LIFE SPAN OF SOLUTIONS TO A PDE MODEL FOR LITHIUM-ION BATTERIES IN HIGH SPACE DIMENSIONS[*]

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Abstract. In this paper we study a system of partial differential equations which models lithium-ion batteries. The system describes the conservation of Lithium and conservation of charges in the solid and electrolyte phases, together with the conservation of energy. The mathematical challenge is due to the fact that the reaction terms in the system involve the hyperbolic sine function along with possible degeneracy in one of the high order terms. We obtain a local existence assertion for the initial boundary problem for the system. In particular, a lower bound for the blow-up time can be derived from our result. We hope that our analysis can lead to a deeper understanding of battery life.

Key words. Lithium-ion batteries, the De Giorgi iteration scheme, existence and boundedness of a solution.

Mathematics Subject Classification. Primary 35A01, 35M10, 35J57, 35K20, 35B50.

1. Introduction. A lithium-ion battery or Li-ion battery (abbreviated as LIB) is a type of rechargeable batteries. The technology was largely developed by John Goodenough, Stanley Whittingham, Rachid Yazami, and Akira Yoshino during the 1970s1980s, and then commercialized by a Sony and Asahi Kasei team led by Yoshio Nishi in 1991. Lithium-ion batteries are commonly used for portable electronics and electric vehicles and are growing in popularity for military, civilian, and aerospace applications. A mathematical model describing the key factors of the battery operation can be very helpful for the design and optimization of battery performance. Based on a macro-homogeneous approach developed by Newman (see [13, 14]) several mathematical models have been developed for these purposes (see [1, 2, 5, 7, 6, 9, 10, 11, 15, 17, 18]) which include the main physics present in charge/discharge processes.

A typical Lithium-ion battery cell (Ω) has three regions: a porous negative electrode (Ω_a), a porous positive electrode (Ω_c), and an electro-blocking separator (Ω_s). See Fig.1 below. In the cell there is stored an electrolyte, which is a concentrated

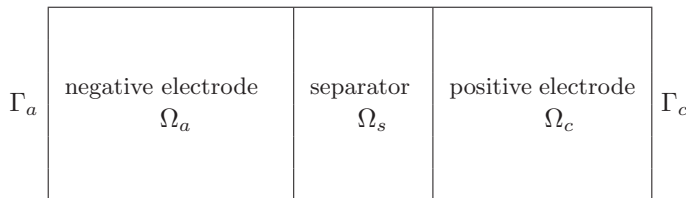| Γ_a | negative electrode Ω_a | separator Ω_s | positive electrode Ω_c | Γ_c |

FIG. 1. *The domain* Ω

solution containing charged species (lithium ions) that move along the cell in response to an electrochemical potential gradient. In the batteries, lithium ions move from the negative electrode through an electrolyte to the positive electrode during discharge, and back when charging. The concentration of the lithium ions in the electrolyte, C,

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the electric potential in the electrodes, Φ_s , and the electric potential measured by a reference Lithium electrode in the electrolyte, Φ_e, can be modeled, for each $t \in (0, T)$, by the following equations expressing the conservation of various physical quantities [19]:

$$-\text{div}(\kappa(C)\nabla\Phi_e) + \text{div}(\alpha_1\kappa(C)\nabla\ln C) - S_e = 0 \;\; \text{in } \Omega, \tag{1.1}$$

$$-\text{div}(\sigma\nabla\Phi_s) + S_e = 0 \;\; \text{in } \Omega' \equiv \Omega_a \cup \Omega_c, \tag{1.2}$$

$$\varepsilon_e\partial_t C - \text{div}(D\nabla C) - \alpha_3 S_e = 0 \;\; \text{in } \Omega. \tag{1.3}$$

We further specify the terms in the system by assuming:

(H1) The function $\kappa(C)$ lies in the space $C[0,\infty)$, satisfying

$$\kappa(0) = 0, \;\; \kappa(C) > 0 \text{ for } C > 0, \;\; \text{and} \;\; \kappa(C) \geq c_0 C^{\alpha_0} \;\; \text{on} \;\; [0, s_0)$$
$$\text{for some} \;\; c_0, s_0, \alpha_0 > 0. \tag{1.4}$$

(H2) $\sigma \in L^\infty(\Omega'), \varepsilon_e \in L^\infty(\Omega), D \in L^\infty(\Omega_T)$ with

$$\underset{\Omega'}{\text{ess inf}} \; \sigma > 0, \;\; \underset{\Omega}{\text{ess inf}} \; \varepsilon_e > 0, \;\; \underset{\Omega_T}{\text{ess inf}} \; D > 0.$$

Moreover,

$$\varepsilon_e \in C(\overline{\Omega}). \tag{1.5}$$

(H3) The term S_e is a function of C, Φ_s, Φ_e, which has the expression

$$S_e = S_e(C, \Phi_s, \Phi_e) = \begin{cases} \alpha_4\sqrt{C}\sinh\left(\alpha_2(\Phi_s - \Phi_e - U(C))\right) & \text{in } \Omega', \\ 0 & \text{in } \Omega_s, \end{cases} \tag{1.6}$$

where U is a bounded smooth function of C.

(H4) The numbers $\alpha_i, i = 1, 2, 3, 4$, in (1.1) and (1.6) are all positive numbers, whose precise values are determined by the physical properties of the materials involved.

To prescribe the initial boundary conditions, we first set

$$\Gamma_a = \partial\Omega \cap \partial\Omega_a, \;\; \Gamma_c = \partial\Omega \cap \partial\Omega_c.$$

We call $\Gamma_a \cup \Gamma_c$ the external boundary of Ω' and denote it by $\partial_{\text{ext}}\Omega'$. Subsequently, we impose

$$\frac{\partial\Phi_e}{\partial\mathbf{n}} = 0 \;\; \text{on } \partial\Omega,$$
$$\frac{\partial\Phi_s}{\partial\mathbf{n}} = 0 \;\; \text{on } \partial\Omega' \setminus \partial_{\text{ext}}\Omega', \tag{1.7}$$
$$-\sigma\frac{\partial\Phi_s}{\partial\mathbf{n}} = I \;\; \text{on } \partial_{\text{ext}}\Omega',$$
$$\frac{\partial C}{\partial\mathbf{n}} = 0 \;\; \text{on } \partial\Omega,$$
$$C(x,0) = C_0(x) \;\; \text{on } \Omega,$$

where $\mathbf{n}$ is the unit outward normal to the boundary and I is the applied current density (a given quantity). As in [19], we assume

$$\int_{\partial_{\text{ext}}\Omega'} I dS = 0. \tag{1.8}$$

This condition means that no charges are generated or consumed within the battery. Equipped with this, we can simplify the problem a little bit. To this end, we observe that (1.8) combing with (1.7) yields

$$\int_{\partial\Omega'} \sigma \frac{\partial\Phi_s}{\partial\mathbf{n}} dS = 0.$$

Consequently, (under suitable assumptions on I) we can find a function $\phi \in W^{1,2}(\Omega') \cap L^\infty(\Omega')$ such that

$$-\mathrm{div}(\sigma\nabla\phi) = 0 \quad \text{in } \Omega',$$
$$\frac{\partial\phi}{\partial\mathbf{n}} = \frac{\partial\Phi_s}{\partial\mathbf{n}} \quad \text{on } \partial\Omega'.$$

We make the following change of dependent variables

$$\Phi_s \leftarrow \Phi_s - \phi, \quad \Phi_e \leftarrow \Phi_e - \alpha_1 \ln C.$$

Accordingly, we represent S_e in terms of the new dependent variables

$$S_e = \alpha_4 \sqrt{C} \sinh\left(\alpha_2(\Phi_s - \Phi_e) + \alpha_2\phi - \alpha_1\alpha_2 \ln C - \alpha_2 U(C)\right)\chi_{\Omega'}$$
$$= \frac{1}{2}\alpha_4\sqrt{C}\left(hC^{-d}e^{\alpha_2(\Phi_s-\Phi_e)} - h^{-1}C^d e^{-\alpha_2(\Phi_s-\Phi_e)}\right)\chi_{\Omega'},$$

where

$$d = \alpha_1\alpha_2 \quad \text{and} \quad h = e^{\alpha_2(\phi - U(C))}.$$

In summary, we obtain the following problem

$$-\mathrm{div}(\kappa(C)\nabla\Phi_e) = S_e\chi_{\Omega'} \quad \text{in } \Omega_T \equiv \Omega \times (0,T), \tag{1.9}$$
$$-\mathrm{div}(\sigma\nabla\Phi_s) = -S_e \quad \text{in } \Omega'_T \equiv \Omega' \times (0,T), \tag{1.10}$$
$$\varepsilon_e\partial_t C - \mathrm{div}(D\nabla C) = \alpha_3 S_e\chi_{\Omega'} \quad \text{in } \Omega_T, \tag{1.11}$$
$$\frac{\partial\Phi_e}{\partial\mathbf{n}} = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.12}$$
$$\frac{\partial\Phi_s}{\partial\mathbf{n}} = 0 \quad \text{on } \partial\Omega' \times (0,T), \tag{1.13}$$
$$\frac{\partial C}{\partial\mathbf{n}} = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.14}$$
$$C(x,0) = C_0(x) \quad \text{on } \Omega, \tag{1.15}$$

where $T > 0$. Note that the introduction of the function f in [19] is unnecessary.

We must point out that LIBs are multiscale and multiphysics systems and deriving accurate mathematical descriptions can be very challenging [1, 11, 18]. Several factors such as the lithium ion concentration in the solid phase and the electric heating [18] are not considered in our model. We wish to investigate these factors in future study.

If we introduce the function

$$G(y_1, y_2, y_3) = y_1 y_2^{-d} e^{\alpha_2 y_3} - y_1^{-1} y_2^d e^{-\alpha_2 y_3} \quad \text{for } (y_1, y_2, y_3) \in (0,\infty) \times (0,\infty) \times \mathbb{R},$$

then we can write S_e in the form

$$S_e = \frac{1}{2}\alpha_4\sqrt{C}G(h, C, \Phi_s - \Phi_e)\chi_{\Omega'}.$$

For simplicity, we assume $U = 1$. Consequently, we can view h as a given function with the property

$$\frac{1}{K} \leq h \leq K \quad \text{for some } K \geq 1. \tag{1.16}$$

The case where U is an increasing function of C can be handled in an entirely similar manner.

