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INVERSION OF A NON-UNIFORM DIFFERENCE OPERATOR AND A STRATEGY FOR NASH-MOSER*

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Abstract. We consider the problem of inverting the linear difference operator $\Delta_\Phi[v] = v \circ \Phi - v$ and obtaining bounds for the inverse operator, where Φ is a non-uniform shift on the circle. This represents the scalar version of a linearized difference operator arising in the construction of periodic solutions to the compressible Euler equations by Nash-Moser methods. We characterize the degeneracies in the linearized operators, thereby describing the complications that can arise in the application of Nash-Moser iteration to quasilinear problems. There are two cases, resonant and non-resonant, which correspond to the rationality or irrationality of the rotation number of Φ, respectively. We introduce a solvability condition which characterizes the range of the difference operator, and obtain uniform bounds for the inverse operator Δ_Φ^{-1} on this range in both cases, but our bounds are not immediately expressible in terms of standard C^r or Sobolev norms. In the resonant case, the bound is in terms of the inverse width of "Arnold tongues". In the non-resonant case the solvability condition simplifies and we translate our estimate into uniform estimates on Sobolev norms with a uniform loss of derivatives, as required for the Nash-Moser method. Our analysis is based on the introduction of the *"ergodic norm"*, which in addition provides an effective rate of convergence in the classical ergodic theorem.

Key words. Nash-Moser, compressible Euler equations, ergodic theorem, Arnold tongues.

Mathematics Subject Classification. 35L65, 37K55.

1. Introduction.

We develop estimates for inverting non-uniform difference operators which arise in authors' program to prove the existence of periodic solutions of the compressible Euler equations. To identify the difficulties that can arise, we address the general case of a non-uniform difference operator on the circle with an arbitrary constant *drift* term. When the operator has sufficiently large drift it has no fixed points, and this general case more accurately addresses the difficulties in the Euler problem than the zero drift case which we addressed in our earlier paper [16]. We consider smooth difference maps on the circle defined by a shift $\Phi(t) = t + \theta + \alpha\phi(t)$. For each such Φ, we define the shift operator by $\mathcal{S}_\Phi[v] = v \circ \Phi$, and the corresponding difference operator $\Delta_\Phi = \mathcal{S}_\Phi - \mathcal{I}$. Here θ is the drift, α the amplitude, and ϕ is the perturbation from pure rotation, assumed to be any C^2 periodic function with non-degenerate zeros, the prototype being $\phi(t) = \sin t + O(\alpha)$, motivated by [11, 14, 15]. The problem of periodic solutions to compressible Euler by Nash-Moser methods leads to the problem of inverting linearized difference operators which are vectorized versions of Δ_Φ.

To begin, we identify a solvability condition for both resonant and non-resonant θ, which provides a necessary and sufficient condition for the existence of an inverse for Δ_Φ. We prove that in the resonant case of rational Poincare rotation number $\rho(\Phi)$, the norm of Δ_Φ^{-1} is on the order of the inverse width of the *"Arnold Tongue"*, a mathematical construct whose width is difficult to estimate. On the other hand, for irrational rotation numbers, we establish that the solvability condition can be expressed in terms of an ergodic average. We then introduce what we call the *"ergodic*

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norm", a new norm in terms of which the difference operator becomes an isometry. The problem then is to estimate the ergodic norm in terms of the classical Sobolev and Hölder norms. We accomplish this by conjugating the shift map Φ to a pure rotation R_ρ. This is based on the conjugation developed in Herman's fundamental paper [8] on circle maps, and we use this to connect the subject of circle maps to our problem of estimating the ergodic norm. This yields bounds for the inverse operator Δ_Φ^{-1} with a loss of derivatives depending on how far the rotation number is from rational. This is precisely the loss of derivatives estimate required to apply a Nash-Moser version of Newton's method. For completeness we record relevant theorems from Herman's paper in the Appendix.

The difference operator Δ_Φ here is based on a nonlinear Burgers model which has no non-trivial kernel. Because of its elemental nature, we believe that this problem is of fundamental importance in its own right. However, we believe that the essential difficulties dealt with in inverting the linearized operator Δ_Φ also accurately reflect essential structural issues which apply in the context of the compressible Euler equations – the case when there is a nontrivial kernel in the nonlinear problem. This analysis suggests a new strategy for expunging parameters in Nash-Moser iterations for quasilinear problems. The idea is that, for fixed ϕ, the Poincare rotation number $\rho(\theta)$ is a Cantor function which depends continuously and monotonically on θ, reflecting the complicated nature of the set of "bad" resonant and near resonant drift angles θ on which Δ_Φ^{-1} is not uniformly bounded. Indeed, the Arnold Tongues, intervals of positive measure in θ where the periodic structure is maintained, get mapped to *rational* rotation numbers, a set of measure zero in ρ. Our new strategy for Nash-Moser, then, is to fix the *rotation number* ρ at the start of a Newton iteration, with the idea to *solve* for the parameter θ which gives that rotation number ρ at each step of Nash-Moser, this being accomplished by appropriate choice of the constant state, a free parameter in the Euler problem. By this method one would obtain the uniform loss of derivatives associated with any fixed rotation number ρ at each Newton step, the result being to effectively expunge a Cantor set of bad values of θ by means of the controllable parameter ρ.

We comment that Nash-Moser methods have been successful in obtaining periodic solutions to semi-linear problems, [3], but the methods seem not to have been successfully applied to quasi-linear problems like compressible Euler. The difference between semi-linear and quasi-linear is that in the semi-linear problem there is a fixed set of characteristics, but in quasi-linear problems the characteristics are different in each linearized shift operator whose inverse must be estimated at each step of the Newton method [3, 15]. Our paper here thus addresses the essential problem of inverting difference operators based on linearized shift operators with bounds uniform in amplitude α, i.e., uniform over the characteristic fields on which they are based. In this sense, the operator Δ_Φ represents the simplest example of the linearized operators which emerge in quasi-linear problems.

In authors' prior work on the problem of constructing periodic solutions of the Compressible Euler Equations by Nash-Moser Newton methods, we realized that the "wall" in these methods is the lack of estimates for the inverses of operators which impose periodicity conditions in terms of "shift operators". The simplest of these is the difference operator $\Delta_\Phi[u] = \mathcal{S}_\Phi[u] - u$, where $\mathcal{S}_\Phi[u]$ is the time one map of the leading order part of the linearization of Burger's equation $u_t + uu_x = 0$ about some fixed periodic solution, [14]. Our program now is to understand the problem of inverting such linear shift operators by isolating phenomena in models simpler than compressible Euler, and we propose that these simple models will uncover the hidden difficulties for Nash-Moser problems involving shifts in more complicated settings. Our starting point is [16] which describes the inverses of difference operators Δ_Φ with vanishing drift $\theta = 0$, and here we address the problem of incorporating a constant drift term $\theta \neq 0$. Somewhat surprisingly, the problem of inverting shift operators with non-zero drift presents a mathematical landscape which is much richer than the case of zero drift resolved in [16].

In the problem without drift, the fixed points of the shift Φ determine fundamental intervals for the difference operator Δ_Φ, and estimates for the norms of the inverses of difference operators are determined by the character of the shift operator at the fixed points. When the drift is non-zero, there are two cases, depending on whether Φ has periodic orbits or not, distinguished by the rationality of the Poincare rotation number $\rho(\Phi)$. When the rotation number is rational, $\rho = 2\pi p/q$, then Φ has periodic orbits of order q. In this case we show that the problem of inverting Δ_Φ can be reduced to our earlier case [16] by viewing the periodic points as fixed points of Φ^q. The set of drift angles θ which produce periodic orbits are known as *Arnold tongues*, and form a set of positive measure when $\alpha > 0$. Our estimate is essentially that the inverse Δ_Φ^{-1} is bounded by the inverse size of the corresponding Arnold tongue. This width is the subject of the Arnold conjecture and we are unable to get rigorous uniform estimates in the resonant case of rational rotation number. We do not know of bounds strong enough to control the rate at which $\|\Delta_\Phi^{-1}\|$ blows up as $\alpha \to 0$, which would pose problems for the Nash-Moser method.

On the other hand, we prove that the norm of the inverses for drifts with irrational rotation number are always uniformly bounded in what we call the *"ergodic norm"*, a norm constructed in terms of ergodic sums. In fact the inverse Δ_Φ^{-1} is an isometry between the ergodic norm of the data and the Lipshitz norm of the solution. We then use ergodic theory to bound the ergodic norm in terms of Sobolev norms with a finite loss of derivative depending on how far the irrational rotation number ρ is from the rationals. From this we obtain a uniform bound on the inverse Δ_Φ^{-1} with a uniform loss of derivative, off a set E_ϵ of arbitrarily small Lebesgue measure $\mu(E_\epsilon) = \epsilon$ containing the rationals, but the bound on Δ_Φ^{-1} tends to infinity as $\epsilon \to 0$. This is precisely the type of estimate utilized by the Nash-Moser method.

To state our results precisely, define

$$\Delta_\Phi v := \mathcal{S}_\Phi v - v = w, \tag{1}$$

where $\mathcal{S}_\Phi$ is defined by

$$\mathcal{S}_\Phi v = v \circ \Phi, \quad \text{so that} \quad \mathcal{S}_\Phi v(t) = v(\Phi(t)), \tag{2}$$

and $\Phi(t)$ has the form

$$\Phi(t) = t + \theta + \alpha\phi(t). \tag{3}$$

Again, we call $\mathcal{S}_\Phi$ the shift operator, ϕ the perturbation, α the amplitude and $\theta \in (0, 2\pi]$ the drift. Assume ϕ is C^2 continuous, has at least two isolated zeros, and has non-zero derivative at any zero. The prototypical problem from [14] is the case $\phi(t) = \sin t + O(\alpha)$.

Recall the following facts from the theory of circle maps $\Phi : \mathbb{S}^1 \to \mathbb{S}^1$. The translation number of the lift (3) (as a map $\mathbb{R} \to \mathbb{R}$) is the average shift, given by

$$\tau(\Phi) = \lim_{n\to\infty} \frac{\Phi^n(t) - t}{n},$$

which is independent of t for C^1 functions Φ, and the rotation number is then given by

$$\rho(\Phi) = \tau(\Phi) \pmod{2\pi},$$

representing the average rotation of the map Φ around the circle [9, 8, 4]. For fixed C^1 perturbation ϕ, and for

$$\alpha < 1/\|\phi'\|_\infty, \tag{4}$$

the translation number $\tau(\Phi)$ is a continuous monotone function of the drift θ, so the rotation is locally continuous and monotone as a function of θ [9, 4]; for this paper we always assume that (4) holds. The rotation number $\rho = \rho(\Phi)$ depends on α, ϕ and θ, but when α and ϕ are fixed we refer to it as $\rho(\theta)$.

The ergodic properties of Φ vary depending on the rationality of rescaled rotation number $\rho(\Phi)/2\pi$. In particular, assuming (4), if $\rho/2\pi \in \mathbb{Q}$, then Φ has periodic orbits, while if $\rho/2\pi \notin \mathbb{Q}$, all orbits are dense and there is a unique invariant measure μ for Φ, satisfying $\mu(\Phi E) = \mu(E)$ for all intervals E. We get uniform bounds by restricting ρ to irrationals satisfying a Diophantine condition of order r, namely

$$\left|\frac{\rho}{2\pi} - \frac{p}{q}\right| \geqslant \frac{C(\rho)}{q^r}, \tag{5}$$

for any integers p and $q > 0$.

In the resonant case of rational rotation number, the inverse Δ_Φ^{-1} in the Lipschitz norm satisfies

$$\|\Delta_\Phi^{-1}\|_{Lip} \leqslant \frac{O(1)}{\delta_{\Phi^q}}, \tag{6}$$

where δ_{Φ^q} is the width of the Arnold tongue corresponding to rotation number $\rho(\Phi) = 2\pi p/q$. Because of the difficulties of estimating δ_{Φ^q} and enforcing the corresponding solvability condition, we conclude that this estimate is not well-suited to the Nash-Moser iteration. We suggest that this identifies the essential difficulties encountered in applying Nash-Moser to periodic solutions of quasi-linear problems. Thus we focus our attention on the non-resonant case of irrational rotation number.

