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LOCAL WELL-POSEDNESS OF A HAMILTONIAN REGULARISATION OF THE SAINT-VENANT SYSTEM WITH UNEVEN BOTTOM*

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Abstract. We prove in this note the local (in time) well-posedness of a broad class of 2×2 symmetrisable hyperbolic system involving additional non-local terms. The latest result implies the local well-posedness of the non dispersive regularisation of the Saint-Venant system with uneven bottom introduced by Clamond et al. [2]. We also prove that, as long as the first derivatives are bounded, singularities cannot appear.

Key words. Dispersionless shallow water equations, nonlinear hyperbolic systems, Hamiltonian regularisation, energy conservation.

Mathematics Subject Classification. 35Q35, 35L65, 37K05, 35B65, 76B15.

1. Introduction and main results. Clamond and Dutykh [1] have recently proposed a new Hamiltonian regularisation of the Saint-Venant (rSV) system with a constant bottom. Such a regularisation avoids some defects of diffusive approximations which flatten the shocks, and the dispersive ones which add spurious oscillations. This Hamiltonian regularisation [1] is of a new kind, non-diffusive and non-dispersive. This new model has been mathematically studied in [11, 12]. Inspired by [1], similar regularisations have been proposed for the inviscid Burgers equation [8], the scalar conservation laws [6] and the barotropic Euler system [7]. A regularisation of the Saint-Venant equations with uneven bottom (rSVub) has also been proposed by Clamond et al. [2]. The latter equations, for the conservation of mass and momentum, can be written in the conservative form

$$h_t \,+\, [\,h\,u\,]_x \,=\, 0, \tag{1a}$$

$$[\,h\,u\,]_t \,+\, \left[\,h\,u^2 \,+\, \tfrac{1}{2}\,g\,h^2 \,+\, \varepsilon\,\mathscr{R}\,\right]_x \,=\, \varepsilon\,g\,h^2\,\eta_x\,d_{xx} \,+\, g\,h\,d_x, \tag{1b}$$

$$\mathscr{R} \,\stackrel{\text{def}}{=}\, 2\,h^3\,u_x^2 \,-\, h^3\,[\,u_t \,+\, u\,u_x \,+\, g\,\eta_x\,]_x \,-\, \tfrac{1}{2}\,g\,h^2\,\left(\eta_x^2 + 2\,\eta_x\,d_x\right). \tag{1c}$$

Here, $u = u(t,x)$ is the depth-averaged horizontal velocity, $h = h(t,x) \stackrel{\text{def}}{=} \eta(t,x) + d(t,x)$ denotes the total water depth, η being the surface elevation from rest and d being the water depth for the unperturbed free surface. We can assume, without losing generality via a change of frame of reference, that the spacial average of the depth $\bar{d}$ is constant in time. In that case, the gravity acceleration $g = g(t)$ may be a function of time. Introducing the Sturm–Liouville operator

$$\mathcal{L}_h \,\stackrel{\text{def}}{=}\, h \,-\, \varepsilon\,\partial_x\,h^3\,\partial_x, \tag{2}$$

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if $h > 0$, the operator $\mathscr{L}_h$ is invertible, then the system (1) can be written on the form

$$h_t + [h\,u]_x = 0, \tag{3a}$$

$$u_t + u\,u_x + g\,\eta_x = \varepsilon\,g\,\mathscr{L}_h^{-1}\left\{h^2\,\eta_x\,d_{xx}\right\} - \varepsilon\,\mathscr{L}_h^{-1}\,\partial_x\,\left\{2\,h^3\,u_x^2 - \tfrac{1}{2}\,g\,h^2\,\left(\eta_x^2 + 2\,\eta_x\,d_x\right)\right\}. \tag{3b}$$