Define the function spaces

$$X_T = L^\infty(\Omega_T) \times L^\infty(\Omega'_T) \times C(\overline{\Omega_T}), \tag{1.17}$$
$$Y_T = L^\infty(0,T;W^{1,2}(\Omega)) \times L^\infty(0,T;W^{1,2}(\Omega')) \times L^2(0,T;W^{1,2}(\Omega)). \tag{1.18}$$

Local existence of a weak solution in $X_T \cap Y_T$ to (1.9)-(1.15) was already established in [19]. The result there asserts that there is positive number T_0 such that problem (1.9)-(1.15) has a solution for $T = T_0$. This is to be expected because a battery can last for only a limited period of time. However, the local existence theorem in [19] did not offer any quantitative information on the size of T_0. The objective of this paper is to fill this gap. To be precise, we have:

THEOREM 1.1. *Let (H1)-(H4) be satisfied. Assume:*
(H5) $N \geq 2$ *and* $\partial\Omega, \partial\Omega'$ *are Lipschitz;*
(H6) $h \in C(\overline{\Omega_T})$ *is such that (1.16) is satisfied;*
(H7) $C_0(x) \in C^{\alpha_5}(\overline{\Omega})$ *for some* $\alpha_5 \in (0,1)$ *with the property*

$$\min_{\overline{\Omega}} C_0(x) > 0;$$

(H8) $d > \frac{1}{2}$*;*
(H9) $|\Omega_s| < |\Omega'|$.
Then there is a positive number T_0 *such that for* $T = T_0$ *problem (1.9)-(1.15) has a weak solution in* $X_T \cap Y_T$ *with the property*

$$\min_{\overline{\Omega_T}} C > 0, \tag{1.19}$$

$$\int_\Omega \Phi_e(x,t)dx + \int_{\Omega'} \Phi_s(x,t)dx = 0. \tag{1.20}$$

Furthermore, the number T_0 *satisfies the equation (4.30) below.*

A careful examination of the proof of (4.30) below reveals how T_0 "explicitly" depends on the given data. At least, we can gain an estimate for T_0 (see (2.10) below). Our method is more constructive than the one employed in [19]. Obviously, T_0 represents a lower bound for the blow-up time of our solution.

Since $\kappa(0) = 0$, we must obtain a positive lower bound for C to avoid degeneracy in (1.9). This also removes the singularity of S_e at $C = 0$. It turns out that condition (H8) is sufficient to do the deed. See Lemma 4.1 below. It is worth noting that (H8) is also the condition which ensures that S_e is a decreasing function of C. Since σ is taken to be 0 on Ω_s, equation (1.10) is only satisfied on Ω'_T. Condition (H9) is assumed to circumvent the difficulty caused by this. In applications, (H9) makes sense because separators Ω_s are rather thin. The reaction term S_e is messy at first glance. A closer examination shows that the function G in S_e is the composition of $b(s) \equiv s - \frac{1}{s}$ with

the term $hC^{-d}e^{\alpha_2(\Phi_s - \Phi_e)}$. The function $b(s)$ has two singular points at 0 and ∞. This suggests that we need to establish

$$\Phi_s - \Phi_e \in L^\infty(\Omega_T'), \quad C \in L^\infty(\Omega_T) \quad \text{with ess} \inf_{\Omega_T} C > 0.$$

It turns out that the above is indeed sufficient for an existence assertion for our problem. We observe

$$\partial_{y_3} G(y_1, y_2, y_3) = \alpha_2 \left(y_1 y_2^{-d} e^{\alpha_2 y_3} + y_1^{-1} y_2^d e^{-\alpha_2 y_3} \right)$$
$$\geq 2\alpha_2 > 0 \quad \text{for } (y_1, y_2, y_3) \in (0, \infty) \times (0, \infty) \times \mathbb{R}, \text{ and} \quad (1.21)$$
$$\partial_{y_2} G(y_1, y_2, y_3) = -\frac{d}{y_2}(y_1 y_2^{-d} e^{\alpha_2 y_3} + y_1^{-1} y_2^d e^{-\alpha_2 y_3})$$
$$\leq -\frac{2d}{y_2} < 0 \quad \text{on } (0, \infty) \times (0, \infty) \times \mathbb{R}.$$

These properties of G will be extensively exploited.

The uniqueness of a solution clearly does not hold. If (Φ_e, Φ_s, C) is a solution, so is $(\Phi_e + g(t), \Phi_s + g(t), C)$ for any $g(t)$. It would be interesting to know if the uniqueness holds under (1.20).

There is a large body of literature devoted to the mathematical study of lithium-ion batteries, and we will not attempt to make a comprehensive review. As far as we know, all the current work is concerned with the so-called (P2D) models in which the space dimension is essentially one with the lonely exception of [19]. Here we only mention [3, 5, 7, 11, 17], where one can find numerical computations and analytic study of this and other related models, with parameters corresponding to actual devices, that help to highlight the structure of models and show the relevance to the applications. As observed in [19], the gap between $N = 1$ and $N > 1$ is substantial in terms of mathematical analysis.

The space dimension N is assumed to be bigger than 2 for the convenience of applying the Sobolev embedding theorem. Our argument still works for $N = 2$ under some minor adjustments. Hölder's inequality is often used without acknowledgment. The same goes for the following two inequalities

$$(|a| + |b|)^\beta \leq 2^{\beta-1} \left(|a|^\beta + |b|^\beta \right) \quad \text{if } \beta > 1, \quad (|a| + |b|)^\beta \leq |a|^\beta + |b|^\beta \quad \text{if } 0 < \beta \leq 1.$$

When no confusion arises, we suppress the dependence of a function on its variables. As usual, the letter c is used to denote a generic positive constant.

This paper is organized as follows. In Section 2 we collect a few known results that have played essential roles in our development. Section 3 is devoted to the study of approximate problems. It is interesting to note that our approximation scheme is designed in such a way that the global existence of a solution holds. The proof of Theorem 1.1 is presented in Section 4.

2. Preliminaries. In this section we collect a few known results that are useful to us.

Our existence theorem is based upon the following fixed point theorem, which is often called the Leray-Schauder fixed point lemma ([8], p.280).

LEMMA 2.1. *Let B be a map from a Banach space $\mathcal{B}$ into itself. Assume:*
(LS1) B is continuous;
(LS2) the images of bounded sets of B are precompact;

(LS3) *there exists a constant c such that*

$$\|z\|_{\mathcal{B}} \leq c$$

for all $z \in \mathcal{B}$ *and* $\sigma \in [0,1]$ *satisfying* $z = \sigma B(z)$.
Then B has a fixed point.

LEMMA 2.2. *Let* $\{y_n\}, n = 0, 1, 2, \cdots$, *be a sequence of positive numbers satisfying the recursive inequalities*

$$y_{n+1} \leq cb^n y_n^{1+\alpha} \quad \text{for some } b > 1, c, \alpha \in (0, \infty). \tag{2.1}$$

If

$$y_0 \leq c^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}},$$

then $\lim_{n \to \infty} y_n = 0$.

This lemma can be found in ([4], p.12).

The following lemma plays a key role in the proof of Theorem 1.1. It is inspired by a result in [20].

LEMMA 2.3. *Let* $a(\tau)$ *be a continuous non-negative function defined on* $[0, T_0]$ *for some* $T_0 > 0$. *Suppose that there exist three positive numbers* ε, δ, m *such that*

$$a(\tau) < \varepsilon e^{ma^{1+\delta}(\tau)} + m \quad \text{for each } \tau \in [0, T_0]. \tag{2.2}$$

Then there exists a positive number s_0 *determined by* m, δ *only such that if* (2.2) *holds for*

$$\varepsilon = \varepsilon_0 \equiv \frac{1}{m(1+\delta)e^{ms_0^{1+\delta}} s_0^{\delta}} \tag{2.3}$$

then we have

$$a(\tau) < s_0 \quad \text{for each } \tau \in (0, T_0], \text{ provided that } a(0) \leq m. \tag{2.4}$$

Proof. Introduce the function $f_\varepsilon(s) = \varepsilon e^{ms^{1+\delta}} - s + m$ on $[0, \infty)$. Then we form the system

$$f'_\varepsilon(s) = \varepsilon m(1+\delta)e^{ms^{1+\delta}} s^{\delta} - 1 = 0, \tag{2.5}$$

$$f_\varepsilon(s) = \varepsilon e^{ms^{1+\delta}} - s + m = 0 \tag{2.6}$$

and solve it for (ε, s). By (2.5), we have

$$\varepsilon = \frac{1}{m(1+\delta)e^{ms^{1+\delta}} s^{\delta}}.$$

Substitute this into (2.6) to derive

$$\frac{1}{m(1+\delta)s^{\delta}} - s + m = 0. \tag{2.7}$$

Clearly, the left-hand side of (2.7) is a strictly decreasing function on $(0, \infty)$ with $f_\varepsilon(0+) = \infty$ and $\lim_{s \to \infty} f_\varepsilon(s) = -\infty$. Thus, (2.7) has a unique solution in $(0, \infty)$. This solution is our s_0, and it depends only on m, δ. Use (2.3) to obtain ε_0. Our assumption implies

$$f_{\varepsilon_0}(a(\tau)) > 0 \quad \text{for each } \tau \in [0, T_0]. \tag{2.8}$$

We easily infer from (2.5) and (2.6) that

$$f_{\varepsilon_0}(s_0) = f'_{\varepsilon_0}(s_0) = 0. \tag{2.9}$$

Moreover, $f_{\varepsilon_0}(s)$ is strictly decreasing on $[0, s_0)$ and strictly increasing on (s_0, ∞). Thus, 0 is the minimum value of $f_{\varepsilon_0}(s)$ on $[0, \infty)$.

It is easy to see from (2.7) that

$$s_0 > m. \tag{2.10}$$

The range of a is a closed interval because it is a continuous function $[0, T_0]$, and this interval is either contained in $[0, s_0)$ or (s_0, ∞) due to (2.8) and (2.9). The latter cannot occur due to the second inequality in (2.4) and (2.10). Thus the lemma follows. □

3. Approximate problems. In this section, we first present our approximation scheme. Then we proceed to prove the existence of a weak solution to our approximate problems. Note that our construction here is entirely different from that in [19] where the authors directly seek C in the function space $Z_M \equiv \{u \in C^{\frac{\beta}{2}, \beta}(\overline{\Omega_T}) : \frac{1}{M} \leq u \text{ and } \|u\|_{C^{\frac{\beta}{2}, \beta}(\overline{\Omega_T})} \leq M\}$ for some $\beta \in (0, 1)$ and $M > 1$.