In the non-resonant case, because the rotation number is irrational, all orbits are dense and our solvability condition for the inverse Δ_Φ^{-1} simplifies dramatically due to the ergodicity of each orbit. Our main theorem (Theorem 5, stated and proved below) states that in this case Δ_Φ is invertible with finite loss of derivatives as measured in classical function spaces, estimates perfectly suited to Nash-Moser methods. We let $W^{k,p}$ denote the usual Sobolev space of 2π-periodic functions whose k-th weak derivative is in L^p, and let $C^{r,\nu}$ denote the space of 2π-periodic functions whose r-th derivative is Hölder continuous of order ν [6].

THEOREM 1. *Let Φ be a fixed shift of the form* (3), *whose rotation number $\rho = \rho(\Phi)$ is an irrational number which satisfies the Diophantine condition* (5) *of order r, and assume w satisfies the solvability condition*

$$\int_0^{2\pi} w \, d\mu = 0, \tag{7}$$

where μ is the invariant measure of Φ. Then, for $1/2 < \nu \leqslant 1$, $p \geqslant 2$, and integers $\ell \geqslant 1$ and $\ell' = \ell + 1$, there exists a 2π-periodic function v, unique to within a constant, which solves $\Delta_\Phi[v] = w$, with estimates

$$\|v\|_{C^{\ell,\nu}} \leqslant K \, \|w\|_{C^{\ell+r,\nu}}, \quad \text{or} \quad \|v\|_{W^{\ell',p}} \leqslant K \, \|w\|_{W^{\ell'+r,p}}. \tag{8}$$

That is, we have uniform bounds on Δ_Φ^{-1} with a uniform loss of r derivatives.

For the proof, given in Theorem 5 below, we introduce what we call the "ergodic norm" $\|\!|\cdot|\!\|$, defined as follows. Let $\mathcal{V}$ denote the set of functions Lipschitz continuous on the circle, let $\mathcal{V}_0$ denote the subset of $\mathcal{V}$ satisfying (7), and define

$$|\!|\!|w|\!|\!| := \sup_{t_0,k} \frac{\left|\sum_{j=0}^{k-1} w(t_j)\right|}{|t_k - t_0|}, \quad \text{and} \quad \mathcal{W} := \Big\{ w \in \mathcal{V} \mid |\!|\!|w|\!|\!| < \infty \Big\}, \tag{9}$$

where $t_j := \Phi^j t_0$. The ergodic norm precisely characterizes the bound on the inverse Δ_Φ^{-1} as stated in the following lemma.

Lemma 1. *The difference operator $\Delta_\Phi = \mathcal{S}_\Phi - \mathcal{I} : \mathcal{V} \to \mathcal{V}_0$ is invertible on the subset $\mathcal{W} \subset \mathcal{V}$, in the following sense: Given $w \in \mathcal{W}$, there is a unique $v \in \mathcal{V}_0$, denoted $\Delta_\Phi^{-1} w$, such that $\Delta_\Phi v = w$, and moreover, this inverse is bounded in the operator norm,*

$$\|\Delta_\Phi^{-1}\|_{\mathcal{W} \to \mathcal{V}_0} = \sup_{w \in \mathcal{W}} \frac{\|\Delta_\Phi^{-1} w\|_{Lip}}{|\!|\!|w|\!|\!|} = 1.$$

Interestingly, the ergodic norm introduced here provides an effective estimate for the rate of convergence of ergodic averages in the classical ergodic theorem, as stated in the following corollary.

Corollary 1. *If $|\!|\!|w|\!|\!| < \infty$, then for every $k \in \mathbb{Z}$, we have*

$$\left| \frac{1}{k} \sum_{j=0}^{k-1} w(t_j) - \frac{1}{2\pi} \int w \, d\mu \right| \leqslant |\!|\!|w|\!|\!| \, \left| \frac{1}{k} (t_k - t_0) \right|.$$

Lemma 1 shows that the inverse Δ_Φ^{-1} is an isometry between the the ergodic norm on $\mathcal{W}$ and the Lipschitz norm on $\mathcal{V}_0$ for each irrational rotation number $\rho \in 2\pi\mathbb{Q}^c$. The main issue is that there is no uniform bound on the ergodic norm in terms of classical Sobolev norms, with a finite loss of derivatives. In Corollary 5 below, we give a counterexample demonstrating that no such bound exists uniformly for irrational ρ. Our idea to prove Theorem 1 is to use the ergodic theory of circle maps to get a uniform estimate by restricting ρ to those values which satisfy a Diophantine condition (5). By this method we are able to identify and remove a Cantor-like set of near-resonant values of θ on which Δ_Φ^{-1} is not uniformly bounded. That is, by expunging those ρ which do not satisfy a Diophantine condition, and then solving for θ, we characterize the desired set of drift values θ for which the norm of the inverse Δ_Φ^{-1} meets a uniform bound in Sobolev norms, with a uniform loss of derivatives. We expect that this estimate will apply to Nash-Moser iterations for quasi-linear problems because of the freedom to choose the constant state, as explained above.

The paper is laid out as follows: in Section 2, we recall the results from our previous paper [16], which is the case of zero drift $\theta = 0$. We also show that the resonant case can be reduced to this case by factoring the difference operator corresponding to finite powers of the shift Φ. In Section 3, we obtain a bound for the inverse in the resonant case subject to a solvability condition. Our inverse in this case is estimated in terms of the inverse width of the corresponding Arnold tongue. Obtaining uniform estimates is problematic in this case for two reasons: it is difficult to interpret the solvability condition, and estimates for the inverse widths of Arnold tongues are not currently available. In Section 4, we consider the case of irrational rotation number, and introduce the ergodic norm, in terms of which we characterize the solution of (1). The solvability condition, which describes the range, reduces to the simple mean zero condition (7), and we use the ergodic theory of circle maps to describe our bounds in terms of finite loss of derivative in Sobolev norms. In Section 5 we discuss our proposed strategy for applying these estimates in a Nash-Moser iteration. In a brief appendix we list relevant results from the theory of circle maps.

2. The case of zero drift. In our previous paper [16] we considered the case of zero drift $\theta = 0$, and we now recall the relevant definitions and results from that paper. We proved that the difference operator Δ_Φ is invertible on its range, with bound

$$\|\Delta_\Phi^{-1}\| \leqslant \frac{K_0}{\alpha}, \quad \text{where} \quad \Delta_\Phi = \mathcal{S}_\Phi - \mathcal{I}, \tag{10}$$

and where the constant K_0 depends only on the perturbation ϕ and not on the amplitude α in the shift

$$\Phi(t) = t + \alpha\phi(t). \tag{11}$$

We begin by recalling the main theorem from [16]. The starting assumption is that $\phi(t)$ is Lipschitz continuous, has at least two zeroes, and near any zero t_* of ϕ we have the Taylor estimate

$$\phi(t) = \phi'(t_*)\,(t - t_*) + O\big((t - t_*)^2\big) \quad \text{with} \quad \phi'(t_*) \neq 0. \tag{12}$$

From this assumption, we identified a solvability condition that deteremines when the target w is in the range of Δ_Φ, namely

$$\sum_{k\in\mathbb{Z}} w(\Phi^k t) = B, \tag{13}$$

B being some constant, for any t between two roots of ϕ. In particular, continuity implies that $w(t_*) = 0$ at any root of ϕ, and this in turn implies convergence of the series in (13). The constant B is the total growth of the solution v over the interval,

$$v(t_{+\infty}) - v(t_{-\infty}) = B. \tag{14}$$

In particular, if ϕ has exactly two zeroes, then these split the circle into the non-overlapping intervals $\{\phi > 0\}$ and $\{\phi < 0\}$, and the roots $t_{-\infty}$ and $t_{+\infty}$ such that $\phi'(t_{-\infty}) > 0$ and $\phi'(t_{+\infty}) < 0$ are the unstable and stable equilibria on each interval separately. This implies that the constant B of (14) is the same for both intervals $\{\phi > 0\}$ and $\{\phi < 0\}$, so (13) must hold for each nontrivial orbit of Φ, with the same constant B.

If w satisfies the solvability conditions (13), (14), then we obtained the explicit formula for the solution v of the equation

$$\Delta_\Phi v = \big(\mathcal{S}_\Phi - \mathcal{I}\big)v = w, \quad \text{so that} \quad v(\Phi t) - v(t) = w(t).$$

The solution v is uniquely determined up to constant $v_{-\infty}$ by the explicit formulae

$$v(t_{-\infty}) = v_{-\infty}, \quad v(t_{+\infty}) = v_{-\infty} + B, \quad \text{and}$$
$$v(t) = v_{-\infty} + \sum_{n<0} w(\Phi^n t), \quad t \neq t_{\pm\infty}. \tag{15}$$

The main theorem from [16], which provides bounds for the solution, can be stated as follows.

THEOREM 2. *For shift Φ given by* (11), *there exist constants α_ϕ and K_ϕ, depending only on ϕ, such that, if $\alpha < \alpha_\phi$ and $w \in C^{0,1}[t_{-\infty}, t_{+\infty}]$ satisfies $w(t_{-\infty}) = w(t_{+\infty}) = 0$ together with the solvability condition* (13), *then the equation*

$$\Delta_\Phi v(t) = v(\Phi t) - v(t) = w(t)$$

has a solution $v \in C^{0,1}[t_{-\infty}, t_{+\infty}]$, uniquely determined up to constant, which satisfies

$$\|v\|_{Lip} \leqslant K_{\alpha\phi} \, \|w\|_{Lip}, \tag{16}$$

so

$$\|\Delta_\Phi^{-1}\|_{Lip} \leqslant K_{\alpha\phi}, \tag{17}$$

where

$$K_{\alpha\phi} = \frac{O(1)}{\|\alpha\phi\|_{Sup}}. \tag{18}$$

We proved the theorem for several norms, each of which must dominate the Lipschitz norm $\|\cdot\|_{Lip}$. In particular, the result holds for any C^p norm, provided $\phi \in C^{p+1}$, and it would also follow for general H^s norms, provided s is large enough.

Recall that we proved the theorem by restricting the equation to the intervals cut out by the zeroes of ϕ, and our solvability condition and solution depend on the fact that the shift Φ defines a monotone discrete dynamical system on each such interval.

Note that, starting with $\phi_0(t) \in C^2$ satisfying our assumptions, we can perturb by a nonzero drift $\alpha\,\gamma$ for any constant γ such that

$$\min \phi_0 < \underline{\phi}_0 \leqslant \gamma \leqslant \overline{\phi}_0 < \max \phi_0,$$

without affecting the result of the theorem. Thus although our result is stated for no drift, this actually includes the case of small drift. We state this as a corollary.

COROLLARY 2. *Suppose that the perturbation ϕ_0 is C^2, has only two zeros, and changes sign at both of these. Then, given a compact interval $\Gamma \subseteq (\min\phi_0, \max\phi_0)$, there is a constant K_Γ such that for any $\gamma \in \Gamma$, the operator*

$$\Delta_\Phi = \mathcal{S}_\Phi - \mathcal{I}, \quad \textit{where} \quad \Phi(t) = t + \alpha\,(\phi_0(t) - \gamma),$$

is invertible on its range, with

$$\|v\| \leqslant \frac{K_\Gamma}{\alpha}\,\|w\|, \quad \textit{so that} \quad \|\Delta_\Phi^{-1}\| \leqslant \frac{K_\Gamma}{\alpha}.$$

Note that because we have assumed only two roots, the two intervals, say $[t_{-\infty}, t_{+\infty}]$ and $[t_{+\infty}, t_{-\infty} + 2\pi]$, form a non-overlapping cover of the whole circle, with $\phi(\cdot)-\gamma>0$ and $\phi(\cdot)-\gamma<0$ on $(t_{-\infty},t_{+\infty})$ and $(t_{+\infty},t_{-\infty}+2\pi)$, respectively. It follows that the forward and backward limits are the same for both intervals modulo 2π, and that

$$t_{+\infty}=\lim_{k\to\infty}\Phi^k t \quad \text{and} \quad t_{-\infty}=\lim_{k\to-\infty}\Phi^k t$$

coincide for all points $t\neq t_{\pm\infty}$ on the circle. In particular, imposing the solvability condition on a single w requires that (13) hold for each point $t\neq t_{\pm\infty}$ (with the same constant). Because the roots $t_{\pm\infty}$ of Φ depend on α and γ, the actual range of Δ_Φ, and corresponding solvability condition, changes in a highly nonlinear and unpredictable way. Even though we know of no procedure to satisfy the solvability condition, our proof only requires that Φ have fixed points, and the conclusion continues to hold.

