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THE INVERSE PROBLEM FOR THE RECONSTRUCTION OF THE WEIGHT FUNCTIONS IN A SOCIO-ECONOMIC SYSTEM MODELLED BY THE DISCRETE THERMOSTATTED KINETIC FRAMEWORK*

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Abstract. The aim of the paper is threefold. First of all, since our main goal is to apply the discrete thermostatted kinetic framework to a socio-economic system whose space of microscopic states is discrete, we discuss the inverse problem (see the Introduction) in the discrete case. Next, we propose an example of a socio-economic system of the above kind, paying special attention to the choice of a particularly meaningful and plausible initial probability distribution on the state space. Finally, we sketch a first simple numerical simulation of the evolution of the system, just in order to show that the appropriate choice of initial conditions leads to a forecast of such evolution which seems to fit the experienced evolutions of western societies at present.

Key words. nonlinear ordinary differential equations, thermostatted framework, modeling, inverse problem.

Mathematics Subject Classification. 37A50, 34A55, 37N40, 91B74.

1. Introduction.

"Complexity" is a notion whose importance in modern science can be hardly overestimated, as it plays a fundamental rôle in almost all reasonably faithful descriptions of systems consisting of a large number of individuals, or particles. As is well-known, a "complex system" is a set of individuals whose pairwise interactions are more or less strongly influenced by surroundings individuals and their interactions [1, 2]. Moreover, these systems are characterized by what can be called "emergent collective behaviours", which can be described as resulting by the ability of individuals to *learn* from past interactions and to elaborate a common strategy [3, 4, 5, 6]. These two features have naturally led to a description of the behaviour of such systems in terms of probabilities [7, 8], based on the concepts and the methods of statistical mechanics and kinetic theory, and in particular on the use of Vlasov's generalization of the Boltzmann equation [9, 10, 11, 12, 13].

The study of complex systems seems to have an ever-growing importance in applications, since it is commonly acknowledged that most of the systems of interest for our everyday's life, from biological systems [14, 15, 16, 17, 18] and disease spreading [19, 20, 21, 22] to human psychology [23, 24], from growth of populations [25, 26] to socially and economically organized human collectivities [27, 28, 29, 30], actually behave like complex systems. And in particular, one of the most interesting applications of the above mentioned description, which — in view of the present period of economic turbulences and social conflicts involving even the most advanced nations — is likely to become soon of a vital importance, is just the study and the (at least stochastic) forecast of the behaviour of socio-economic systems, in order to achieve some reliable hints about the conditions assuring their stability and the possible causes of their instability. As we shall see in some more details in the final Section of this paper, at the moment the examples of socio-economic systems taken in consideration in the

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literature are quite simplified and treated in quite schematic terms (so that they are often referred to as "toy models"), but nevertheless they are quite useful to establish a general frame for the problem of *stability* of evolution [31, 32, 33, 34].

The present paper offers one more of such examples, trying to recover the most likely probability distributions for the interaction rates and the transition probabilites after interactions (see Section 2 and Section 4) and, in spite of the unavoidably simplified picture we give of the economic structure of human societies, to capture the essential features of commercial exchanges. Also, it seems of particular interest to adopt the so-called *thermostatted kinetic framework* [35]. A dissipative term, called *thermostat*, is considered sucht that some macroscopic quantities keeps constant furing the evolution of the system [36, 37]. Finally, a particular attention has been paid to the so-called "inverse problem" [38, 39, 40, 41], arising from the need to define an *interaction domain* for each particle. More precisely, in the thermostatted framework of kinetic theory each particle is in principle allowed to interact with all other particles. To avoid this rather unplausible and confusing circumstance, in the moments of the probability distribution function on the set of possible states of particles a *weight* function, modeling the interaction domain of each particle, is introduced [42]. The existence and uniqueness of the best weight function is not guaranteed, so that it requires a selection criterion, the maximum entropy principle of Jayne [43], based on the Shannon entropy [44, 45]. As the weight function allows to identify, among all the active particles, the ones that actually fall into the interaction domains of each other, and, as a consequence, to redefine the thermostatted framework, its selection is of the greatest interest for modeling applications. Accordingly, Section 3 is devoted to a discussion of the inverse problem and to its application to the case in which the space of microscopic states is discrete. This is the case which seems to us of particular interest in view of an effective application of the kinetic framework to the real socio-economic organization of present western societies, to describe and forecast their evolution. Accordingly, relying upon such a discrete picture, we propose in 4 an example of a simple socio-economic structure, which seems to be particularly fit to giving a likely portrait of present human collectivities. In this connection, Subsection 4.1 offers a preliminary attempt to draw — by means of simple numerical simulations — the evolution of such a system. And, as far as we can see, the forecast obtained seems to reproduce rather faithfully the socio-economic evolutions we are nowadays experiencing in almost all western countries.

2. The thermostatted framework. Let $\mathcal{C}$ be a *complex system* homogeneous with respect to the mechanical variables, i.e. *space* and *velocity*. The microscopic state of *particles* is described by a scalar variable u which takes values in a real discrete subset, i.e. $u \in I_u = \{u_1, u_2, \ldots, u_n\} \subseteq \mathbb{R}$. The system is divided into n *functional subsystems* [46] such that particles belonging to the same functional subsystem share the same strategy.