To design our approximation scheme, we pick

$$\tau \in (0, 1) \tag{3.1}$$

and define

$$\begin{aligned}
\kappa_\tau(s) &= \kappa(s^+ + \tau), \\
\theta_\tau(s) &= \begin{cases} \frac{1}{\tau} & \text{if } s \geq \frac{1}{\tau}, \\ s & \text{if } \kappa(s) \leq \frac{1}{\tau} \end{cases} \quad \text{for } s \in \mathbb{R}.
\end{aligned} \tag{3.2}$$

We approximate $G(y_1, y_2, y_3)$ by

$$\begin{aligned}
G_\tau(y_1, y_2, y_3) &= y_1(\theta_\tau(y_2) + \tau)^{-d} e^{\alpha_2 y_3} - y_1^{-1} \theta_\tau^d(y_2) e^{-\alpha_2 y_3} \\
&\text{for } (y_1, y_2, y_3) \in (0, \infty) \times [0, \infty) \times \mathbb{R}.
\end{aligned} \tag{3.3}$$

Our approximate problems are:

$$-\text{div}\,[\kappa_\tau(C)\nabla\Phi_e] + \tau\Phi_e = \frac{1}{2}\alpha_4 H_\tau(h, C^+, \Phi_s - \Phi_e)\chi_{\Omega'} \quad \text{in } \Omega_T, \tag{3.4}$$

$$-\text{div}\,(\sigma\nabla\Phi_s) + \tau\Phi_s = -\frac{1}{2}\alpha_4 H_\tau(h, C^+, \Phi_s - \Phi_e) \text{ in } \Omega'_T, \tag{3.5}$$

$$\varepsilon_e \partial_t C - \text{div}\,(D\nabla C) = \frac{1}{2}\alpha_3\alpha_4 H_\tau(h, C^+, \Phi_s - \Phi_e)\chi_{\Omega'} \quad \text{in } \Omega_T, \tag{3.6}$$

$$\frac{\partial \Phi_e}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{3.7}$$

$$\frac{\partial \Phi_s}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega' \times (0, T), \tag{3.8}$$

$$\frac{\partial C}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega' \times (0, T), \tag{3.9}$$

$$C(x, 0) = C_0(x) \quad \text{on } \Omega, \tag{3.10}$$

where

$$H_\tau(y_1, y_2, y_3) = \sqrt{\theta_\tau(y_2)} G_\tau(y_1, y_2, y_3). \tag{3.11}$$

THEOREM 3.1. *Let (H1)-(H7) be satisfied. For each $T > 0$ there is a weak solution (Φ_e, Φ_s, C) in the space $X_T \cap Y_T$ to problem (3.4)-(3.10), where X_T and Y_T are defined in (1.17) and (1.18), respectively.*

The proof of this theorem is divided into a couple of lemmas.

LEMMA 3.2. *Let $C = C(x, t) \in C(\overline{\Omega_T})$ be given. For each $t \in [0, T]$ there is a unique weak solution $(\varphi, \psi) = (\varphi(x, t), \psi(x, t))$ in $\left(C(\overline{\Omega}) \cap W^{1,2}(\Omega)\right)^2$ to the boundary value problem*

$$-\mathrm{div}\left(\kappa_\tau(C)\nabla\varphi\right) + \tau\varphi = \frac{1}{2}\alpha_4 H_\tau(h, C^+, \psi - \varphi)\chi_{\Omega'} \quad in\ \Omega, \tag{3.12}$$

$$-\mathrm{div}\left(\sigma\nabla\psi\right) + \tau\psi = -\frac{1}{2}\alpha_4 H_\tau(h, C^+, \psi - \varphi) \quad in\ \Omega', \tag{3.13}$$

$$\frac{\partial\varphi}{\partial\mathbf{n}} = 0 \quad on\ \partial\Omega, \tag{3.14}$$

$$\frac{\partial\psi}{\partial\mathbf{n}} = 0 \quad on\ \partial\Omega'. \tag{3.15}$$

Proof. We first establish the uniqueness assertion. Suppose that there exist two weak solutions, say, $(\varphi_1, \psi_1), (\varphi_2, \psi_2)$. Then we have

$$-\mathrm{div}\left(\kappa_\tau(C)\nabla(\varphi_1 - \varphi_2)\right) + \tau(\varphi_1 - \varphi_2)$$
$$= \frac{1}{2}\alpha_4\sqrt{\theta_\tau(C^+)}\left(G_\tau(h, C^+, \psi_1 - \varphi_1) - G_\tau(h, C^+, \psi_2 - \varphi_2)\right)\chi_{\Omega'} \quad in\ \Omega, \tag{3.16}$$
$$-\mathrm{div}\left(\sigma\nabla(\psi_1 - \psi_2)\right) + \tau(\psi_1 - \psi_2)$$
$$= -\frac{1}{2}\alpha_4\sqrt{\theta_\tau(C^+)}\left(G_\tau(h, C^+, \psi_1 - \varphi_1) - G_\tau(h, C^+, \psi_2 - \varphi_2)\right) \quad in\ \Omega'. \tag{3.17}$$

Note from (3.3) that for each fixed τ we have

$$G_\tau(h, C^+, \varphi) \in L^\infty(\Omega') \quad whenever\ \varphi \in L^\infty(\Omega').$$

Use $\varphi_1 - \varphi_2$ as a test function in (3.16), $\psi_1 - \psi_2$ in (3.17), then add up the two resulting equations to derive

$$\int_\Omega \kappa_\tau(C)|\nabla(\varphi_1 - \varphi_2)|^2 dx + \tau\int_\Omega |\varphi_1 - \varphi_2|^2 dx$$
$$+ \int_{\Omega'} \sigma|\nabla(\psi_1 - \psi_2)|^2 dx + \tau\int_{\Omega'} |\psi_1 - \psi_2|^2 dx$$
$$= -\frac{1}{2}\alpha_4\int_{\Omega'} \sqrt{\theta_\tau(C^+)}\left(G_\tau(h, C^+, \psi_1 - \varphi_1) - G_\tau(h, C^+, \psi_2 - \varphi_2)\right)\left[(\psi_1 - \varphi_1) - (\psi_2 - \varphi_2)\right]dx$$
$$\leq 0.$$

The last step is due to (1.21). Thus,

$$\psi_1 = \psi_2, \quad \varphi_1 = \varphi_2.$$

For the existence, we define an operator $\mathbb{T}$ from $\mathbb{B} \equiv L^\infty(\Omega') \times L^\infty(\Omega)$ into itself as follows: Given $(u, v) \in \mathbb{B}$, we say $\mathbb{T}(u, v) = (\psi, \varphi)$ if ψ, φ are the respective solutions of the following two problems

$$-\operatorname{div}(\kappa_\tau(C)\nabla\varphi) + \tau\varphi = \frac{1}{2}\alpha_4 H_\tau(h, C^+, u - v)\chi_{\Omega'} \quad \text{in } \Omega, \tag{3.18}$$

$$\frac{\partial\varphi}{\partial\mathbf{n}} = 0 \quad \text{on } \partial\Omega, \tag{3.19}$$

$$-\operatorname{div}(\sigma\nabla\psi) + \tau\psi = -\frac{1}{2}\alpha_4 H_\tau(h, C^+, u - v) \quad \text{in } \Omega', \tag{3.20}$$

$$\frac{\partial\psi}{\partial\mathbf{n}} = 0 \quad \text{on } \partial\Omega'. \tag{3.21}$$

Observe from our assumptions that $\kappa_\tau(C)$ is bounded above and bounded away from 0 below and the right hand side term in (3.18) lies in $L^\infty(\Omega)$. We can easily conclude from classical theory for linear elliptic equations that problem (3.18)-(3.19) has a unique weak solution φ in $W^{1,2}(\Omega) \cap C^\beta(\overline{\Omega})$ for some $\beta \in (0, 1)$. Similarly, problem (3.20)-(3.21) has a unique weak solution ψ in $W^{1,2}(\Omega') \cap C^\beta(\overline{\Omega'})$. Therefore, the operator $\mathbb{T}$ is well-defined. Obviously, any fixed point of $\mathbb{T}$ is a weak solution to problem (3.12)-(3.15). The existence of such a fixed point will be established by applying Lemma 2.1, the Leray-Schauder fixed point lemma. To this end, we can infer from the previously-mentioned classical results that $\mathbb{T}$ is continuous and maps bounded sets into precompact ones. We still need to show that there is a positive number c such that

$$\|(v\chi_{\Omega'}, u)\|_{\infty,\Omega} \leq c \tag{3.22}$$

for all $(u, v) \in \mathbb{B}$ and $\delta \in (0, 1)$ satisfying $(u, v) = \delta\mathbb{T}(u, v)$. The preceding equation is equivalent to

$$-\operatorname{div}(\kappa_\tau(C)\nabla u) + \tau u = \frac{1}{2}\delta\alpha_4 H_\tau(h, C^+, v - u)\chi_{\Omega'} \quad \text{in } \Omega, \tag{3.23}$$

$$\frac{\partial u}{\partial\mathbf{n}} = 0 \quad \text{on } \partial\Omega, \tag{3.24}$$

$$-\operatorname{div}(\sigma\nabla v) + \tau v = -\frac{1}{2}\delta\alpha_4 H_\tau(h, C^+, v - u)\chi_{\Omega'} \quad \text{in } \Omega', \tag{3.25}$$

$$\frac{\partial v}{\partial\mathbf{n}} = 0 \quad \text{on } \partial\Omega'. \tag{3.26}$$

To see (3.22), we use $(u^+)^n$, where $n \geq 1$, as a test function in (3.23) and keep in mind (3.24) to get

$$\tau \int_\Omega (u^+)^{n+1}\, dx \leq \frac{1}{2}\delta\alpha_4 \int_{\Omega'} H_\tau(h, C^+, v - u)\, (u^+)^n\, dx. \tag{3.27}$$

Similarly, we can infer from (3.25) and (3.26) that

$$\tau \int_{\Omega'} (v^+)^{n+1}\, dx \leq -\frac{1}{2}\delta\alpha_4 \int_{\Omega'} H_\tau(h, C^+, v - u)\, (v^+)^n\, dx.$$

Add this to (3.27) to derive

$$
\tau \int_\Omega \left(u^+ \right)^{n+1} dx + \tau \int_{\Omega'} \left(v^+ \right)^{n+1} dx
$$

$$
\leq -\frac{1}{2}\delta\alpha_4 \int_{\Omega'} H_\tau(h, C^+, v - u) \left(\left(v^+ \right)^n - \left(u^+ \right)^n \right) dx
$$

$$
= -\frac{1}{2}\delta\alpha_4 \int_{\Omega'} \sqrt{\theta_\tau(C^+)} \left[G_\tau(h, C^+, v - u) - G_\tau(h, C^+, 0) \right] \left(\left(v^+ \right)^n - \left(u^+ \right)^n \right) dx
$$

$$
+\frac{1}{2}\delta\alpha_4 \int_{\Omega'} H_\tau(h, C^+, 0) \left(\left(v^+ \right)^n - \left(u^+ \right)^n \right) dx. \tag{3.28}
$$

Recall from (1.21) that $G_\tau(y_1, y_2, y_3)$ is increasing in y_3. Thus,

$$
\left[G_\tau(h, C^+, v - u) - G_\tau(h, C^+, 0) \right] \left(\left(v^+ \right)^n - \left(u^+ \right)^n \right) \geq 0 \ \text{ in } \Omega'.
$$