The lift of the shift *function* $\Phi:\mathbb{R}\to\mathbb{R}$ is nonlinear and 2π-periodic, so it projects to a circle map $\mathbb{S}^1\to\mathbb{S}^1$, where $\mathbb{S}^1$ denotes the unit circle parameterized by angle $\theta\in[0,2\pi)$. Our function space X is some Banach space of periodic functions, or functions on the circle, $X=\{u:\mathbb{S}^1\to\mathbb{R}\}$ with appropriate norm, say C^p or H^s norm. Given 2π-periodic $\Phi:\mathbb{R}\to\mathbb{R}$, we define the shift *operator*

$$\mathcal{S}_\Phi : X\to X \quad \text{by} \quad \mathcal{S}_\Phi(u)=u\circ\Phi, \quad \text{so that} \quad \mathcal{S}_\Phi(u)(t)=u(\Phi t),$$

which is linear and bounded on X. At this point we have not made any assumptions beyond smoothness on Φ.

Next, given two shift functions Φ_1 and Φ_2, our shift operators satisfy

$$\mathcal{S}_{\Phi_1}\big[\mathcal{S}_{\Phi_2}u\big]=\mathcal{S}_{\Phi_1}\big[u\circ\Phi_2\big]=u\circ\Phi_2\circ\Phi_1=\mathcal{S}_{\Phi_2\circ\Phi_1}u,$$

so that $\mathcal{S}_{\Phi_1}\mathcal{S}_{\Phi_2}=\mathcal{S}_{\Phi_2\circ\Phi_1}$, and in particular

$$(\mathcal{S}_\Phi)^k=\mathcal{S}_{\Phi^k},$$

for $k\in\mathbb{N}$. Moreover, for Φ invertible, which is always the case for us, this last expression holds for all integers $k\in\mathbb{Z}$.

The above identities express the fact that the shift operators on X form an algebra (non-commutative with identity), and in particular we can make sense of the partial Neumann series,

$$\mathcal{S}_{\Phi^q}-\mathcal{I}=\big(\mathcal{S}_\Phi-\mathcal{I}\big)\big(\mathcal{I}+\mathcal{S}_\Phi+\cdots+\mathcal{S}_{\Phi^{q-1}}\big), \tag{19}$$

and these last two factors commute. In the next section we will use this factorization in the resonant case to reduce the case of nonzero drift to that of zero drift, by showing that some power of Φ has a fixed point when the winding number $\rho(\Phi)$ is rational.

3. Resonant Case. In this section we address the problem of estimating the inverse of Δ_Φ in (1), in the case when the rotation number is rational, $\rho=2\pi p/q$, taken in lowest terms. To start, we show that for such ρ, Φ^q has a fixed point, and so our problem reduces to the previous case of no drift. From this we deduce a solvability condition which characterizes the domain and range of (1) when Φ is replaced by $\Psi=\Phi^q$. An argument based on (19) together with the zero drift case [16] shows that Δ_Φ is invertible. To apply Theorem (2) to get a bound on the inverse of Δ_Φ we need a bound on the amplitude of Ψ, which we denote by δ_Ψ, c.f. (16)-(18). We then give an argument that the amplitude δ_Ψ is on the order of the size of the Arnold Tongue, an open set of drifts angles around each θ with rational rotation number, on which the rotation number is constant, a set of positive measure in θ when $\alpha > 0$, [2]. This, then, would imply that the inverse of Δ_Φ is on the order of the inverse size of the Arnold Tongue. The size of Arnold tongues is the subject of the Arnold conjecture. Based on these arguments, the norm of the inverse Δ_Ψ^{-1} is on the order $O(\alpha^q)$. We end the section with a brief discussion of Arnold tongues and the conjecture, which to our knowledge remains unsolved. The difficulty in analyzing the solvability condition and the problem of estimating the sizes of Arnold Tongues clarified in this section for the resonant case of rational rotation number, leads us to establish the bounds we seek on the inverse of Δ_Φ in the non-resonant case of irrational rotation number. This is accomplished in the next section.

3.1. Periodic Points. Suppose that the shift Φ has the form (3), namely

$$\Phi(t) = t + \theta + \alpha\phi(t),$$

with nonzero drift θ and amplitude α satisfying

$$\alpha \ll \theta/2\pi < 1,$$

so we are out of the zero drift regime of Corollary 2. That is, in this case, the shift Φ has no fixed points, so we cannot apply our earlier result directly. Regarding $\Phi : \mathbb{R} \to \mathbb{R}$ as (the lift of) a degree one circle diffeomorphism, or *circle map*, the translation number $\tau(\Phi) \in \mathbb{R}$ of Φ and corresponding rotation number $\rho(\Phi) \in [0, 2\pi)$ are defined by

$$\tau(\Phi) = \lim_{n\to\infty} \frac{\Phi^n t - t}{n} = \lim_{n\to\infty} \frac{t_n - t_0}{n}, \quad \text{and} \quad \rho(\Phi) = \tau(\Phi) \pmod{2\pi},$$

respectively. It is known [9, 4] that if $\rho(\Phi)$ is a rational multiple of 2π,

$$\rho(\Phi) = \frac{2\pi p}{q},$$

in lowest terms, then the map Φ has periodic points of period q, so that the q-fold composition Φ^q has fixed points.

Furthermore, $\Phi^{q'}$ has no fixed points for $q' < q$. It follows that

$$\Phi^q(t) = t + q\,\theta + O(\alpha),$$

and thus $q\theta = O(\alpha)$. Therefore $\Psi := \Phi^q$ can be written

$$\Psi(t) = t + \alpha\,\widetilde{\psi},$$

where $\widetilde{\psi}(t)$ depends on α, θ and $\phi(t)$. Since $\Psi(t)$ has fixed points, and therefore has zero drift, our previous result can be applied.

We cannot apply the results directly to $\Psi = t + \alpha\widetilde{\psi}$ because $\widetilde{\psi}$ is not $O(1)$ as $\alpha \to 0$, so we need to rescale to apply the theorem. To identify what play the roles of α, ϕ and Φ in terms of our iterated map, we write

$$\Psi(t) = t + \delta_\Psi\,\psi(t), \tag{20}$$

where the *amplitude* δ_Ψ is defined to be the smallest value of $\Psi - t$ at critical points of $\Psi(t) - t$,

$$\delta_\Psi = \min_{\Psi'(t)=1} |\Psi(t) - t|, \tag{21}$$

and $\psi(t)$ is $\widetilde{\psi}(t)$ after rescaling,

$$\psi(t) := (\Psi(t) - t)/\delta_\Psi,$$

so $\psi = O(1)$. In applying our previous results, δ_Ψ and ψ now play the roles of α and ϕ, respectively.

Theorem 2 gives us an immediate estimate of the inverse Δ_Ψ^{-1} on each subinterval invariant under Ψ in terms of δ_Ψ. It remains to estimate the inverse Δ_Φ^{-1} in terms of Δ_Ψ^{-1} on the full circle, and then to estimate δ_Ψ in terms of α.

The iteration $\Psi = \Phi^q$ has either $2q$ or q fixed points, as shown in the two cases in Figure 1. Generically, Φ^q changes sign and there are $2q$ fixed points, while if Φ^q has one sign, there are only q fixed points [9, 4]. We now apply Theorem 2 on each of these separate intervals.

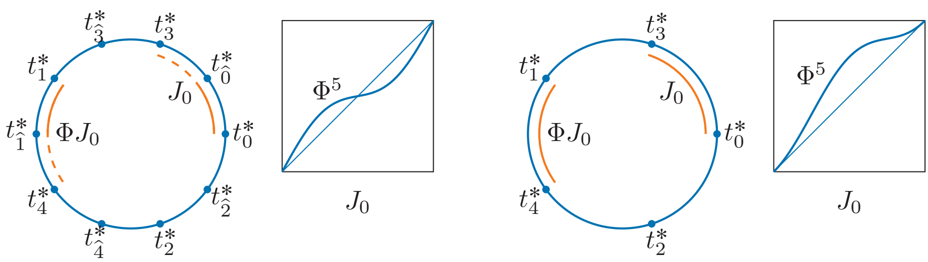

FIG. 1. *Orbit of a periodic point*

To be specific, assume that there is some q such that Φ has a periodic point of period q, or equivalently, the q-th power $\Psi := \Phi^q$ has a fixed point t_0^*. That is,

$$\Psi t_0^* = \Phi^q t_0^* = t_0^*, \quad \text{while} \quad \Phi^j t_0^* \neq t_0^*, \quad 1 \leqslant j < q.$$

The orbit of t_0^*, which is the set $\{t_j^* = \Phi^j t_0^* \mid j = 0, \dots, q-1\}$, cuts the circle $\mathbb{S}^1$ into q non-overlapping intervals $J_j = \Phi^j J_0$, where $J_0 = [t_0^*, t_m^*]$, where t_m^* minimizes $t_k^* - t_0^*$ (mod 2π). It follows that each J_j is invariant under $\Psi = \Phi^q$, namely $\Psi(J_j) = J_j$, and we can write

$$\mathbb{S}^1 = \bigcup_{j=0}^{q-1} J_j = \bigcup_{j=0}^{q-1} \Phi^j J_0, \quad \text{with} \quad \bigcap J_j = \{t_j^*\}.$$

In Figure 2 left, we plot the first $q = 5$ iterations of the shift map corresponding to $\theta = 2\pi/5$, observing that $\Phi^q = t + O(\alpha^2)$ (mod 2π), and on the right, we plot $\Phi^q - t - 2\pi p$ as a regular plot, and as a perturbation of the unit circle in a polar plot. By perturbing the drift θ as necessary, we assume that the periodic points are generic, so we are in the left case of Figure 1.

Since by assumption, $\Phi(t) - t - \theta$ changes sign monotonically (as a function of t) at each zero, the same is true for its q-th power, that is

$$\Psi'(t_0) \neq 1 \quad \text{whenever} \quad \Psi t_0 = t_0.$$

Thus the left picture in Figure 1, the generic case, is structurally stable under small perturbations. The right picture of Figure 1 corresponds to a bifurcation in the rotation number of Φ.

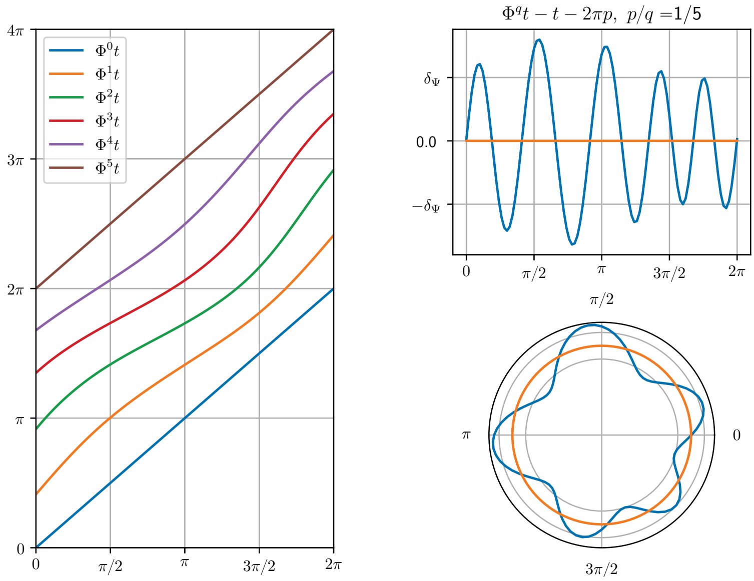

FIG. 2. *Iterations of shift map,* $\theta/2\pi = 1/5$

In order to reduce the general case of rational rotation number ρ to case the case of zero drift considered in Theorem 2, we next identify the intervals between fixed points associated with powers of Φ.

