{2}\int_\Omega\int_0^1(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}(z(x,\rho,t))^2\,d\rho\,dx,$$

and consequently we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\xi(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\right)=&\frac{\xi(t)}{2}\int_\Omega z^2(x,0,t)\,dx\\&-\frac{\xi(t)(1-\tau'(t))}{2}\int_\Omega z^2(x,1,t)\,dx\\&+\frac{\xi'(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx.\end{aligned} \tag{2.15}$$

From (2.9), (2.13), (2.14), (2.15) and using the conditions (2.6) and (2.7), we obtain

$$\begin{aligned}E'(t)=&\frac{\xi(t)}{2}\int_\Omega u_t^2\,dx-\frac{\xi(t)(1-\tau'(t))}{2}\int_\Omega z^2(x,1,t)\,dx\\&+\frac{\xi'(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\\&-\mu_1(t)\int_\Omega u_t^2\,dx-\mu_2(t)\int_\Omega z(x,1,t)u_t\,dx.\end{aligned} \tag{2.16}$$

Due to Young's inequality, we have

$$\mu_2(t)\int_\Omega z(x,1,t)u_t\,dx\le\frac{|\mu_2(t)|}{2\sqrt{1-d}}\int_\Omega u_t^2\,dx+\frac{|\mu_2(t)|\sqrt{1-d}}{2}\int_\Omega z^2(x,1,t)\,dx. \tag{2.17}$$

Now inserting (2.17) into (2.16), we obtain

$$\begin{aligned}E'(t)\le&-\left(\mu_1(t)-\frac{\xi(t)}{2}-\frac{|\mu_2(t)|}{2\sqrt{1-d}}\right)\int_\Omega u_t^2\,dx\\&-\left(\frac{\xi(t)}{2}-\frac{\xi(t)\tau'(t)}{2}-\frac{|\mu_2(t)|\sqrt{1-d}}{2}\right)\int_\Omega z^2(x,1,t)\,dx\\&+\frac{\xi'(t)\tau(t)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\\\le&-\mu_1(t)\left(1-\frac{\bar\xi}{2}-\frac{\beta}{2\sqrt{1-d}}\right)\int_\Omega u_t^2\,dx\\&-\mu_1(t)\left(\frac{\bar\xi(1-\tau'(t))}{2}-\frac{\beta\sqrt{1-d}}{2}\right)\int_\Omega z^2(x,1,t)\,dx\\\le&\ 0.\end{aligned}$$

Hence, the proof is complete. □

**3. Global solution.** In this section, our goal is to prove existence and uniqueness of solutions to the system (2.5) - (2.8). This is the content of Theorem 3.2.

We begin by introducing the vector function $U = (u, v, \varphi, \psi, z)^T$, where $\varphi(x,t) = u_t(x,t)$ and $\psi(x,t) = v_t(x,t)$. The system (2.5)-(2.8) can be written as

$$\begin{cases} U_t - \mathcal{A}(t)U = 0, \\ U(0) = U_0 = (u_0, v_0, u_1, v_1, f_0(\cdot, -, \tau(0)))^T, \end{cases} \tag{3.1}$$

where the operator $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t)\,U = \begin{pmatrix} \varphi(x,t) \\ \psi(x,t) \\ au_{xx}(x,t) - \mu_1(t)\varphi(x,t) - \mu_2(t)z(x,1,t) \\ bv_{xx}(x,t) \\ -\frac{1-\tau'(t)\rho}{\tau(t)} z_\rho(x,\rho,t) \end{pmatrix}. \tag{3.2}$$

Now, taking into account the conditions (1.2)-(1.3), as well as previous results presented in [6, 15, 17, 27], we introduce the set

$$X_* = \{(u,v) \in H^1(\Omega) \times H^1(]L_1, L_2[)/u(0) = u(L_3) = 0, u(L_i) = v(L_i), \\ au_x(L_i) = bv_x(L_i), i = 1, 2\}.$$

We define the phase space as

$$\mathcal{H} = X_* \times L^2(\Omega) \times L^2(]L_1, L_2[) \times L^2(\Omega \times ]0,1[)$$

equipped with the inner product

$$\langle U, \hat{U} \rangle_{\mathcal{H}} = \int_\Omega (\varphi\hat{\varphi} + au_x\hat{u}_x)\,dx + \int_{L_1}^{L_2} \left(\psi\hat{\psi} + bv_x\hat{v}_x\right)\,dx + \xi(t)\tau(t)\int_\Omega\int_0^1 z\hat{z}\,d\rho\,dx, \tag{3.3}$$

for $U = (u, v, \varphi, \psi, z)^T$ and $\hat{U} = (\hat{u}, \hat{v}, \hat{\varphi}, \hat{\psi}, \hat{z})^T$.