Use this in (3.28) to derive

$$
\tau \int_\Omega \left(u^+ \right)^{n+1} dx + \tau \int_{\Omega'} \left(v^+ \right)^{n+1} dx
$$

$$
\leq \frac{1}{2}\alpha_4 \int_{\Omega'} \sqrt{\theta_\tau(C^+)} \left| G_\tau(h, C^+, 0) \right| \left(\left(v^+ \right)^n + \left(u^+ \right)^n \right) dx
$$

$$
\leq \alpha_4 \int_{\Omega'} \sqrt{\theta_\tau(C^+)} \left| G_\tau(h, C^+, 0) \right| w^n dx,
$$

where

$$
w = \max\{ u^+, v^+ \chi_{\Omega'} \}.
$$

Consequently,

$$
\tau \int_\Omega w^{n+1} dx \leq \tau \int_\Omega \left(u^+ \right)^{n+1} dx + \tau \int_{\Omega'} \left(v^+ \right)^{n+1} dx
$$

$$
\leq \alpha_4 \int_{\Omega'} \sqrt{\theta_\tau(C^+)} \left| G_\tau(h, C^+, 0) \right| w^n dx
$$

$$
\leq \alpha_4 \| w \|_{n+1,\Omega}^n \| H_\tau(h, C^+, 0) \|_{n+1,\Omega},
$$

from whence follows

$$
\| w \|_{n+1,\Omega} \leq \frac{\alpha_4}{\tau} \| H_\tau(h, C^+, 0) \|_{n+1,\Omega}.
$$

Taking $n \to \infty$ yields

$$
\max\{ \| u^+ \|_{\infty,\Omega}, \| v^+ \|_{\infty,\Omega'} \} \leq \frac{\alpha_4}{\tau} \| H_\tau(h, C^+, 0) \|_{\infty,\Omega}. \tag{3.29}
$$

In an entirely similar manner, we can also prove

$$
\max\{ \| u^- \|_{\infty,\Omega}, \| v^- \|_{\infty,\Omega'} \} \leq \frac{\alpha_4}{\tau} \| H_\tau(h, C^+, 0) \|_{\infty,\Omega}. \tag{3.30}
$$

Combining this with (3.29) yields (3.22). The proof is complete. $\square$

To continue the proof of Theorem 3.1, we define an operator $\mathbb{A}$ from $C(\overline{\Omega_T})$ into $C(\overline{\Omega_T})$ as follows: For each $C \in C(\overline{\Omega_T})$, we first solve system (3.4)-(3.5) coupled

with boundary conditions (3.7) and (3.8) for (Φ_e, Φ_s). Then use the pair to form the problem

$$\varepsilon_e \partial_t B - \mathrm{div}\, (D\nabla B) = \frac{1}{2}\alpha_3\alpha_4 H_\tau(h, C^+, \Phi_s - \Phi_e) \;\; \text{in } \Omega_T, \tag{3.31}$$

$$\frac{\partial B}{\partial \mathbf{n}} = 0 \;\; \text{on } \partial\Omega \times (0, T), \tag{3.32}$$

$$B(x, 0) = C_0(x) \;\; \text{on } \Omega. \tag{3.33}$$

Classical regularity for linear parabolic equations ([12], Chap.III) asserts that problem (3.31)-(3.33) has a unique weak solution B in the space $C([0, T]; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega))$. Since we have assumed (1.5), B is Hölder continuous on $\overline{\Omega_T}$. We define $B = \mathbb{A}(C)$. It is easy to see that $\mathbb{A}$ is well-defined and continuous and maps bounded sets into precompact ones. To be able to use Lemma 2.1, we still need to show that

$$\|C\|_{\infty, \Omega_T} \le c \tag{3.34}$$

for all $\delta \in [0, 1]$ and $C \in C(\overline{\Omega_T})$ satisfying $C = \delta\mathbb{A}(C)$. This equation is equivalent to the problem

$$-\mathrm{div}\,[\kappa_\tau(C)\nabla\Phi_e] + \tau\Phi_e = \frac{1}{2}\alpha_4 H_\tau(h, C^+, \Phi_s - \Phi_e)\chi_{\Omega'} \;\; \text{in } \Omega_T, \tag{3.35}$$

$$-\mathrm{div}\,(\sigma\nabla\Phi_s) + \tau\Phi_s = -\frac{1}{2}\alpha_4 H_\tau(h, C^+, \Phi_s - \Phi_e) \;\; \text{in } \Omega'_T, \tag{3.36}$$

$$\varepsilon_e \partial_t C - \mathrm{div}\,(D\nabla C) = \frac{1}{2}\alpha_3\alpha_4\delta H_\tau(h, C^+, \Phi_s - \Phi_e)\chi_{\Omega'} \;\; \text{in } \Omega_T, \tag{3.37}$$

$$\frac{\partial\Phi_e}{\partial\mathbf{n}} = \frac{\partial C}{\partial\mathbf{n}} = 0 \;\; \text{on } \partial\Omega \times (0, T), \tag{3.38}$$

$$\frac{\partial\Phi_s}{\partial\mathbf{n}} = 0 \;\; \text{on } \partial\Omega' \times (0, T), \tag{3.39}$$

$$C(x, 0) = \delta C_0(x) \;\; \text{on } \Omega. \tag{3.40}$$

To see (3.34), we use C^- as a test function in (3.37) to get

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega \varepsilon_e (C^-)^2 dx - \int_\Omega D|\nabla C^-|^2 dx = 0,$$

from whence follows

$$C \ge 0 \;\; \text{a.e. on } \Omega_T.$$

Thus, we can replace C^+ in (3.35)-(3.37) by C. We will do this in the subsequent calculations without acknowledgment.

An upper bound for C is established in the following lemma.

LEMMA 3.3. *For each $q > 1 + \frac{N}{2}$ there is a positive number c depending on N, q, Ω and the given data in system (1.9)-(1.11) such that*

$$\max_{\Omega_T} C \le c(1 + T)^{\frac{2N}{(N+2)}} T^{\frac{2(2q-2-N)}{q(N+2)}} \left\| e^{\alpha_2(\Phi_s - \Phi_e)} \right\|_{q, \Omega'_T}^2 + 4\|C_0\|_{\infty, \Omega} + 2.$$

For our later purpose, the number c in the above inequality will be made independent of τ.

Proof. We employ the well known De Giorgi iteration scheme [4]. Let q be given as in the lemma. Select

$$k \geq \max\{2\|C_0\|_{\infty,\Omega}, 1\} \tag{3.41}$$

as below. Define

$$k_n = k - \frac{k}{2^{n+1}}, \quad n = 0, 1, \cdots.$$

Consequently,

$$k \geq k_{n+1} \geq k_n \geq \frac{k}{2} \geq \|C_0\|_{\infty,\Omega} \quad \text{for each } n \in \{0, 1, \cdots\}. \tag{3.42}$$

We use $(C - k_{n+1})^+$ as a test function in (3.37) to derive

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \varepsilon_e[(C - k_{n+1})^+]^2 dx + \int_\Omega D|\nabla(C - k_{n+1})^+|^2 dx$$
$$= \frac{1}{2}\alpha_3\alpha_4\delta\int_{\Omega'} H_\tau(h, C, \Phi_s - \Phi_e)(C - k_{n+1})^+ dx. \tag{3.43}$$

Note that $G_\tau(y_1, y_2, y_3)$ is decreasing in y_2. Thus,

$$[G_\tau(h, C, \Phi_s - \Phi_e) - G_\tau(h, k_{n+1}, \Phi_s - \Phi_e)](C - k_{n+1})^+ \leq 0.$$

With this in mind, we can estimate the last integral in (3.43) as follows:

$$\int_{\Omega'} H_\tau(h, C, \Phi_s - \Phi_e)(C - k_{n+1})^+ dx$$
$$= \int_{\Omega'} \sqrt{\theta_\tau(C)}\left(G_\tau(h, C, \Phi_s - \Phi_e) - G_\tau(h, k_{n+1}, \Phi_s - \Phi_e)\right)(C - k_{n+1})^+ dx$$
$$+ \int_{\Omega'} H_\tau(h, k_{n+1}, \Phi_s - \Phi_e)(C - k_{n+1})^+ dx$$
$$\leq \frac{\|h\|_{\infty,\Omega_T'}}{(\theta_\tau(k_{n+1}) + \tau)^d}\int_{\Omega'} \sqrt{\theta_\tau(C)}e^{\alpha_2(\Phi_s - \Phi_e)}(C - k_{n+1})^+ dx. \tag{3.44}$$

Here we have used (3.3), the definition of G_τ. We easily verify from (3.1), (3.2), (3.41), and (3.42) that

$$\frac{1}{\theta_\tau(k_{n+1}) + \tau} \leq 2.$$

Substitute (3.44) into (3.43), integrate the resulting inequality with respect to t, and keep in mind the fact that $(C - k_{n+1})^+|_{t=0} = 0$ due to (3.42) to deduce

$$\sup_{0 \leq t \leq T}\int_\Omega \varepsilon_e[(C - k_{n+1})^+]^2 dx + \int_{\Omega_T} D|\nabla(C - k_{n+1})^+|^2 dxdt$$
$$\leq c\int_{\Omega_T'} \sqrt{\theta_\tau(C)}e^{\alpha_2(\Phi_s - \Phi_e)}(C - k_{n+1})^+ dxdt. \tag{3.45}$$

Note that

$$
\begin{aligned}
\left[(C - k_n)^+\right]^{\frac{3}{2}} &\geq \sqrt{(C - k_n)^+}(C - k_{n+1})^+ \\
&\geq \sqrt{C\left(1 - \frac{k_n}{k_{n+1}}\right)}(C - k_{n+1})^+ \\
&\geq \frac{1}{2^{\frac{n+2}{2}}}\sqrt{\theta_\tau(C)}(C - k_{n+1})^+,
\end{aligned}
$$

from whence follows

$$
\begin{aligned}
\int_{\Omega_T'} \sqrt{\theta_\tau(C)}e^{\alpha_2(\Phi_s - \Phi_e)}(C - k_{n+1})^+ \, dxdt &\leq 2^{\frac{n+2}{2}}\int_{\Omega_T'} e^{\alpha_2(\Phi_s - \Phi_e)}\left[(C - k_n)^+\right]^{\frac{3}{2}} \, dxdt \\
&\leq 2^{\frac{n+2}{2}}\left\|e^{\alpha_2(\Phi_s - \Phi_e)}\right\|_{q,\Omega_T'} y_n^{\frac{q-1}{q}},
\end{aligned} \tag{3.46}
$$

where

$$
y_n = \int_{\Omega_T} \left[(C - k_n)^+\right]^{\frac{3q}{2(q-1)}} \, dxdt.
$$