The regularised Saint-Venant system admits a Hamiltonian structure, thus, it necessarily conserves[1] the corresponding energy for smooth solutions. The energy equation of (3) writes

$$\begin{aligned}&\left[\tfrac{1}{2}\,h\,u^2 + \tfrac{1}{2}\,\varepsilon\,h^3\,u_x^2 + \tfrac{1}{2}\,g\,\eta^2 + \tfrac{1}{2}\,\varepsilon\,g\,h^2\,\eta_x^2\right]_t \\ &+ \left[\left(\tfrac{1}{2}\,h\,u^2 + g\,h\,\eta + \tfrac{1}{2}\,\varepsilon\,h^3\,u_x^2 + \tfrac{1}{2}\,\varepsilon\,g\,h^2\,\eta_x^2 + \varepsilon\,\mathscr{R}\right)u + \varepsilon\,g\,h^3\,\eta_x\,u_x\right]_x \\ &= \tfrac{1}{2}\,\dot{g}\left(\eta^2 + \varepsilon\,h^2\,\eta_x^2\right) - g\,\eta\,d_t - \varepsilon\,g\,h^2\,\eta_x\,d_{xt}.\end{aligned} \tag{4}$$

Note that, injecting (3b) in (1c), one obtains the alternative definition of $\mathscr{R}$

$$\begin{aligned}\mathscr{R} &= \left(1 + \varepsilon\,h^3\,\partial_x\,\mathscr{L}_h^{-1}\,\partial_x\right)\left\{2\,h^3\,u_x^2 - \tfrac{1}{2}\,g\,h^2\,\left(\eta_x^2 + 2\eta_x d_x\right)\right\} \\ &\quad - \varepsilon\,h^3\,\partial_x\,\mathscr{L}_h^{-1}\,\left\{g\,h^2\,\eta_x\,d_{xx}\right\}.\end{aligned}$$

The rSV and rSVub equations can be compared with the Serre–Green–Naghdi and the two-component Camassa–Holm equations. The local well-posedness of those equations have been studied in the literature (see, e.g., [9, 5]; see also [3] for higher-order Camassa–Holm equations). Liu et al. [11] have proved the local well-posedness of the rSV equations introduced in [1] for constant depth. Liu et al. [11] have constructed some small initial data, such that the corresponding solutions blow-up in finite time. The goal of the present note is to prove the local (in time) well-posedness of the rSVub equations. To this aim, we prove first the local (in time) well-posedness of a general 2×2 symmetrisable hyperbolic system. Then, using some estimates of the operator $\mathscr{L}_h^{-1}$, we prove that the system (3) is locally well-posed in the Sobolev space $H^s(\mathbb{R})$ for any real number $s \geqslant 2$. We also prove that if the L^∞-norm of the first derivatives remain bounded, then the singularities cannot appear in finite time.

In order to state the main results of this note, let d be a smooth function of t and x with

$$h \stackrel{\text{def}}{=} \eta + d, \quad \bar{d} \stackrel{\text{def}}{=} \lim_{|x|\to+\infty} d(t,x) > 0, \quad \text{and} \quad \inf_{(t,x)\in\mathbb{R}^+\times\mathbb{R}} d(t,x) > 0, \tag{5}$$

defining the Sobolev space $H^s \stackrel{\text{def}}{=} H^s(\mathbb{R}) \stackrel{\text{def}}{=} \left\{u,\ \int_{\mathbb{R}}\left(1+\xi^2\right)^s|\hat{u}(\xi)|^2\,\mathrm{d}\xi < +\infty\right\}$ where $\hat{u}$ is the Fourier transform of u, then

Theorem 1. *Let $\tilde{m} \geqslant s \geqslant 2$, $0 < g \in C^1([0,+\infty[)$, $d - \bar{d} \in C([0,+\infty], H^{s+1}) \cap C^1([0,+\infty], H^s)$ and let $W_0 = (\eta_0, u_0)^\top \in H^s$ satisfying $\inf_{x\in\mathbb{R}} h_0(x) \geqslant h^* > 0$, then there exist $T > 0$ and a unique solution $W = (\eta, u) \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$ of (3) satisfying the non-zero depth condition $\inf_{(t,x)\in[0,T]\times\mathbb{R}} h(t,x) > 0$. Moreover, if the maximal time of existence $T_{max} < +\infty$, then*

$$\lim_{t\to T_{max}} \|W\|_{H^s} = +\infty \qquad or \qquad \inf_{(t,x)\in[0,T_{max}[\times\mathbb{R}} h(t,x) = 0. \tag{6}$$

[1] If the energy changes, it is necessarily due to exterior forces (the moving bottom) and not due to the dynamic itself.

