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GLOBAL ASYMPTOTIC STABILITY OF THE RAREFACTION WAVES TO THE CAUCHY PROBLEM FOR THE GENERALIZED ROSENAU-KORTEWEG-DE VRIES-BURGERS EQUATION*

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Abstract. In this paper, we investigate the asymptotic behavior of solutions to the Cauchy problem with the far field condition for the generalized Rosenau-Korteweg-de Vries-Burgers equation. When the corresponding Riemann problem for the hyperbolic part admits a Riemann solution which consists of single rarefaction wave, it is proved that the solution of the Cauchy problem tends toward the rarefaction wave as time goes to infinity. We can further obtain the same global asymptotic stability of the rarefaction wave to the generalized Rosenau-Benjamin-Bona-Mahony-Burgers equation with a third-order dispersive term as the former one.

Key words. Rosenau-Burgers equation, Rosenau-Benjamin-Bona-Mahony-Burgers equation, Rosenau-Korteweg-de Vries-Burgers equation, convex flux, asymptotic behavior, rarefaction wave.

Mathematics Subject Classification. Primary 35Q35; Secondary 35B40, 35G20, 35G25, 35L65, 35Q53.

1. Introduction and main theorems. In this paper, we consider the asymptotic behavior of solutions to the Cauchy problem for the generalized Rosenau-Korteweg-de Vries-Burgers equation

$$\begin{cases} \partial_t\big(u+\nu\,\partial_x^4 u\big)+\partial_x\big(f(u)-\mu\,\partial_x u+\delta\,\partial_x^2 u\big)=0 & \big(t>0,\ x\in\mathbb{R}\big), \\ u(0,x)=u_0(x)\to u_\pm & (x\to\pm\infty). \end{cases} \tag{1.1}$$

Here, $u=u(t,x)$ is the unknown function of $t>0$ and $x\in\mathbb{R}$, and μ, ν are positive constants, $\delta\in\mathbb{R}$ is a constant, u_0 is the initial data, and $u_\pm\in\mathbb{R}$ are the prescribed far field states. We suppose that f is a smooth function.

The equation in (1.1) can be applied to the physics in nonlinear waves such as behavior of shallow water and so on (see [1]). In fact, when $\delta=0$ and $f(u)=\alpha\,u+u^2/2$ ($\alpha\in\mathbb{R}$), then the equation in (1.1) becomes the Rosenau-Burgers equation

$$\partial_t\big(u+\nu\,\partial_x^4 u\big)+\partial_x\left(\alpha\,u+\frac{1}{2}\,u^2-\mu\,\partial_x u\right)=0, \tag{1.2}$$

when $\delta=\mu=0$ and $f(u)=\alpha\,u+u^2/2$ ($\alpha\in\mathbb{R}$), then the equation in (1.1) becomes the Rosenau equation

$$\partial_t\big(u+\nu\,\partial_x^4 u\big)+\partial_x\left(\alpha\,u+\frac{1}{2}\,u^2\right)=0, \tag{1.3}$$

when $\nu=0$ and $f(u)=(\alpha/2)\,u^2$ ($\alpha\in\mathbb{R}$), then becomes the Korteweg-de Vries-Burgers equation

$$\partial_t u+\partial_x\left(\frac{\alpha}{2}\,u^2-\mu\,\partial_x u+\delta\,\partial_x^2 u\right)=0, \tag{1.4}$$

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when $\nu = \delta = 0$ and $f(u) = u^2/2$, then becomes the viscous Burgers equation

$$\partial_t u + \partial_x \left(\frac{1}{2} u^2 - \mu \, \partial_x u \right) = 0, \tag{1.5}$$

when $\nu = \delta = 0$ and $f(u) = u^2/2$, then becomes the non-viscous Burgers equation (hyperbolic Burgers equation)

$$\partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = 0, \tag{1.6}$$

and when $\mu = \nu = 0$ and $f(u) = (\alpha/2) \, u^2$ $(\alpha \in \mathbb{R})$, then becomes the Korteweg-de Vries equation

$$\partial_t u + \partial_x \left(\frac{\alpha}{2} u^2 + \delta \, \partial_x^2 u \right) = 0. \tag{1.7}$$

The equation (1.1) is also related to the following Benjamin-Bona-Mahony-Burgers equation

$$\partial_t \big(u - \nu \, \partial_x^2 u \big) + \partial_x \left(\alpha \, u + \frac{1}{2} u^2 - \mu \, \partial_x u \right) = 0, \tag{1.8}$$

the Benjamin-Bona-Mahony-Burgers equation with third-order dispersive and fourth-order dissipative terms

$$\partial_t \big(u - \nu \, \partial_x^2 u \big) + \partial_x \left(\alpha \, u + \frac{1}{2} u^2 - \mu \, \partial_x u + \delta \, \partial_x^2 u + \beta \, \partial_x^3 u \right) = 0, \tag{1.9}$$

and the Korteweg-de Vries Burgers-Kuramoto equation

$$\partial_t u + \partial_x \left(\alpha \, u + \frac{1}{2} u^2 - \mu \, \partial_x u + \delta \, \partial_x^2 u + \beta \, \partial_x^3 u \right) = 0, \tag{1.10}$$

where α, $\delta \in \mathbb{R}$, $\beta > 0$, $\mu > 0$ and $\nu > 0$.

There are many results concerning the existence and time-decay properties of solutions, the stability of nonlinear waves (that is, rarefaction waves, shock waves (travelling wave), viscous contact waves and and multiwave pattern of rarefaction waves and viscous contact waves) and the other mathematical structure of the models (1.2)-(1.10) (and, (1.15) and (1.16) in Remark 1.3) (for the related works, see Amick-Bona-Schonbek [2], Andreiev-Egorova-Lange-Teschl [3], Benjamin-Bona-Mahony [4], Bona-Schonbek [5], Bona-Rajopadhye-Schonbek [6], Duan-Fan-Kim-Xie [7], Duan-Zhao [8], Egorova-Grunert-Teschl [9], Egorova-Teschl [10], Harabetian [11], Hattori-Nishihara [13], Il'in-Oleĭnik [15], Kondo-Webler [17]-[20], Matsumura-Nishihara [24]-[26], Matsumura-Yoshida [27], [28], Mei [29], [30], Mei-Schmeiser [31], Naumkin [32], Nishihara-Rajopadhye [33], Osher-Ralston [34], Peregrine [35], Rajopadhye [36], Rashindinia-Nikan-Khoddam [37], Ruan-Gao-Chen [38], Wang [39], Wang-Zhu [40], Xu-Li [41], Yin-Zhao-Kim [42], Yoshida [43]-[53], Zhao-Xuan [54] and so on).

This paper is devoted to the study of stability of rarefaction wave of the solution to (1.1). Therefore we deal with the case where the flux function f is fully convex, that is,

$$f''(u) > 0 \quad (u \in \mathbb{R}), \tag{1.11}$$

and $u_- < u_+$. Then, since the corresponding Riemann problem

$$\begin{cases} \partial_t u + \partial_x\big(\, f(u)\,\big) = 0 & \big(\, t > 0,\ x \in \mathbb{R}\,\big), \\ u(0,x) = u_0^{\mathrm{R}}(x) := \begin{cases} u_- & (x < 0), \\ u_+ & (x > 0) \end{cases} \end{cases} \tag{1.12}$$

turns out to admit a single rarefaction wave solution, we expect that the solution of the Cauchy problem (1.1) tends toward the rarefaction wave as time goes to infinity (see Lax [22]). Here, the rarefaction wave connecting u_- to u_+ is given by

$$u^{\mathrm{r}}\left(\frac{x}{t}\,;\,u_-,\,u_+\right) = \begin{cases} u_- & \big(\, x \le f'(u_-)\,t\,\big), \\ (f')^{-1}\left(\dfrac{x}{t}\right) & \big(\, f'(u_-)\,t \le x \le f'(u_+)\,t\,\big), \\ u_+ & \big(\, x \ge f'(u_+)\,t\,\big). \end{cases} \tag{1.13}$$

In particular, we also expect that if $u_- = u_+ =: \tilde{u}$, then the solution of the Cauchy problem (1.1) tends toward the constant state $\tilde{u}$ as time goes to infinity.