Writing $\Psi = \Phi^q$, it is easy to see inductively that for all t_0,

$$\Psi'(t_0) = \prod_{k=0}^{q-1} \Phi'(t_k), \quad t_k = \Phi^k t_0, \tag{22}$$

and in particular we have $\Psi'(t_k^*) = \Psi'(t_0^*)$ for each k. Since $\Psi'(t_0^*) \neq 1$, it follows that $\Psi : J_0 \to J_0$ has a second fixed point, which we label $t_{\hat{0}}^*$, and in particular Φ has exactly $2q$ periodic points of period q. Thus the interval J_0 (and each J_k) splits into two subintervals,

$$J_k = J_k' \cup J_k'', \quad J_k' = [t_k^*, t_{\hat{k}}^*], \quad J_k'' = [t_{\hat{k}}^*, t_{k+m}^*],$$

for $k = 0, \ldots, q-1$, each of which is invariant under $\Psi = \Phi^q$. Moreover, by our labeling system, we have for each k,

$$\Phi J_k' = J_{k+1}' \quad \text{and} \quad \Phi J_k'' = J_{k+1}''. \tag{23}$$

For the purpose of estimating the inverse, it suffices, as in [16], to assume, without loss of generality, that $\Psi' t_k^* > 1$ and $\Psi' t_{\hat{k}}^* < 1$, the case when each fixed point t_k^* is unstable and each $t_{\hat{k}}^*$ stable.

3.2. Inverting Δ_Φ. Although the shift Φ has no fixed points, because we assume here the rotation number ρ is rational, we've seen that Φ has periodic orbits of period q, so that its q'th iterate $\Psi = \Phi^q$ has fixed points, which cut the circle $\mathbb{S}^1$ into $2q$ non-overlapping intervals J_k', J_k''. We now invert $\mathcal{S}_\Psi - \mathcal{I}$ on each of these intervals, and combine these to get the full inverse of $\mathcal{S}_\Phi - \mathcal{I}$. Recall the partial Neumann series (19) and write

$$\left(\mathcal{S}_\Phi - \mathcal{I}\right)^{-1} = \left(\mathcal{S}_\Psi - \mathcal{I}\right)^{-1}\left(\mathcal{I} + \mathcal{S}_\Phi + \cdots + \mathcal{S}_{\Phi^{q-1}}\right). \tag{24}$$

Our strategy is to obtain uniform estimates based on this factorization.

Recall that in inverting the difference $\mathcal{S}_\Phi - \mathcal{I}$, we are solving the equation

$$\left(\mathcal{S}_\Phi - \mathcal{I}\right)v = w$$

for v given w, and so referring to (19), this is equivalent to solving

$$\left(\mathcal{S}_\Psi - \mathcal{I}\right) v = \widehat{w}, \quad \text{where} \quad \widehat{w} = \sum_{\ell=0}^{q-1} \mathcal{S}_{\Phi^\ell}\, w. \tag{25}$$

Here $\widehat{w}$ represents the target function w summed across the intervals J_k.

We now apply Theorem 2 to the function Ψ on each of the intervals J'_k and J''_k. In order to do so, we must first apply the solvability conditions (13) and second, understand the size of the bound which is $O\left(\frac{1}{\delta_\Psi}\right)$, where δ_Ψ is the amplitude, defined in (21). Applying (13), (14) to each subinterval, we get, for each $t' \in J_k'^{\,o}$, $t'' \in J_k''^{\,o}$,

$$\sum_{j\in\mathbb{Z}} \widehat{w}(\Psi^j t') = B'_k = v(t^*_{\hat{k}}) - v(t^*_k), \tag{26}$$

$$\sum_{j\in\mathbb{Z}} \widehat{w}(\Psi^j t'') = B''_k = v(t^*_{\hat{k}}) - v(t^*_{k+m}), \tag{27}$$

since each fixed point t^*_k of Ψ is unstable. Lipshitz continuity of $\widehat{w}$, together with $\widehat{w}(t^*) = 0$ implies convergence of the sums.

We now use (25) to express these conditions in w, namely

$$B'_k = \sum_{j\in\mathbb{Z}} \mathcal{S}_{\Psi^j}\widehat{w}(t') = \sum_{j\in\mathbb{Z}}\sum_{\ell=0}^{q-1} \mathcal{S}_{\Psi^j}\mathcal{S}_{\Phi^\ell} w(t') = \sum_{n\in\mathbb{Z}} \mathcal{S}_{\Phi^n} w(t'),$$

where we have set $n = qj + \ell$. Thus for all $t' \in J_k'^{\,o}$, and similarly $t'' \in J_k''^{\,o}$, we have

$$B'_k = \sum_{n\in\mathbb{Z}} w(\Phi^n t') \quad \text{and} \quad B''_k = \sum_{n\in\mathbb{Z}} w(\Phi^n t'').$$

Since each $t_k \in J_k$ can be written as $\Phi^k t_0$ with $t_0 \in J_0$, the sum $\sum w(\Phi^n t)$ is independent of k, so that we have $B'_k = B'_0$ and $B''_k = B''_0$ for each k. Also, adding equations (26) and relabelling gives

$$\sum_{k=0}^{q-1} B'_k = \sum_{k=0}^{q-1} B''_k,$$

and we conclude that for each k, $B'_k = B''_k = B$, and our solvability condition is the same as our previous one in [16], namely

$$\sum_{n\in\mathbb{Z}} w(\Phi^n t) = B, \quad t \neq t^*, \quad \text{and} \quad w(t^*) = 0, \tag{28}$$

where B is a fixed constant, and t^* are the periodic points of Φ, which is also the limit set of the discrete dynamical system induced by Φ. With a slight abuse of notation, we call the periodic points of Φ, which are the fixed points of Ψ, the *limit set* of Φ. We call the expression $\sum_{n\in\mathbb{Z}} w(\Phi^n t)$ the *orbital sum* of w, so our solvability condition is simply that w vanish on the limit set of Φ, and all orbital sums are constant.

The solvability condition establishes the range of the operator Δ_Φ, and assuming this, we now have an explicit solution $v(t)$ of (1), unique up to constant. Namely, from (15), we have for $t \in J_k'^{\,o} \cup J_k''^{\,o}$,

$$\begin{aligned} v(t) &= v_{-\infty} + \sum_{k>0} \widehat{w}(\Psi^{-k}t) = v_{-\infty} + \sum_{k>0} \mathcal{S}_{\Psi^{-k}} \sum_{j=0}^{q-1} w(\Phi^j t) \\ &= v_{-\infty} + \sum_{n>0} \mathcal{S}_{\Phi^{-n}} w(t) = v_{-\infty} + \sum_{n>0} w(\Phi^{-n}t). \end{aligned}$$

Again this expression is independent of k, so it holds for all t not in the limit set of Φ. The solution takes on the value $v_{-\infty}$ on the backward ω-limit set $\omega(\Phi^{-1})$, and $v_{-\infty} + B$ on the forward ω-limit set $\omega(\Phi)$. The following theorem now characterizes the solutions of (1) for continuous w.

THEOREM 3. *Assume that Φ has rotation number $\rho(\Phi) = 2\pi p/q$ in lowest terms. Suppose that w is continuous and satisfies the solvability condition* (28)*, namely, w vanishes on the limit set of Φ, and all non-degenerate orbital sums of w are constant. Then the equation*

$$\big(\mathcal{S}_\Phi - \mathcal{I}\big)\, v = w$$

has a continuous solution, given uniquely up to constant $v_{-\infty}$ by

$$\begin{aligned} v(t_{-\infty}) &= v_{-\infty}, && t_{-\infty} \in \alpha(\Phi), \\ v(t_{+\infty}) &= v_{-\infty} + B, && t_{+\infty} \in \omega(\Phi), \\ v(t) &= v_{-\infty} + \sum_{n>0} w(\Phi^{-n}t), && \textit{otherwise.} \end{aligned} \tag{29}$$

On the other hand, if the solvability condition (28) *fails, i.e., any two orbital sums are distinct, then there is no continuous solution for that w.*

3.3. Estimate for Δ_Φ^{-1} in the resonant case. Theorem 3 shows that the inverse Δ_Φ^{-1} is well defined on its range, namely the set of Lipshitz continuous w which meet the solvability condition (28). We now obtain an estimate for the solution operator Δ_Φ^{-1} on its range. So assume a fixed rational rotation number $\rho(\Phi) = 2\pi p/q$ in lowest terms, let ϕ be a given smooth function, and consider α a small perturbation and θ a controllable parameter, respectively. We show that for each each $\alpha > 0$ small, and each rotation number $\rho = 2\pi p/q$, there is an open interval of θ for which the map Φ has periodic points of order q, and the same rotation number ρ. This is the so-called *Arnold tongue*, and Arnold's conjecture is that the size of this open interval is big $O(\alpha^q)$, [2, 9, 5, 7]. To clarify the issues, we assume ϕ is an analytic function. In this case we prove that the width of the Arnold tongue is $O(\delta_\Psi)$. Here δ_Ψ is the amplitude defined above in (21), corresponding to the value of θ at the "middle" of the Arnold tongue when $\rho = 2\pi p/q$. We begin with the following corollary of Theorems 2, 3:

Corollary 3. *The norm of* Δ_Φ^{-1} *is given by*

$$\|\Delta_\Phi^{-1}\|_{Lip} \leqslant O(1)\frac{q}{\delta_\Psi}. \tag{30}$$

Proof. By (17), (18) with δ_Ψ given by (21) in place of α, we have

$$\|\Delta_\Psi^{-1}\| \leqslant \frac{O(1)}{\delta_\Psi}.$$

Also, $\mathcal{S}_\Phi$ is bounded, so each $\mathcal{S}_{\Phi^k}$ is bounded, $k \geqslant 1$, and we have

$$\|\Delta_\Phi^{-1}\| \leqslant \|\Delta_\Psi^{-1}\|\left(\|I\| + \cdots + \|\mathcal{S}_{\Phi^{k-1}}\|\right) \leqslant O(1)\frac{q}{\delta_\Psi},$$

after using (24). □

In order to estimate δ_Ψ, we need to develop an expansion of Ψ in terms of our original parameter α and perturbation $\phi(t)$. To do this we make simplifying assumptions which allow us to develop the asymptotic expansion. This will in turn allow us show that δ_Ψ is of the order of the width of the Arnold tongue corresponding to rotation number ρ.

Assuming now that ϕ is analytic, the k-th Fourier mode exists and decays like α^k. This means that after incorporating the 0-mode into the constant shift θ, the leading order term of $\phi(t)$ is a 1-mode. By rescaling α and translating around the circle, we can thus take

$$\phi(t) = \sin t + \alpha\,\varphi(t), \tag{31}$$

without loss of generality, and as above, we set

$$\Phi t = t + \theta + \alpha\,\phi(t). \tag{32}$$

Denote the constant term of $\varphi(t)$ by

$$\varphi_0 = \frac{1}{2\pi}\int_0^{2\pi} \varphi(t)\,dt.$$

With these assumptions, we develop an asymptotic expression for the iteration $\Psi = \Phi^q$ and use this to identify the values of θ for which the rotation number is $2\pi p/q$.