Using the energy equation (4) and some estimates, the blow-up criteria (6) can be improved.

THEOREM 2. *For any interval* $[0,T] \subset [0,T_{max}[$ *(T_{max} is the life span of the smooth solution), there exists* $C>0$*, such that* $\forall t \in [0,T]$ *we have*

$$\mathscr{E}(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \left[\, \tfrac{1}{2}\, h\, u^2 \,+\, \tfrac{1}{2}\, \varepsilon\, h^3\, u_x^2 \,+\, \tfrac{1}{2}\, g\, \eta^2 \,+\, \tfrac{1}{2}\, \varepsilon\, g\, h^2\, \eta_x^2 \,\right] \mathrm{d}x \;\leqslant\; C. \tag{7}$$

Moreover, if $T_{max} < +\infty$*, then*

$$\lim_{t \to T_{max}} \|W_x\|_{L^\infty} \;=\; +\infty. \tag{8}$$

Section 2 is devoted to prove the local well-posedness of a general 2×2 system. The proofs of Theorems 1 and 2 are given in Section 3.

2. Local well-posedness of a general 2×2 system. We prove here the local well-posedness of a class of systems with non-local operators in the H^s space with $s > 3/2$. Let d be a smooth function such that (5) holds. Let also $N \geqslant 1$ be a natural number and $G \stackrel{\text{def}}{=} (g_1, \cdots, g_N)$ be a smooth function of t and x, possibly depending on d, such that

$$\begin{aligned} g_\infty(t) \;&\stackrel{\text{def}}{=}\; g_1(t,\infty) \;=\; \lim_{|x|\to\infty} g_1(t,x) \;>\; 0 \quad \text{and} \\ g_{\inf} \;&\stackrel{\text{def}}{=}\; \inf_{(t,x)\in\mathbb{R}^+\times\mathbb{R}} g_1(t,x) \;>\; 0. \end{aligned} \tag{9}$$

Let $f(d,h)$ be a positive function and let f_1, f_2 be functions of d, h, u, η_x, u_x and G. Let also a, b, c, f_3, f_4 be functions of d, h, u and G. We consider the symmetrisable hyperbolic system

$$\eta_t \;+\; a(d,h,u,G)\,\eta_x \;+\; b(d,h,u,G)\,u_x \;=\; \mathfrak{A}_1\, f_1 \;+\; \mathfrak{A}_3\, f_3, \tag{10a}$$

$$u_t \;+\; g_1\, f(d,h)\, b(d,h,u,G)\,\eta_x \;+\; c(d,h,u,G)\,u_x \;=\; \mathfrak{A}_2\, f_2 \;+\; \mathfrak{A}_4\, f_4, \tag{10b}$$

where the $\mathfrak{A}_j$ are linear operators depending on h and u. In order to obtain the well-posedness of the system (10) in H^s with $s > 3/2$, we define $W \stackrel{\text{def}}{=} (\eta, u)^T$, $G_0 \stackrel{\text{def}}{=} (g_\infty, 0, \cdots, 0)$ and

$$B(W) \;\stackrel{\text{def}}{=}\; \begin{pmatrix} a(d,h,u,G) & b(d,h,u,G) \\ g_1\, f(d,h)\, b(d,h,u,G) & c(d,h,u,G) \end{pmatrix},$$

$$F(W) \;\stackrel{\text{def}}{=}\; \begin{pmatrix} \mathfrak{A}_1\, f_1(d,h,u,h_x,u_x,G) \;+\; \mathfrak{A}_3\, f_3(d,h,u,G) \\ \mathfrak{A}_2\, f_2(d,h,u,h_x,u_x,G) \;+\; \mathfrak{A}_4\, f_4(d,h,u,G) \end{pmatrix},$$

the system (10) can be written as

$$W_t \;+\; B(W)\,W_x \;=\; F(W), \qquad W(0,x) \;=\; W_0(x). \tag{11}$$

We assume that:

(A1) For $s \leqslant \tilde{m} \in \mathbb{N}$, we have

- $d - \bar{d}, g_1 - g_\infty, g_2, g_3, \cdots, g_N \;\in\; C(\mathbb{R}^+, H^s)$ and $d - \bar{d}, g_1 - g_\infty \;\in\; C^1(\mathbb{R}^+, H^{s-1})$;
- $f \in C^{\tilde{m}+2}(]0,+\infty[^2)$ and for all $h_1, h_2 > 0$ we have $f(h_1,h_2) > 0$;

- $f_1, f_2 \in C^{\tilde{m}+2}(]0,+\infty[^2\times\mathbb{R}^3\times]0,+\infty[\times\mathbb{R}^{N-1})$;
- $a, b, c, f_3, f_4 \in C^{\tilde{m}+2}(]0,+\infty[^2\times\mathbb{R}\times]0,+\infty[\times\mathbb{R}^{N-1})$;
- $f_1(\bar{d},\bar{d},0,0,0,G_0) = f_2(\bar{d},\bar{d},0,0,0,G_0) = f_3(\bar{d},\bar{d},0,G_0) = f_4(\bar{d},\bar{d},0,G_0) = 0$.

(A2) For all $r \in [s-1, s]$, if $\phi \in H^r$ and $\psi \in H^{r-1}$, then

$$\begin{aligned} \|\mathfrak{A}_1\,\psi\|_{H^r} \;+\; \|\mathfrak{A}_2\,\psi\|_{H^r} \;&\leqslant\; C(s,r,d,\|W\|_{H^r})\,\|\psi\|_{H^{r-1}}, \\ \|\mathfrak{A}_3\,\phi\|_{H^r} \;+\; \|\mathfrak{A}_4\,\phi\|_{H^r} \;&\leqslant\; C(s,r,d,\|W\|_{H^r})\,\|\phi\|_{H^r}. \end{aligned}$$

(A3) If $\phi, W, \tilde{W} \in H^s$ and $\psi \in H^{s-1}$, then

$$\begin{aligned} \|(\mathfrak{A}_1(W)-\mathfrak{A}_1(\tilde{W}))\,\psi\|_{H^{s-1}} \;+\; \|(\mathfrak{A}_2(W)-\mathfrak{A}_2(\tilde{W}))\,\psi\|_{H^{s-1}} \;&\leqslant\; C\,\|W-\tilde{W}\|_{H^{s-1}}, \\ \|(\mathfrak{A}_3(W)-\mathfrak{A}_3(\tilde{W}))\,\phi\|_{H^{s-1}} \;+\; \|(\mathfrak{A}_4(W)-\mathfrak{A}_4(\tilde{W}))\,\phi\|_{H^{s-1}} \;&\leqslant\; C\,\|W-\tilde{W}\|_{H^{s-1}}, \end{aligned}$$

where $C = C\left(s, d, \|W\|_{H^s}, \|\tilde{W}\|_{H^s}, \|\phi\|_{H^s}, \|\psi\|_{H^{s-1}}\right)$.

Note that if h is far from zero (i.e., $\inf h > 0$), then $g_1 f(d,h)$ is positive and far from zero. Then, the system (10) is symmetrisable and hyperbolic. The main result of this section is the following theorem:

THEOREM 3. *For $s > 3/2$ and under the assumptions* (A1), (A2) and (A3), if $W_0 \in H^s$ satisfy the non-emptiness condition

$$\inf_{x\in\mathbb{R}} h_0(x) \;=\; \inf_{x\in\mathbb{R}} \left(\eta_0(x) \,+\, d(0,x)\right) \;\geqslant\; h^* >\; 0, \tag{12}$$

then there exist $T > 0$ and a unique solution $W \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$ of the system (11). Moreover, if the maximal existence time $T_{max} < +\infty$, then

$$\inf_{(t,x)\in[0,T_{max}[\times\mathbb{R}} h(t,x) \;=\; 0 \qquad \text{or} \qquad \lim_{t\to T_{max}} \|W\|_{H^s} \;=\; +\infty. \tag{13}$$