The domain $D(\mathcal{A}(t))$ of $\mathcal{A}(t)$ is defined by

$$D(\mathcal{A}(t)) = \{(u, v, \varphi, \psi, z)^T \in \mathcal{H}/(u,v) \in \left(H^2(\Omega) \times H^2(]L_1, L_2[)\right) \cap X_*, \\ \varphi \in H^1(\Omega), \psi \in H^1(]L_1, L_2[), z \in L^2\left(]0, L[; H_0^1(]0,1[)\right), \varphi = z(\cdot, 0)\}. \tag{3.4}$$

Notice that the domain of the operator $\mathcal{A}(t)$ does not dependent on time $t$, i.e.,

$$D(\mathcal{A}(t)) = D(\mathcal{A}(0)), \quad \forall t > 0. \tag{3.5}$$

A general theory for non autonomous operators given by equations of type (3.1) has been developed using semigroup theory, see [10], [11] and [23]. The simplest way to prove existence and uniqueness results is to show that the triplet $\{(\mathcal{A}, \mathcal{H}, Y)\}$, with $\mathcal{A} = \{\mathcal{A}(t)/t \in [0,T]\}$, for some fixed $T > 0$ and $Y = \mathcal{A}(0)$, forms a CD-systems (or constant domain system, see [10] and [11]). More precisely, the following theorem, which is due to Tosio Kato gives the existence and uniqueness results and is proved in Theorem 1.9 of [10] (see also Theorem 2.13 of [11] or [1]). For convenience let state Kato's result here.

Theorem 3.1. *Assume that*

(i) $Y = D(\mathcal{A}(0))$ *is dense subset of* $\mathcal{H}$,

(ii) (3.5) *holds,*

(iii) *for all* $t \in [0,T]$, $\mathcal{A}(t)$ *generates a strongly continuous semigroup on* $\mathcal{H}$ *and the family* $\mathcal{A}(t) = \{\mathcal{A}(t)/t \in [0,T]\}$ *is stable with stability constants* $C$ *and* $m$ *independent of* $t$ *(i.e., the semigroup* $(S_t(s))_{s\geq 0}$ *generated by* $\mathcal{A}(t)$ *satisfies* $\|S_t(s)u\|_{\mathcal{H}} \leq Ce^{ms}\|u\|_{\mathcal{H}}$, *for all* $u \in \mathcal{H}$ *and* $s \geq 0$*),*

(iv) $\partial_t \mathcal{A}(t)$ *belongs to* $L^\infty_*([0,T], B(Y,\mathcal{H}))$, *which is the space of equivalent classes of essentially bounded, strongly measurable functions from* $[0,T]$ *into the set* $B(Y,\mathcal{H})$ *of bounded linear operators from* $Y$ *into* $\mathcal{H}$*. Then, problem* (3.1) *has a unique solution* $U \in C([0,T],Y) \cap C^1([0,T],\mathcal{H})$ *for any initial datum in* $Y$*.*

Using the time-dependent inner product (3.3) and the Theorem 3.1 we get the following result of existence and uniqueness of global solutions to the problem (3.1).

Theorem 3.2 (Global solution). *For any initial datum* $U_0 \in \mathcal{H}$ *there exists a unique solution* $U$ *satisfying*

$$U \in C([0,+\infty[,\mathcal{H})$$

*for problem* (3.1)*. Moreover, if* $U_0 \in D(\mathcal{A}(0))$*, then*

$$U \in C([0,+\infty[, D(\mathcal{A}(0))) \cap C^1([0,+\infty[,\mathcal{H}).$$

*Proof.* Our goal is then to check the assumptions (i)-(iv) of Theorem 3.1 to the problem (3.1). First, we show that $D(\mathcal{A}(0))$ is dense in $\mathcal{H}$. The proof will follow the method used in [21] with the appropriate changes imposed by the nature of our problem. Let $\hat{U} = (\hat{u},\hat{v},\hat{\varphi},\hat{\psi},\hat{z})^T \in \mathcal{H}$ be orthogonal to all the elements of $D(\mathcal{A}(0))$, namely

$$\begin{aligned} 0 = \langle U,\hat{U}\rangle_{\mathcal{H}} = \int_\Omega (\varphi\hat{\varphi} + a u_x \hat{u}_x)\,dx + \int_{L_1}^{L_2} \left(\psi\hat{\psi} + b v_x \hat{v}_x\right)\,dx \\ + \xi(t)\tau(t)\int_\Omega\int_0^1 z\hat{z}\,d\rho\,dx, \end{aligned} \tag{3.6}$$

for $U = (u,v,\varphi,\psi,z)^T \in D(\mathcal{A}(0))$.

We first take $u = v = \varphi = \psi = 0$ and $z \in C_0^\infty(\Omega\times]0,1[)$. As $U = (0,0,0,0,z)^T \in D(\mathcal{A}(0))$ and therefore, from (3.6) we deduce that

$$\int_\Omega\int_0^1 z\hat{z}\,d\rho\,dx = 0.$$

Since $C_0^\infty(\Omega\times]0,1[)$ is dense in $L^2(\Omega\times]0,1[)$, then, it follows that $\hat{z} = 0$. Similarly, for $\varphi \in C_0^\infty(\Omega)$, we have $U = (0,0,\varphi,0,0)^T \in D(\mathcal{A}(0))$, which implies from (3.6) that

$$\int_\Omega \varphi\hat{\varphi}\,dx = 0.$$

So, as above, it follows that $\hat{\varphi} = 0$. In the same way, by taking $\psi \in C_0^\infty(]L_1,L_2[)$, we get from (3.6)

$$\int_{L_1}^{L_2} \psi\hat{\psi}\,dx = 0$$

and by density of $C_0^\infty(]L_1,L_2[)$ in $L^2(]L_1,L_2[)$, we obtain $\hat{\psi} = 0$.