Lemma 2. *For the shift* Φ *given by* (32)*, with*

$$\theta_0 := 2\pi p/q, \quad \text{and} \quad |\theta - \theta_0| \leqslant K\,\alpha^2,$$

the q*-th iterate of* Φ *has the expansion*

$$\Psi t = \Phi^q t = t + q\,\theta + \alpha^2 q\Big(\varphi_0 - \frac{1}{4\tan(\theta/2)}\Big) + \alpha^3\,F(t,\alpha,\theta), \tag{33}$$

with $F(t,\alpha,\theta) = O(1)$ *depending on* K *and* ϕ*. In order for* Φ *to have rotation number* $\rho(\Phi) = \theta_0$*, it is necessary and sufficient that* Φ *has a periodic point* t_* *of period* q*. This holds if and only if* $\Phi^q t_* = t_*$*, and evaluating* (33) *at* t_**, we have*

$$q\,\theta + \alpha^2\,q\Big(\varphi_0 - \frac{1}{4\tan(\theta/2)}\Big) + \alpha^3\,F(t_*,\alpha,\theta) = 2\pi p. \tag{34}$$

We postpone the proof of Lemma 2 to the end of the section.

According to the lemma, Φ has rotation number $\rho(\Phi) = \theta_0 = 2\pi p/q$ if and only if (34) holds for some $t_* \in \mathbb{S}^1$. For given $\phi(t)$ and fixed θ_0, this condition generates a closed set in the (α, θ)-plane, which is an interval for each fixed α, known as the *Arnold tongue* corresponding to $\rho = 2\pi p/q$. Arnold's conjecture is that the width of this Arnold tongue is $O(\alpha^q)$, so without loss of generality it suffices to take $K = 1$ in Lemma 2, so that F is bounded in terms of ϕ alone.

In order to ensure (34) holds for some t_*, we now average it to find the *"middle"* $\overline{\theta}$ of the Arnold tongue. In (34), the only t dependence is in F, so we define the function $\overline{\theta}(\alpha)$ implicitly by setting

$$q\,\overline{\theta} + \alpha^2\, q\Big(\varphi_0 - \frac{1}{4\tan(\overline{\theta}/2)}\Big) + \alpha^3 \langle F(\alpha, \overline{\theta})\rangle = 2\pi p, \tag{35}$$

where $\langle F\rangle$ is given by

$$\langle F(\alpha, \theta)\rangle := \frac{1}{2\pi}\int_0^{2\pi} F(t, \alpha, \theta)\, dt,$$

and we can express (35) in the equivalent form

$$\langle \Psi t - t\rangle = 2\pi p.$$

The mean value theorem immediately implies the existence of some t_* satisfying (34) for $\theta = \overline{\theta}(\alpha)$, so that $\overline{\theta}(\alpha)$ is indeed in the interior of the tongue.

In the following theorem we show that the width of the tongue is of the order of $O(1)\delta_\Psi/q$ for α small enough, where δ_Ψ is defined by (21) for $\theta = \overline{\theta}(\alpha)$. Taken together with (30) of Corollary 3, this means that our bound on $\|\Delta_\Phi^{-1}\|$ is $O(1)$ times the width of the Arnold tongue. In particular, if Arnold's conjecture were shown to hold, our estimate would be

$$\|\Delta_\Phi^{-1}\| = O(1)\,\alpha^{-q},$$

for α small enough.

Theorem 4. *For fixed rotation number $\theta_0 = 2\pi p/q$, let $\overline{\theta}(\alpha)$ be the middle of the Arnold tongue and define δ_Ψ according to* (21)*. Then for α small enough, the Arnold tongue has radius between $\delta_\Psi/2q$ and $2\delta_\Psi/q$.*

Proof. For fixed α, let $\overline{\Psi}$ denote the function $\Psi = \Phi^q$ obtained by setting $\theta = \overline{\theta}(\alpha)$, so that

$$\langle \overline{\Psi}(t) - t\rangle = 2\pi p,$$

and let Ψ denote the function obtained for $\theta = \overline{\theta}(\alpha) + \epsilon$. Using the expansion (33), we then get

$$\Psi(t) - t = \overline{\Psi}(t) - t + q\,\epsilon + O(\alpha^2),$$

and in particular,

$$\frac{\partial(\Psi - t)}{\partial\epsilon}\bigg|_{\epsilon,\alpha=0} = q.$$

It follows that provided α is small enough, and if $|\epsilon| < \delta_\Psi/2q$, then $\Psi - t$ has the same number of zeroes as $\overline{\Psi} - t$, while if $|\epsilon| > 2\,\delta_\Psi/q$, $\Psi - t$ has a different number of zeroes. Since the rotation number is characterizes by the number of zeroes, it follows that while $|\epsilon| < \delta_\Psi/2q$, $\theta = \overline{\theta}(\alpha) + \epsilon$ remains in the tongue, while if $|\epsilon| > 2\,\delta_\Psi/q$, θ has left the tongue. The result now follows by monotonicity of Ψ as a function of θ for α small enough. □

We can obtain a picture of the Arnold tongues numerically, as follows. First, for fixed α, solve (35) to get $\overline{\theta}$. This is easily accomplished by using a secant method, in which the initial guesses for the iteration are

$$\theta_0 = \frac{2\pi p}{q} \quad \text{and} \quad \theta_1 = \theta_0 + \alpha^2\Big(\frac{1}{4\tan(\theta_0/2)} - \varphi_0\Big).$$

Having found $\overline{\theta}$, we perturb this by $\epsilon > 0$ to find the bifurcation point, at which the number of zeroes of $\Psi(t)$ changes; this is one edge of the tongue. Similarly perturbing by $\epsilon < 0$ locates the other edge of the tongue. Figure 3 shows a picture of the first tongues, computed for the beginning of the *Farey sequence* of rationals,

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{4},\ \frac{3}{4},\ \frac{1}{5},\ \dots,$$

which approximately orders the tongues by width. The left panel shows the tongues only, while the right also shows the middles $\overline{\theta}(\alpha)$ drawn as curves of fixed width.[1]

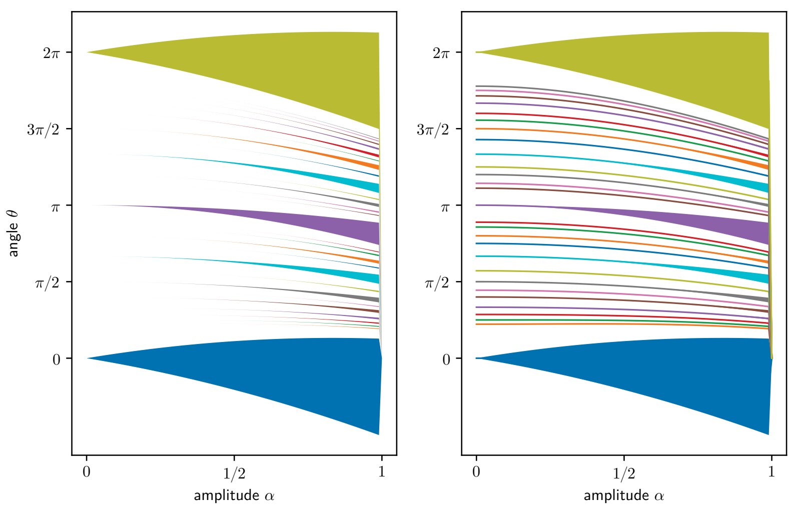

FIG. 3. *Arnold tongues: uncentered and centered*

We end this section with the proof of Lemma 2. Our strategy is to assume (32), (31), namely

$$\Phi t = t + \theta + \alpha\,\phi(t), \quad \text{where}$$
$$\phi(t) = \sin t + \alpha\,\varphi(t),$$

[1]This picture was generated by Becca Rosenblum as part of an REU project at UMass with the second author.

and bootstrap to get the leading order asymptotic expansion of powers of the shift Φ. Here we regard $\varphi(t)$ as encoding higher order interaction effects in the nonlinear evolution operator, consistent with [11, 14]. We proceed with the details.

Proof of Lemma 2. Using (32), we write

$$\Phi^2 t = \Phi t + \theta + \alpha\phi(\Phi t) = t + 2\theta + \alpha\big(\phi(t) + \phi(\Phi t)\big),$$

and inductively,

$$\Phi^k t = t + k\theta + \alpha \sum_{j=0}^{k-1} \phi\big(\Phi^j t\big), \tag{36}$$

and so in particular,

$$\Phi^q t = t + q\theta + \alpha \sum_{j=0}^{q-1} \phi\big(\Phi^j t\big), \quad \text{with} \quad q\theta \approx 2\pi p.$$

We will see that this last sum vanishes to order $O(\alpha)$, as seen in Figure 2.

Assuming differentiability of φ, we bootstrap to get the leading terms of the k-th iteration $\Phi^k t$. First, we write (36) as

$$\Phi^j t = t + j\theta + O(\alpha),$$

which in turn yields

$$\Phi^k t = t + k\theta + \alpha \sum_{j<k} \phi(t + j\theta) + O(\alpha^2),$$

and substituting this back into (36) yields

$$\begin{aligned}
\Phi^k t &= t + k\theta + \alpha \sum_{j<k} \phi\Big(t + j\theta + \alpha\textstyle\sum_{\ell<j}\phi(t+\ell\theta)\Big) + O(\alpha^3)\\
&= t + k\theta + \alpha \sum_{j<k} \phi(t + j\theta + \alpha S_j) + O(\alpha^3)\\
&= t + k\theta + \alpha \sum_{j<k} \sin(t + j\theta + \alpha S_j) + \alpha^2 \sum_{j<k} \varphi(t + j\theta) + O(\alpha^3),
\end{aligned}$$

where we have used (31) and written

$$S_j = S_j(\theta) := \sum_{\ell<j} \sin(t + \ell\theta) \quad \text{for} \quad j = 0, \dots q. \tag{37}$$

One final expansion now yields

$$\begin{aligned}
\Phi^k t &= t + k\theta + \alpha\, S_k(\theta)\\
&\quad + \alpha^2 \sum_{j<k} \Big[\cos(t + j\theta) S_j + \varphi(t + j\theta)\Big] + O(\alpha^3).
\end{aligned} \tag{38}$$

Considering the nonlinear evolution, the terms in (38) respectively represent translation, compression and rarefaction, and second order self-interaction effects. We note that the translation is by a zeroth order constant, compression is first order and a 1-mode, and the leading interaction term is quadratic, as expected.

We now calculate

$$\begin{aligned} S_k(\theta) &= \sum_{j<k} \sin(t+j\theta) = \sum_{j<k} \operatorname{Im}\bigl(e^{i(t+j\theta)}\bigr) \\ &= \operatorname{Im}\Bigl(e^{it}\sum_{j<k}(e^{i\theta})^j\Bigr) = \operatorname{Im}\Bigl(e^{it}\frac{1-e^{ik\theta}}{1-e^{i\theta}}\Bigr), \end{aligned} \tag{39}$$

and similarly for $\sum_{j<q}\cos(t+j\theta)$. In particular, when $\theta = \theta_0 = 2\pi p/q$, we have $S_q(\theta_0) = 0$ and the $O(\alpha)$ part of $\Psi = \Phi^q$ vanishes. That is, if we write

$$\theta = \theta_0 + \alpha\theta_1 + \alpha^2\theta_2 + \dots,$$

then

$$\Phi^q t = t + 2\pi p + \alpha\bigl(q\theta_1 + S_q(\theta_0)\bigr) + O(\alpha^2),$$

so we must take $\theta_1 = 0$, which in turn yields $\theta = \theta_0 + O(\alpha^2)$.