Let $f_i(t) := f(t, u_i) : [0, +\infty[\to \mathbb{R}^+$ be the *distribution function* of the ith functional subsystem, for $i \in \{1, 2, \ldots, n\}$, and $\mathbf{f}(t) = (f_1(t), f_2(t), \ldots, f_n(t))$ the *distribution function vector* of the overall system.

The evolution of the system is based on the definition of the *interaction rate* $\eta_{hk} : I_u \times I_u \to \mathbb{R}^+$ between the particle u_h and the particle u_k, and the *transition probability density* $B^i_{hk} := B(u_i, u_h, u_k) : I_u \times I_u \times I_u \to \mathbb{R}^+$ which models the probability that an active particle in the state u_h falls into the state u_i after interacting with a particle in the state u_k, with the property:

$$\sum_{i=1}^{n} B^i_{hk} = 1, \qquad \forall h, k \in \{1, 2, \dots, n\}. \tag{2.1}$$

The *distribution function* of the whole system can be written as follows:

$$f(t, u) = \sum_{i=1}^{n} f_i(t)\delta\left(u - u_i\right),$$

where δ denotes the Delta distribution function.

The *pth-order moment* of the system is:

$$\mathbb{E}_p\left[\mathbf{f}\right](t) = \sum_{i=1}^{n} u_i^p f_i(t) = \sum_{i=1}^{n} u_i^p f(t, u_i), \quad p \in \mathbb{N}.$$

The local density, the linear activity-momentum and the activation energy are obtained for $p = 0$, $p = 1$ and $p = 2$, respectively.

Bearing all what is stated above in mind, the *evolution equation* of the ith functional subsystem reads:

$$\frac{df_i}{dt} = J_i[\mathbf{f}](t) = G_i[\mathbf{f}](t) - L_i[\mathbf{f}](t), \quad i \in \{1, 2, \dots, n\},$$

where the operator $\mathbf{J}[\mathbf{f}] = \left(J_i[\mathbf{f}]\right)_i$ models the *conservative interactions* between the active particles. Specifically

$$\mathbf{G}[\mathbf{f}] = \left(G_i[\mathbf{f}]\right)_i = \left(\sum_{h=1}^{n}\sum_{k=1}^{n} \eta_{hk}\, B^i_{hk}\; f_h(t)\, f_k(t)\right)_i,$$

denotes the *gain-term* and

$$\mathbf{L}[\mathbf{f}] = \left(L_i[\mathbf{f}]\right)_i = \left(f_i(t)\sum_{k=1}^{n} \eta_{ik}\, f_k(t)\right)_i,$$

the *loss-term*. More precisely, $G_i[\mathbf{f}]$, for $i \in \{1, 2, \dots, n\}$, models the number of particles u_h that acquire the state u_i after interacting with particles u_k, whereas $L_i[\mathbf{f}]$, for $i \in \{1, 2, \dots, n\}$, models the number of particles that leave the state u_i. Basically, $G_i[\mathbf{f}] - L_i[\mathbf{f}]$, for $i \in \{1, 2, \dots, n\}$, is a *net-flux* related to the ith functional subsystem.

The system $\mathcal{C}$ is assumed to be subjected to the following *external force field*:

$$\mathbf{F}(t) = \left(F_1(t), F_2(t), \dots, F_n(t)\right) : [0, T] \to \left(\mathbb{R}^+\right)^n,$$

and a *discrete thermostat* [35] is introduced to in order to keep constant the p-th order moment of the system. Accordingly, the *thermostatted evolution equation of the ith functional subsystem* f_i now reads:

$$\frac{df_i}{dt} = J_i[\mathbf{f}] + F_i(t) - \alpha f_i(t), \quad i \in \{1, 2, \dots, n\},$$

where the term $\alpha f_i(t)$, called *thermostat term*, makes the evolution dissipative. Specifically, by straightforward calculations:

$$\alpha = \alpha\left(\mathbf{J}[\mathbf{f}], \mathbb{E}_p[\mathbf{f}], \mathbf{f}\right) = \sum_{i=1}^{n} \frac{u_i^p\left(J_i[\mathbf{f}] + F_i\right)}{\mathbb{E}_p[\mathbf{f}]}.$$

The related *Cauchy problem for the discrete thermostatted framework* reads:

$$\begin{cases} \dfrac{d\mathbf{f}}{dt} = \mathbf{J}[\mathbf{f}] + \mathbf{T_F}[\mathbf{f}] \quad t \in [0,T] \quad T > 0 \\ \\ \mathbf{f}(0,u) = \mathbf{f}^0(u), \end{cases} \tag{2.2}$$

where $\mathbf{f}^0$ denotes the *initial data* of the Cauchy problem and $\mathbf{T_F}[\mathbf{f}]$ is the operator:

$$\mathbf{T_F}[\mathbf{f}] = \mathbf{F} - \sum_{i=1}^{n} \left(\frac{u_i^p\left(J_i[\mathbf{f}] + F_i\right)}{\mathbb{E}_p[\mathbf{f}]}\right)\mathbf{f}.$$

The Cauchy problem (2.2) consists in a system of n nonlinear ordinary differential equations, with quadratic nonlinearity.