Recall from the Sobolev embedding theorem that

$$
\begin{aligned}
&\int_{\Omega_T} \left[(C - k_{n+1})^+\right]^{2+\frac{4}{N}} \, dxdt \\
&\leq \int_0^T \left(\int_\Omega \left[(C - k_{n+1})^+\right]^2 \, dx\right)^{\frac{2}{N}} \left(\int_\Omega \left[(C - k_{n+1})^+\right]^{\frac{2N}{N-2}} \, dx\right)^{\frac{N-2}{N}} \, dt \\
&\leq c\left(\sup_{0 \leq t \leq T} \int_\Omega \left[(C - k_{n+1})^+\right]^2 \, dx\right)^{\frac{2}{N}} \int_{\Omega_T} \left|\nabla(C - k_{n+1})^+\right|^2 \, dxdt \\
&\quad + c\left(\sup_{0 \leq t \leq T} \int_\Omega \left[(C - k_{n+1})^+\right]^2 \, dx\right)^{\frac{2}{N}} \int_0^T \int_\Omega \left[(C - k_{n+1})^+\right]^2 \, dxdt \\
&\leq c(1 + T)2^{\frac{(n+2)(N+2)}{2N}}\left\|e^{\alpha_2(\Phi_s - \Phi_e)}\right\|_{q,\Omega_T'}^{\frac{N+2}{N}} y_n^{\frac{(q-1)(N+2)}{qN}}.
\end{aligned} \tag{3.47}
$$

The last step is due to (3.45) and (3.46). We easily see that

$$
y_n \geq \int_{\Omega_T \cap \{C \geq k_{n+1}\}} (k_{n+1} - k_n)^{\frac{3q}{2(q-1)}} \, dxdt = \frac{k^{\frac{3q}{2(q-1)}}}{2^{\frac{3q(n+2)}{2(q-1)}}} |\{C \geq k_{n+1}\}|.
$$

Furthermore, our assumption on q implies that

$$
2 + \frac{4}{N} > \frac{3q}{2(q-1)}, \quad \beta \equiv \frac{3}{4} - \frac{3qN}{4(N+2)(q-1)} > 0.
$$

Keeping the preceding three inequalities and (3.47) in mind, we estimate that

$$
\begin{aligned}
y_{n+1} &\leq \left(\int_{\Omega_T} \left[(C - k_{n+1})^+\right]^{2+\frac{4}{N}} \, dxdt\right)^{\frac{3qN}{4(N+2)(q-1)}} |\{C \geq k_{n+1}\}|^{\frac{1+4\beta}{4}} \\
&\leq \frac{c(1+T)^{\frac{3qN}{4(N+2)(q-1)}}\left\|e^{\alpha_2(\Phi_s - \Phi_e)}\right\|_{q,\Omega_T'}^{\frac{3q}{4(q-1)}} 2^{\frac{3q(n+2)}{8(q-1)}\left(\frac{1}{2}+\beta\right)}}{k^{\frac{3q(1+4\beta)}{8(q-1)}}} y_n^{1+\beta}.
\end{aligned}
$$

By Lemma 2.2, if we choose k so large that

$$y_0 \leq \int_{\Omega_T} C^{\frac{3q}{2(q-1)}}\, dxdt \leq \frac{ck^{\frac{3q(1+4\beta)}{8\beta(q-1)}}}{(1+T)^{\frac{3qN}{4\beta(N+2)(q-1)}} \left\| e^{\alpha_2(\Phi_s - \Phi_e)} \right\|_{q,\Omega_T'}^{\frac{3q}{4(q-1)\beta}}}$$

then

$$C \leq k.$$

In view of (3.41), it is enough for us to take

$$k = c(1+T)^{\frac{2N}{(N+2)(1+4\beta)}} \left\| e^{\alpha_2(\Phi_s - \Phi_e)} \right\|_{q,\Omega_T'}^{\frac{2}{1+4\beta}} \|C\|_{\frac{3q}{2(q-1)},\Omega_T}^{\frac{4\beta}{1+4\beta}} + 2\|C_0\|_{\infty,\Omega} + 1.$$

That is,

$$\|C\|_{\infty,\Omega_T} \leq c(1+T)^{\frac{2N}{(N+2)(1+4\beta)}} \left\| e^{\alpha_2(\Phi_s - \Phi_e)} \right\|_{q,\Omega_T'}^{\frac{2}{1+4\beta}} \|C\|_{\frac{3q}{2(q-1)},\Omega_T}^{\frac{4\beta}{1+4\beta}} + 2\|C_0\|_{\infty,\Omega} + 1$$

$$\leq c(1+T)^{\frac{2N}{(N+2)(1+4\beta)}} T^{\frac{8\beta(q-1)}{3q(1+4\beta)}} \left\| e^{\alpha_2(\Phi_s - \Phi_e)} \right\|_{q,\Omega_T'}^{\frac{2}{1+4\beta}} \|C\|_{\infty,\Omega_T}^{\frac{4\beta}{1+4\beta}} + 2\|C_0\|_{\infty,\Omega} + 1$$

$$\leq \frac{1}{2}\|C\|_{\infty,\Omega_T} + c(1+T)^{\frac{2N}{N+2}} T^{\frac{8\beta(q-1)}{3q}} \left\| e^{\alpha_2(\Phi_s - \Phi_e)} \right\|_{q,\Omega_T'}^{2} + 2\|C_0\|_{\infty,\Omega} + 1.$$

This implies the desired result.☐

To conclude the proof of Theorem 3.1, we can easily infer from (3.29), (3.30), and (3.11) that

$$\|\Phi_e\|_{\infty,\Omega_T} + \|\Phi_s\|_{\infty,\Omega_T'} \leq c(\tau).$$

This combined with Lemma 3.3 yields (3.34). Now we are ready to invoke Lemma 2.1. Upon doing so, we obtain a fixed point of $\mathbb{A}$, which produces a weak solution to problem (3.4)-(3.10). This completes the proof of Theorem 3.1.

4. Proof of Theorem 1.1. In this section, we offer the proof of Theorem 1.1. For this purpose, we denote the solutions to problem (3.4)-(3.10) constructed in Theorem 3.1 by $(\Phi_e^{(\tau)}, \Phi_s^{(\tau)}, C^{(\tau)})$ with $\tau = \frac{1}{k}$, where $k = 1, 2, \cdots$. The plan is to derive enough a priori estimates for the sequence to justify passing to the limit in problem (3.4)-(3.10). The generic positive constant c in the subsequent calculations will be independent of τ. The key to our development is the following lemmas, which asserts that $C^{(\tau)}$ is bounded away from 0 below.

LEMMA 4.1. *For each $q > 1 + \frac{N}{2}$ there is a positive number c such that*

$$\frac{1}{C^{(\tau)}} \leq \frac{1}{\min_{\overline{\Omega}} C_0} + c(1+T)^{\frac{2N}{(N+2)(2d-1)}} \left\| e^{-\alpha_2(\Phi_s^{(\tau)} - \Phi_e^{(\tau)})} \right\|_{q,\Omega_T'}^{\frac{2}{2d-1}}.$$

Proof. Pick

$$\frac{2}{k} \leq \min_{\overline{\Omega}} C_0 \qquad\qquad\qquad (4.1)$$

as below. Set

$$k_n = k - \frac{k}{2^{n+1}}, \quad n = 0, 1, \cdots.$$

Recall that $G_\tau(y_1, y_2, y_3)$ is decreasing in y_2. Thus,

$$M_\tau \equiv \left[G_\tau(h, C^{(\tau)}, \Phi_s^{(\tau)} - \Phi_e^{(\tau)}) - G_\tau(h, k_{n+1}^{-1}, \Phi_s^{(\tau)} - \Phi_e^{(\tau)}) \right] \left(\left(C^{(\tau)} \right)^{-\frac{1}{2}} - k_{n+1}^{\frac{1}{2}} \right)^+ \geq 0.$$

Use $\left(\left(C^{(\tau)} \right)^{-\frac{1}{2}} - k_{n+1}^{\frac{1}{2}} \right)^+$ as a test function in (3.37) and apply the above inequality and the definition of G_τ to get

$$\frac{d}{dt} \int_\Omega \varepsilon_e \int_{C^{(\tau)}}^{k_{n+1}^{-1}} \left(s^{-\frac{1}{2}} - k_{n+1}^{\frac{1}{2}} \right)^+ ds dx + 8 \int_\Omega D \left| \nabla \left(k_{n+1}^{-\frac{1}{4}} - \left(C^{(\tau)} \right)^{\frac{1}{4}} \right)^+ \right|^2 dx$$

$$= -\frac{1}{2} \alpha_3 \alpha_4 \delta \int_{\Omega'} H_\tau(h, C^{(\tau)}, \Phi_s^{(\tau)} - \Phi_e^{(\tau)}) \left(\left(C^{(\tau)} \right)^{-\frac{1}{2}} - k_{n+1}^{\frac{1}{2}} \right)^+ dx$$

$$= -\frac{1}{2} \alpha_3 \alpha_4 \delta \int_{\Omega'} \sqrt{\theta_\tau(C^{(\tau)})} M_\tau dx$$

$$\quad - \frac{1}{2} \alpha_3 \alpha_4 \delta \int_{\Omega'} \sqrt{\theta_\tau(C^{(\tau)})} G_\tau(h, k_{n+1}^{-1}, \Phi_s^{(\tau)} - \Phi_e^{(\tau)}) \left(\left(C^{(\tau)} \right)^{-\frac{1}{2}} - k_{n+1}^{\frac{1}{2}} \right)^+ dx$$

$$\leq \frac{1}{2} \alpha_3 \alpha_4 \delta \theta_\tau^d \left(k_{n+1}^{-1} \right) \int_{\Omega' \cap \{C^{(\tau)} \leq k_{n+1}^{-1}\}} h^{-1} e^{-\alpha_2(\Phi_s^{(\tau)} - \Phi_e^{(\tau)})} dx$$

$$\leq \frac{c}{k^d} \int_{\Omega' \cap \{C^{(\tau)} \leq k_{n+1}^{-1}\}} e^{-\alpha_2(\Phi_s^{(\tau)} - \Phi_e^{(\tau)})} dx.$$

The last step is due to (4.1) and (1.16). Using (4.1) again yields $\left. \left(\left(C^{(\tau)} \right)^{-\frac{1}{2}} - k_{n+1}^{\frac{1}{2}} \right)^+ \right|_{t=0} = 0.$ With this in mind, we integrate the above inequality to derive

$$\sup_{0 \leq t \leq T} \int_\Omega \varepsilon_e \int_{C^{(\tau)}}^{k_{n+1}^{-1}} \left(s^{-\frac{1}{2}} - k_{n+1}^{\frac{1}{2}} \right)^+ ds dx$$

$$+ \int_{\Omega_T} D \left| \nabla \left(k_{n+1}^{-\frac{1}{4}} - \left(C^{(\tau)} \right)^{\frac{1}{4}} \right)^+ \right|^2 dx dt$$

$$\leq \frac{c}{k^d} \int_{\Omega' \cap \{C^{(\tau)} \leq k_{n+1}^{-1}\}} e^{-\alpha_2(\Phi_s^{(\tau)} - \Phi_e^{(\tau)})} dx dt$$

$$\leq \frac{c}{k^d} \| e^{-\alpha_2(\Phi_s^{(\tau)} - \Phi_e^{(\tau)})} \|_{q, \Omega_T'} \left| \Omega_T' \cap \left\{ C^{(\tau)} \leq k_{n+1}^{-1} \right\} \right|^{\frac{q-1}{q}}. \tag{4.2}$$