REMARKS. (i) Theorem 3 holds also for periodic domains; (ii) The right-hand side of (11) can be replaced by a finite sum on the form

$$F(W) \;=\; \begin{pmatrix} \mathfrak{A}_1\,f_1 \;+\; \mathfrak{A}_3\,f_3 \\ \mathfrak{A}_2\,f_2 \;+\; \mathfrak{A}_4\,f_4 \end{pmatrix} \;+\; \begin{pmatrix} \mathfrak{B}_1\,k_1 \;+\; \mathfrak{B}_3\,k_3 \\ \mathfrak{B}_2\,k_2 \;+\; \mathfrak{B}_4\,k_4 \end{pmatrix} \;+\; \cdots, \tag{14}$$

where the additional terms satisfy also the conditions (A1), (A2) and (A3); (iii) Under some additional assumptions, the blow-up criteria (13) can be improved (see Theorem 2, for example); (iv) If for some $2 \leqslant i \leqslant N$, the function g_i appears only on f_1 and f_2, then, due to (A2), the assumption $g_i \in C(\mathbb{R}^+, H^s)$ can be replaced by $g_i \in C(\mathbb{R}^+, H^{s-1})$.

In order to prove the local well-posedness of (11), we consider

$$\partial_t\, W^{n+1} \;+\; B(W^n)\,\partial_x\, W^{n+1} \;=\; F(W^n), \qquad W^n(0,x) \;=\; (\eta_0(x), u_0(x))^T, \tag{15}$$

where $n \geqslant 0$ and $W^0(t,x) = (\eta_0(x), u_0(x))^\top$. The idea of the proof is to solve the linear system (15), then, taking the limit $n \to \infty$, we obtain a solution of (11). Note that we have assumed that g_1 and f are positive, so $g_1 f > 0$, then the system (15) is hyperbolic; it is an important point to solve each iteration in (15). Note that a symmetriser of the matrix $B(W)$ is $A(W) \stackrel{\text{def}}{=} \begin{pmatrix} g_1\,f(d,h) & 0 \\ 0 & 1 \end{pmatrix}$.

Let $(\cdot,\,\cdot)$ be the scalar product in L^2 and let the energy of (15) be defined as

$$E^{n+1}(t) \;\stackrel{\text{def}}{=}\; \left(\Lambda^s\, W^{n+1},\; A^n\,\Lambda^s\, W^{n+1}\right)\;\; \forall t \geqslant 0.$$

If $g_1 f$ is bounded and far from 0, then $E^n(t)$ is equivalent to $\|W^n\|_{H^s}$. In order to prove Theorem 3, the following results are needed.

Theorem 4. *Let $s > 3/2$, $h^* > 0$ and $R > 0$, then there exist $K, T > 0$ such that: if the initial data $(\eta_0, h_0) \in H^s$ satisfy*

$$\inf_{x\in\mathbb{R}} h_0(x) \;\geqslant\; 2\,h^*, \qquad E^n(0) \;<\; R, \tag{16}$$

and $W^n \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$, satisfying for all $t \in [0,T]$

$$h^n \;\geqslant\; h^*, \qquad \|(W^n)_t\|_{H^{s-1}} \;\leqslant\; K, \qquad E^n(t) \;\leqslant\; R, \tag{17}$$

then there exists a unique $W^{n+1} \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$ solution of (15) such that

$$h^{n+1} \;\geqslant\; h^*, \qquad \|(W^{n+1})_t\|_{H^{s-1}} \;\leqslant\; K, \qquad E^{n+1}(t) \;\leqslant\; R. \tag{18}$$

The proof of Theorem 4 is classic (it can be done following Guelmame et al. [7], Israwi [9], Liu et al. [11] and using the following lemmas).

Let Λ be defined such that $\widehat{\Lambda f} = (1+\xi^2)^{\frac{1}{2}}\hat{f}$, and let $[A,B] \stackrel{\text{def}}{=} AB - BA$ be the commutator of the operators A and B. We have the following lemma.