Let $\mathcal{R}_{\mathbf{f}}^p = \mathcal{R}_{\mathbf{f}}^p\left(\mathbb{R}^+; \mathbb{E}_p^0\right) = \left\{\mathbf{f} \in C\left([0,T]; (\mathbb{R}^+)^n\right) : \mathbb{E}_p[\mathbf{f}](t) = \mathbb{E}_p^0\right\}$, with $\mathbb{E}_p > 0$ fixed. The existence and uniqueness of a solution to the Cauchy problem (2.2) is obtained under suitable assumptions (see [35] and references therein). Furthermore the existence and uniqueness of the *nonequilibrium stationary solution* related to the problem (2.2) is proved in [47].

This paper deals with a socio-economic system modelled by the *discrete weighted thermostatted framework* [42].

It is assumed that a particle $u_h \in I_u$ has an *interaction domain* $I_{u_h} = \{u_{k_1}, u_{k_2}, \dots, u_{k_h}\} \subseteq I_u$, which contains the particles u_k which are able to interact with u_h. Let $w(t, u_h, u_k) := w_{hk}(t) : [0, +\infty[\times I_u \times I_{u_h} \to \mathbb{R}^+$ be a positive function that weights the interactions among the particles. The *weight* function w_{hk} is such that

$$\sum_{k=1}^{n} w_{hk}(t) = 1, \quad \forall h \in \{1, 2, \dots, n\}.$$

The *weighted pth-order moment* of the system at the time $t > 0$ is defined as:

$$\mathbb{E}_p^w[\mathbf{f}](t) = \sum_{h=1}^{n}\sum_{k=1}^{n} u_k^p\, w_{hk}(t)\, f_k(t), \tag{2.3}$$

where $\mathbb{E}_0^w[\mathbf{f}]$, $\mathbb{E}_1^w[\mathbf{f}]$ and $\mathbb{E}_2^w[\mathbf{f}]$ denote the *weighted density*, the *weighted linear momentum* and the *weighted activation energy*, respectively.

The *discrete thermostatted evolutin equation* for the ith functional subsystem, for $i \in \{1, 2, \dots, n\}$ and $t > 0$, reads:

$$\begin{aligned} \frac{df_i}{dt}(t) &= J_i[\mathbf{f}] + F_i(t) - \alpha f_i(t) \\ &= \left(G_i[\mathbf{f}](t) - L_i[\mathbf{f}](t)\right) + F_i(t) - \alpha f_i(t), \end{aligned}$$

where (see [46, 42]) α denotes the *thermostat term* which is introduced in order to keep constant the pth-order moment (2.3), $G_i[\mathbf{f}](t)$ denotes the following *gain-term operator*:

$$G_i[\mathbf{f}](t) = \sum_{h=1}^{n} \sum_{k=1}^{n} \eta_{hk}\, w_{hk}(t)\, B_{hk}^{i}\, f_h(t) f_k(t),$$

and $L_i[\mathbf{f}](t)$ the following *loss-term operator*:

$$L_i[\mathbf{f}](t) = f_i(t) \sum_{k=1}^{n} \eta_{ik}\, w_{ik}(t)\, f_k(t).$$

The thermostat term α is obtained by imposing the conservation of the weighted pth-order moment (2.3). Accordingly one has:

$$\begin{aligned}\frac{d}{dt}\left(\sum_{h=1}^{n}\sum_{k=1}^{n} u_k^p\, w_{hk}\,(t) f_k(t)\right) &= 0 \\ &= \sum_{h=1}^{n}\sum_{k=1}^{n} u_k^p (w'_{hk}(t) f_k(t) + w_{hk}(t) f'_k(t)) \\ &= \sum_{h=1}^{n}\sum_{k=1}^{n} u_k^p [w'_{hk}(t) f_k(t) + w_{hk}(t)(J_k[\mathbf{f}] + F_k(t) - \alpha f_k(t))],\end{aligned}$$

and then

$$\alpha(\mathbf{F}, \mathbf{J}, w, \mathbb{E}_p^w, p) = \sum_{h=1}^{n}\sum_{k=1}^{n} \left(\frac{u_k^p\, w_{hk}(t)\,(J_k[\mathbf{f}](t) + F_k(t)) + u_k^p\, w'_{hk}(t)\, f_k(t)}{\mathbb{E}_p^w[\mathbf{f}]} \right),$$

where $w(t) = (w_{hk}(t)) \in \mathbb{R}^{n,n}$.