Finally, for $(u,v)\in C_0^\infty(\Omega\times]L_1,L_2[)$ (then $(u_x,v_x)\in C_0^\infty(\Omega\times]L_1,L_2[)$) we obtain

$$a\int_\Omega u_x\hat{u}_x\,dx+b\int_{L_1}^{L_2}v_x\hat{v}_x\,dx=0.$$

Since $C_0^\infty(\Omega\times]L_1,L_2[)$ is dense in $L^2(\Omega\times]L_1,L_2[)$, we deduce that $(\hat{u}_x,\hat{v}_x)=(0,0)$ because $(\hat{u},\hat{v})\in X_*$.

We consequently have

$$D(\mathcal{A}(0)) \text{ is dense in } \mathcal{H}. \tag{3.7}$$

Now, we show that the operator $\mathcal{A}(t)$ generates a $C_0-$semigroup in $\mathcal{H}$ for a fixed $t$.

We calculate $\langle\mathcal{A}(t)U,U\rangle_t$ for a fixed $t$. Take $U=(u,v,\varphi,\psi,z)^T\in D(\mathcal{A}(t))$. Then

$$\begin{aligned}\langle\mathcal{A}(t)U,U\rangle_t=&-\mu_1(t)\int_\Omega\varphi^2\,dx-\mu_2(t)\int_\Omega z(x,1)\varphi\,dx\\&-\frac{\xi(t)}{2}\int_\Omega\int_0^1(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}z^2(x,\rho)\,d\rho\,dx.\end{aligned}$$

Since

$$(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}z^2(x,\rho)=\frac{\partial}{\partial\rho}\left((1-\tau'(t)\rho)\,z^2(x,\rho)\right)+\tau'(t)z^2(x,\rho),$$

we have

$$\int_0^1(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}z^2(x,\rho)\,d\rho=(1-\tau'(t))\,z^2(x,1)-z^2(x,0)+\tau'(t)\int_0^1z^2(x,\rho)\,d\rho.$$

Therefore

$$\begin{aligned}\langle\mathcal{A}(t)U,U\rangle_t=&-\mu_1(t)\int_\Omega\varphi^2\,dx-\mu_2(t)\int_\Omega z(x,1)\varphi\,dx+\frac{\xi(t)}{2}\int_\Omega\varphi^2\,dx\\&-\frac{\xi(t)(1-\tau'(t))}{2}\int_\Omega z^2(x,1)\,dx-\frac{\xi(t)\tau'(t)}{2}\int_\Omega\int_0^1z^2(x,\rho)\,d\rho\,dx.\end{aligned}$$

Now, taking into account the inequality (2.17), we deduce

$$\begin{aligned}\langle\mathcal{A}(t)U,U\rangle_t\le&-\mu_1(t)\left(1-\frac{\bar{\xi}}{2}-\frac{\beta}{2\sqrt{1-d}}\right)\int_\Omega\varphi^2\,dx\\&-\mu_1(t)\left(\frac{\bar{\xi}(1-\tau'(t))}{2}-\frac{\beta\sqrt{1-d}}{2}\right)\int_\Omega z^2(x,1)\,dx\\&+\frac{\xi(t)|\tau'(t)|}{2\tau(t)}\tau(t)\int_\Omega\int_0^1z^2(x,\rho)\,d\rho\,dx.\end{aligned}$$

Thus, we have

$$\begin{aligned}\langle\mathcal{A}(t)U,U\rangle_t\le&-\mu_1(t)\left(1-\frac{\bar{\xi}}{2}-\frac{\beta}{2\sqrt{1-d}}\right)\int_\Omega\varphi^2\,dx\\&-\mu_1(t)\left(\frac{\bar{\xi}(1-\tau'(t))}{2}-\frac{\beta\sqrt{1-d}}{2}\right)\int_\Omega z^2(x,1)\,dx\\&+\kappa(t)\langle U,U\rangle_t,\end{aligned}$$

where

$$\kappa(t) = \frac{\sqrt{1+\tau'(t)^2}}{2\tau(t)}.$$

From (2.12) we conclude that

$$\langle \mathcal{A}(t)U, U\rangle_t - \kappa(t)\langle U, U\rangle_t \leq 0, \tag{3.8}$$

which means that operator $\tilde{\mathcal{A}}(t) = \mathcal{A}(t) - \kappa(t)I$ is dissipative.

Now, we prove the surjectivity of the operator $\lambda I - \mathcal{A}(t)$ for fixed $t > 0$ and $\lambda > 0$. For this purpose, let $F = (f_1, f_2, f_3, f_4, f_5)^T \in \mathcal{H}$. We seek $U = (u, v, \varphi, \psi, z)^T \in D(\mathcal{A}(t))$ which is solution of

$$(\lambda I - \mathcal{A}(t))\, U = F,$$

that is, the entries of $U$ satisfy the system of equations

$$\lambda u - \varphi = f_1, \tag{3.9}$$
$$\lambda v - \psi = f_2, \tag{3.10}$$
$$\lambda\varphi - a u_{xx} + \mu_1(t)\varphi + \mu_2(t)z(x,1) = f_3, \tag{3.11}$$
$$\lambda\psi - b v_{xx} = f_4, \tag{3.12}$$
$$\lambda z + \frac{1-\tau'(t)\rho}{\tau(t)} z_\rho = f_5. \tag{3.13}$$