Our main results of the present paper are as follows.

THEOREM 1.1 (Main Theorem I). *Assume the far field states $u_\pm$ satisfy $u_- = u_+ = \tilde{u}$, and the convective flux $f \in C^1(\mathbb{R})$. Further assume the initial data satisfy $u_0 - \tilde{u} \in L^2$ and $\partial_x u_0 \in H^2$. Then the Cauchy problem* (1.1) *has a unique global in time solution u satisfying*

$$\begin{cases} u - \tilde{u} \in C^0\big(\,[\,0,\,\infty\,)\,;H^3\,\big), \\ \partial_x u \in L^2\big(\,0,\,\infty\,;H^1\,\big), \\ \partial_t u \in L^2\big(\,0,\,\infty\,;H^2\,\big), \end{cases}$$

and the asymptotic behavior

$$\lim_{t\to\infty}\left(\,\sup_{x\in\mathbb{R}}|\,u(t,x) - \tilde{u}\,| + \sup_{x\in\mathbb{R}}|\,\partial_x u(t,x)\,| + \sup_{x\in\mathbb{R}}|\,\partial_x^2 u(t,x)\,|\,\right) = 0.$$

THEOREM 1.2 (Main Theorem II). *Assume the far field states $u_\pm$ satisfy $u_- < u_+$, and the convective flux $f \in C^6(\mathbb{R})$ satisfy* (1.11). *Further assume the initial data satisfy $u_0 - u_0^{\mathrm{R}} \in L^2$ and $\partial_x u_0 \in H^2$. Then the Cauchy problem* (1.1) *has a unique global in time solution u satisfying*

$$\begin{cases} u - u^{\mathrm{r}} \in C^0\big(\,[\,0,\,\infty\,)\,;H^3\,\big), \\ \partial_x u \in L^2_{\mathrm{loc}}\big(\,0,\,\infty\,;H^1\,\big), \\ \partial_t u \in L^2_{\mathrm{loc}}\big(\,0,\,\infty\,;H^2\,\big), \end{cases}$$

and the asymptotic behavior

$$\begin{cases} \displaystyle\lim_{t\to\infty}\sup_{x\in\mathbb{R}}\left|\,u(t,x) - u^{\mathrm{r}}\left(\frac{x}{t}\,;\,u_-,\,u_+\right)\right| = 0, \\ \displaystyle\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|\,\partial_x u(t,x) - \partial_x u^{\mathrm{r}}\,(\,t,\,x\,;\,u_-,\,u_+\,)\,| = 0, \\ \displaystyle\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|\,\partial_x^2 u(t,x) - \partial_x^2 u^{\mathrm{r}}\,(\,t,\,x\,;\,u_-,\,u_+\,)\,| = 0, \end{cases}$$

where $\partial_x u^{\mathrm{r}}$ and $\partial_x^2 u^{\mathrm{r}}$ are given by

$$\partial_x u^{\mathrm{r}}\,(\,t,\,x\,;\,u_-,\,u_+\,) = \begin{cases} 0 & \bigl(\,x < f'(u_-)\,t\,\bigr), \\ \dfrac{1}{f''\Bigl(\,(f')^{-1}\left(\frac{x}{t}\right)\,\Bigr)}\,\dfrac{1}{t} & \bigl(\,f'(u_-)\,t < x < f'(u_+)\,t\,\bigr), \\ 0 & \bigl(\,x > f'(u_+)\,t\,\bigr), \end{cases} \tag{1.14}$$

$$\partial_x^2 u^{\mathrm{r}}\,(\,t,\,x\,;\,u_-,\,u_+\,) = \begin{cases} 0 & \bigl(\,x < f'(u_-)\,t\,\bigr), \\ \dfrac{1}{\Bigl(f''\Bigl(\,(f')^{-1}\left(\frac{x}{t}\right)\,\Bigr)\Bigr)^3}\,\dfrac{x}{t^3} - \dfrac{1}{f''\Bigl(\,(f')^{-1}\left(\frac{x}{t}\right)\,\Bigr)}\,\dfrac{1}{t^2} & \bigl(\,f'(u_-)\,t < x < f'(u_+)\,t\,\bigr), \\ 0 & \bigl(\,x > f'(u_+)\,t\,\bigr). \end{cases} \tag{1.15}$$

REMARK 1.3. The equation in (1.1) is also related to the following generalized Rosenau-Benjamin-Bona-Mahony-Burgers equation with a third-order dispersive term (see [1] and so on)

$$\partial_t\bigl(\,u - \nu_1\,\partial_x^4 u + \nu_2\,\partial_x^4 u\,\bigr) + \partial_x\bigl(\,f(u) - \mu\,\partial_x u + \delta\,\partial_x^2 u\,\bigr) = 0, \tag{1.16}$$

where $\mu > 0$, $\nu_1 > 0$, $\nu_2 > 0$ and $\delta \in \mathbb{R}$ are constants. We note that when $\delta = 0$ and $f(u) = \alpha\,u + u^2/2$ ($\alpha \in \mathbb{R}$), then the equation in (1.16) becomes the Rosenau-Benjamin-Bona-Mahony-Burgers equation

$$\partial_t\bigl(\,u - \nu_1\,\partial_x^4 u + \nu_2\,\partial_x^4 u\,\bigr) + \partial_x\left(\,\alpha\,u + \frac{1}{2}\,u^2 - \mu\,\partial_x u\,\right) = 0, \tag{1.17}$$

and when $\delta = \mu = 0$ and $f(u) = \alpha\,u + u^2/2$ ($\alpha \in \mathbb{R}$), then the equation in (1.1) does the Rosenau-Benjamin-Bona-Mahony equation

$$\partial_t\bigl(\,u - \nu_1\,\partial_x^4 u + \nu_2\,\partial_x^4 u\,\bigr) + \partial_x\left(\,\alpha\,u + \frac{1}{2}\,u^2\,\right) = 0. \tag{1.18}$$

REMARK 1.4. If we also consider the following Cauchy problem for the generalized Rosenau-Benjamin-Bona-Mahony-Burgers equation with a third-order dispersive term

$$\begin{cases} \partial_t\bigl(\,u - \nu_1\,\partial_x^4 u + \nu_2\,\partial_x^4 u\,\bigr) + \partial_x\bigl(\,f(u) - \mu\,\partial_x u + \delta\,\partial_x^2 u\,\bigr) = 0 & \bigl(\,t > 0,\ x \in \mathbb{R}\,\bigr), \\ u(0,x) = u_0(x) \to u_\pm & (x \to \pm\infty), \end{cases}$$

then we can further obtain quite the same statements as Theorems 1.1 and 1.2. Because the proofs of them are similarly given as Theorems 1.1 and 1.2, we omit the details here.

Because the proof of Theorem 1.1 is easier than that for Theorem 1.2, we only show Theorem 1.2 in the following sections.

This paper is organized as follows. In Section 2, we construct the approximation of the rarefaction wave and prepare the basic properties of the rarefaction wave and the approximated one. We reformulate the problem in terms of the deviation from the asymptotic state in Section 3. In order to show the asymptotics, we establish the *a priori* estimates by using the technical energy method in Section 4.