We compute the next term in (38) similarly:

$$\begin{aligned} \sum_{j<k}\cos(t+j\theta)S_j(\theta) &= \frac{1}{2}\sum_{j<k}\sum_{\ell<j}\bigl(\sin(2t+j\theta+\ell\theta) - \sin(j\theta-\ell\theta)\bigr) \\ &= \frac{1}{2}\operatorname{Im}\Bigl[\sum_{j<k} e^{ij\theta}\sum_{\ell<j}\bigl(e^{2it}e^{i\ell\theta} - e^{-i\ell\theta}\bigr)\Bigr] \\ &= \frac{1}{2}\operatorname{Im}\Bigl[\sum_{j<k} e^{ij\theta}\Bigl(e^{2it}\frac{1-e^{ij\theta}}{1-e^{i\theta}} - \frac{1-e^{-ij\theta}}{1-e^{-i\theta}}\Bigr)\Bigr] \\ &= \frac{1}{2}\operatorname{Im}\Bigl[e^{2it}\Bigl(\frac{1-e^{ik\theta}}{(1-e^{i\theta})^2} - \frac{1-e^{2ik\theta}}{(1-e^{i\theta})(1-e^{2i\theta})}\Bigr) \\ &\qquad - \frac{1-e^{ik\theta}}{(1-e^{-i\theta})(1-e^{i\theta})} + \frac{k}{1-e^{-i\theta}}\Bigr], \end{aligned}$$

and note again that all but the last term vanish for $k = q$ if $\theta = \theta_0 = 2\pi p/q$. The last term simplifies as

$$\frac{k}{2}\operatorname{Im}\frac{1}{1-e^{-i\theta}} = \frac{k}{2}\frac{-\sin\theta}{(1-e^{-i\theta})(1-e^{i\theta})} = \frac{-k\sin\theta}{4(1-\cos\theta)} = \frac{-k}{4\tan(\theta/2)}.$$

Finally, suppose that $\varphi(t)$ can be expressed as a sum of Fourier modes,

$$\varphi(t) = \varphi_0 + \sum_{\ell\geqslant 1} a_\ell\cos(\ell t) + b_\ell\sin(\ell t).$$

Then, exactly as in (39), the last sum in (38) degenerates, namely

$$\sum_{j<q}\varphi(t+j\theta) = q\varphi_0.$$

Putting all these sums together yields (33), thus completing the proof of the lemma. □

Although we have assumed analyticity for convenience, this argument applies under the assumption that ϕ has a Fourier series expansion of order q.

In view of these computations and Arnold's conjecture, we expect that if we were to assume analyticity and continue this bootstrap and expand technique, then any expansion in powers α^k for $k \leqslant q-1$ will similarly yield only 0-th order terms. These in turn affect the location of the middle of the tongue but not the width. In general we expect that the first non-cancelling term will appear in the $O(\alpha^q)$ term. Thus our expectation is that the width of the tongue will be at most $O(\alpha^q)$, although it may prove to be a higher power of α in general.

We have obtained a bound on the inverse Δ_Φ^{-1} for an open set of θ with fixed rational rotation number $\rho = 2\pi p/q$. At first sight it appears that we could now apply these estimates in Nash-Moser, but it remains to address the following issues. First, despite Arnold's conjecture, we do not know of a uniform bound away from zero for the width δ_Ψ of the tongue in terms of powers of α. More troubling is that we do not have a clear understanding of the solvability condition (28) beyond its abstract statement. Specifically, in order to effectively check this solvability condition, it appears that we need an *a priori* description of each of the orbits of Φ. For these reasons we now consider the non-resonant case of irrational rotation number, where the results are definitive.

4. Non-resonant case. We now condsider the non-resonant case, which occurs when the rotation number $\rho(\Phi)$ is irrational. In this case $\Phi : \mathbb{S}^1 \to \mathbb{S}^1$ has no periodic orbits, and indeed all orbits $\{t_k \mid t_k := \Phi^k t_0,\ k \in \mathbb{Z}\}$ are dense. We are only interested in continuous solutions. To construct these, fix an arbitrary t_0, and solve the difference equation (1) directly, so that

$$v(t_k) = v(t_0) + \sum_{j=0}^{k-1} w(t_j), \quad k \geqslant 1, \tag{40}$$

and similarly for $k < 0$. Having thus defined v on one orbit, the strategy of this section is to extend it to all of $\mathbb{S}^1$ by continuity. This defines v only up to arbitrary constant $v(t_0)$, a constant which can be fixed by fixing a particular value $v(t_*)$ or choosing an average value of v.

In order to ensure that the solution so defined be Lipschitz continuous, we impose the solvability condition

$$\left|\sum_{j=0}^{k-1} w(t_j)\right| \leqslant K \left|t_k - t_0\right|, \quad \text{for all} \quad k \in \mathbb{Z},\ t_0 \in \mathbb{S}^1, \tag{41}$$

on the data w, where K is a fixed constant. For each such w, we use (40) to define a Lipschitz continuous solution v, and we seek bounds on the solution operator.

Our first task then is to show that (40) yields a well-defined solution which is Lipschitz continuous. Thus, for fixed t_0, which determines all t_k, and given any t and subsequence $t_{j_n} \to t$, inequality (41) implies

$$\left|v(t_{j_n}) - v(t_{j_m})\right| = \left|\sum_{k=j_m}^{j_n-1} w(t_k)\right| \leqslant K \left|t_{j_n} - t_{j_m}\right|.$$

Thus the subsequence $\{v(t_{j_n})\}$ is Cauchy, so has a unique limit, and $v(t)$ is indeed well-defined. To verify continuity of v, if $t = \lim t_{j_n}$ and $\tau = \lim t_{l_m}$ are sequences converging to different limits, then for ϵ, δ small, and for n and m sufficiently large,

$$\begin{aligned}\Big|v(t)-v(\tau)\Big| &\leqslant \Big|v(t)-v(t_{j_n})\Big| + \Big|v(t_{j_n})-v(t_{l_m})\Big| + \Big|v(t_{l_m})-v(\tau)\Big| \\ &\leqslant 2\,\epsilon + K\,\Big(|t-\tau| + 2\,\delta\Big),\end{aligned}$$

implying that the solution v is Lipschitz continuous with constant K. To show that the solution v so constructed is independent of t_0, note that since all orbits are dense, by linearity the only continuous solution to

$$\big(\mathcal{S}_\Phi - \mathcal{I}\big)v(t) = v(\Phi t) - v(t) = 0$$

is $v(t) = v_0$, for some constant v_0 which is independent of t_0. Thus v is well defined and uniquely determined to within a constant.

Having shown the existence of a unique continuous solution up to arbitrary constant, we now wish to find a bound for the solution operator $w \mapsto v$. We first show that the domain of this solution operator is the set of Lipshitz continuous functions $\{w\}$ satisfying (41), and the range is the set of Lipshitz continuous $\{v\}$ of zero ergodic mean satisfying (7). Because we are solving the equation explicitly, this Lipschitz estimate is the strongest possible, and we will see that this implies a hierarchy of estimates in Sobolev spaces.

To establish norms, let $\mathcal{V}$ denote the set of Lipschitz continuous functions on the circle,

$$\mathcal{V} := \Big\{v : \mathbb{S}^1 \to \mathbb{R} \mid \|v\|_{Lip} < \infty\Big\}, \quad \|v\|_{Lip} := \sup_{t\neq\tau} \frac{|v(t)-v(\tau)|}{|t-\tau|}, \tag{42}$$

and regard $\mathcal{S}_\Phi : \mathcal{V} \to \mathcal{V}$, which is bounded in the operator norm by $\|\Phi\|_{Lip}$, so we take the domain of Δ_Φ to be $\mathcal{V}$. Note that $\|\cdot\|_{Lip}$ is not a norm because it vanishes on the set of constant functions, which is exactly the (continuous) kernel of $\Delta_\Phi = \mathcal{S}_\Phi - \mathcal{I}$. Thus we let $\mathcal{V}_0$ denote the set $\mathcal{V}/\!\sim$, where $v_1 \sim v_2$ iff $\|v_1 - v_2\|_{Lip} = 0$, and without confusion we equivalently let

$$\mathcal{V}_0 = \big\{v \in \mathcal{V} \mid v(t_0) = v_0\big\}, \tag{43}$$

for arbitrary fixed t_0 and v_0.

For the range, we use the solvability condition (41) to define the norm on the range of Δ_Φ directly. That is, set

$$|\!|\!|w|\!|\!| := \sup_{t_0,k} \frac{\Big|\sum_{j=0}^{k-1} w(t_j)\Big|}{|t_k - t_0|}, \quad \text{and} \quad \mathcal{W} := \Big\{w \in \mathcal{V} \mid |\!|\!|w|\!|\!| < \infty\Big\},$$

where again $t_j := \Phi^j t_0$. Evidently $|\!|\!|\cdot|\!|\!|$ satisfies the triangle inequality, and since in addition it scales with and dominates a multiple of the sup-norm, $\|w\|_\infty \leqslant |\!|\!|w|\!|\!|/\inf|\Phi t - t|$ (by taking $k = 1$), so $|\!|\!|w|\!|\!| = 0$ implies $w = 0$, and it follows that $|\!|\!|\cdot|\!|\!|$ is a norm. The completeness of $|\!|\!|\cdot|\!|\!|$ follows from the explicit formula (40), as follows: given a sequence $w^{(n)}$ which is Cauchy in $|\!|\!|\cdot|\!|\!|$, define $v^{(n)} \in \mathcal{V}_0$ by (40). It then follows that $v^{(n)}$ is Cauchy in $\mathcal{V}_0$, so converges to some v. Setting $w := (\mathcal{S}_\Phi - \mathcal{I})v$ gives the limit of $w^{(n)}$.

Note that the condition (41) is mainly a restriction on w for values $t_k \approx t_0$, and can be interpreted as a "zero average" condition on w. Moreover, by construction, the difference $\Delta_\Phi = \mathcal{S}_\Phi - \mathcal{I}$ is a bounded operator $\mathcal{V}_0 \to \mathcal{W}$, but the individual operators $\mathcal{S}_\Phi$ and $\mathcal{I}$ are unbounded from $\mathcal{V}_0 \to \mathcal{W} \subset \mathcal{V}$ because $|||\cdot|||$ is not bounded by $\|\cdot\|_{Lip}$.

We have built the spaces $\mathcal{V}_0$ and $\mathcal{W}$ so that the operator $\Delta_\Phi : \mathcal{V}_0 \to \mathcal{W}$ is bounded and invertible with bounded inverse, by solving the equation (1) explicitly via formula (40). Recall that this construction is only possible if the shift Φ has irrational rotation number, the case when there are no periodic orbits and all orbits are dense. We are able to do this essentially because the target norm $|||\cdot|||$ is based on the structure of the shift Φ through the iterates t_k. We summarize the foregoing in a lemma.

Lemma 3. *The difference operator $\Delta_\Phi = \mathcal{S}_\Phi - \mathcal{I} : \mathcal{V}_0 \to \mathcal{V}_0$ is invertible on the subset $\mathcal{W} \subset \mathcal{V}_0$, in the following sense: given $w \in \mathcal{W}$, there is a unique $v \in \mathcal{V}_0$, denoted $\Delta_\Phi^{-1} w$, such that $\Delta_\Phi v = w$, and moreover, this inverse is bounded in the operator norm,*

$$\|\Delta_\Phi^{-1}\|_{\mathcal{W}\to\mathcal{V}_0} = \sup_{w\in\mathcal{W}} \frac{\|\Delta_\Phi^{-1} w\|_{Lip}}{|||w|||} = 1.$$

We have exhibited spaces and norms on which the operator is an isometry from $(\mathcal{V}_0, \|\cdot\|_{Lip})$ to $(\mathcal{W}, |||\cdot|||)$. It remains to estimate the norm $|||\cdot|||$ in terms of classical norms, such as C^k norms, in order to quantify the loss of smoothness in the inversion process. Note that the sums $\sum w(t_j)$ resemble ergodic averages, so we look to apply ergodic techniques [9, 17].

According to ergodic theory, the shift map Φ has an invariant measure, that is a measure μ such that for each interval E, we have $\mu(\Phi E) = \mu(E)$. This is obtained by conjugating Φ to a pure rotation,

$$h \circ \Phi = R_\rho \circ h, \tag{44}$$

where R_ρ is translation by the rotation number ρ of Φ, assumed here to be irrational. According to [8], we can assume the conjugation h is C^1 or higher by assuming Φ is sufficiently smooth (at least C^3). Since Lebesgue measure λ is invariant for R_ρ, the invariant measure μ is obtained by pulling back Lebesgue measure: that is, define $\mu(E) := \lambda(h\,E)$, so that

$$\mu(\Phi\,E) = \lambda(h\,\Phi\,E) = \lambda(R_\rho\,h\,E) = \lambda(h\,E) = \mu(E).$$

Recall the ergodic theorem states that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n-1} w(\Phi^j t_0) = \frac{1}{2\pi} \int_0^{2\pi} w\,d\mu. \tag{45}$$

The following lemma shows that a necessary condition for solvability of (1) is that this ergodic average vanish.