Suppose that we have found $u$ and $v$ with the appropriated regularity. Therefore, from (3.9) and (3.10) we have

$$\varphi = \lambda u - f_1, \tag{3.14}$$
$$\psi = \lambda v - f_2. \tag{3.15}$$

It is clear that $\varphi \in H^1(\Omega)$ and $\psi \in H^1(]L_1, L_2[)$. Furthermore, if $\tau'(t) \neq 0$, following the same approach as in [21], we obtain

$$z(x,\rho) = \varphi(x)e^{\sigma(\rho,t)} + \tau(t)e^{\sigma(\rho,t)} \int_0^\rho \frac{f_5(x,s)}{1-s\tau'(s)} e^{-\sigma(s,t)}\, ds,$$

where

$$\sigma(\rho,t) = \lambda\frac{\tau(t)}{\tau'(t)} \ln(1-\rho\tau'(t)),$$

is solution of (3.13) satisfying

$$z(x,0) = \varphi(x). \tag{3.16}$$

Otherwise,

$$z(x,\rho) = \varphi(x)e^{-\lambda\tau(t)\rho} + \tau(t)e^{-\lambda\tau(t)\rho} \int_0^\rho f_5(x,s)e^{\lambda\tau(t)s}\, ds$$

is solution of (3.13) satisfying (3.16). From now on, for pratical purposes we will consider $\tau'(t) \neq 0$ (the case $\tau'(t) = 0$ is analogous). Taking into account (3.14) we have

$$\begin{aligned} z(x,1) &= \varphi e^{\sigma(1,t)} + \tau(t)e^{\sigma(1,t)} \int_0^1 \frac{f_5(x,s)}{1-s\tau'(s)} e^{-\sigma(s,t)}\, ds \\ &= (\lambda u - f_1)\, e^{\sigma(1,t)} + \tau(t)e^{\sigma(1,t)} \int_0^1 \frac{f_5(x,s)}{1-s\tau'(s)} e^{-\sigma(s,t)}\, ds \\ &= \lambda u e^{\sigma(1,t)} - f_1 e^{\sigma(1,t)} + \tau(t)e^{\sigma(1,t)} \int_0^1 \frac{f_5(x,s)}{1-s\tau'(s)} e^{-\sigma(s,t)}\, ds. \end{aligned} \tag{3.17}$$

Substituting (3.14) and (3.17) into (3.11), and (3.15) into (3.12), we obtain

$$\begin{cases} \eta u - a u_{xx} = g_1, \\ \lambda^2 v - b v_{xx} = g_2, \end{cases} \tag{3.18}$$

where

$$\eta := \lambda^2 + \lambda\mu_1(t) + \lambda\mu_2(t)e^{\sigma(1,t)},$$

$$\begin{aligned} g_1 :=& f_3 + \lambda f_1 + \mu_1(t) f_1 + \mu_2(t) f_1 e^{\sigma(1,t)} \\ &- \mu_2(t)\tau(t)e^{\sigma(1,t)} \int_0^1 \frac{f_5(x,s)}{1-s\tau'(s)} e^{-\sigma(s,t)}\, ds, \end{aligned}$$

$$g_2 := f_4 + \lambda f_2.$$

In order to solve (3.18), we use a standard procedure, considering variational problem

$$\Upsilon((u,v),(\tilde{u},\tilde{v})) = L(\tilde{u},\tilde{v}), \tag{3.19}$$

where the bilinear form

$$\Upsilon : X_* \times X_* \to \mathbb{R}$$

and the linear form

$$L : X_* \to \mathbb{R}$$

are defined by

$$\begin{aligned} \Upsilon((u,v),(\tilde{u},\tilde{v})) =& \eta \int_\Omega u\tilde{u}\, dx + a \int_\Omega u_x \tilde{u}_x\, dx + \lambda^2 \int_{L_1}^{L_2} v\tilde{v}\, dx \\ &+ b \int_{L_1}^{L_2} v_x \tilde{v}_x\, dx - a\,[u_x \tilde{u}]_{\partial\Omega} - b\,[v_x\tilde{v}]_{L_1}^{L_2} \end{aligned}$$

and

$$L(\tilde{u},\tilde{v}) = \int_\Omega g_1 \tilde{u}\, dx + \int_{L_1}^{L_2} g_2 \tilde{v}\, dx.$$

It is easy to verify that $\Upsilon$ is continuous and coercive, and $L$ is continuous, so by applying the Lax-Milgram Theorem, we obtain a solution $(u,v) \in X_*$ for (3.18). In addition, it follows from (3.11) and (3.12) that $(u,v) \in H^2(\Omega) \times H^2(]L_1, L_2[)$ and so $(u, v, \varphi, \psi, z) \in D(\mathcal{A}(t))$.

Therefore, the operator $\lambda I - \mathcal{A}(t)$ is surjective for any $\lambda > 0$ and $t > 0$. Since $\kappa(t) > 0$, it follows that

$$\lambda I - \tilde{\mathcal{A}}(t) = (\lambda + \kappa(t))\, I - \mathcal{A}(t) \text{ is surjective} \tag{3.20}$$

for any $\lambda > 0$ and $t > 0$.