Some Notation. We denote by C generic positive constants unless they need to be distinguished. In particular, use $C_{\alpha,\beta,\cdots}$ when we emphasize the dependency on $\alpha,\ \beta,\ \cdots$.

For function spaces, $L^p = L^p(\mathbb{R})$ and $H^k = H^k(\mathbb{R})$ denote the usual Lebesgue space and k-th order Sobolev space on the whole space $\mathbb{R}$ with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^k}$, respectively. We also defne the bounded C^m-class $\mathscr{B}^m$ and the bounded C^∞-class $\mathscr{B}^\infty$ as

$$f \in \mathscr{B}^m(\Omega) \Longleftrightarrow f \in C^m(\Omega),\ \sup_{\Omega} \sum_{k=0}^{m} \left| D^k f \right| < \infty,$$

$$\begin{aligned} f \in \mathscr{B}^\infty(\Omega) &\Longleftrightarrow \forall n \in \mathbb{N},\ f \in \mathscr{B}^n(\Omega) \\ &\Longleftrightarrow \forall n \in \mathbb{N},\ f \in C^n(\Omega),\ \sup_{\Omega} \sum_{k=0}^{n} \left| D^k f \right| < \infty, \end{aligned}$$

respectively, where $\Omega \subset \mathbb{R}^d$ and D^k denote all of the k-th order derivatives.

2. Preliminaries. In this section, we prepare the several lemmas concerning the basic properties of the rarefaction wave for the proof of the main Theorem 1.2. Since the rarefaction wave u^{r} is not smooth enough, we construct a smooth approximated one. To do that, we first consider the rarefaction wave solution w^r to the Riemann problem for the non-viscous Burgers equation

$$\begin{cases} \partial_t w + \partial_x \left(\dfrac{1}{2} w^2 \right) = 0 & \big(t > 0,\ x \in \mathbb{R} \big), \\ w(0,x) = w_0^{\mathrm{R}}(\, x\,;\ w_-,\ w_+\,) := \begin{cases} w_+ & (x > 0), \\ w_- & (x < 0), \end{cases} \end{cases} \tag{2.1}$$

where $w_\pm \in \mathbb{R}\,(w_- < w_+)$ are the prescribed far field states. The unique global weak solution $w = w^{\mathrm{r}}\,(\,x/t\,;\ w_-,\ w_+\,)$ of (2.1) is explicitly given by

$$w^{\mathrm{r}}\left(\frac{x}{t}\,;\ w_-,\ w_+ \right) = \begin{cases} w_- & \big(x \le w_- t \big), \\ \dfrac{x}{t} & \big(w_- t \le x \le w_+ t \big), \\ w_+ & \big(x \ge w_+ t \big). \end{cases} \tag{2.2}$$

Next, under the condition $f''(u) > 0$ $(u \in \mathbb{R})$ and $u_- < u_+$, the rarefaction wave solution $u = u^{\mathrm{r}}\,(\,x/t\,;\ u_-,\ u_+\,)$ of the Riemann problem (1.2) for hyperbolic conservation law is exactly given by

$$u^{\mathrm{r}}\left(\frac{x}{t}\,;\ u_-,\ u_+ \right) = (\lambda)^{-1}\left(w^{\mathrm{r}}\left(\frac{x}{t}\,;\ \lambda_-,\ \lambda_+ \right) \right) \tag{2.3}$$

which is nothing but (1.6), where $\lambda_\pm := \lambda(u_\pm) = f'(u_\pm)$. We define a smooth approximation of $w^{\mathrm{r}}(\,x/t\,;\ w_-,\ w_+\,)$ by the unique $\mathscr{B}^\infty$-solution

$$w = w(\,t,\ x\,;\ w_-,\ w_+\,) \in \mathscr{B}^\infty\big(\,[0,\ \infty) \times \mathbb{R}\,\big)$$

to the Cauchy problem for the following non-viscous Burgers equation as

$$\begin{cases} \partial_t w + \partial_x \left(\dfrac{1}{2} w^2 \right) = 0 & \big(t > 0,\ x \in \mathbb{R} \big), \\ w(0,x) = w_0(x) := \dfrac{w_- + w_+}{2} + \dfrac{w_+ - w_-}{2} K_q \displaystyle\int_0^{\epsilon x} \frac{\mathrm{d}y}{(1+y^2)^q} & (x \in \mathbb{R}), \end{cases} \tag{2.4}$$

where K_q is a positive constant such that

$$K_q \int_0^\infty \frac{\mathrm{d}y}{(1+y^2)^q} = 1 \quad \left(q > \frac{1}{2} \right),$$

and ϵ is a positive parameter of the initial condition w_0, and therefore the solution to (2.4) depends on ϵ. By applying the method of characteristics, we get the following formula

$$\begin{cases} w(t,x) = w_0\big(x_0(t,x) \big) = \dfrac{\lambda_- + \lambda_+}{2} + \dfrac{\lambda_+ - \lambda_-}{2}\, K_q \displaystyle\int_0^{\epsilon x_0(t,x)} \frac{\mathrm{d}y}{(1+y^2)^q}, \\ x = x_0(t,x) + w_0\big(x_0(t,x) \big)\, t. \end{cases} \tag{2.5}$$

By making use of (2.5) similarly as in [24], we can obtain the properties of the smooth approximation $w(\,t,\,x\,;\,w_-,\,w_+\,)$ in the next lemma.

Lemma 2.1. *Assume that the far field states satisfy $w_- < w_+$. Then the classical solution $w \in \mathscr{B}^\infty\big([\,0,\,\infty) \times \mathbb{R} \big)$ given by (2.4) satisfies the following properties:*

(1) *$w_- < w(t,x) < w_+$ and $\partial_x w(t,x) > 0$ for $t > 0$, $x \in \mathbb{R}$.*

(2) *For any $r \in [\,1,\,\infty\,]$, there exists a positive constant $C_{q,r}$ such that*

$$\begin{aligned}
&\|\, \partial_x w(t) \,\|_{L^r}^r \le C_{q,r}\, \min\big\{ \epsilon^{r-1}\, \tilde{w}^r,\, \tilde{w}\,(1+t)^{-r+1} \big\} \quad \big(t \ge 0 \big),\\
&\|\, \partial_t w(t) \,\|_{L^r}^r \le C_{q,r}\, \tilde{\tilde{w}}^r\, \min\big\{ \epsilon^{r-1}\, \tilde{w}^r,\, \tilde{w}\,(1+t)^{-r+1} \big\} \quad \big(t \ge 0 \big),\\
&\|\, \partial_x^2 w(t) \,\|_{L^r}^r\\
&\le C_{q,r}\, \min\Big\{ \epsilon^{2r-1}\, \tilde{w}^r,\, \epsilon^{(r-1)(1-\frac{1}{2q})}\, \tilde{w}^{-\frac{r-1}{2q}}\, (1+t)^{-r-\frac{r-1}{2q}} \Big\} \quad \big(t \ge 0 \big),\\
&\|\, \partial_x^3 w(t) \,\|_{L^r}^r \le C_{q,r}\, \min\big\{ \epsilon^{3r-1}\, \tilde{w}^r,\, a(1+t,\epsilon,\tilde{w}) \big\} \quad \big(t \ge 0 \big),\\
&\|\, \partial_x^4 w(t) \,\|_{L^r}^r \le C_{q,r}\, \min\big\{ \epsilon^{4r-1}\, \tilde{w}^r,\, b(1+t,\epsilon,\tilde{w}) \big\} \quad \big(t \ge 0 \big),
\end{aligned}$$