In conclusion, the *weighted discrete thermostatted kinetic theory framework* reads:

$$\frac{df_i}{dt}(t) = J_i[\mathbf{f}] + F_i(t) - \sum_{h=1}^{n}\sum_{k=1}^{n} \left(\frac{u_k^p\, w_{hk}(t)\,(J_k[\mathbf{f}](t) + F_k(t)) + u_k^p\, w'_{hk}(t)\, f_k(t)}{\mathbb{E}_p^w[\mathbf{f}]} \right) f_i(t).$$

Let $\mathbf{f}^0 = (f_0^1, f_0^2, \ldots, f_n^0) \in \mathbb{R}^n$ be the initial data and $\mathbf{J}[\mathbf{f}](t) = (J_1[\mathbf{f}](t), J_2[\mathbf{f}](t), \ldots, J_n[\mathbf{f}](t))$. The existence and uniqueness of the solution of the related *Cauchy problem*

$$\begin{cases} \dfrac{d\mathbf{f}}{dt}(t) = \mathbf{J}[\mathbf{f}](t) + \mathbf{F}(t) - \alpha(\mathbf{F}, \mathbf{J}, w, \mathbb{E}_p^w, p)\, \mathbf{f}(t) \quad t \in [0, +\infty[\\ \\ \mathbf{f}(0) = \mathbf{f}^0 \end{cases} \tag{2.4}$$

can be proved as in [35] under the following assumptions:
A1 There exists $\eta > 0$ such that $\eta_{hk} \le \eta$, for $h, k \in \{1, 2, \ldots, n\}$;
A2 $F_i(t) \le F$, for $i \in \{1, 2, \ldots, n\}$;
A3 $u_h \ge 1$, for $h \in \{1, 2, \ldots, n\}$;
A4 $\mathbb{E}_p^w[\mathbf{f}_0](t) = \mathbb{E}_p^w$;
A5 $\sum_{k=1}^{n} w_{hk}(t) = 1$, $\forall h \in \{1, 2, \ldots, n\}$.

3. The inverse problem related to the discrete framework. As in [42], this paper deals with the case $p = 0$ of (2.4), where:

$$\rho_w[\mathbf{f}](t) := \mathbb{E}_0^w[\mathbf{f}](t) = \mathbb{E}_0^w[\mathbf{f}^0](t) = \sum_{h=1}^{n}\sum_{k=1}^{n} w_{hk}(t) f_k^0 = \mathbb{E}_0^w.$$

In [42], the *inverse problem* related to the discrete weighted thermostatted framework is defined. The density is the *known quantity*, whereas the weight functions w_{hk}, for $h, k\{1, 2, \ldots, n\}$, are the *unknown quantities*. In particular, the weights are assumed to have the following form:

$$w_{hk}(t) = \mu_k\, f_h(t), \quad \mu_k \in \mathbb{R}, \quad h, k \in \{1, 2, \ldots, n\}.$$

Two inverse problems are taken into account in [42]. First the *weighted global density case* is studied, and it writes:

$$\rho_w[\mathbf{f}_0](t) = \sum_{h=1}^{n}\sum_{k=1}^{n} \left(f_h(t) f_k(t)\right)\, \mu_k = \sum_{h=1}^{n}\sum_{k=1}^{n} K_{hk}(t)\, \mu_k, \tag{3.1}$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n$ and the *kernel* $\mathbf{K}[\mathbf{f}](t) = (K_{hk}(t)) \in \mathbb{R}^{n\times n}$ is the following matrix:

$$K_{hk}(t) := f_h(t)\, f_k(t).$$

The *Shannon entropy* related to (3.1) is, [43, 44, 45]:

$$H[\mu] = -\sum_{i=1}^{n} \mu_i\, ln\mu_i.$$

Then the inverse problem rewrites as the the following *optimization problem*:

$$\mu_H = \arg \max_{\mu\in\mathcal{H}(\mathbf{K},\rho_w)} H[\mu],$$

where

$$\mathcal{H}(\mathbf{K}, \rho_w) := \left\{ \mu \in \mathbb{R}^n : \rho_w[\mathbf{f}^0](t) = \sum_{h=1}^{n}\sum_{k=1}^{n} K_{hk}(t)\, \mu_k,\ \mu_k \geq 0,\ \sum_{k=1}^{n} \mu_k = 1 \right\}.$$

If it is assumed that $\mathbb{E}_0^w[\mathbf{f}](t) = n$ then

$$\sum_{k=1}^{n} \mu_k = 1.$$

Theorem 4.1 in [42] assures about existence and uniqueness of the solutions of (3.1) under suitable assumptions.

Furthermore in [42], the *weighted local density case* is presented. Specifically:

$$\mathbb{E}_p^w[\mathbf{f}](t) = \sum_{h=1}^{n}\sum_{k=1}^{n} u_k^p\, w_{hk}(t)\, f_k(t) = \sum_{h=1}^{n}\left(\sum_{k=1}^{n} u_k^p\, w_{hk}(t)\, f_k(t)\right)$$
$$= \sum_{h=1}^{n} \mathbb{E}_p^w[\mathbf{f}, u_h](t).$$

According to (3.1) the following inverse problem is defined:

$$\rho_w[\mathbf{f}, u_h](t) = \sum_{k=1}^{n} w_{hk}(t) f_k(t), \tag{3.2}$$

where $\rho_w[\mathbf{f}, u_h](t) := \mathbb{E}_0^w[\mathbf{f}, u_h](t)$, for $h \in \{1, 2, \dots, n\}$, is assumed to be known, and $w_{hk}(t)$, for $h, k \in \{1, 2, \dots, n\}$, are the unknown functions. Since $w_{hk}(t) = \mu_k f_h(t)$ and $\mu_k \in \mathbb{R}$, the inverse problem (3.2) thus rewrites:

$$\rho_w[\mathbf{f}, u_h](t) = \sum_{k=1}^{n} (f_h(t) f_k(t))\, \mu_k, \quad h \in \{1, 2, \dots, n\}. \tag{3.3}$$