To complete the proof of (iii), it suffices to show that

$$\frac{\|\Phi\|_t}{\|\Phi\|_s} \le e^{\frac{c}{2\tau_0}|t-s|}, \quad t, s \in [0,T], \tag{3.21}$$

where $\Phi = (u, v, \varphi, \psi, z)^T$, $c$ is a positive constant and $\|\cdot\|$ is the norm associated the inner product (3.3). For all $t, s \in [0,T]$, we have

$$\|\Phi\|_t^2 - \|\Phi\|_s^2 e^{\frac{c}{\tau_0}|t-s|} = \left(1 - e^{\frac{c}{\tau_0}|t-s|}\right)\left[\int_\Omega \left(\varphi^2 + a u_x^2\right) dx + \int_{L_1}^{L_2} \left(\psi^2 + b v_x^2\right) dx\right]$$
$$+ \left(\xi(t)\tau(t) - \xi(s)\tau(s) e^{\frac{c}{\tau_0}|t-s|}\right) \int_\Omega \int_0^1 z^2(x,\rho)\, d\rho\, dx.$$

It is clear that $1 - e^{\frac{c}{\tau_0}|t-s|} \le 0$. Now we will prove $\xi(t)\tau(t) - \xi(s)\tau(s)e^{\frac{c}{\tau_0}|t-s|} \le 0$ for some $c > 0$. In order to do this, first observe that

$$\tau(t) = \tau(s) + \tau'(r)(t-s),$$

for some $r \in (s,t)$. Since $\xi$ is a non increasing function and $\xi > 0$, we get

$$\xi(t)\tau(t) \le \xi(s)\tau(s) + \xi(s)\tau'(r)(t-s),$$

which implies

$$\frac{\xi(t)\tau(t)}{\xi(s)\tau(s)} \le 1 + \frac{|\tau'(r)|}{\tau(s)}|t-s|.$$

Using (1.5) and that $\tau'$ is bounded, we deduce

$$\frac{\xi(t)\tau(t)}{\xi(s)\tau(s)} \le 1 + \frac{c}{\tau_0}|t-s| \le e^{\frac{c}{\tau_0}|t-s|},$$

which proves (3.21) and therefore $(iii)$ follows.

Moreover, as $\kappa'(t) = \frac{\tau'(t)\tau''(t)}{2\tau(t)\sqrt{1+\tau'(t)^2}} - \frac{\tau'(t)\sqrt{1+\tau'(t)^2}}{2\tau(t)^2}$ is bounded on $[0,T]$ for all $T > 0$ (by (1.5) and (2.11)) we have

$$\frac{d}{dt}\mathcal{A}(t)U = \begin{pmatrix} 0 \\ 0 \\ -\mu_1'(t)\varphi - \mu_2'(t) z(\cdot,1) \\ 0 \\ \frac{\tau''(t)\tau(t)\rho - \tau'(t)(\tau'(t)\rho - 1)}{\tau(t)^2} z_\rho \end{pmatrix},$$

with $\frac{\tau''(t)\tau(t)\rho - \tau'(t)(\tau'(t)\rho-1)}{\tau(t)^2}$ bounded on $[0,T]$ by (1.5) and (2.11). Thus

$$\frac{d}{dt}\tilde{\mathcal{A}}(t) \in L_*^\infty([0,T], B(D(\mathcal{A}(0)), \mathcal{H})), \tag{3.22}$$

where $L_*^\infty([0,T], B(D(\mathcal{A}(0)),\mathcal{H}))$ is the space of equivalence classes of essentially bounded, strongly measurable functions from $[0,T]$ into $B(D(\mathcal{A}(0)),\mathcal{H})$. Here $B(D(\mathcal{A}(0)),\mathcal{H})$ is the set of bounded linear operators from $D(\mathcal{A}(0))$ into $\mathcal{H}$.

Then, (3.8), (3.20) and (3.21) imply that the family $\tilde{\mathcal{A}} = \left\{\tilde{\mathcal{A}}(t) : t \in [0,T]\right\}$ is a stable family of generators in $\mathcal{H}$ with stability constants independent of $t$, by Proposition 1.1 from [10]. Therefore, the assumptions $(i)-(iv)$ of Theorem 3.1 are verified by (3.5), (3.7), (3.8), (3.20), (3.21) and (3.22). Thus, the problem

$$\begin{cases} \tilde{U}_t = \tilde{\mathcal{A}}(t)\tilde{U}, \\ \tilde{U}(0) = U_0 \end{cases} \tag{3.23}$$

has a unique solution $\tilde{U} \in C\left([0,+\infty[, D(\mathcal{A}(0))\right) \cap C^1\left([0,+\infty[, \mathcal{H}\right)$ for $U_0 \in D(\mathcal{A}(0))$. The requested solution of (3.1) is then given by

$$U(t) = e^{\int_0^t \kappa(s)\,ds}\tilde{U}(t)$$

because

$$\begin{aligned} U_t(t) &= \kappa(t)e^{\int_0^t \kappa(s)\,ds}\tilde{U}(t) + e^{\int_0^t \kappa(s)\,ds}\tilde{U}_t(t) \\ &= e^{\int_0^t \kappa(s)\,ds}\left(\kappa(t) + \tilde{\mathcal{A}}(t)\right)\tilde{U}(t) \\ &= \mathcal{A}(t)e^{\int_0^t \kappa(s)\,ds}\tilde{U}(t) \\ &= \mathcal{A}(t)U(t) \end{aligned}$$

which concludes the proof. □

**4. Exponential stability.** This section is dedicated to study the asymptotic behavior fo the solutions. The main goal is to obtain the stability of the solutions to the system (2.5)-(2.8). This is the content of Theorem 4.4 where we show that the solution of the problem (2.5)-(2.8) is exponentially stable. Our effort consists in building a suitable Lyapunov functional by the energy method. So, in order to achieve this goal, in the sequence we present several technical lemmas.