where

$$\tilde{w} := \frac{w_+ - w_-}{2} > 0, \quad \tilde{\tilde{w}} := \max\{\, |\, w_- \,|,\, |\, w_+ \,|\, \},$$

$$\begin{aligned} a(t,\epsilon,\tilde{w}) := {}& \epsilon^{3r}\, \tilde{w}^r\, (1+\epsilon\, \tilde{w}\, t)^{1-4r} + \epsilon^{2(r-1)(1-\frac{1}{2q})}\, \tilde{w}^{-\frac{r-1}{q}}\, t^{-1-(r-1)\left(1+\frac{1}{q}\right)}\\ &+ \epsilon^{(2r-1)(1-\frac{1}{2q})}\, \tilde{w}^{-\frac{2r-1}{2q}}\, t^{-r-\frac{2r-1}{2q}}, \end{aligned}$$

and

$$\begin{aligned} b(t,\epsilon,\tilde{w}) := {}& \epsilon^{3r}\, \tilde{w}^r\, (1+\epsilon\, \tilde{w}\, t)^{1-5r} + \epsilon^{(3r-2)(1-\frac{1}{2q})}\, \tilde{w}^{-\frac{3r-2}{2q}}\, t^{-r\left(1+\frac{3}{2q}\right)+\frac{1}{q}}\\ &+ \epsilon^{(3r-1)(1-\frac{1}{2q})}\, \tilde{w}^{-\frac{3r-1}{2q}}\, t^{-r-\frac{3r-1}{2q}}. \end{aligned}$$

(3) *It follows that*

$$\begin{cases} \displaystyle\lim_{t\to\infty} \sup_{x\in\mathbb{R}} \left| w(t,x) - w^{\mathrm{r}}\left(\frac{x}{t}\right) \right| = 0,\\ \displaystyle\lim_{t\to\infty} \sup_{x\in\mathbb{R}} \left|\, \partial_x w(t,x) - \partial_x w^{\mathrm{r}}(t,x) \,\right| = 0, \end{cases}$$

where $\partial_x w^{\mathrm{r}}$ is given by

$$\partial_x w^{\mathrm{r}}\,(\,t,\,x\,;\,w_-,\,w_+\,) = \begin{cases} 0 & \big(\,x \le w_-\,t\,\big), \\ \dfrac{1}{t} & \big(\,w_-\,t \le x \le w_+\,t\,\big), \\ 0 & \big(\,x \ge w_+\,t\,\big). \end{cases}$$

We now define the approximation for the rarefaction wave $u^{\mathrm{r}}\,(\,x/t\,;\,u_-,\,u_+\,)$ by

$$U^{\mathrm{r}}(\,t,\,x\,;\,u_-,\,u_+\,) = (\lambda)^{-1}\big(\,w(\,t,\,x\,;\,\lambda_-,\,\lambda_+\,)\,\big) \in \mathscr{B}^5\big(\,[\,0,\,\infty) \times \mathbb{R}\,\big), \tag{2.6}$$

where $\lambda \in C^5(\mathbb{R})$. Noting (2.6) and using Lemma 2.1, we also obtain the properties of U^{r} in the next lemma.

Lemma 2.2. *Let $q = 1$. Assume that the far field states satisfy $u_- < u_+$, and the flux function $f \in C^6(\mathbb{R})$, $f''(u) > 0$ $(\,u \in [\,u_-,\,u_+\,]\,)$. Then we have the following properties.*
(1) *$U^{\mathrm{r}}(t,x)$ defined by (2.6) is the unique $\mathscr{B}^5$-global in space-time solution of the Cauchy problem*

$$\begin{cases} \partial_t U^{\mathrm{r}} + \partial_x\big(\,f(U^{\mathrm{r}})\,\big) = 0 \quad \big(\,t>0,\,x\in\mathbb{R}\,\big), \\ U^{\mathrm{r}}(0,x) = (\lambda)^{-1}\left(\dfrac{\lambda_- + \lambda_+}{2} + \dfrac{\lambda_+ - \lambda_-}{2}\,K_q \displaystyle\int_0^{\epsilon x} \frac{\mathrm{d}y}{(1+y^2)^q}\right) \quad (x\in\mathbb{R}), \\ \displaystyle\lim_{x\to\pm\infty} U^{\mathrm{r}}(t,x) = u_\pm \quad \big(\,t\ge 0\,\big). \end{cases}$$

(2) *$u_- < U^{\mathrm{r}}(t,x) < u_+$ and $\partial_x U^{\mathrm{r}}(t,x) > 0$ for $t>0$, $x\in\mathbb{R}$.*
(3) *For any $r \in [\,1,\,\infty\,]$, there exists a positive constant $C_{\lambda_\pm,r}$ such that*

$$\begin{aligned}
&\|\,\partial_x U^{\mathrm{r}}(t)\,\|_{L^r}^r \le C_{\lambda_\pm,r}\,\min\big\{\,\epsilon^{r-1},\,(1+t)^{-r+1}\,\big\} \quad \big(\,t\ge 0\,\big), \\
&\|\,\partial_t U^{\mathrm{r}}(t)\,\|_{L^r}^r \le C_{\lambda_\pm,r}\,\min\big\{\,\epsilon^{r-1},\,(1+t)^{-r+1}\,\big\} \quad \big(\,t\ge 0\,\big), \\
&\|\,\partial_x^2 U^{\mathrm{r}}(t)\,\|_{L^r}^r \le C_{\lambda_\pm,r}\,\min\Big\{\,\epsilon^{2r-1},\,\epsilon^{\frac{r-1}{2}}\,(1+t)^{-\frac{3r-1}{2}}\,\Big\} \quad \big(\,t\ge 0\,\big), \\
&\|\,\partial_x^3 U^{\mathrm{r}}(t)\,\|_{L^r}^r \le C_{\lambda_\pm,r}\,\min\big\{\,\epsilon^{3r-1},\,\epsilon^{r-1}\,(1+t)^{-2r+1}\,\big\} \quad \big(\,t\ge 0\,\big), \\
&\|\,\partial_x^4 U^{\mathrm{r}}(t)\,\|_{L^r}^r \le C_{\lambda_\pm,r}\,\min\Big\{\,\epsilon^{4r-1},\,\epsilon^{\frac{3r-2}{2}}\,(1+t)^{-\min\left\{\frac{3}{2}r+1,\,\frac{5}{2}r-1\right\}}\,\Big\} \quad \big(\,t\ge 0\,\big).
\end{aligned}$$

(4) *It follows that*

$$\begin{cases} \displaystyle\lim_{t\to\infty}\sup_{x\in\mathbb{R}}\left|\,U^{\mathrm{r}}(t,x) - u^{\mathrm{r}}\left(\frac{x}{t}\right)\right| = 0, \\ \displaystyle\lim_{t\to\infty}\sup_{x\in\mathbb{R}}\,|\,\partial_x U^{\mathrm{r}}(t,x) - \partial_x u^{\mathrm{r}}(t,x)\,| = 0, \\ \displaystyle\lim_{t\to\infty}\sup_{x\in\mathbb{R}}\,\big|\,\partial_x^2 U^{\mathrm{r}}(t,x) - \partial_x^2 u^{\mathrm{r}}(t,x)\,\big| = 0, \end{cases}$$

where $\partial_x u^{\mathrm{r}}$ and $\partial_x^2 u^{\mathrm{r}}$ are defined by (1.) and (1.), respectively.

Because the proofs of Lemmas 2.1 and 2.2 are given in [12], [13], [23], [24], [27], [40], [43], [50], [53] and so on, we omit the proofs here.