Lemma 1 (Kato and Ponce [10]). *If $r \geqslant 0$, then*

$$\|f\,g\|_{H^r} \;\lesssim\; \|f\|_{L^\infty}\,\|g\|_{H^r} \;+\; \|f\|_{H^r}\,\|g\|_{L^\infty}, \tag{19}$$

$$\|[\Lambda^r,\, f]\; g\|_{L^2} \;\lesssim\; \|f_x\|_{L^\infty}\,\|g\|_{H^{r-1}} \;+\; \|f\|_{H^r}\,\|g\|_{L^\infty}. \tag{20}$$

Lemma 2. *Let $k \in \mathbb{N}^*$, $F \in C^{m+2}(\mathbb{R}^k)$ with $F(0,\cdots,0) = 0$ and $0 \leqslant s \leqslant m$, then there exists a continuous function $\tilde{F}$, such that for all $f = (f_1,\cdots,f_k) \in H^s \cap W^{1,\infty}$ we have*

$$\|F(f)\|_{H^s} \;\leqslant\; \tilde{F}\left(\|f\|_{W^{1,\infty}}\right)\,\|f\|_{H^s}. \tag{21}$$

Proof. The case $k=1$ has been proved in [4]. Here, we prove the inequality (21) by induction (on s). Note that

$$F(f_1,\cdots,f_k) \;=\; F(0,f_2,\cdots,f_k) \;+\; \int_0^{f_1} F_{f_1}(g_1,f_2,\cdots,f_k)\,\mathrm{d}g_1$$

$$=\; F(0,0,f_3,\cdots,f_k) + \int_0^{f_1} F_{f_1}(g_1,f_2,\cdots,f_k)\,\mathrm{d}g_1 + \int_0^{f_2} F_{f_2}(0,g_2,f_3,\cdots,f_k)\,\mathrm{d}g_2 \;+\; \cdots$$

$$=\; \int_0^{f_1} F_{f_1}(g_1,f_2,\cdots,f_k)\,\mathrm{d}g_1 \;+\; \cdots \;+\; \int_0^{f_k} F_{f_k}(0,\cdots,0,g_k)\,\mathrm{d}g_k.$$

This implies that

$$\|F(f_1,\cdots,f_k)\|_{L^2} \;\lesssim\; \|f\|_{L^2}, \tag{22}$$

which is (21) for $s=0$. For $s\in]0,1[$, let

$$
\begin{aligned}
&\left|F(f_1(x+y),\cdots,f_k(x+y)) \,-\, F(f_1(x),\cdots,f_k(x))\right| \\
\leqslant\ & \left|F(f_1(x+y),\cdots,f_k(x+y)) \,-\, F(f_1(x),f_2(x+y),\cdots,f_k(x+y))\right| \\
&+\ \left|F(f_1(x),f_2(x+y),\cdots,f_k(x+y)) \,-\, F(f_1(x),f_2(x),f_3(x+y),\cdots,f_k(x+y))\right| \\
&+\ \cdots\ +\ \left|F(f_1(x),\cdots,f_{k-1}(x),f_k(x+y)) - F(f_1(x),\cdots,f_k(x))\right| \\
\leqslant\ & \sum_{i=1}^{k} |f_i(x+y) \,-\, f_i(x)|\ \|F_{f_i}\|_{L^\infty}.
\end{aligned}
$$

The last inequality, with the definition $H^s \stackrel{\text{def}}{=} \left\{f\in L^2,\ \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x+y)-f(x)|^2}{|y|^{1+2\,s}}\,\mathrm{d}x\,\mathrm{d}y<+\infty\right\}$, implies (21) for $s\in]0,1[$. For $s\geqslant 1$, the proof is done by induction. Using (19) and (22), we obtain $\|F(f)\|_{H^s}\lesssim \left\|\sum_{i=1}^{k}F_{f_i}(f)\,\partial_x f_i\right\|_{H^{s-1}}+\|F(f)\|_{L^2}\lesssim \|f\|_{H^s}+\sum_{i=1}^{k}\|F_{f_i}(f)\|_{H^{s-1}}$. Using the induction and the last inequality, we obtain (21) for all $s\geqslant 0$. $\square$