LEMMA 4.1. *Let $(u, v, z)$ be a solution of (2.5)-(2.8), then for any $\varepsilon_1 > 0$, we have the estimate*

$$\begin{aligned} \frac{d}{dt}\mathcal{I}_1(t) \le &-\left(a - \mu_1^2(0)c_1^2\varepsilon_1\right)\int_\Omega u_x^2\,dx - b\int_{L_1}^{L_2} v_x^2\,dx \\ &+\left(1 + \frac{1}{2\varepsilon_1}\right)\int_\Omega u_t^2\,dx + \int_{L_1}^{L_2} v_t^2\,dx + \frac{\beta^2}{2\varepsilon_1}\int_\Omega z^2(x,1,t)\,dx, \end{aligned} \tag{4.1}$$

*where $c_1$ is the Poincaré's constant and*

$$\mathcal{I}_1(t) = \int_\Omega uu_t\,dx + \int_{L_1}^{L_2} vv_t\,dx. \tag{4.2}$$

*Proof.* By differentiating $\mathcal{I}_1(t)$ and using (2.5), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{I}_1(t) &= \int_\Omega u_t^2\,dx - a\int_\Omega u_x^2\,dx - \mu_1(t)\int_\Omega uu_t\,dx - \mu_2(t)\int_\Omega uz(x,1,t)\,dx \\ &\quad + \int_{L_1}^{L_2} v_t^2\,dx - b\int_{L_1}^{L_2} v_x^2\,dx \\ &\le \int_\Omega u_t^2\,dx - a\int_\Omega u_x^2\,dx + \left|\mu_1(t)\int_\Omega uu_t\,dx\right| + \left|\mu_2(t)\int_\Omega uz(x,1,t)\,dx\right| \\ &\quad + \int_{L_1}^{L_2} v_t^2\,dx - b\int_{L_1}^{L_2} v_x^2\,dx.\end{aligned}$$

From hypothesis (H1) and (H2), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{I}_1(t) \le \int_\Omega u_t^2\,dx - a\int_\Omega u_x^2\,dx + \mu_1(0)\left|\int_\Omega uu_t\,dx\right| \\ + \beta\mu_1(0)\left|\int_\Omega uz(x,1,t)\,dx\right| + \int_{L_1}^{L_2} v_t^2\,dx - b\int_{L_1}^{L_2} v_x^2\,dx.\end{aligned} \tag{4.3}$$

By using the conditions (2.6) and (2.7), we obtain

$$\begin{aligned}u^2(x,t) = \left(\int_0^x u_x(s,t)\,ds\right)^2 \le L_1\int_0^{L_1} u_x^2(x,t)\,dx, \quad x\in[0,L_1], \\ u^2(x,t) \le (L_3-L_2)\int_{L_2}^{L_3} u_x^2(x,t)\,dx, \quad x\in[L_2,L_3],\end{aligned}$$

which imply the following Poincaré's inequality

$$\int_\Omega u^2(x,t)\,dx \le c_1^2\int_\Omega u_x^2\,dx, \quad x\in\Omega, \tag{4.4}$$

where $c_1 = \max\{L_1, L_3-L_2\}$ is the Poincaré's constant. Now using Young's inequality and (4.4), we have

$$\mu_1(0)\left|\int_\Omega uu_t\,dx\right| \le \frac{\varepsilon_1\mu_1^2(0)c_1^2}{2}\int_\Omega u_x^2\,dx + \frac{1}{2\varepsilon_1}\int_\Omega u_t^2\,dx \tag{4.5}$$

and

$$\beta\mu_1(0)\left|\int_\Omega uz(x,1,t)\,dx\right| \le \frac{\varepsilon_1\mu_1^2(0)c_1^2}{2}\int_\Omega u_x^2\,dx + \frac{\beta^2}{2\varepsilon_1}\int_\Omega z^2(x,1,t)\,dx. \tag{4.6}$$

Finally, substituting (4.5) and (4.6) into (4.3) completes the proof. □

Now, inspired by [18], we introduce the function

$$q(x) = \begin{cases} x - \dfrac{L_1}{2}, & x\in[0,L_1], \\ \dfrac{L_2-L_3-L_1}{2(L_2-L_1)}(x-L_1) + \dfrac{L_1}{2}, & x\in[L_1,L_2], \\ x - \dfrac{L_2+L_3}{2}, & x\in[L_2,L_3]. \end{cases} \tag{4.7}$$

It is easy to see that $q(x)$ is bounded, i.e., $|q(x)| \leq M$, where

$$M = \max\left\{\frac{L_1}{2}, \frac{L_3 - L_2}{2}\right\}.$$

We have the following result.

LEMMA 4.2. *Let $(u, v, z)$ be a solution of (2.5)-(2.8), then for any $\varepsilon_2 > 0$, the following estimates holds true*

$$\begin{aligned}\frac{d}{dt}\mathcal{I}_2(t) \leq & \left(\frac{1}{2} + \frac{1}{2\varepsilon_2}\right)\int_\Omega u_t^2\,dx + \left(\frac{a}{2} + M^2\mu_1^2(0)\varepsilon_2\right)\int_\Omega u_x^2\,dx \\ & + \frac{\beta^2}{2\varepsilon_2}\int_\Omega z^2(x,1,t)\,dx - \frac{1}{4}\left[L_1 u_t^2(L_1,t) + (L_3 - L_2)u_t^2(L_2,t)\right] \\ & - \frac{a}{4}\left[L_1 u_x^2(L_1,t) + (L_3 - L_2)u_x^2(L_2,t)\right],\end{aligned} \tag{4.8}$$