3. Reformulation of the problem. In this section, we reformulate our problem (1.1) in terms of the deviation from the asymptotic state. Putting ϕ as

$$u(t,x) = U^{\mathrm{r}}(t,x) + \phi(t,x), \tag{3.1}$$

we reformulate the problem (1.1) in terms of the deviation ϕ from U^{r} as

$$\begin{cases} \partial_t \phi + \nu\, \partial_t \partial_x^4 \phi + \partial_x \big(f(\phi + U^{\mathrm{r}}) - f(U^{\mathrm{r}}) \big) \\ \quad -\mu\, \partial_x^2 \phi + \delta\, \partial_x^3 \phi = F(U^{\mathrm{r}}) \quad \big(t > 0,\ x \in \mathbb{R} \big), \\ \phi(0,x) = \phi_0(x) := u_0(x) - U^{\mathrm{r}}(0,x) \to 0 \quad (x \to \pm\infty), \end{cases} \tag{3.2}$$

where

$$F(U^{\mathrm{r}}) := -\nu\, \partial_t \partial_x^4 U^{\mathrm{r}} + \mu\, \partial_x^2 U^{\mathrm{r}} - \delta\, \partial_x^3 U^{\mathrm{r}}.$$

Then we look for the unique global in time solution ϕ which has the asymptotic behavior

$$\sum_{k=0}^{2} \sup_{x \in \mathbb{R}} \big| \partial_x^k \phi(t,x) \big| \to 0 \qquad (t \to \infty). \tag{3.3}$$

Here we note that $\phi_0 \in H^3$ by the assumptions on u_0 and Lemma 2.2. Then the corresponding theorem for ϕ to Theorem 1.2 we should prove is as follows.

THEOREM 3.1 (Global Existence). *Assume the far field states $u_\pm$ satisfy $u_- < u_+$, and the convective flux $f \in C^6(\mathbb{R})$ satisfy* (1.11). *Further assume the initial data satisfy $\phi_0 \in H^3$. Then the Cauchy problem* (3.2) *has a unique global in time solution ϕ satisfying*

$$\begin{cases} \phi \in C^0\big([0,\infty) ; H^3 \big), \\ \partial_x \phi \in L^2\big(0,\infty ; H^1 \big), \\ \partial_t \phi \in L^2\big(0,\infty ; H^2 \big), \end{cases}$$

and the asymptotic behavior

$$\lim_{t \to \infty} \sum_{k=0}^{2} \sup_{x \in \mathbb{R}} \big| \partial_x^k \phi(t,x) \big| = 0.$$

In order to obtain Theorem 3.1, we prepare the local existence theorem precisely. To do that, we formulate the problem (3.2) at general initial time $\tau \geq 0$:

$$\begin{cases} \partial_t \phi + \nu\, \partial_t \partial_x^4 \phi + \partial_x \big(f(\phi + U^{\mathrm{r}}) - f(U^{\mathrm{r}}) \big) \\ \quad -\mu\, \partial_x^2 \phi + \delta\, \partial_x^3 \phi = F(U^{\mathrm{r}}) \quad \big(t > \tau,\ x \in \mathbb{R} \big), \\ \phi(\tau,x) = \phi_\tau(x) := u_\tau(x) - U^{\mathrm{r}}(\tau,x) \to 0 \quad (x \to \pm\infty). \end{cases} \tag{3.4}$$

THEOREM 3.2 (Local Existence). *For any $M > 0$, there exists a positive constant $t_0 = t_0(M)$ not depending on τ such that if $\phi_\tau \in H^3$ and*

$$\| \phi_\tau \|_{H^3} \leq M,$$

then the Cauchy problem (3.4) *has a unique solution* ϕ *on the time interval* $[\,\tau,\,\tau+t_0(M)\,]$ *satisfying*

$$\begin{cases} \phi\in C^0\big(\,[\,\tau,\,\tau+t_0\,]\,;H^3\big),\\ \partial_x\phi\in L^2\big(\,\tau,\,\tau+t_0\,;H^1\big),\\ \partial_t\phi\in L^2\big(\,\tau,\,\tau+t_0\,;H^2\big),\\ \displaystyle\sup_{t\in[\tau,\tau+t_0]}\|\,\phi(t)\,\|_{H^3}\le 2\,M. \end{cases}$$

Because the proof of Theorem 3.2 is standard, we omit the details here (cf. [42], [54]). The *a priori* estimates we establish in Section 4 are the following.

Theorem 3.3 (*A Priori* Estimates). *Under the same assumptions as in Theorem 3.1, for any initial data* $\phi_0\in H^3$*, there exists a positive constant* C_{ϕ_0} *such that if the Cauchy problem* (3.1) *has a solution* ϕ *on the time interval* $[\,0,\,T\,]$ *satisfying*

$$\begin{cases} \phi\in C^0\big(\,[\,0,\,T\,]\,;H^3\big),\\ \partial_x\phi\in L^2\big(\,0,\,T\,;H^1\big),\\ \partial_t\phi\in L^2\big(\,0,\,T\,;H^2\big), \end{cases}$$

for some positive constant T*, then it holds that*

$$\begin{aligned} &\|\,\phi(t)\,\|_{H^3}^2+\int_0^t\Big\|\,\big(\,\sqrt{\partial_xU^{\mathrm{r}}}\,\phi\,\big)(\tau)\,\Big\|_{L^2}^2\,\mathrm{d}\tau\\ &+\int_0^t\|\,\partial_x\phi(\tau)\,\|_{H^1}^2\,\mathrm{d}\tau+\int_0^t\|\,\partial_t\phi(\tau)\,\|_{H^2}^2\,\mathrm{d}\tau\le C_{\phi_0}\quad\big(\,t\in[\,0,\,T\,]\big). \end{aligned}\tag{3.5}$$

Combining the local existence Theorem 3.2 together with the *a priori* estimates, Theorem 3.3, we can obtain global existence Theorem 3.1. In fact, we can obtain the unique global in time solutions ϕ to (3.2) in Theorem 3.1 satisfying

$$\begin{cases} \phi\in C^0\big(\,[\,0,\,\infty)\,;H^3\,\big),\\ \partial_x\phi\in L^2\big(\,0,\,\infty\,;H^1\,\big),\\ \partial_t\phi\in L^2\big(\,0,\,\infty\,;H^2\,\big), \end{cases}$$

and

$$\begin{aligned} &\sup_{t\ge0}\|\,\phi(t)\,\|_{H^3}^2+\int_0^\infty\Big\|\,\big(\,\sqrt{\partial_xU^{\mathrm{r}}}\,\phi\,\big)(t)\,\Big\|_{L^2}^2\,\mathrm{d}t\\ &\quad+\int_0^\infty\|\,\partial_x\phi(t)\,\|_{H^1}^2\,\mathrm{d}t+\int_0^\infty\|\,\partial_t\phi(t)\,\|_{H^2}^2\,\mathrm{d}t<\infty. \end{aligned}\tag{3.6}$$