Let $\rho_w[\mathbf{f}](t) = (\rho_w[\mathbf{f}, u_1](t), \rho_w[\mathbf{f}, u_2](t), \dots, \rho_w[\mathbf{f}, u_n](t)) \in \mathbb{R}^n$, then the inverse problem (3.3) can be written in the following vector form:

$$\rho_w[\mathbf{f}](t) = \mathbf{K}[\mathbf{f}](t)\, \mu, \tag{3.4}$$

where $\mu = (\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{R}^n$ and the *kernel* $\mathbf{K}[\mathbf{f}](t) = (K_{hk}(t)) \in \mathbb{R}^{n \times n}$ is the following matrix:

$$K_{hk}(t) := f_h(t)\, f_k(t).$$

If it is assumed that $\mathbb{E}_0^w[\mathbf{f}, u_h](t) = 1$ then

$$\sum_{k=1}^{n} \mu_k = 1.$$

If the set $\mathcal{F} = \{f_1(t), f_2(t), \dots, f_n(t)\}$ is composed of *linearly independent* functions, then the inverse problem (3.4) has, in virtue of Theorem 5.1 of [42], a unique solution $w_{hk}(t)$, for $h, k \in \{1, 2, \dots, n\}$ such that

$$w_{hk}(t) = \left(\sum_{l=1}^{n} K_{kl}^{-1}(t) \rho_w[\mathbf{f}, u_l](t) \right) f_h(t), \tag{3.5}$$

where $\mathbf{K}^{-1}[\mathbf{f}](t) = (K_{kl}^{-1}(t))$ denotes the inverse matrix of $\mathbf{K}[\mathbf{f}](t)$.

The set $\mathcal{F}$ may be composed of *linearly dependent* functions when some *singularities* occur. Then the matrix $\mathbf{K}[\mathbf{f}](t)$ is only positive semidefinite, so that the inverse matrix of $\mathbf{K}[\mathbf{f}](t)$ is not defined. Consequently, the inverse problem (3.4) is an ill-posed inverse problem. In this framework the *maximum entropy principle of Shannon* [43, 44, 45] is applied.

The inverse problem (3.4) thus consists in the following *optimization problem*:

$$\mu_H = \arg \max_{\mu \in \mathcal{H}(\mathbf{K}, \rho_w^l)} H[\mu],$$

where

$$\mathcal{H}(\mathbf{K}, \rho_w) := \left\{ \mu \in \mathbb{R}^n : \rho_w[\mathbf{f}](t) = \mathbf{K}[\mathbf{f}](t)\, \mu,\, \mu_k \geq 0,\, \sum_{k=1}^{n} \mu_k = 1 \right\}.$$

In this more general case, in virtue of Theorem 5.2 in [42], the solution w_{hk}, for $h, k \in \{1, 2, \ldots, n\}$, of the inverse problem (3.4) takes the form:

$$w_{hk}(t) = \frac{\exp\left(-f_k(t)\sum_{i=1}^{n}\lambda_i f_i(t)\right)}{\sum_{k=1}^{n}\exp\left(-f_k(t)\sum_{i=1}^{n}\lambda_i f_i(t)\right)} f_h(t), \tag{3.6}$$

where $\lambda \in \mathbb{R}^n_+$ is the solution of the following equation:

$$-\nabla_\lambda \ln Z[\mathbf{f}](\lambda, t) = \rho^l_w[\mathbf{f}](t),$$

and $Z[\mathbf{f}](\lambda, t)$ is the partition function defined as follows:

$$Z[\mathbf{f}](\lambda, t) = \sum_{k=1}^{n}\exp\left(-f_k(t)\sum_{i=1}^{n}\lambda_i f_i(t)\right).$$

4. A socio-economic system. This section deals with a particular mathematical picture of the *socio-economic system* modelled by framework (2.4). In this example, the particles are the *agents* of the system and the activity variable u has the specific meaning of *wealth*.

Let $n \geq 1$ be fixed in $\mathbb{N}$. The wealth is represented by the scalar variable u which attains its value in $\{1, 2, \ldots, n\}$. Therefore the previous assumption **A3** is satisfied. Thus the system is divided into n functional subsystems with respect to the wealth, each described by the related distribution function $f_i(t)$.