*and*

$$\begin{aligned}\frac{d}{dt}\mathcal{I}_3(t) \leq & \frac{L_2 - L_3 - L_1}{4(L_2 - L_1)}\left(\int_{L_1}^{L_2} v_t^2\,dx + b\int_{L_1}^{L_2} v_x^2\,dx\right) \\ & + \frac{1}{4}\left[L_1 v_t^2(L_1,t) + (L_3 - L_2)v_t^2(L_2,t)\right] \\ & + \frac{b}{4}\left[L_1 v_x^2(L_1,t) + (L_3 - L_2)v_x^2(L_2,t)\right],\end{aligned} \tag{4.9}$$

*where*

$$\mathcal{I}_2(t) = -\int_\Omega q(x)u_x u_t\,dx \quad \textit{and} \quad \mathcal{I}_3(t) = -\int_{L_1}^{L_2} q(x)v_x v_t\,dx. \tag{4.10}$$

*Proof.* By differentiating $\mathcal{I}_2(t)$ and using (2.5), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{I}_2(t) = & -\int_\Omega q(x)u_{xt}u_t\,dx - a\int_\Omega q(x)u_{xx}u_x\,dx \\ & + \mu_1(t)\int_\Omega q(x)u_x u_t\,dx + \mu_2(t)\int_\Omega q(x)u_x z(x,1,t)\,dx.\end{aligned}$$

Now, integrating by parts and considering the hypothesis (H1) and (H2), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{I}_2(t) \leq & \frac{1}{2}\int_\Omega q'(x)u_t^2\,dx - \frac{1}{2}\left[q(x)u_t^2\right]_{\partial\Omega} + \frac{a}{2}\int_\Omega q'(x)u_x^2\,dx - \frac{a}{2}\left[q(x)u_x^2\right]_{\partial\Omega} \\ & + \mu_1(0)\left|\int_\Omega q(x)u_x u_t\,dx\right| + \beta\mu_1(0)\left|\int_\Omega q(x)u_x z(x,1,t)\,dx\right| \\ \leq & \frac{1}{2}\int_\Omega u_t^2\,dx - \frac{1}{2}\left[q(x)u_t^2\right]_{\partial\Omega} + \frac{a}{2}\int_\Omega u_x^2\,dx - \frac{a}{2}\left[q(x)u_x^2\right]_{\partial\Omega} \\ & + \mu_1(0)M\left|\int_\Omega u_x u_t\,dx\right| + \beta\mu_1(0)M\left|\int_\Omega u_x z(x,1,t)\,dx\right|.\end{aligned} \tag{4.11}$$

On the other hand, by using the boundary conditions (2.7), we get

$$\begin{aligned}\frac{1}{2}\left[q(x)u_t^2\right]_{\partial\Omega} &= \frac{1}{4}\left[L_1 u_t^2(L_1,t) + (L_3 - L_2)u_t^2(L_2,t)\right], \\ -\frac{a}{2}\left[q(x)u_x^2\right]_{\partial\Omega} &\leq -\frac{a}{4}\left[L_1 u_x^2(L_1,t) + (L_3 - L_2)u_x^2(L_2,t)\right].\end{aligned}$$

Inserting the above two equalities into (4.11) and by Young's inequality, we conclude that (4.11) gives (4.8).

By the same argument, taking the derivative of $\mathcal{I}_3(t)$, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{I}_3(t) &= \frac{1}{2}\int_{L_1}^{L_2} q'(x)v_t^2\,dx - \frac{1}{2}\left[q(x)v_t^2\right]_{L_1}^{L_2} + \frac{b}{2}\int_{L_1}^{L_2} q'(x)v_x^2\,dx - \frac{b}{2}\left[q(x)v_x^2\right]_{L_1}^{L_2}\\ &= \frac{L_2-L_3-L_1}{4(L_2-L_1)}\left(\int_{L_1}^{L_2} v_t^2\,dx + b\int_{L_1}^{L_2} v_x^2\,dx\right)\\ &\quad+\frac{1}{4}\left[L_1v_t^2(L_1,t)+(L_3-L_2)v_t^2(L_2,t)\right]\\ &\quad+\frac{b}{4}\left[L_1v_x^2(L_1,t)+(L_3-L_2)v_x^2(L_2,t)\right]\end{aligned}$$

Hence, the proof is complete. □

As in [12], taking into account the last lemma, we introduce the functional

$$\mathcal{J}(t) = \bar{\xi}\tau(t)\int_\Omega\int_0^1 e^{-2\tau(t)\rho}z^2(x,\rho,t)\,d\rho\,dx. \tag{4.12}$$

For this functional we have the estimate (4.13) given by the following lemma.

LEMMA 4.3 ([12, Lemma 3.7]). *Let $(u,v,z)$ be a solution of* (2.5)-(2.8). *Then the functional $\mathcal{J}(t)$ satisfies*

$$\frac{d}{dt}\mathcal{J}(t) \le -2\mathcal{J}(t) + \bar{\xi}\int_\Omega u_t^2\,dx. \tag{4.13}$$

Now we are in position to prove our result of stability.