Proof of Theorem 3. Using Theorem 4, one obtains that (W^n) is uniformly bounded in $C([0,T],H^s)\cap C^1([0,T],H^{s-1})$ and satisfies $h^n\geqslant h^*$. Defining

$$
\tilde{E}^{n+1}(t)\ \stackrel{\text{def}}{=}\ \left(\Lambda^{s-1}\,(W^{n+1}\,-\,W^n),\ A^n\,\Lambda^{s-1}\,(W^{n+1}\,-\,W^n)\right), \tag{23}
$$

and using (15), one obtains

$$
\begin{aligned}
\tilde{E}^{n+1}_t\ &= 2\,\left(\Lambda^{s-1}\left(F^n-F^{n-1}+(B^{n-1}-B^n)W^n_x\right),\ A^n\,\Lambda^{s-1}\,(W^{n+1}\,-\,W^n)\right) \\
&-\ 2\,\left([\Lambda^{s-1},B^n](W^{n+1}-W^n)_x,\ A^n\,\Lambda^{s-1}(W^{n+1}-W^n)\right) \\
&+\ \left(\Lambda^{s-1}\left((A^nB^n)_x(W^{n+1}-W^n)\right),\ \Lambda^{s-1}\,(W^{n+1}\,-\,W^n)\right) \\
&+\ \left(\Lambda^{s-1}\,(W^{n+1}\,-\,W^n),\ (A^n)_t\,\Lambda^{s-1}\,(W^{n+1}\,-\,W^n)\right).
\end{aligned} \tag{24}
$$

Using (A2), (A3), (19) and (21), one obtains

$$
\|F^n-F^{n-1}\|_{H^{s-1}}+\|(B^{n-1}-B^n)\partial_xW^n\|_{H^{s-1}}\ \lesssim\ \|W^n-W^{n-1}\|_{H^{s-1}}\ \lesssim\ \sqrt{\tilde{E}^n}, \tag{25}
$$

where "$\mathscr{A}\lesssim\mathscr{B}$" means $\mathscr{A}\leqslant C\mathscr{B}$, with $C>0$ is a constant independent of n. Using (20) and (21), we obtain

$$
\left\|[\Lambda^{s-1},B^n](W^{n+1}-W^n)_x\right\|_{L^2}\ \lesssim\ \|W^{n+1}-W^n\|_{H^{s-1}}\ \lesssim\ \sqrt{\tilde{E}^{n+1}}. \tag{26}
$$

From (19) and (21), it follows that

$$
\|(A^nB^n)_x(W^{n+1}-W^n)\|_{H^{s-1}}\ \lesssim\ \|W^{n+1}-W^n\|_{H^{s-1}}\ \lesssim\ \sqrt{\tilde{E}^{n+1}}. \tag{27}
$$

Combining the estimates above, we obtain that $\tilde{E}^{n+1}_t\lesssim\tilde{E}^{n+1}+\tilde{E}^n$, and using that $\tilde{E}^n(0)=0$, we obtain $\tilde{E}^{n+1}\leqslant\left(\mathrm{e}^{Ct}-1\right)\tilde{E}^n$ where $C>0$ does not depend on n. Taking $T>0$ small enough, it follows that

$$
\|W^{n+1}-W^n\|_{H^{s-1}}\lesssim\ \tilde{E}^{n+1}\ \leqslant\ \tfrac{1}{2}\,\tilde{E}^n\ \leqslant\ \tfrac{1}{2^n}\,\tilde{E}^1. \tag{28}
$$

Finally, taking the limit $n\to\infty$ in the weak formulation of (15) and using (A3), we obtain a solutions of the system (10). This completes the proof of Theorem 3. $\square$

3. Proof of Theorems 1 and 2. The system (3) is written in the form (10) by replacing the right-hand side of (10), as in (14), taking $N = 4$ and $G(t,x) = (g, d_x, d_{xx}, d_t)$, $a(d,h,u,g,d_x,d_{xx},d_t) = c(d,h,u,g,d_x,d_{xx},d_t) = u$, $b(d,h,u,g,d_x,d_{xx},d_t) = h$, $f(d,h) = h^{-1}$, $f_1 = f_4 = k_1 = k_3 = k_4 = 0$, $f_2(d,h,u,h_x,u_x,g,d_x,d_{xx},d_t) = 2h^3u_x^2 - \frac{1}{2}gh^2(\eta_x^2 + 2\eta_x d_x)$, $f_3(d,h,u,g,d_x,d_{xx},d_t) = -d_t - ud_x$, $k_2(d,h,u,h_x,u_x,g,d_x,d_{xx},d_t) = gh^2\eta_x d_{xx}$, $\mathfrak{A}_1 = \mathfrak{A}_4 = \mathfrak{B}_1 = \mathfrak{B}_3 = \mathfrak{B}_4 = 0$, $\mathfrak{A}_2 = -\varepsilon\mathcal{L}_h^{-1}\partial_x$ and $\mathfrak{A}_3 = 1$, $\mathfrak{B}_2 = \varepsilon\mathcal{L}_h^{-1}$. Then, in order to prove Theorem 1, the following lemma is needed:

LEMMA 3 (Liu et al. [11]). *Let $0 < h_{\inf} \leqslant h \in W^{1,\infty}$, then the operator $\mathcal{L}_h$ is an isomorphism from H^2 to L^2 and if $0 \leqslant s \leqslant \tilde{m} \in \mathbb{N}$, then*

$$\left\|\mathcal{L}_h^{-1}\psi\right\|_{H^{s+1}} + \left\|\mathcal{L}_h^{-1}\partial_x\psi\right\|_{H^{s+1}} \leqslant C\,\|\psi\|_{H^s}\left(1 + \left\|h - \bar{d}\right\|_{H^s}\right), \tag{29}$$

where C depends on s, ε, $h_{\inf}$, $\|h - \bar{d}\|_{W^{1,\infty}}$ and not on $\|h - \bar{d}\|_{H^s}$.

Proof of Theorem 1. In order to prove Theorem 1, it suffices to verify (A1)–(A3). The assumption (A1) is obviously satisfied and (A2) follows from Lemma 3. In order to prove (A3), let $W, \tilde{W}, \psi \in H^s$. Using Lemma 3 and (19), we obtain

$$\begin{aligned}\left\|\left(\mathcal{L}_h^{-1} - \mathcal{L}_{\tilde{h}}^{-1}\right)\psi\right\|_{H^{s-1}} &= \left\|\mathcal{L}_h^{-1}\left(\mathcal{L}_{\tilde{h}} - \mathcal{L}_h\right)\mathcal{L}_{\tilde{h}}^{-1}\psi\right\|_{H^{s-1}}\\ \lesssim \left\|\left(\mathcal{L}_{\tilde{h}} - \mathcal{L}_h\right)\mathcal{L}_{\tilde{h}}^{-1}\psi\right\|_{H^{s-2}} &\lesssim \left\|h - \tilde{h}\right\|_{H^{s-1}} \leqslant \left\|W - \tilde{W}\right\|_{H^{s-1}}.\end{aligned}$$

where the constants depend on $s, d, \|W\|_{H^s}, \|\tilde{W}\|_{H^s}, \|\psi\|_{H^{s-1}}$. The same proof can be used with the operator $\mathfrak{A}_2$. □

Proof of Theorem 2. Using the characteristics $\chi(0,x) = x$ and $\chi_t(t,x) = u(t,\chi(t,x))$, the conservation of the mass (3a) becomes

$$\mathrm{d}h/\mathrm{d}t + u_x h = 0, \qquad \Longrightarrow \qquad h_0(x)\,\mathrm{e}^{-t\,\|u_x\|_{L^\infty}} \leqslant h(t,x) \leqslant h_0(x)\,\mathrm{e}^{t\,\|u_x\|_{L^\infty}}. \tag{30}$$

The energy equation (4) implies that

$$\mathscr{E}'(t) \leqslant (|\dot{g}|/g + 1)\,\mathscr{E}(t) + \tfrac{1}{2}\,g\int_{\mathbb{R}}\left(d_t^2 + \varepsilon\,h^2\,d_{xt}^2\right)\mathrm{d}x, \tag{31}$$

since h is bounded, the inequality (7) follows by Gronwall's lemma.

In order to prove the blow-up criterion, we first suppose that $\|W_x\|_{L^\infty}$ is bounded and we show that the scenario (6) is impossible. The equation (30) implies that h is bounded and far from 0. Using $\|W\|_{L^\infty} \leqslant \|W\|_{H^1} \lesssim \mathscr{E}(t)$, one obtains that $\|W\|_{W^{1,\infty}}$ is bounded on any interval $[0,T]$. Using Lemma 3 and doing some classical energy estimates (see [7, 9, 11]), we can prove that $\|W\|_{H^s}$ is also bounded. This ends the proof of Theorem 2. □

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