In this first step, the external force field $\mathbf{F}(t)$ acting on the system will be completely disregarded, *i.e.* we assume

$$\mathbf{F}(t) = \mathbf{0}, \quad \forall t > 0.$$

The parameters of the framework (2.4) need now to be defined. The interaction rate η_{hk}, for $h, k \in \{1, 2, \ldots, n\}$, which describes the fequency of interactions between the agent with wealth u_h and the agent with the wealth u_k, is assumed to have the following form:

$$\eta_{hk} = \eta_{max} \exp\left(-\gamma\, |h - k|\right), \tag{4.1}$$

where η_{max} and γ are positive fixed coefficients that depend on the particular considered socio-economic system. Specifically:

- The interactions between agents with "similar" wealth are more frequent.
- If $h = k$ then:

  $$\eta_{hh} = \eta_{max};$$

  this is related to the empirical assumption that the *self-interaction rate* is maximum.
- We assume $\gamma \geq 1$, because if $|h - k| \geq 1$, the related interaction rate should be very small. This is related to the empirical assumption that in a socio-economic system the interactions between agents with "similar" wealth, i.e. "small" $|h - k|$, are much more frequent than in the case in which they are far away from each other, i.e. when $|h - k|$ is "large".

The transition probability density B^i_{hk} describes the probability that the agent with wealth u_h "moves" to the wealth u_i after an interaction with the agent with wealth u_k. This parameter is assumed to have the following form:

$$B^i_{hk} = c_{ihk}\,\frac{1}{s}\,g\left(|h-i|\right), \quad i,h,k \in \{1,2,\dots,n\}, \tag{4.2}$$

where g is a non-increasing function of $|h-i|$ and s, and the parameters c_{ihk}, for $i,h,k \in \{1,2,\dots,n\}$, are positive real numbers, depending on the particular considered socio-economic system. In particular:

- The probability that an agent passes from a wealth state to another one is higher for "near" wealth states. This is related to the empirical assumption that a "very poor" agent can very hardly become "very rich".
- The coefficients c_{ihk} are chosen such that the relation (2.1) is satisfied, i.e.

$$1 = \sum_{i=1}^{n} B^i_{hk} = \sum_{i=1}^{n} c_{ihk}\,\frac{1}{s}\,g\left(|h-i|\right).$$

  Specifically, for $h = k$:

$$c_{ihh} := \begin{cases} 0 & i \neq h \\ \\ 1 & i = h, \end{cases} \tag{4.3}$$

  and,

$$\frac{1}{s}\,g\left(|h-i|\right) = \exp\left(-s\,|h-i|\right), \tag{4.4}$$

  so that

$$\begin{aligned}\sum_{i=1}^{n} B^i_{hh} &= \sum_{i=1}^{n} c_{ihh}\,\exp\left(-s\,|h-i|\right)\\ &= B^1_{hh} + B^2_{hh} + \cdots + B^{h-1}_{hh} + B^h_{hh} + B^{h+1}_{hh} + \cdots + B^n_{hh}\\ &= 0 + 0 + \cdots + 0 + 1 + 0 + \cdots + 0\\ &= 1.\end{aligned}$$

- The coefficient s is assumed to satisfy the condition $s \geq 1$. Indeed if $|h-i| \gg 1$, the related probability (i.e. the probability that the agent with wealth u_h acquires the wealth u_i after an interaction with the agent u_k) has to be "small". This is related to the empirical assumption that an agent u_h, after an interaction with another agent u_k, may pass more probably to a new wealth state u_i "near" to the initial wealth state u_h than to farther ones.

In order to complete the definition of the framework (2.4), the intial data $\mathbf{f}^0$ needs to be assigned, since the weights w_{hk}, for $h,k \in \{1,2,\dots,n\}$, are solutions of the inverse problem (3.4). In this scheme, the weighted local density $\sum_{k=1}^{n} w_{hk}\,f^0_k = \rho_{w,h}$, for $h \in \{1,2,\dots,n\}$, and the vector $\rho^l_w[\mathbf{f}](t) = (\rho_{w,1}, \rho_{w,2}, \dots, \rho_{w,n})$ are kept constant during the evolution.

In this first step the initial data $\mathbf{f}^0$ is the *binomial distribution*. In particular, for $i \in \{1,2,\dots,n\}$:

$$f^0_i = n \binom{n-1}{i-1} \left(\frac{1}{2}\right)^{n-1}. \tag{4.5}$$

Straightforward calculations yield:

$$\begin{aligned}\sum_{i=1}^{n} f_i^0 &= \sum_{i=1}^{n} n \binom{n-1}{i-1} \left(\frac{1}{2}\right)^{n-1} \\ &= n \sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{1}{2}\right)^{n-1} \\ &= n,\end{aligned}$$

then Theorem 5.1 and Theorem 5.2 [42] can be applied.

Then the Cauchy problem (2.4) related to the current socio-economic application can be written explicitly.

The local weighted density related to the initial data is, for $h \in \{1, 2, \ldots, n\}$:

$$\rho_{w,h}[\mathbf{f}^0] = \sum_{k=1}^{n} w_{hk}\, f_k^0.$$

Suppose that the local weighted density of the socio-economic system $\mathcal{C}$ is required to be constant during the evolution, i.e, for $h \in \{1, 2, \ldots, n\}$:

$$\rho_{w,h}[\mathbf{f}](t) = \rho_{w,h}[\mathbf{f}^0], \quad t > 0.$$

To this aim, the required weights w_{hk}, for $h, k \in \{1, 2, \ldots, n\}$, must be found from the following conditions:

(i) The weights have the form

$$w_{hk}(t) = \mu_k\, f_h(t),$$

for $h, k \in \{1, 2, \ldots, n\}$, where $\mu_k \in \mathbb{R}$, for $k \in \{1, 2, \ldots, n\}$;

(ii) The weights are solution to the inverse problem (3.4).