Note that on the set $\Omega_T \cap \{ C^{(\tau)} \leq k_{n+1}^{-1} \}$ we have

$$\int_{C^{(\tau)}}^{k_{n+1}^{-1}} \left(\frac{1}{\sqrt{s}} - \sqrt{k_{n+1}} \right)^+ ds = k_{n+1}^{-\frac{1}{2}} - 2\sqrt{\theta_\tau(C^{(\tau)})} + \sqrt{k_{n+1}} C^{(\tau)}$$

$$= \left(k_{n+1}^{-\frac{1}{4}} - (k_{n+1} C^{(\tau)})^{\frac{1}{4}} \left(C^{(\tau)} \right)^{\frac{1}{4}} \right)^2$$

$$\geq \left[\left(k_{n+1}^{-\frac{1}{4}} - \left(C^{(\tau)} \right)^{\frac{1}{4}} \right)^+ \right]^2. \tag{4.3}$$

Recall from the Sobolev embedding theorem that

$$
y_{n+1}
$$
$$
\equiv \int_{\Omega_T}\left[\left(k_{n+1}^{-\frac{1}{4}}-\left(C^{(\tau)}\right)^{\frac{1}{4}}\right)^{+}\right]^{2+\frac{4}{N}}dxdt
$$

$$
\leq \int_0^T\left(\int_\Omega\left[\left(k_{n+1}^{-\frac{1}{4}}-\left(C^{(\tau)}\right)^{\frac{1}{4}}\right)^{+}\right]^2 dx\right)^{\frac{2}{N}}\left(\int_\Omega\left[\left(k_{n+1}^{-\frac{1}{4}}-\left(C^{(\tau)}\right)^{\frac{1}{4}}\right)^{+}\right]^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}}dt
$$

$$
\leq c\left(\sup_{0\leq t\leq T}\int_\Omega\left[\left(k_{n+1}^{-\frac{1}{4}}-\left(C^{(\tau)}\right)^{\frac{1}{4}}\right)^{+}\right]^2 dx\right)^{\frac{2}{N}}\int_{\Omega_T}\left|\nabla\left(k_{n+1}^{-\frac{1}{4}}-\left(C^{(\tau)}\right)^{\frac{1}{4}}\right)^{+}\right|^2 dxdt
$$

$$
+c\left(\sup_{0\leq t\leq T}\int_\Omega\left[\left(k_{n+1}^{-\frac{1}{4}}-\left(C^{(\tau)}\right)^{\frac{1}{4}}\right)^{+}\right]^2 dx\right)^{\frac{2}{N}}\int_0^T\int_\Omega\left[\left(k_{n+1}^{-\frac{1}{4}}-\left(C^{(\tau)}\right)^{\frac{1}{4}}\right)^{+}\right]^2 dxdt
$$

$$
\leq \frac{c(1+T)}{k^{\frac{d(N+2)}{N}}}\left\|e^{-\alpha_2(\Phi_s^{(\tau)}-\Phi_e^{(\tau)})}\right\|_{q,\Omega_T'}^{\frac{N+2}{N}}\left|\Omega_T'\cap\left\{C^{(\tau)}\leq k_{n+1}^{-1}\right\}\right|^{\frac{(q-1)(N+2)}{qN}}. \qquad (4.4)
$$

The last step is due to (4.3) and (4.2). Observe that

$$
y_n \geq \int_{\left\{C^{(\tau)}\leq k_{n+1}^{-1}\right\}}\left[\left(k_n^{-\frac{1}{4}}-\left(C^{(\tau)}\right)^{\frac{1}{4}}\right)^{+}\right]^{2+\frac{4}{N}}dxdt
$$

$$
\geq \left(k_n^{-\frac{1}{4}}-k_{n+1}^{-\frac{1}{4}}\right)^{2+\frac{4}{N}}\left|\Omega_T'\cap\left\{C^{(\tau)}\leq k_{n+1}^{-1}\right\}\right|
$$

$$
\geq \left(\frac{k_{n+1}-k_n}{4k_n^{\frac{1}{4}}k_{n+1}}\right)^{2+\frac{4}{N}}\left|\Omega_T'\cap\left\{C^{(\tau)}\leq k_{n+1}^{-1}\right\}\right|
$$

$$
\geq \frac{c}{2^{\frac{2(n+2)(N+2)}{N}}k^{\frac{N+2}{2N}}}\left|\Omega_T'\cap\left\{C^{(\tau)}\leq k_{n+1}^{-1}\right\}\right|.
$$

Use this in (4.4) to get

$$
y_{n+1}\leq \frac{c(1+T)}{k^{\frac{(N+2)}{N}\left(d-\frac{1+\lambda}{2}\right)}}\left\|e^{-\alpha_2(\Phi_s^{(\tau)}-\Phi_e^{(\tau)})}\right\|_{q,\Omega_T'}^{\frac{N+2}{N}}2^{\frac{2(n+2)(N+2)(1+\lambda)}{N}}y_n^{1+\lambda},
$$

where

$$
\lambda = \frac{(q-1)(N+2)}{qN}-1>0
$$

due to our assumption. According to Lemma 2.2, if we choose k so large that

$$
y_0 = \int_{\Omega_T}\left[\left(\left(\frac{k}{2}\right)^{-\frac{1}{4}}-\left(C^{(\tau)}\right)^{\frac{1}{4}}\right)^{+}\right]^{2+\frac{4}{N}}dxdt
$$

$$
\leq \left(\frac{k}{2}\right)^{-\frac{N+2}{2N}}|\Omega_T|\leq \frac{k^{\frac{(N+2)}{N\lambda}\left(d-\frac{1+\lambda}{2}\right)}}{c(1+T)^{\frac{1}{\lambda}}\left\|e^{-\alpha_2(\Phi_s^{(\tau)}-\Phi_e^{(\tau)})}\right\|_{q,\Omega_T'}^{\frac{N+2}{N\lambda}}} \qquad (4.5)
$$

then

$$C^{(\tau)} \geq \frac{1}{k}.$$

Under (H8), (4.5) is equivalent to

$$k \geq c(1+T)^{\frac{2N}{(N+2)(2d-1)}} \left\| e^{-\alpha_2(\Phi_s^{(\tau)} - \Phi_e^{(\tau)})} \right\|_{q,\Omega_T'}^{\frac{2}{2d-1}}.$$

This combined with (4.1) implies the desired result. The proof is complete. □

LEMMA 4.2. *We have*

$$\int_\Omega \left(\left(\Phi_e^{(\tau)} \right)^2 + \left(\Phi_s^{(\tau)} \right)^2 \chi_{\Omega'} \right) dx \leq c \left(L^{(\tau)} \right)^{\frac{3d+3+2\alpha_0}{2}} \int_{\Omega'} H_\tau^2(h, C^{(\tau)}, 0) dx, \qquad (4.6)$$

where α_0 is given as in (1.4) *and*

$$L^{(\tau)} = L^{(\tau)}(T) = \max_{\overline{\Omega}_T} \frac{1}{C^{(\tau)}}.$$

Proof. Integrate (3.35) over Ω and take into account the boundary condition (3.38) to get

$$\tau \int_\Omega \Phi_e^{(\tau)} dx = \frac{1}{2}\alpha_4 \int_{\Omega'} H_\tau(h, C^{(\tau)}, \Phi_s^{(\tau)} - \Phi_e^{(\tau)}) dx. \qquad (4.7)$$

In the same manner, we can deduce from (3.36) and (3.39) that

$$\tau \int_{\Omega'} \Phi_s^{(\tau)} dx = -\frac{1}{2}\alpha_4 \int_{\Omega'} H_\tau(h, C^{(\tau)}, \Phi_s^{(\tau)} - \Phi_e^{(\tau)}) dx.$$

Add this to (4.7) to obtain

$$\int_\Omega \Phi_e^{(\tau)} dx + \int_{\Omega'} \Phi_s^{(\tau)} dx = 0. \qquad (4.8)$$

Use $\Phi_e^{(\tau)}$ as a test function in (3.35), then $\Phi_s^{(\tau)}$ in (3.36), and add the two resulting equations to derive

$$\int_\Omega \kappa_\tau(C^{(\tau)}) |\nabla \Phi_e^{(\tau)}|^2 dx + \int_{\Omega'} \sigma |\nabla \Phi_s^{(\tau)}|^2 dx + \tau \int_\Omega \left(\Phi_e^{(\tau)} \right)^2 dx + \tau \int_{\Omega'} \left(\Phi_s^{(\tau)} \right)^2 dx$$

$$+ \frac{1}{2}\alpha_4 \int_{\Omega'} H_\tau(h, C^{(\tau)}, \Phi_s^{(\tau)} - \Phi_e^{(\tau)})(\Phi_s^{(\tau)} - \Phi_e^{(\tau)}) dx = 0. \qquad (4.9)$$

The mean value theorem asserts that there is an ξ lying between 0 and $\Phi_s^{(\tau)} - \Phi_e^{(\tau)}$ such that

$$G_\tau(h, C^{(\tau)}, \Phi_s^{(\tau)} - \Phi_e^{(\tau)}) = G_\tau(h, C^{(\tau)}, \Phi_s^{(\tau)} - \Phi_e^{(\tau)}) - G_\tau(h, C^{(\tau)}, 0) + G_\tau(h, C^{(\tau)}, 0)$$

$$= \partial_{y_3} G_\tau(h, C^{(\tau)}, \xi)(\Phi_s^{(\tau)} - \Phi_e^{(\tau)}) + G_\tau(h, C^{(\tau)}, 0).$$

Note from (1.21) that

$$\partial_{y_3} G_\tau(h, C^{(\tau)}, \xi) = \alpha_2 \left(h(\theta_\tau(C^{(\tau)}) + \tau)^{-d} e^{\alpha_2 \xi} + h^{-1} \theta_\tau^d(C^{(\tau)}) e^{-\alpha_2 \xi} \right)$$

$$\geq \frac{2\alpha_2 \theta_\tau^{\frac{d}{2}}(C^{(\tau)})}{(\theta_\tau(C^{(\tau)}) + \tau)^{\frac{d}{2}}}. \tag{4.10}$$