Further by using (3.6), we have that

$$\begin{aligned} \int_0^\infty\left|\frac{\mathrm{d}}{\mathrm{d}t}\|\,\partial_x\phi(t)\,\|_{L^2}^2\right|\mathrm{d}t&=2\int_0^\infty\left|\int_{-\infty}^\infty\partial_x\phi\,\partial_t\partial_x\phi\,\mathrm{d}x\right|\mathrm{d}t\\ &\le\int_0^\infty\big(\,\|\,\partial_x\phi(t)\,\|_{L^2}^2+\|\,\partial_t\partial_x\phi(t)\,\|_{L^2}^2\,\big)\,\mathrm{d}t<\infty, \end{aligned}\tag{3.7}$$

$$\int_0^\infty \left| \frac{\mathrm{d}}{\mathrm{d}t} \| \partial_x^2 \phi(t) \|_{L^2}^2 \right| \mathrm{d}t = 2 \int_0^\infty \left| \int_{-\infty}^\infty \partial_x^2 \phi \, \partial_t \partial_x^2 \phi \, \mathrm{d}x \right| \mathrm{d}t$$
$$\leq \int_0^\infty \left(\| \partial_x^2 \phi(t) \|_{L^2}^2 + \| \partial_t \partial_x^2 \phi(t) \|_{L^2}^2 \right) \mathrm{d}t < \infty. \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$\int_0^\infty \left| \frac{\mathrm{d}}{\mathrm{d}t} \| \partial_x \phi(t) \|_{L^2}^2 \right| \mathrm{d}t < \infty, \quad \int_0^\infty \left| \frac{\mathrm{d}}{\mathrm{d}t} \| \partial_x^2 \phi(t) \|_{L^2}^2 \right| \mathrm{d}t < \infty. \tag{3.9}$$

We immediately have from (3.9) that

$$\lim_{t \to \infty} \| \partial_x \phi(t) \|_{L^2} = 0, \quad \lim_{t \to \infty} \| \partial_x^2 \phi(t) \|_{L^2} = 0. \tag{3.10}$$

Further from (3.10), by using the Sobolev inequality, we obtain the desired asymptotic behavior (3.3) as follows.

$$\lim_{t \to \infty} \sup_{x \in \mathbb{R}} | \phi(t,x) | \leq \sqrt{2} \lim_{t \to \infty} \| \phi(t) \|_{L^2}^{\frac{1}{2}} \| \partial_x \phi(t) \|_{L^2}^{\frac{1}{2}} = 0,$$

$$\lim_{t \to \infty} \sup_{x \in \mathbb{R}} | \partial_x \phi(t,x) | \leq \sqrt{2} \lim_{t \to \infty} \| \partial_x \phi(t) \|_{L^2}^{\frac{1}{2}} \| \partial_x^2 \phi(t) \|_{L^2}^{\frac{1}{2}} = 0,$$

$$\lim_{t \to \infty} \sup_{x \in \mathbb{R}} | \partial_x^2 \phi(t,x) | \leq \sqrt{2} \lim_{t \to \infty} \| \partial_x^2 \phi(t) \|_{L^2}^{\frac{1}{2}} \| \partial_x^3 \phi(t) \|_{L^2}^{\frac{1}{2}} = 0.$$

Thus Theorem 3.1 is proved.

4. *A priori* **estimates.** In this section, we show the *a priori* estimate for ϕ, $\partial_x \phi$ and $\partial_x^2 \phi$ in Theorem 3.3. To do that, we prepare the following basic estimate.

PROPOSITION 4.1. *There exists a positive constant C_{ϕ_0} such that*

$$\| \phi(t) \|_{H^2}^2 + \int_0^t \int_{-\infty}^\infty \int_0^\phi \left(f'(\eta + U^{\mathrm{r}}) - f'(U^{\mathrm{r}}) \right) \mathrm{d}\eta \, \partial_x U^{\mathrm{r}} \, \mathrm{d}x \mathrm{d}\tau$$
$$+ \int_0^t \| \partial_x \phi(\tau) \|_{L^2}^2 \, \mathrm{d}\tau \leq C_{\phi_0} \quad \big(t \in [\,0,\, T\,] \big).$$

Proof of Proposition 4.1. Multiplying the equation in (3.2) by ϕ and integrating it with respect to x, we have, after integration by parts, that

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \| \phi(t) \|_{L^2}^2 + \frac{\nu}{2} \frac{\mathrm{d}}{\mathrm{d}t} \| \partial_x^2 \phi(t) \|_{L^2}^2$$
$$+ \int_{-\infty}^\infty \int_0^\phi \left(f'(\eta + U^{\mathrm{r}}) - f'(U^{\mathrm{r}}) \right) \mathrm{d}\eta \, \partial_x U^r \, \mathrm{d}x \tag{4.1}$$
$$+ \mu \| \partial_x \phi(t) \|_{L^2}^2 = \int_{-\infty}^\infty \phi \, F(U^{\mathrm{r}}) \, \mathrm{d}x.$$

By making use of the Sobolev inequality and the Young inequality, we estimate the right-hand-side of (4.1) as follows.

$$
\begin{aligned}
\left| \int_{-\infty}^{\infty} \phi\, F(U^{\mathrm{r}})\, \mathrm{d}x \right| &\le \nu \left| \int_{-\infty}^{\infty} \partial_x \phi\, \partial_t \partial_x^3 U^{\mathrm{r}}\, \mathrm{d}x \right| \\
&\quad + \sup_{x\in\mathbb{R}} |\, \phi \,| \left| \int_{-\infty}^{\infty} \left(\mu\, \partial_x^2 U^{\mathrm{r}} - \delta\, \partial_x^3 U^{\mathrm{r}} \right) \mathrm{d}x \right| \\
&\le \frac{\mu}{4} \|\, \partial_x \phi \,\|_{L^2}^2 + C_{\mu,\nu} \|\, \partial_t \partial_x^3 U^{\mathrm{r}} \,\|_{L^2}^2 \\
&\quad + C_{\delta,\mu} \|\, \phi \,\|_{L^2}^{\frac{1}{2}} \|\, \partial_x \phi \,\|_{L^2}^{\frac{1}{2}} \Big(\|\, \partial_x^2 U^{\mathrm{r}} \,\|_{L^1}^{\frac{4}{3}} + \|\, \partial_x^3 U^{\mathrm{r}} \,\|_{L^1}^{\frac{4}{3}} \Big) \\
&\le \frac{\mu}{2} \|\, \partial_x \phi \,\|_{L^2}^2 + C_{\delta,\mu,\nu} \left(1 + \|\, \phi \,\|_{L^2}^2 \right) \Big(\|\, \partial_x^2 U^{\mathrm{r}} \,\|_{L^1}^{\frac{4}{3}} + \|\, \partial_x^3 U^{\mathrm{r}} \,\|_{L^1}^{\frac{4}{3}} \Big).
\end{aligned}
\tag{4.2}
$$

Substituting (4.2) into (4.1), integrating the resulting inequality with respect to t, noting for $U^{\mathrm{r}} \in \mathscr{B}^5\big([\,0,\, \infty) \times \mathbb{R} \big)$ that

$$
\|\, \partial_x^2 U^{\mathrm{r}} \,\|_{L^1}^{\frac{4}{3}} \in L_t^1(0,\, \infty), \quad \|\, \partial_x^3 U^{\mathrm{r}} \,\|_{L^1}^{\frac{4}{3}} \in L_t^1(0,\, \infty),
$$

$$
\begin{aligned}
\partial_t \partial_x^3 U^{\mathrm{r}} &= -f^{(4)}(U^{\mathrm{r}})\, |\, \partial_x U^{\mathrm{r}} \,|^4 - 3\, f'''(U^{\mathrm{r}})\, |\, \partial_x U^{\mathrm{r}} \,|^2\, \partial_x^2 U^{\mathrm{r}} \\
&\quad - 3\, f''(U^{\mathrm{r}})\, |\, \partial_x^2 U^{\mathrm{r}} \,|^2 - 4\, f''(U^{\mathrm{r}})\, \partial_x U^{\mathrm{r}}\, \partial_x^3 U^{\mathrm{r}} - f'(U^{\mathrm{r}})\, \partial_x^4 U^{\mathrm{r}},
\end{aligned}
$$