Lemma 4. *Assume ρ is irrational. If $|||w||| < \infty$, then equation* (7) *holds, namely*

$$\int_0^{2\pi} w\,d\mu = \int_0^{2\pi} \big(v(\Phi t) - v(t)\big)\,d\mu = 0.$$

Proof. The condition $|||w||| < \infty$ implies existence of a function v such that $w(t) = v(\Phi t) - v(t)$, so the lemma follows by invariance of μ. □

Our goal is now to show that $\int_0^{2\pi} w\, d\mu = 0$, together with smoothness conditions on w, are *sufficient* to imply $|||w||| < \infty$, which in turn implies invertibility of Δ_Φ. To do this, we bound the norm $|||w|||$ in terms of C^r or H^s norms, allowing for a loss of derivatives. This is accomplished in the following theorem.

THEOREM 5. *Suppose that the shift Φ has rotation number ρ satisfying the diophantine condition*

$$\left|\frac{\rho}{2\pi} - \frac{p}{q}\right| \geqslant \frac{C(\rho)}{q^r}, \tag{46}$$

for all integers p and $q > 0$, and assume Φ is smooth enough that the conjugation h in (44) is $C^{r,\nu}$ or $H^{r+\nu}$, respectively, for $1/2 < \nu \leqslant 1$. Assume further that $w \in C^{r,\nu}$ or $H^{r+\nu}$, respectively, with $\int_0^{2\pi} w\, d\mu = 0$. Then $|||w|||$ is finite, with

$$|||w||| \leqslant K\, \|w\|_{C^{r,\nu}} \quad \textit{or} \quad |||w||| \leqslant K\, \|w\|_{H^{r+\nu}}, \tag{47}$$

respectively, and in particular, the solution $v = \Delta_\Phi^{-1} w$ exists and is Lipschitz, with $\|v\|_{Lip} = |||w|||$.

Proof. We begin by using the conjugation above to transfer the nonlinearity in Φ over to h, effectively reducing the shift to the case of pure irrational rotation. That is, given Φ, there is an h such that (44) holds, namely $h \circ \Phi = R_\rho \circ h$, and assume that Φ is smooth enough that both h and h^{-1} are at least C^1. Then, using the conjugation, write $\tau = h(t)$, so that $t = h^{-1}\tau$, and

$$t_j = \Phi^j\, t_0 = \Phi^j\, h^{-1}\tau_0 = h^{-1} R_\rho^j \tau_0 = h^{-1}\big(\tau_0 + j\rho\big), \tag{48}$$

so that in the variable τ, the shift is pure rotation: $\tau_j = \tau_0 + j\,\rho$. We transfer this to w, by setting $W = w \circ h^{-1}$, yielding

$$w(t_j) = W(\tau_j) = W(\tau_0 + j\rho), \quad \text{and} \quad w = W \circ h.$$

Since h is bi-Lipshitz, we have

$$|\tau - \tau'| \leqslant \|h\|_{Lip}\, |t - t'| \quad \text{and} \quad |t - t'| \leqslant \|h^{-1}\|_{Lip}\, |\tau - \tau'|,$$

so

$$\frac{1}{\|h\|_{Lip}} \frac{\left|\sum_{j=0}^{k-1} w(t_j)\right|}{|t_k - t_0|} \leqslant \frac{\left|\sum_{j=0}^{k-1} W(\tau_j)\right|}{|\tau_k - \tau_0|} \leqslant \|h^{-1}\|_{Lip} \frac{\left|\sum_{j=0}^{k-1} w(t_j)\right|}{|t_k - t_0|}. \tag{49}$$

It follows that the quantity $\sup \frac{|\sum W(\tau_j)|}{|\tau_k - \tau_0|}$ is equivalent to the norm $|||w|||$, where the denominator is measured on the circle,

$$|\tau_k - \tau_0| = |k\,\rho| \bmod 2\pi = \min_{m \in \mathbb{Z}} |k\,\rho - 2\pi m|.$$

Now for integer $q \neq 0$ and $k > 1$, we estimate the sum (49) for a single q-mode, $W_q(\tau) = e^{i\,q\,\tau}$. For this W_q, we have

$$\begin{aligned} \frac{\left|\sum_{j=0}^{k-1} W_q(\tau_j)\right|}{|\tau_k - \tau_0|} &= \frac{\left|e^{i\,q\,\tau_0} \sum_{j=0}^{k-1} e^{i\,q\,j\,\rho}\right|}{|k\,\rho| \bmod 2\pi} \\ &= \left|\frac{1 - e^{i\,q\,k\,\rho}}{1 - e^{i\,q\,\rho}}\right| \frac{1}{\min_{m\in\mathbb{Z}} |k\,\rho - 2\pi m|}. \end{aligned}$$

If m_k denotes the integer closest to $k\,\rho/2\pi$, write

$$\frac{\left|1-e^{i\,q\,k\,\rho}\right|}{\min_{m\in\mathbb{Z}}|k\,\rho-2\pi m|}=\left|\frac{e^{i\,2\pi\,q\,m_k}-e^{i\,q\,k\,\rho}}{k\,\rho-2\pi\,m_k}\right|=|i\,q\,e^{i\,q\,\xi}|=|q|,$$

by the mean value theorem. Thus

$$\frac{\left|\sum_{j=0}^{k-1}W_q(\tau_j)\right|}{|\tau_k-\tau_0|}=\frac{|q|}{\left|1-e^{i\,q\,\rho}\right|},\tag{50}$$

which is independent of k. We now impose the diophantine condition (46) on the (scaled) irrational rotation number $\rho/2\pi$: namely, $\rho/2\pi$ is diophantine of order r if there is some constant $C=C(\rho)$, such that, for any integers p and $q>0$,

$$\left|\frac{\rho}{2\pi}-\frac{p}{q}\right|\geqslant\frac{C}{q^r}.$$

Since m_1 is the integer closest to $\rho/2\pi$, again using the mean value theorem, it follows that

$$\begin{aligned}\frac{|1-e^{i\,q\,\rho}|}{|q|}&=\frac{|e^{i\,2\pi\,m_1}-e^{i\,q\,\rho}|}{|q|}\\&=\frac{|2\pi\,i\,e^{i\,2\pi\,\xi}(m_1-q\,\rho/2\pi)|}{|q|}\\&=2\pi\left|\frac{m_1}{q}-\frac{\rho}{2\pi}\right|\geqslant\frac{2\pi C}{|q|^r},\end{aligned}\tag{51}$$

so that the q-mode W_q satisfies

$$\frac{\left|\sum_{j=0}^{k-1}W_q(\tau_j)\right|}{|\tau_k-\tau_0|}\leqslant\frac{|q|^r}{2\pi\,C(\rho)}.$$

Assuming W is C^r, it can be written as a Fourier series,

$$W(\tau)=\sum_{q\neq0}\gamma_q\,e^{i\,q\,\tau}=\sum_{q\neq0}\gamma_q\,W_q(\tau).$$

It follows that

$$\frac{\left|\sum_{j=0}^{k-1}W(\tau_j)\right|}{|\tau_k-\tau_0|}\leqslant\frac{1}{2\pi\,C(\rho)}\sum_{q\neq0}|\gamma_q|\,|q|^r,$$

which together with (49) provides a bound for $|||w|||$, namely

$$|||w|||\leqslant\|h\|_{Lip}\sup_{\tau_0,k}\frac{\left|\sum_{j=0}^{k-1}W(\tau_j)\right|}{|\tau_k-\tau_0|}\leqslant\frac{\|h\|_{Lip}}{2\pi\,C(\rho)}\sum_{q\neq0}|\gamma_q|\,|q|^r.\tag{52}$$

In particular, if w is in $C^{r,\nu}$ or $H^{r+\nu}$ for $\nu>1/2$, then the right hand side of (52) is bounded by a constant times $\|w\|_{C^{r,\nu}}$ or $\|w\|_{H^{r+\nu}}$, respectively. This completes the proof of the theorem. □

As an immediate consequence of Lemma 4 and the theorem, we have the following interpretation. The norm $|||\cdot|||$ quantifies the (effective) rate of convergence of ergodic sums to the spatial average, providing an error estimate for the ergodic theorem.

COROLLARY 4. *If $|||w||| < \infty$, then for every $k \in \mathbb{Z}$, we have*

$$\left|\frac{1}{k}\sum_{j=0}^{k-1} w(t_j) - \frac{1}{2\pi}\int w\,d\mu\right| \leqslant |||w|||\,\left|\frac{1}{k}(t_k - t_0)\right|.$$

Theorem 5 shows that in solving equation (1), the inverse loses a finite amount of smoothness, depending on the diophantine condition (46). We now show that, for arbitrary rotation number, we cannot limit the amount of smoothness that is lost. To this end, suppose that $\rho/2\pi$ is a Liouville number, which is a number that does not satisfy any diophantine condition (46). In other words, for each n, there exist p_n and q_n such that

$$\left|\frac{\rho}{2\pi} - \frac{p_n}{q_n}\right| \leqslant \frac{1}{q_n^n}. \tag{53}$$

We can generate one such by fixing a base b and setting

$$\rho = 2\pi\sum_{k=0}^{\infty}\frac{d_k}{b^{k!}}, \quad \text{with} \quad d_k \in \big\{0,\ 1, \dots\ b-1\big\},$$

which is clearly irrational. Indeed, setting

$$q_n := b^{n!} \quad \text{and} \quad p_n := q_n\sum_{k=0}^{n}\frac{d_k}{b^{k!}} \in \mathbb{Z},$$

we calculate

$$\frac{\rho}{2\pi} - \frac{p_n}{q_n} \leqslant (b-1)\sum_{k=n+1}^{\infty}\frac{1}{b^{k!}} < \frac{b-1}{b^{(n+1)!}}\sum_{j=0}^{\infty}\frac{1}{b^j} = \frac{b}{b^{(n+1)!}} = \frac{1}{(b^{n!})^n} = \frac{1}{q_n^n}.$$

The following corollary provides the counterexample.