With these in mind, we can estimate the last integral in (4.9) as follows:

$$\frac{1}{2}\alpha_4 \int_{\Omega'} H_\tau(h, C^{(\tau)}, \Phi_s^{(\tau)} - \Phi_e^{(\tau)})(\Phi_s^{(\tau)} - \Phi_e^{(\tau)})dx$$

$$= \frac{1}{2}\alpha_4 \int_{\Omega'} \sqrt{\theta_\tau(C^{(\tau)})} \partial_{y_3} G(h, C^{(\tau)}, \xi)(\Phi_s^{(\tau)} - \Phi_e^{(\tau)})^2 dx$$

$$+ \frac{1}{2}\alpha_4 \int_{\Omega'} \sqrt{\theta_\tau(C^{(\tau)})} G_\tau(h, C^{(\tau)}, 0)(\Phi_s^{(\tau)} - \Phi_e^{(\tau)})dx$$

$$\geq \alpha_2\alpha_4 \int_{\Omega'} \frac{\theta_\tau^{\frac{d+1}{2}}(C^{(\tau)})}{(\theta_\tau(C^{(\tau)}) + \tau)^{\frac{d}{2}}}(\Phi_s^{(\tau)} - \Phi_e^{(\tau)})^2 dx + \frac{1}{2}\alpha_4 \int_{\Omega'} H_\tau(h, C^{(\tau)}, 0)(\Phi_s^{(\tau)} - \Phi_e^{(\tau)})dx$$

$$\geq \frac{1}{2}\alpha_2\alpha_4 \int_{\Omega'} \frac{\theta_\tau^{\frac{d+1}{2}}(C^{(\tau)})}{(\theta_\tau(C^{(\tau)}) + \tau)^{\frac{d}{2}}}(\Phi_s^{(\tau)} - \Phi_e^{(\tau)})^2 dx - \frac{\alpha_4}{8\alpha_2} \int_{\Omega'} \frac{(\theta_\tau(C^{(\tau)}) + \tau)^{\frac{d}{2}}}{\theta_\tau^{\frac{d-1}{2}}(C^{(\tau)})} G_\tau^2(h, C^{(\tau)}, 0)dx.$$

Plug this into (4.9) to derive

$$\int_\Omega \kappa_\tau(C^{(\tau)})|\nabla\Phi_e^{(\tau)}|^2 dx + \int_{\Omega'} \sigma|\nabla\Phi_s^{(\tau)}|^2 dx + \tau \int_\Omega \left(\Phi_e^{(\tau)}\right)^2 dx + \tau \int_{\Omega'} \left(\Phi_s^{(\tau)}\right)^2 dx$$

$$+ \frac{1}{2}\alpha_2\alpha_4 \int_{\Omega'} \frac{\theta_\tau^{\frac{d+1}{2}}(C^{(\tau)})}{(\theta_\tau(C^{(\tau)}) + \tau)^{\frac{d}{2}}}(\Phi_s^{(\tau)} - \Phi_e^{(\tau)})^2 dx$$

$$\leq \frac{\alpha_4}{8\alpha_2} \int_{\Omega'} \frac{(\theta_\tau(C^{(\tau)}) + \tau)^{\frac{d}{2}}}{\theta_\tau^{\frac{d-1}{2}}(C^{(\tau)})} G_\tau^2(h, C^{(\tau)}, 0)dx. \tag{4.11}$$

Note that the function $\frac{s^{\frac{d+1}{2}}}{(s+\tau)^{\frac{d}{2}}}$ is increasing on $[0, \infty)$. Thus,

$$\frac{\theta_\tau^{\frac{d+1}{2}}(C^{(\tau)})}{(\theta_\tau(C^{(\tau)}) + \tau)^{\frac{d}{2}}} \geq \frac{\theta_\tau^{\frac{d+1}{2}}(m_c)}{(\theta_\tau(m_c) + \tau)^{\frac{d}{2}}}, \quad \text{where } m_c = \min_{\overline{\Omega_T}} C^{(\tau)} = \frac{1}{L^{(\tau)}}.$$

We derive from (4.11) that

$$\int_{\Omega'} (\Phi_s^{(\tau)} - \Phi_e^{(\tau)})^2 dx \leq \frac{(\theta_\tau(m_c) + \tau)^{\frac{d}{2}}}{4\alpha_2^2 \theta_\tau^{\frac{d+1}{2}}(m_c)} \int_{\Omega'} \frac{(\theta_\tau(C^{(\tau)}) + \tau)^{\frac{d}{2}}}{\theta_\tau^{\frac{d-1}{2}}(C^{(\tau)})} G_\tau^2(h, C^{(\tau)}, 0)dx$$

$$\leq \frac{(\theta_\tau(m_c) + \tau)^d}{4\alpha_2^2 \theta_\tau^{d+1}(m_c)} \int_{\Omega'} H_\tau^2(h, C^{(\tau)}, 0)dx$$

$$\leq c\left(L^{(\tau)}\right)^{d+1} \int_{\Omega'} H_\tau^2(h, C^{(\tau)}, 0)dx. \tag{4.12}$$

Here we have used the obvious fact that $m_c \leq \min_{\overline{\Omega}} C_0(x)$ and assumed $\frac{1}{\tau} \geq \min_{\overline{\Omega}} C_0(x)$. In view of (1.4), we have

$$\kappa_\tau(C^{(\tau)}) \geq \min\{c_0(C^{(\tau)} + \tau)^{\alpha_0}, \min_{s \in [s_0, \infty)} \kappa(s)\} \geq \min\{c_0 m_c^{\alpha_0}, \min_{s \in [s_0, \infty)} \kappa(s)\}.$$

Without loss of generality, assume $c_0 m_c^{\alpha_0} \leq \min\{\min_{s \in [s_0, \infty)} \kappa(s), \inf_{\Omega'} \sigma\}$. We obtain from (4.11) that

$$\int_\Omega |\nabla \Phi_e^{(\tau)}|^2 dx + \int_{\Omega'} |\nabla \Phi_s^{(\tau)}|^2 dx \leq \frac{c}{m_c^{\alpha_0}} \int_{\Omega'} \frac{(\theta_\tau(C^{(\tau)}) + \tau)^{\frac{d}{2}}}{\theta_\tau^{\frac{d-1}{2}}(C^{(\tau)})} G_\tau^2(h, C^{(\tau)}, 0) dx$$

$$\leq \frac{c(m_c + \tau)^{\frac{d}{2}}}{m_c^{\frac{d+1+2\alpha_0}{2}}} \int_{\Omega'} H_\tau^2(h, C^{(\tau)}, 0) dx$$

$$\leq c\left(L^{(\tau)}\right)^{\frac{d+1+2\alpha_0}{2}} \int_{\Omega'} H_\tau^2(h, C^{(\tau)}, 0) dx. \quad (4.13)$$

Note from (4.8) that

$$\left(\int_\Omega \Phi_e^{(\tau)} dx\right)^2 + \left(\int_{\Omega'} \Phi_s^{(\tau)} dx\right)^2$$

$$= \frac{1}{2}\left[\left(\int_\Omega (\Phi_e^{(\tau)} + \Phi_s^{(\tau)} \chi_{\Omega'}) dx\right)^2 + \left(\int_\Omega (\Phi_e^{(\tau)} - \Phi_s^{(\tau)} \chi_{\Omega'}) dx\right)^2\right]$$

$$= \frac{1}{2}\left(\int_{\Omega'} (\Phi_e^{(\tau)} - \Phi_s^{(\tau)}) dx\right)^2 + \frac{1}{2}\left(\int_{\Omega \setminus \Omega'} \Phi_e^{(\tau)} dx\right)^2. \quad (4.14)$$

We estimate from the Sobolev inequality that

$$\int_\Omega \left(\left(\Phi_e^{(\tau)}\right)^2 + \left(\Phi_s^{(\tau)}\right)^2 \chi_{\Omega'}\right) dx$$

$$\leq 2\int_\Omega \left(\Phi_e^{(\tau)} - \frac{1}{|\Omega|}\int_\Omega \Phi_e^{(\tau)} dx\right)^2 dx + 2\int_{\Omega'} \left(\Phi_s^{(\tau)} - \frac{1}{|\Omega'|}\int_{\Omega'} \Phi_s^{(\tau)} dx\right)^2 dx$$

$$+ \frac{2}{|\Omega|}\left(\int_\Omega \Phi_e^{(\tau)} dx\right)^2 + \frac{2}{|\Omega'|}\left(\int_{\Omega'} \Phi_s^{(\tau)} dx\right)^2$$

$$\leq c\int_\Omega |\nabla \Phi_e^{(\tau)}|^2 dx + c\int_{\Omega'} |\nabla \Phi_s^{(\tau)}|^2 dx + \frac{2}{|\Omega'|}\left[\left(\int_\Omega \Phi_e^{(\tau)} dx\right)^2 + \left(\int_{\Omega'} \Phi_s^{(\tau)} dx\right)^2\right]$$

$$\leq c\left(L^{(\tau)}\right)^{\frac{3d+3+2\alpha_0}{2}} \int_{\Omega'} H_\tau^2(h, C^{(\tau)}, 0) dx + \frac{1}{|\Omega'|}\left(\int_{\Omega \setminus \Omega'} \Phi_e^{(\tau)} dx\right)^2. \quad (4.15)$$

The last step is due to (4.13), (4.12), and (4.14). We easily check

$$\frac{1}{|\Omega'|}\left(\int_{\Omega \setminus \Omega'} \Phi_e^{(\tau)} dx\right)^2 \leq \frac{|\Omega_s|}{|\Omega'|}\int_\Omega \left(\Phi_e^{(\tau)}\right)^2 dx.$$

Making use of this in (4.15) and keep (H9) in mind yield the lemmas. □

LEMMA 4.3. *For each $q > \frac{N}{2}$ there is a positive number c such that*

$$\text{ess sup}_\Omega |\Phi_e^{(\tau)}| + \text{ess sup}_{\Omega'} |\Phi_s^{(\tau)}| \leq c\left(L^{(\tau)}\right)^{\frac{qN\alpha_0}{2q-N} + \frac{3d+3+2\alpha_0}{4}} \|H_\tau(h, C^{(\tau)}, 0)\|_{q, \Omega'}. \quad (4.16)$$

Proof. Let q be given as in the lemma. Set

$$b = \|H_\tau(h, C^{(\tau)}, 0)\|_{q, \Omega'}.$$

Fix $n > 0$. We use $\left(\left(\Phi_e^{(\tau)} \right)^+ + b \right)^n$ as a test function in (3.35) to get

$$\frac{4n}{(n+1)^2} \int_\Omega \kappa_\tau(C) \left| \nabla \left(\left(\Phi_e^{(\tau)} \right)^+ + b \right)^{\frac{n+1}{2}} \right|^2 dx + \tau \int_\Omega \Phi_e^{(\tau)} \left(\left(\Phi_e^{(\tau)} \right)^+ + b \right)^n dx$$

$$= \frac{1}{2}\alpha_4 \int_{\Omega'} H_\tau(h, C^{(\tau)}, \Phi_s^{(\tau)} - \Phi_e^{(\tau)}) \left(\left(\Phi_e^{(\tau)} \right)^+ + b \right)^n dx. \tag{4.17}$$