There exists only one solution of the framework (2.4), for a fixed initial data. Then two cases may occur: either the functions $f_i(t)$, for $i \in \{1, 2, \ldots, n\}$, are linearly independent or they are linearly dependent. In the former case the solution is of the form (3.5), i.e., for $h, k \in \{1, 2, \ldots, n\}$:

$$w_{hk}(t) = \left(\sum_{l=1}^{n} K_{kl}^{-1}(t) \rho_w[\mathbf{f}, u_l](t) \right) f_h(t).$$

In the latter cases, the solution is of the form (3.6), i.e., for $h, k \in \{1, 2, \ldots, n\}$:

$$w_{hk}(t) = \frac{\exp\left(-f_k(t) \sum_{i=1}^{n} \lambda_i f_i(t) \right)}{\sum_{k=1}^{n} \exp\left(-f_k(t) \sum_{i=1}^{n} \lambda_i f_i(t) \right)}\, f_h(t),$$

where $\lambda \in \mathbb{R}_+^n$ is the solution of the following equation:

$$-\nabla_\lambda \ln Z[\mathbf{f}](\lambda, t) = \rho_w^l[\mathbf{f}](t),$$

TABLE 1
The values of the transition probability B^i_{hk}

B^1_{13}=8/9	B^2_{13}=991/9000	B^3_{13}=1/1000
B^1_{23}=991/9000	B^2_{23}=8/9	B^3_{23}=1/1000
B^1_{31}=1/1000	B^2_{31}=991/9000	B^3_{31}=8/9
B^1_{32}=1/9	B^2_{32}=3/9	B^3_{32}=5/9
B^1_{12}=9999/10000	B^2_{12}=1/10000	B^3_{12}=0

TABLE 2
Lagrangian multipliers

γ	λ_1	λ_2	λ_3
1	35.9910	0.0982	0.8010
3	4.5348	-0.6961	0.9670
5	4.6503	0.4068	-1.1149
7	4.2404	2.1624	2.1485

and $Z[\mathbf{f}](\lambda, t)$ is the partition function defined as follows:

$$Z[\mathbf{f}](\lambda, t) = \sum_{k=1}^{n} \exp\left(-f_k(t)\sum_{i=1}^{n} \lambda_i f_i(t)\right).$$

Then:

$$-\nabla_\lambda \ln \sum_{k=1}^{n} \exp\left(-f_k(t)\sum_{i=1}^{n} \lambda_i f_i(t)\right) = \sum_{k=1}^{n} \left(f_i(t) f_k(t)\right) \mu_k. \tag{4.6}$$

4.1. Numerical analysis for $n = 3$. Let be $n = 3$, i.e. three functional subsystems are taken into account such that:

- f_1 is related to the functional subsystem consisting of *poor* people;
- f_2 is related to the functional subsystem called *middle-class*;
- f_3 is related to the functional subsystem consisting of *rich* people.

The parameter $\eta_{max} = 1$ (see (4.1)) is chosen. Four values of the parameter γ (4.1) are considered:

(i) $\gamma = 1$;
(ii) $\gamma = 3$;
(iii) $\gamma = 5$;
(iv) $\gamma = 7$.

It is worth stressing that if $\gamma_1 > \gamma_2$ then $\exp(-\gamma_2 |h-k|) > \exp(-\gamma_1 |h-k|)$, i.e. the interaction rate between the functional subsystems "rich" and "poor" is bigger in the case γ_2 than the case γ_1.

The transition probability density B^i_{hk} follows the previous scheme (4.2), (4.3) and (4.4); its values are listed in Table 1.

Bearing definition (4.5) in mind, the initial data is:

$$\mathbf{f}^0 = \left(\frac{3}{4}, \frac{3}{2}, \frac{3}{4}\right).$$

Using the Matlab routine *Ode45*, the solutions f_1, f_2, f_3 are obtained (see figure 1), by solving the system (2.4). Recently some results of stability with respect to the parameters of the system have been proved in [48].

Specifically, the three equations related to the three current functional subsystems read:

$$\frac{df_1}{dt}(t) = J_1[f_1, f_2, f_3](t) - \alpha\, f_1(t)$$
$$\frac{df_2}{dt}(t) = J_2[f_1, f_2, f_3](t) - \alpha\, f_2(t)$$
$$\frac{df_3}{dt}(t) = J_3[f_1, f_2, f_3](t) - \alpha\, f_3(t).$$

Bearing figure 1 in mind, the first functional subsystem, i.e. the "poor" part of the population, increases during the evolution of the system, according to the choice of the transition probability densities B^i_{hk} and the interaction rate γ, whereas the second functional subsystem, i.e. the "middle class", decreases with the same velocity. It is worth stressing that the third functional subsystem, i.e. the "rich" part of the population, does not move too much from the initial value, except in the first case $\gamma = 1$, when the interaction rates are larger.

The particular evolution of the system is related to the structure of the equations, which is related to the typical scheme of the "free market economy". Due to the rules on economic exchanges commonly applied in these systems, the middle class, represented by function f_2, decreases as the poor part increases. Since the rich part does not move too much from the initial value, the transition of many agents from the middle class to the poor class is quite evident.