THEOREM 4.4. *Let $U(t) = (u(t), v(t), \varphi(t), \psi(t), z(t))$ be the solution of* (2.5)-(2.8) *with initial data $U_0 \in D(\mathcal{A}(0))$ and let $E(t)$ be the energy of $U$. Assume that the hypothesis* (1.5), (1.6), *(H1), (H2) and*

$$\max\{1,\frac{a}{b}\} < \frac{L_1+L_3-L_2}{2(L_2-L_1)} \tag{4.14}$$

*hold. Then there exist positive constants c and $\alpha$ such that*

$$E(t) \le cE(0)e^{-\alpha t}, \quad \forall t \ge 0. \tag{4.15}$$

*Proof.* Let us define the Lyapunov functional

$$\mathcal{L}(t) = NE(t) + \sum_{i=1}^{3} N_i\mathcal{I}_i(t) + \mathcal{J}(t), \tag{4.16}$$

where $N$, $N_i$, $i = 1,2,3$ are positive real numbers which will be chosen later. By the Lemma 2.1, there exists a positive constant $K$ such that

$$\frac{d}{dt}E(t) \le -K\left[\int_\Omega u_t^2\,dx + \int_\Omega z^2(x,1,t)\,dx\right]. \tag{4.17}$$

It follows from the transmission conditions (2.6) that

$$a^2 u_x^2(L_i,t) = b^2 v_x^2(L_i,t), \quad i = 1,2. \tag{4.18}$$

Using the estimates (4.1), (4.8), (4.9), (4.13), (4.17) and the inequality (4.18), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \leq &- \left[KN - \left(1 + \frac{1}{2\varepsilon_1}\right)N_1 - \left(\frac{1}{2} + \frac{1}{2\varepsilon_2}\right)N_2 - \bar{\xi}\right]\int_\Omega u_t^2\,dx \\ &- \left(KN - \frac{\beta^2}{2\varepsilon_1}N_1 - \frac{\beta^2}{2\varepsilon_2}N_2\right)\int_\Omega z^2(x,1,t)\,dx \\ &- \left[\left(a - \mu_1^2(0)c_1^2\varepsilon_1\right)N_1 - \left(\frac{a}{2} + M^2\mu_1^2(0)\varepsilon_2\right)N_2\right]\int_\Omega u_x^2\,dx \\ &+ \left[N_1 + \frac{L_2 - L_3 - L_1}{4(L_2 - L_1)}N_3\right]\int_{L_1}^{L_2} v_t^2\,dx \\ &- \left[N_1 - \frac{L_2 - L_3 - L_1}{4(L_2 - L_1)}N_3\right] b\int_{L_1}^{L_2} v_x^2\,dx \\ &- (N_2 - N_3)\left[\frac{L_1}{4}u_t^2(L_1,t) + \frac{L_3 - L_2}{4}u_t^2(L_2,t)\right] \\ &- \left(N_2 - \frac{a}{b}N_3\right)\frac{a}{4}\left[\frac{L_1}{4}u_t^2(L_1,t) + \frac{L_3 - L_2}{4}u_t^2(L_2,t)\right] - 2\mathcal{J}(t).\end{aligned} \tag{4.19}$$

Now we observe that under assumption (4.14), we can always find real constants $N_1, N_2$ and $N_3$ in such way that

$$N_1 + \frac{L_2 - L_3 - L_1}{4(L_2 - L_1)}N_3 < 0, \quad N_2 > \max\left\{1, \frac{a}{b}\right\}N_3, \quad N_1 > \frac{N_2}{2}.$$

After that, we pick the positive constants $\varepsilon_1$ and $\varepsilon_2$ small enough that

$$\mu_1^2(0)c_1^2\varepsilon_1 N_1 + M^2\mu_1^2(0)\varepsilon_2 N_2 < a\left(N_1 - \frac{N_2}{2}\right).$$

Finally, since $\xi(t)\tau(t)$ is non-negative and bounded, we choose $N$ large enough that (4.19) is turned into the following estimate

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) &\leq -\eta_1 \int_\Omega \left(u_t^2 + u_x^2\right)\,dx - \eta_1 \int_{L_1}^{L_2} \left(v_t^2 + v_x^2\right)\,dx \\ &\quad - \eta_1 \int_\Omega z^2(x,\rho,t)\,dx - \eta_1 \int_\Omega z^2(x,1,t)\,dx \\ &\leq -\eta_1 \int_\Omega \left(u_t^2 + u_x^2\right)\,dx - \eta_1 \int_{L_1}^{L_2} \left(v_t^2 + v_x^2\right)\,dx - \eta_1 \int_\Omega z^2(x,\rho,t)\,dx,\end{aligned}$$

for a certain positive constant $\eta_1$.

This implies by (2.9) that there exists $\eta_2 > 0$ such that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\eta_2 E(t), \quad \forall t \geq 0. \tag{4.20}$$

On the other hand, it is not hard to see that for large enough $N$, we have $\mathcal{L}(t) \sim E(t)$, i.e. there exists two positive constants $\gamma_1$ and $\gamma_2$ such that

$$\gamma_1 E(t) \leq \mathcal{L}(t) \leq \gamma_2 E(t), \quad \forall t \geq 0. \tag{4.21}$$

Combining (4.20) and (4.21), we obtain

$$\frac{d}{dt}\mathcal{L}(t) \leq -\alpha\mathcal{L}(t), \quad \forall t \geq 0$$

which leads to

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\alpha t}, \quad \forall t \geq 0. \tag{4.22}$$

The desired result (4.15) follows by using estimates (4.21) and (4.22). Then, the proof of Theorem 4.4 is complete. □

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