$$
\begin{aligned}
\|\, \partial_t \partial_x^3 U^{\mathrm{r}} \,\|_{L^2}^2 \le C_{u_\pm} \big(&\|\, \partial_x U^{\mathrm{r}} \,\|_{L^8}^8 \|\, \partial_x U^{\mathrm{r}} \,\|_{L^\infty}^4 \|\, \partial_x^2 U^{\mathrm{r}} \,\|_{L^2}^2 + \|\, \partial_x^2 U^{\mathrm{r}} \,\|_{L^4}^4 \\
&+ \|\, \partial_x U^{\mathrm{r}} \,\|_{L^\infty}^2 \|\, \partial_x^3 U^{\mathrm{r}} \,\|_{L^2}^2 + \|\, \partial_x^4 U^{\mathrm{r}} \,\|_{L^2}^2 \big) \in L_t^1(0,\, \infty)
\end{aligned}
\tag{4.3}
$$

from Lemma 2.2,

$$
\|\, \partial_x \phi \,\|_{L^2}^2 \le \|\, \phi \,\|_{L^2} \|\, \partial_x^2 \phi \,\|_{L^2} \le \frac{1}{2} \left(\|\, \phi \,\|_{L^2}^2 + \|\, \partial_x^2 \phi \,\|_{L^2}^2 \right)
$$

by the integration by parts, and using the Gronwall inequality, we obtain the desired estimate.

Thus, we complete the proof of Proposition 4.1.

We should emphasize from Proposition 4.1, by using the Sobolev inequality, that we can get not only the uniform boundedness of ϕ but also that of $\partial_x \phi$ in the next lemma (cf. [14], [16], [21]).

LEMMA 4.2. *There exists a positive constant C_{ϕ_0} such that*

$$
\sup_{t\in[0,T],\, x\in\mathbb{R}} |\, \phi(t,x) \,| \le C_{\phi_0}, \qquad \sup_{t\in[0,T],\, x\in\mathbb{R}} |\, \partial_x \phi(t,x) \,| \le C_{\phi_0}.
$$

By the uniform boundedness of ϕ in Lemma 4.2, the second term on the left-hand side of the *a priori* estimate in Proposition 4.1 can be replaced by the left-hand side of the following inequality as

$$
\begin{aligned}
&\int_0^t \int_{-\infty}^{\infty} \int_0^{\phi} \left(f'(\eta + U^{\mathrm{r}}) - f'(U^{\mathrm{r}}) \right) \mathrm{d}\eta\, \partial_x U^{\mathrm{r}}\, \mathrm{d}x \mathrm{d}\tau \\
&\ge C_{\phi_0}^{-1} \int_0^t \left\|\, \left(\sqrt{\partial_x U^{\mathrm{r}}}\, \phi \right)(\tau) \,\right\|_{L^2}^2 \mathrm{d}\tau.
\end{aligned}
\tag{4.4}
$$

We can further obtain

$$\int_0^t \| \partial_x \big(f(\phi + U^{\mathrm{r}}) - f(U^{\mathrm{r}}) \big)(\tau) \|_{L^2}^2 \, \mathrm{d}\tau \le C_{\phi_0} \quad \big(t \in [0, T] \big), \tag{4.5}$$

from Lemma 2.2 and $\| \partial_x \phi \|_{L^2}^2 \in L^1_t(0, \infty)$ by Proposition 4.1, and

$$\| \partial_x^2 U^{\mathrm{r}} \|_{H^1}^2 \in L^1_t(0, \infty), \quad \| \partial_t \partial_x^2 U^{\mathrm{r}} \|_{L^2}^2 \in L^1_t(0, \infty) \tag{4.6}$$

from Lemma 2.2.

Next, we show the *a priori* estimate for $\partial_x \phi$ in the next proposition.

Proposition 4.3. *There exists a positive constant C_{ϕ_0} such that*

$$\| \partial_x \phi(t) \|_{L^2}^2 + \int_0^t \| \partial_t \phi(\tau) \|_{H^2}^2 \, \mathrm{d}\tau \le C_{\phi_0} \quad \big(t \in [0, T] \big).$$

Proof of Proposition 4.3. Multiplying the equation in (3.2) by $\partial_t \phi$ and integrating it with respect to x, we have, after integration by parts, that

$$\begin{aligned} &\frac{\mu}{2} \frac{\mathrm{d}}{\mathrm{d}t} \| \partial_x \phi(t) \|_{L^2}^2 + \| \partial_t \phi(t) \|_{L^2}^2 + \nu \| \partial_t \partial_x^2 \phi(t) \|_{L^2}^2 \\ &= -\delta \int_{-\infty}^{\infty} \partial_t \phi \, \partial_x^3 \phi \, \mathrm{d}x - \int_{-\infty}^{\infty} \partial_t \phi \, \partial_x \big(f(\phi + U^{\mathrm{r}}) - f(U^{\mathrm{r}}) \big) \, \mathrm{d}x \\ &+ \int_{-\infty}^{\infty} \partial_t \phi \, F(U^{\mathrm{r}}) \, \mathrm{d}x. \end{aligned} \tag{4.7}$$

By using the integration by parts and the Young inequality, we can estimate the each terms on the right-hand side of (4.7) as follows:

$$\begin{aligned} \left| \delta \int_{-\infty}^{\infty} \partial_t \phi \, \partial_x^3 \phi \, \mathrm{d}x \right| &\le |\, \delta \,| \int_{-\infty}^{\infty} |\, \partial_t \partial_x^2 \phi \,| \, |\, \partial_x \phi \,| \, \mathrm{d}x \\ &\le \frac{\nu}{4} \| \partial_t \partial_x^2 \phi \|_{L^2}^2 + C_{\delta, \nu} \, \| \partial_x \phi \|_{L^2}^2 , \end{aligned} \tag{4.8}$$

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} \partial_t \phi \, \partial_x \big(f(\phi + U^{\mathrm{r}}) - f(U^{\mathrm{r}}) \big) \, \mathrm{d}x \right| \\ &\le \frac{1}{2} \| \partial_t \phi \|_{L^2}^2 + \frac{1}{2} \, \| \partial_x \big(f(\phi + U^{\mathrm{r}}) - f(U^{\mathrm{r}}) \big) \|_{L^2}^2 , \end{aligned} \tag{4.9}$$

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} \partial_t \phi \, F(U^{\mathrm{r}}) \, \mathrm{d}x \right| \\ &\le \int_{-\infty}^{\infty} |\, \partial_t \partial_x^2 \phi \,| \, |\, \partial_t \partial_x^2 U^{\mathrm{r}} \,| \, \mathrm{d}x + \int_{-\infty}^{\infty} |\, \partial_t \phi \,| \, \big|\, \mu \, \partial_x^2 U^{\mathrm{r}} - \delta \, \partial_x^3 U^{\mathrm{r}} \,\big| \, \mathrm{d}x \\ &\le \frac{\nu}{4} \| \partial_t \partial_x^2 \phi \|_{L^2}^2 + C_\nu \, \| \partial_x \phi \|_{L^2}^2 + \frac{1}{4} \| \partial_t \phi \|_{L^2}^2 + C_{\delta, \mu, \nu} \| \partial_x^2 U^{\mathrm{r}} \|_{H^1}^2 . \end{aligned} \tag{4.10}$$

Substituting (4.8)-(4.10) into (4.7), and using (4.5),(4.6) and

$$\| \partial_t \partial_x \phi \|_{L^2}^2 \le \| \partial_t \phi \|_{L^2} \| \partial_t \partial_x^2 \phi \|_{L^2} \le \frac{1}{2} \big(\| \partial_t \phi \|_{L^2}^2 + \| \partial_t \partial_x^2 \phi \|_{L^2}^2 \big)$$

by using the integration by parts and the Young inequality, we obtain the desired estimate.