Changing the value of the interaction rate γ, the *crossing point* between the percentages of middle class and poor class, i.e. between the functions f_1 and f_2, moves along the time-line (see figure 1). This means that the parameters of the systems, i.e. the interaction rate η and the transition probability density B^i_{hk}, does not change the behaviour of the functions and of the overall socio-economic system, but only its "speed".

The rich part of the population, represented by the function f_3, does not move too much from the initial data. This means that:

- People from the poor part or the middle class move rarely into the rich part;
- Rich people remain in their class.

Finally, from the viewpoint of the rich part of the society, the assumed mobility due to the free-market system is severely limited.

In order to find out the weights w_{hk}, the entropy method has to be used since the matrix $\mathbf{K}$ is *ill-conditioned*, then the method (3.5) fails. As a matter of fact, after some iterations of the Matlab routine for the inverse matrix, the instability predominates and the values of the weights are completely ill-conditioned.

Then the entropy method (3.6) is used in order to find the weights w_{hk}. Indeed, by using the Matlab routine *fsolve*, the nonlinear equation (4.6) is solved and the langrangian multipliers λ_1, λ_2 and λ_3 are obtained (see Table 2), depending on the values of the parameter γ.

5. Conclusions and research perspectives. The results exposed in the previous Sections, mainly in view of the example shown in Section 4 and of the simple numerical simulations proposed in Section 4.1, give rather precise hints about the possible development of research about the evolution of complex systems, with particular reference to socio-economic collectivities. We know the way in which the Cauchy problem for the discrete thermostatted framework can be set and solved under assumptions that seem to be rather plausible in view of its application to these systems,

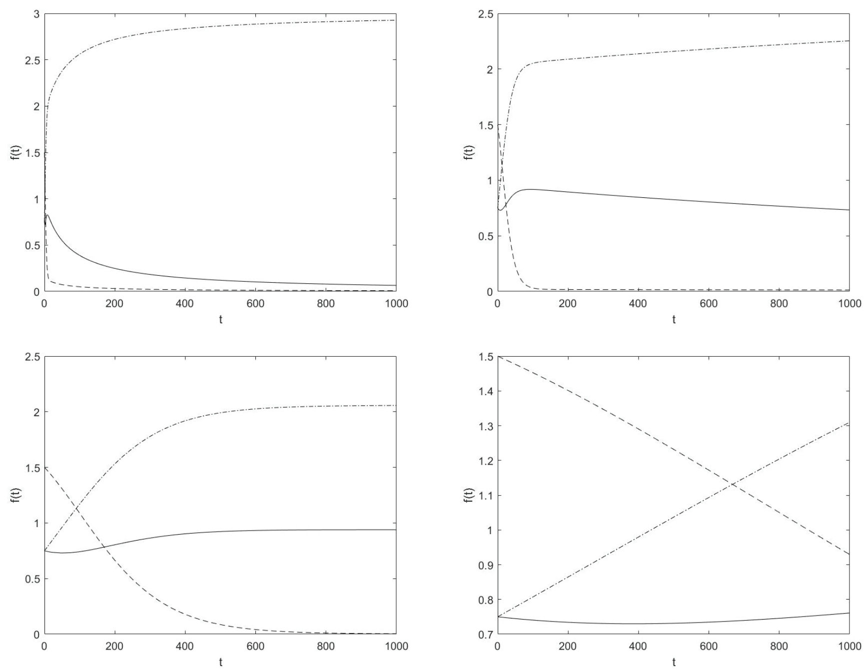

FIG. 1. *From top left to bottom right* $\gamma = 1$, $\gamma = 3$, $\gamma = 5$, $\gamma = 7$. f_1 *(poor class) dot-dashed,* f_2 *(middle-class) dashed,* f_3 *(rich class) full.*

and know how to set the inverse problem associated to our framework. But we have also found much more. The example offered in Section 4, though still too simple to capture all the relevant features of real socio-economic systems, is nevertheless able to point out some of their important aspects: in particular, it starts from a distribution which depicts a society in which wealth is sufficiently shared, and the “extreme” classes (that of poor people and that of the wealthy) contain few individuals compared with the “middle” classes. And the simple simulations performed in Section 4.1 shows that the adoption of a strict “market logic”, according to which only private exchanges between individuals with a similar wealth take place, leads to a “draining” of the middle class toward the lowest one (while the highest class remains almost unchanged).

In view of these results, the perspectives for future reseach are quite clear. Two ways are to be followed: from a purely theoretical viewpoint, to examine the rôle of thermostat term, and the study of assumptions under which the inverse problem can be solved; but, what is even more important, the way toward effective application to real socio-economic problems of the present time requires a deeper analysis of the models and more complete simulations. In this connection, it would be of the greatest interest to replace the scalar variable representing wealth with a vector variable (recently introduced in [49]) taking into account *different kinds* of wealth as well as different aspects of what could be called “the social position”. On the other hand, the formulation of this type of models would require a side statistical research to determine the most realistic forms for the initial distributions, the interaction rates and the transition probabilities.

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