COROLLARY 5. *Suppose the rotation number of Φ is Liouville, so that* (53) *holds. Then for any N, we can find a function w so that $w \in C^N$ but $w \notin \mathcal{W}$; that is $|||w||| = \infty$.*

Proof. According to (50), (51), we have for each q, with $W_q = e^{i\,q\,\rho}$,

$$\frac{\left|\sum_{j=0}^{k-1} W_q(\tau_j)\right|}{|\tau_k - \tau_0|} = \frac{|q|}{|1 - e^{i\,q\,\rho}|} = \left(2\pi\,\min_m\left|\frac{m}{q} - \frac{\rho}{2\pi}\right|\right)^{-1}.$$

Thus for ρ satisfying (53) and each n, we have

$$\frac{\left|\sum_{j=0}^{k-1} W_{\pm q_n}(\tau_j)\right|}{|\tau_k - \tau_0|} \geqslant \frac{q_n^n}{2\pi}. \tag{54}$$

Thus, setting $W = \sum \gamma_q W_q$, and $w = W \circ h$, with h bi-Lipschitz, by (49) it follows that if $\|w\| < \infty$, the Fourier coefficients γ_{q_n} of W must decay faster than q_n^{-n}, and we cannot bound the exponent. On the other hand, we have

$$W = \sum \gamma_q W_q \in H^s \quad \text{iff} \quad \sum |\gamma_q|^2 q^{2s} < \infty,$$

so Fourier coefficients decay like $o(q^{-s})$, where s is a fixed finite number. Thus, given the sequence q_n satisfying (53), we choose $s > N + 1/2$ and set

$$W_*(\tau) := \sum_{n=1}^{\infty} q_n^{-(s+1)} W_{\pm q_n}(\tau), \tag{55}$$

so that $W_* \in H^s \subset C^N$. To show that $\|w\| = \infty$, we set $\tau_0 = 0$, fix $k > 1$ and introduce $\theta_{\pm n} \in \mathbb{C}$ by

$$\theta_{\pm n} := \frac{1}{q_n^n} \frac{\sum_{j=0}^{k-1} W_{\pm q_n}(\tau_j)}{\tau_k - \tau_0},$$

so that (54) can be written as

$$\frac{\sum_{j=0}^{k-1} W_{\pm q_n}(\tau_j)}{\tau_k - \tau_0} = \theta_{\pm n}\, q_n^n, \quad \text{with} \quad |\theta_{\pm n}| \geqslant \frac{1}{2\pi}.$$

Then, with W_* given by (55), we have

$$\frac{\left|\sum_{j=0}^{k-1} W_*(\tau_j)\right|}{|\tau_k - \tau_0|} = \Big| \sum_{n=1}^{\infty} q_n^{n-(s+1)}\, \theta_{\pm n} \Big|,$$

which cannot converge because the terms of the sum don't converge to zero. This implies that $\|w\| = \infty$. ∎

Having obtained Lipshitz bounds for the solution v of (1), we now bootstrap to obtain bounds on higher derivatives of v. Recall the equivalence of norms $\|\cdot\|_{Lip} = \|\cdot\|_{C^{0,1}} = \|\cdot\|_{W^{1,\infty}}$.

COROLLARY 6. *Suppose that the rotation number $\rho = \rho(\Phi)$ of the shift Φ is diophantine of order r as in (46), and that the conjugation h in (44) is smooth enough. Let constants $1/2 < \nu \leqslant 1$, $2 \leqslant p \leqslant \infty$ and integer $\ell \geqslant 1$ be given. If the input w of (1) satisfies $\int w\, d\mu = 0$ and $w \in C^{\ell+r,\nu}$ or $w \in H^{\ell+r+\nu}$, then the solution v has ℓ Lipshitz derivatives, with bounds*

$$\|v\|_{C^{\ell,1}} \leqslant K\, \|w\|_{C^{\ell+r,\nu}} \quad \text{or} \quad \|v\|_{W^{\ell+1,\infty}} \leqslant K\, \|w\|_{H^{\ell+r+\nu}}, \tag{56}$$

respectively. Moreover, the regularity of the solution can be quantified by a simple loss of r derivatives in Hölder and/or Sobolev spaces, via the estimates

$$\|v\|_{C^{\ell,\nu}} \leqslant K\, \|w\|_{C^{\ell+r,\nu}}, \quad \text{or} \quad \|v\|_{W^{\ell',p}} \leqslant K\, \|w\|_{W^{\ell'+r,p}}, \tag{57}$$

respectively, with $\ell' = \ell + 1$. Here the constants K depend on Φ through its dependence on ρ and h.

Proof. We translate the original equation (1) with shift Φ to an equation with pure rotation: that is, we use the conjugation to directly rewrite the equation. Using (48), define $W = w \circ h^{-1}$ and $V = v \circ h^{-1}$, so that the difference equation (1) becomes

$$V(\tau + \rho) - V(\tau) = W(\tau), \tag{58}$$

with solution

$$V(\tau + k\,\rho) = V(\tau) + \sum_{j=0}^{k-1} W(\tau + j\,\rho),$$

as above. Since (58) is linear with constant shift, differentiation shows that derivatives solve the same equation. Thus the results of the theorem apply directly to the derivatives $W^{(\ell)}$ and $V^{(\ell)}$, and by assuming Φ (and thus h) is smooth enough, the norms on v, w can be taken as equivalent to the corresponding norms on V and W as in (49). Thus applying the theorem after differentiating, we have

$$\|V^{(\ell)}\|_{C^{0,1}} \leqslant K\,\|W^{(\ell)}\|_{C^{r,\nu}}, \quad \text{or} \quad \|V^{(\ell)}\|_{W^{1,\infty}} \leqslant K\,\|W^{(\ell)}\|_{W^{r+\nu,2}},$$

respectively, and the equivalence of norms immediately implies (56). Since $\|\cdot\|_{C^{\ell,\nu}} \leqslant K\,\|\cdot\|_{C^{\ell,1}}$, while also

$$\|\cdot\|_{W^{\ell',p}} \leqslant K\,\|\cdot\|_{W^{\ell+1,\infty}} \quad \text{and} \quad \|\cdot\|_{W^{\ell+r+\nu,2}} \leqslant K\,\|\cdot\|_{W^{\ell'+r,p}}.$$

the inequalities (57) now follow. □

5. Strategy for Nash-Moser Iteration. The philosophy of the Nash-Moser iteration is that a finite number of derivatives can be lost at each step in the approximations generated by Newton's method. The Nash-Moser method proceeds by smoothing the approximation at each step. The amount of required smoothing decreases rapidly as the iteration continues because of the fast (quadratic) convergence of Newton's method [10, 15, 1].

In Nash-Moser problems, the linearized operator in the Newton step is typically not uniformly invertible except possibly off complicated sets of small measure of a controllable parameter. In such cases, an extra step is required, namely, the expunging of a small set of "bad" values of this parameter. Off this bad set, the inverse exists with bounds sufficient to complete the Newton step. Convergence then requires that the measure of the union of expunged sets be smaller than the total measure of the set allowable parameters, so the Nash-Moser method then converges on a set of positive measure. This expunging process is generally technical, and requires removal of increasingly small sets at each step of the Newton iteration, and the measure of such expunged sets must be carefully estimated.

In the current work we invert the simplest case of a linearized difference operator which arises when imposing a periodicity condition in a quasilinear problem. We have shown that this operator is invertible with a uniform loss of derivatives on an appropriate set of rotation numbers $\rho(\Phi)$, any one of which which can be imposed by solving for the drift angle θ.

Even though our difference operator is based on a nonlinear Burgers model which has no non-trivial kernel, we believe that the essential difficulties dealt with in inverting the linearized operators here, accurately reflect essential structural issues which apply to the problem of inverting the linearized operators in authors' Nash-Moser approach to constructing periodic solutions of the compressible Euler equations – the case when there is a nontrivial kernel in the nonlinear problem. The free parameter in the Euler problem is a pair of analogous drift angles $(\overline{\theta}, \underline{\theta})$, which are determined by the underlying constant state. In our previous attempts to apply Nash-Moser to the problem of finding periodic solutions of the compressible Euler equations [15], we have been faced with the problem of having to expunge a finite measure set of the drift angles at each step. This fails to provide convergence because there are infinitely many steps.

The results of the present paper suggest a new proposal for implementing Nash-Moser for Euler that overcomes this problem. Because the nonlinear Euler problem is quasilinear, the constant states (0-modes) automatically satisfy the linearized equation, and we can thus regard the *constant state* as the free parameter for fixing the drift angle. The new idea here is to obtain a desired (fixed) rotation number $\rho(\Phi)$ by *solving* for θ (or equivalently the 0-mode v_0). By this proposal, we can take the rotation number ρ as fixed *a priori*, making our linearized operator invertible with uniform loss of derivatives, for the appropriate values of the parameter θ which generate that rotation number ρ. Based on our theory here, we need only expunge at the beginning of the iteration when we choose the fixed rotation number ρ. For a given loss of derivatives r, we choose ρ to meet the diophantine condition (46), thus bounding the measure of the set of excluded ρ. Therefore in essence, based on our theory here, the expunging in the Newton method can be implemented by solving for our free parameter θ, or equivalently v_0, so that the correct rotation number is achieved at each step. Indeed, the map $\theta \mapsto \Phi \mapsto \rho$ is a continuous Cantor function which cannot be inverted directly, but we can solve for θ in terms of ρ at each step because the map $\theta \mapsto \rho$ is continuous and monotone. In this way the expunging of θ is done in terms of ρ, *implicitly* at each step, rather than *explicitly*.

By this procedure, a new constant state is selected at each step of the Newton iteration, to obtain a sequence of constant states uniformly bounded by the amplitude parameter α. From this sequence we can extract a convergent subsequence of constant states to obtain the limiting constant state of the nonlinear solution. The cost of implicitly expunging at each step is then the extraction of a convergent subsequence of constant states, but the Nash-Moser iteration would proceed and converge as usual for the other modes.

From the beginning [11, 14, 13], authors recognized the difficulty of inverting the linearized operators which arise in the Newton step for periodic solutions of quasilinear problems. In these cases, the nonlinear operator has a kernel because the linearized operators have a similar form to the nonlinear operator. Namely, the linearized operators are inhomogeneous difference operators expressed in terms of shift operators, the simplest case of which is the setting in this paper. We earlier identified the Liapunov-Schmidt decomposition as a standard way of projecting orthogonally off the kernel of the linearized operator, in an attempt to make the linearized operators invertible on the complement of the kernel [13]. In light of the results of this paper, this standard Liapunov-Schmidt method fails because the projection cannot be uniformly defined in θ due to the singularity of the mapping $\theta \mapsto \rho$. This new proposed strategy to expunge constant states can thus be regarded as bypassing a Liapunov-Schmidt decomposition. That is, for a positive measure set of *rotation numbers* ρ, rather than drift parameter θ, the linearized operator is actually invertible with a *uniform* loss of derivatives.

What still needs to be accomplished to apply this program of constructing periodic solutions of the compressible Euler equations, is to establish a vectorized version of the results here. That is, we need a theory for inverting linearized shift operators which impose periodicity, appropriately vectorized to account for the fact that the 3×3 Euler problem has two nonlinear characteristic fields and a linearly degenerate field, which we view as a coupling of scalar nonlinear characteristic fields like those addressed here.

6. Appendix: Relevant Theorems for Circle Maps. For completeness we record the following theorems from Herman's fundamental paper [8], which we rely on in the proof of Theorem 5[2]. This provides the bridge between the subject of circle maps and our problem of inverting shift operators on the circle.

To help the reader, note that Herman uses the following notation: $D^r(\mathbb{S}^1)$ denotes the space of functions f which are C^r diffeomorphisms of the circle $\mathbb{S}^1$, and which can be written $f = id + \phi$, where $\phi : \mathbb{S}^1 \to \mathbb{R}^1$ is a C^r map on the circle. The derivative of f is denoted Df, and Herman introduces the seminorms

$$H_1(f) = \sup_{n\in\mathbb{Z}} \|D(f^n)\|_0, \qquad H_r(f) = \sup_{n\in\mathbb{Z}} \|D(f^n)\|_{C^{r-1}},$$

where $\|\cdot\|_0$ denotes the supnorm. This removes the constant term from the C^r norm, in analogy with our treatment of the Lipshitz norm in (42), (43).

Theorem IV.6.1.1 (page 48): A necessary and sufficient condition for $f \in D^1(\mathbb{S}^1)$ to be C^1 conjugate to a rotation R_ρ with rotation number $\rho = \rho(f)$ is that $H_1(f) < \infty$.

Proposition IV.6.2 (page 50): Let $f = g^{-1} \circ R_\rho \circ g$ with $g \in D^r(\mathbb{S}^1)$, and $\rho \in \mathbb{R}\backslash\mathbb{Q}$, $1 < r < \infty$. Then we have

$$\|Dg\|_{C^{r-1}} + \|Dg^{-1}\|_0 \leqslant (r+1)H_r(f).$$

Corollary IV.6.3.2 (Page 51): Let $f \in D^r(\mathbb{S}^1)$ with $r \geqslant 1$, such that $\rho(f) = \rho \in \mathbb{R}\backslash\mathbb{Q}$, and $H_r(f) < \infty$; then

$$f = h^{-1} \circ R_\rho \circ h,$$

where $h \in D^{r-1}(\mathbb{S}^1)$ and $D^{r-1}(h)$ is Lipschitz. This implies $D^{r-1}(h^{-1})$ is Lipschitz.

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