Thus, we complete the proof of Proposition 4.3.

We further show the *a priori* estimate for $\partial_x\phi$, $\partial_x^2\phi$ and $\partial_x^3\phi$ as follows.

PROPOSITION 4.4. *There exists a positive constant C_{ϕ_0} such that*

$$\|\,\partial_x\phi(t)\,\|_{H^2}^2+\int_0^t\|\,\partial_x^2\phi(\tau)\,\|_{L^2}^2\,\mathrm{d}\tau\le C_{\phi_0}\quad\big(t\in[\,0,\,T\,]\big).$$

Proof of Proposition 4.4. Multiplying the equation in (3.2) by $-\partial_x^2\phi$ and integrating it with respect to x, we have, after integration by parts, that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\,\partial_x\phi(t)\,\|_{L^2}^2+\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\,\partial_x^3\phi(t)\,\|_{L^2}^2+\mu\,\|\,\partial_x^2\phi(t)\,\|_{L^2}^2\\&=\int_{-\infty}^{\infty}\partial_x^2\phi\,\partial_x\big(\,f(\phi+U^{\mathrm{r}})-f(U^{\mathrm{r}})\,\big)\,\mathrm{d}x-\int_{-\infty}^{\infty}\partial_x^2\phi\,F(U^{\mathrm{r}})\,\mathrm{d}x.\end{aligned}\tag{4.11}$$

The second term on the right-hand side of (4.11) becomes

$$\begin{aligned}&-\int_{-\infty}^{\infty}\partial_x^2\phi\,F(U^{\mathrm{r}})\,\mathrm{d}x=\nu\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\partial_x^2\phi\,\partial_x^4U^{\mathrm{r}}\,\mathrm{d}x\\&\quad-\nu\int_{-\infty}^{\infty}\partial_t\partial_x^2\phi\,\partial_x^4U^{\mathrm{r}}\,\mathrm{d}x-\int_{-\infty}^{\infty}\partial_x^2\phi\,\big(\,\mu\,\partial_x^2U^{\mathrm{r}}-\delta\,\partial_x^3U^{\mathrm{r}}\,\big)\,\mathrm{d}x.\end{aligned}\tag{4.12}$$

Substituting (4.12) into (4.11), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\,\partial_x\phi(t)\,\|_{L^2}^2+\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\,\partial_x^3\phi(t)\,\|_{L^2}^2\\&-\nu\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\partial_x^2\phi\,\partial_x^4U^{\mathrm{r}}\,\mathrm{d}x+\mu\,\|\,\partial_x^2\phi(t)\,\|_{L^2}^2\\&=\int_{-\infty}^{\infty}\partial_x^2\phi\,\partial_x\big(\,f(\phi+U^{\mathrm{r}})-f(U^{\mathrm{r}})\,\big)\,\mathrm{d}x\\&\quad-\nu\int_{-\infty}^{\infty}\partial_t\partial_x^2\phi\,\partial_x^4U^{\mathrm{r}}\,\mathrm{d}x-\int_{-\infty}^{\infty}\partial_x^2\phi\,\big(\,\mu\,\partial_x^2U^{\mathrm{r}}-\delta\,\partial_x^3U^{\mathrm{r}}\,\big)\,\mathrm{d}x.\end{aligned}\tag{4.13}$$

Each terms on the right-hand side of (4.13) are estimated as follows.

$$\begin{aligned}&\left|\int_{-\infty}^{\infty}\partial_x^2\phi\,\partial_x\big(\,f(\phi+U^{\mathrm{r}})-f(U^{\mathrm{r}})\,\big)\,\mathrm{d}x\right|\\&\le\frac{\mu}{4}\|\,\partial_x^2\phi\,\|_{L^2}^2+C_\mu\,\Big\|\,\partial_x\big(\,f(\phi+U^{\mathrm{r}})-f(U^{\mathrm{r}})\,\big)\,\Big\|_{L^2}^2,\end{aligned}\tag{4.14}$$

$$\left|\nu\int_{-\infty}^{\infty}\partial_t\partial_x^2\phi\,\partial_x^4U^{\mathrm{r}}\,\mathrm{d}x\right|\le\frac{\nu}{2}\|\,\partial_t\partial_x^2\phi\,\|_{L^2}^2+\frac{\nu}{2}\|\,\partial_x^4U^{\mathrm{r}}\,\|_{L^2}^2,\tag{4.15}$$

$$\left|\int_{-\infty}^{\infty}\partial_x^2\phi\,\big(\,\mu\,\partial_x^2U^{\mathrm{r}}-\delta\,\partial_x^3U^{\mathrm{r}}\,\big)\,\mathrm{d}x\right|\le\frac{\mu}{4}\|\,\partial_x^2\phi\,\|_{L^2}^2+C_{\mu,\delta}\,\|\,\partial_x^2U^{\mathrm{r}}\,\|_{H^1}^2.\tag{4.16}$$

Substituting (4.13) into (4.14)-(4.16), integrating the resultant formula with respect to t, noting

$$\| \partial_x^2 \phi \|_{L^2}^2 \leq \| \partial_x \phi \|_{L^2} \| \partial_x^3 \phi \|_{L^2} \leq \frac{1}{2} \left(\| \partial_x \phi \|_{L^2}^2 + \| \partial_x^3 \phi \|_{L^2}^2 \right),$$

and using (4.5), (4.6) and Propositions 4.1 and 4.3, we arrive at

$$\begin{aligned} &\frac{1}{4} \| \partial_x \phi(t) \|_{L^2}^2 + \frac{\min\{1,\, \nu\}}{2} \| \partial_x^2 \phi(t) \|_{L^2}^2 + \frac{\nu}{4} \| \partial_x^3 \phi(t) \|_{L^2}^2 \\ &+ \mu \int_0^t \| \partial_x^2 \phi(\tau) \|_{L^2}^2 \, \mathrm{d}\tau \\ &\leq C_{\phi_0} + \nu \left| \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau} \int_{-\infty}^{\infty} \partial_x^2 \phi \, \partial_x^4 U^{\mathrm{r}} \, \mathrm{d}x \mathrm{d}\tau \right| \quad \big(t \in [\,0,\, T\,] \big). \end{aligned} \tag{4.17}$$

The second term on the right-hand side of (4.17) is easily estimated as

$$\begin{aligned} &\nu \left| \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau} \int_{-\infty}^{\infty} \partial_x^2 \phi \, \partial_x^4 U^{\mathrm{r}} \, \mathrm{d}x \mathrm{d}\tau \right| \\ &\leq \nu \left| \int_{-\infty}^{\infty} \partial_x^2 \phi \, \partial_x^4 U^{\mathrm{r}} \, \mathrm{d}x \right| + \nu \left| \int_{-\infty}^{\infty} \partial_x^2 \phi_0 \, \partial_x^4 U^{\mathrm{r}}(0) \, \mathrm{d}x \right| \leq C_{\phi_0}. \end{aligned} \tag{4.18}$$

Substituting (4.18) into (4.17), we obtain the desired estimate.

Thus, we complete the proof of Proposition 4.4.

Remark 4.5. Similarly to Lemma 4.2, we have the uniform boundedness of $\partial_x^2 \phi$ by using Proposition 4.4 that

$$\sup_{t \in [0,T],\, x \in \mathbb{R}} | \partial_x^2 \phi(t,x) | \leq C_{\phi_0}.$$

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