By (4.7),

$$\tau \int_\Omega \left(\Phi_e^{(\tau)} \right)^- dx = \tau \int_\Omega \left(\Phi_e^{(\tau)} \right)^+ dx - \frac{1}{2}\alpha_4 \int_{\Omega'} H_\tau(h, C^{(\tau)}, \Phi_s^{(\tau)} - \Phi_e^{(\tau)}) dx.$$

With this in mind, we calculate

$$\tau \int_\Omega \Phi_e^{(\tau)} \left(\left(\Phi_e^{(\tau)} \right)^+ + b \right)^n dx$$

$$= \tau \int_\Omega \left(\Phi_e^{(\tau)} \right)^+ \left(\left(\Phi_e^{(\tau)} \right)^+ + b \right)^n dx - b^n \tau \int_\Omega \left(\Phi_e^{(\tau)} \right)^- dx$$

$$= \tau \int_\Omega \left(\Phi_e^{(\tau)} \right)^+ \left[\left(\left(\Phi_e^{(\tau)} \right)^+ + b \right)^n - b^n \right] dx$$

$$+ \frac{1}{2}\alpha_4 b^n \int_{\Omega'} H_\tau(h, C^{(\tau)}, \Phi_s^{(\tau)} - \Phi_e^{(\tau)}) dx.$$

Plug this into (4.17) to deduce

$$\frac{4n}{(n+1)^2} \int_\Omega \kappa_\tau(C) \left| \nabla \left(\left(\Phi_e^{(\tau)} \right)^+ + b \right)^{\frac{n+1}{2}} \right|^2 dx$$

$$+ \tau \int_\Omega \left(\Phi_e^{(\tau)} \right)^+ \left[\left(\left(\Phi_e^{(\tau)} \right)^+ + b \right)^n - b^n \right] dx$$

$$= \frac{1}{2}\alpha_4 \int_{\Omega'} H_\tau(h, C^{(\tau)}, \Phi_s^{(\tau)} - \Phi_e^{(\tau)}) \left[\left(\left(\Phi_e^{(\tau)} \right)^+ + b \right)^n - b^n \right] dx. \tag{4.18}$$

Similarly, we can derive from (3.25) and (3.26) that

$$\frac{4n}{(n+1)^2} \int_{\Omega'} \sigma \left| \nabla \left(\left(\Phi_s^{(\tau)} \right)^+ + b \right)^{\frac{n+1}{2}} \right|^2 dx + \tau \int_\Omega \left(\Phi_s^{(\tau)} \right)^+ \left[\left(\left(\Phi_s^{(\tau)} \right)^+ + b \right)^n - b^n \right] dx$$

$$= -\frac{1}{2}\alpha_4 \int_{\Omega'} H_\tau(h, C^{(\tau)}, \Phi_s^{(\tau)} - \Phi_e^{(\tau)}) \left[\left(\left(\Phi_s^{(\tau)} \right)^+ + b \right)^n - b^n \right] dx. \tag{4.19}$$

Since $G_\tau(y_1, y_2, y_3)$ is increasing in y_3, we have

$$N_\tau \equiv \left[G_\tau(h, C^{(\tau)}, \Phi_s^{(\tau)} - \Phi_e^{(\tau)}) - G_\tau(h, C^{(\tau)}, 0) \right]$$

$$\times \left[\left(\left(\Phi_s^{(\tau)} \right)^+ + b \right)^n - \left(\left(\Phi_e^{(\tau)} \right)^+ + b \right)^n \right]$$

$$\geq 0.$$

Add (4.19) to (4.18) and keep the above inequality and (1.21) in mind to deduce

$$\frac{4n}{(n+1)^2} \int_\Omega \kappa_\tau(C) \left| \nabla \left(\left(\Phi_e^{(\tau)} \right)^+ + b \right)^{\frac{n+1}{2}} \right|^2 dx$$

$$+ \frac{4n}{(n+1)^2} \int_{\Omega'} \sigma \left| \nabla \left(\left(\Phi_s^{(\tau)} \right)^+ + b \right)^{\frac{n+1}{2}} \right|^2 dx$$

$$\leq -\frac{1}{2}\alpha_4 \int_{\Omega'} \sqrt{\theta_\tau(C^{(\tau)})} N_\tau dx$$

$$+ \frac{1}{2}\alpha_4 \int_{\Omega'} H_\tau(h, C^{(\tau)}, 0) \left[\left(\left(\Phi_s^{(\tau)} \right)^+ + b \right)^n - \left(\left(\Phi_e^{(\tau)} \right)^+ + b \right)^n \right] dx$$

$$\leq \frac{1}{2}\alpha_4 b \left\| \left(\left(\Phi_s^{(\tau)} \right)^+ + b \right)^n - \left(\left(\Phi_e^{(\tau)} \right)^+ + b \right)^n \right\|_{\frac{q}{q-1}, \Omega'}$$

$$\leq \alpha_4 \left\| (W + b)^{n+1} \right\|_{\frac{q}{q-1}, \Omega}, \tag{4.20}$$

where

$$W_\tau = \max\{ \left(\Phi_e^{(\tau)} \right)^+, \left(\Phi_s^{(\tau)} \right)^+ \chi_{\Omega'} \}. \tag{4.21}$$

By the Sobolev inequality, we have

$$\left\| (W_\tau + b)^{n+1} \right\|_{\frac{N}{N-2}, \Omega}$$

$$= \left(\int_{\left\{ \left(\Phi_e^{(\tau)} \right)^+ \geq \left(\Phi_s^{(\tau)} \right)^+ \chi_{\Omega'} \right\}} \left(\left(\Phi_e^{(\tau)} \right)^+ + b \right)^{\frac{(n+1)N}{N-2}} dx \right)^{\frac{N-2}{N}}$$

$$+ \left(\int_{\left\{ \left(\Phi_e^{(\tau)} \right)^+ < \left(\Phi_s^{(\tau)} \right)^+ \chi_{\Omega'} \right\}} \left(\left(\Phi_s^{(\tau)} \right)^+ \chi_{\Omega'} + b \right)^{\frac{(n+1)N}{N-2}} dx \right)^{\frac{N-2}{N}}$$

$$\leq \left(\int_\Omega \left(\left(\Phi_e^{(\tau)} \right)^+ + b \right)^{\frac{n+1}{2} \frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} + \left(\int_{\Omega'} \left(\left(\Phi_s^{(\tau)} \right)^+ + b \right)^{\frac{n+1}{2} \frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}}$$

$$\leq c \int_\Omega \left| \nabla \left(\left(\Phi_e^{(\tau)} \right)^+ + b \right)^{\frac{n+1}{2}} \right|^2 dx + c \int_\Omega \left(\left(\Phi_e^{(\tau)} \right)^+ + b \right)^{n+1} dx$$

$$+ c \int_{\Omega'} \left| \nabla \left(\left(\Phi_s^{(\tau)} \right)^+ + b \right)^{\frac{n+1}{2}} \right|^2 dx + c \int_{\Omega'} \left(\left(\Phi_s^{(\tau)} \right)^+ + b \right)^{n+1} dx$$

$$\leq \left(\frac{c(n+1)^2 \alpha_4}{4nm_c^{\alpha_0}} + c|\Omega|^{\frac{1}{q}} \right) \left\| (W_\tau + b)^{n+1} \right\|_{\frac{q}{q-1}, \Omega}. \tag{4.22}$$

The last step is due to (4.20) and (4.21). Remember

$$\ell \equiv \frac{N}{N-2} \frac{q-1}{q} > 1.$$

Then let $n + 1 = \ell^i$, $i = 1, 2, \cdots$. Subsequently,

$$\frac{n+1}{n} \leq \frac{\ell}{\ell - 1}.$$

Take the $(n+1)^{\text{th}}$ root of both sides of (4.22) to get

$$
\begin{aligned}
\|W_\tau + b\|_{\frac{q\ell i+1}{q-1},\Omega} &= \|W_\tau + b\|_{\frac{N(n+1)}{N-2},\Omega} \\
&\leq \left(\frac{c(n+1)^2 \alpha_4}{4n m_c^{\alpha_0}} + c|\Omega|^{\frac{1}{q}} \right)^{\frac{1}{n+1}} \|W_\tau + b\|_{\frac{q(n+1)}{q-1},\Omega} \\
&\leq \left(c \left(L^{(\tau)} \right)^{\alpha_0} \right)^{\frac{1}{n+1}} (n+1)^{\frac{1}{n+1}} \|W_\tau + b\|_{\frac{q(n+1)}{q-1},\Omega} \\
&= \left(c \left(L^{(\tau)} \right)^{\alpha_0} \right)^{\frac{1}{\ell i}} \ell^{\frac{i}{\ell i}} \|W_\tau + b\|_{\frac{q\ell i}{q-1},\Omega} .
\end{aligned}
$$

Iterate on i to yield

$$
\|W_\tau + b\|_{\frac{q\ell i+1}{q-1},\Omega} \leq \left(c \left(L^{(\tau)} \right)^{\alpha_0} \right)^{\sum_{j=1}^{i} \frac{1}{\ell j}} \ell^{\sum_{j=1}^{i} \frac{j}{\ell j}} \|W_\tau + b\|_{\frac{q\ell}{q-1},\Omega} .
$$

Take $i \to \infty$ to get

$$
\operatorname{ess\,sup}_\Omega (W_\tau + b) \leq c \left(L^{(\tau)} \right)^{\frac{\alpha_0}{\ell-1}} \|W_\tau + b\|_{\frac{q\ell}{q-1},\Omega} . \tag{4.23}
$$

The interpolation inequality in ([8], p.146) asserts

$$
\|W_\tau + b\|_{\frac{q\ell}{q-1},\Omega} \leq \varepsilon \operatorname{ess\,sup}_\Omega (W_\tau + b) + \frac{1}{\varepsilon^{\frac{q\ell-q+1}{q-1}}} \|W_\tau + b\|_{1,\Omega} , \quad \varepsilon > 0.
$$

Use this in (4.23) and choose ε appropriately in the resulting inequality to get

$$
\begin{aligned}
\operatorname{ess\,sup}_\Omega W_\tau &\leq \operatorname{ess\,sup}_\Omega (W_\tau + b) \\
&\leq c \left(L^{(\tau)} \right)^{\frac{q\ell\alpha_0}{(\ell-1)(q-1)}} \|W_\tau + b\|_{1,\Omega} \\
&\leq c \left(L^{(\tau)} \right)^{\frac{qN\alpha_0}{2q-N}} \left(\|W_\tau\|_{1,\Omega} + \|H_\tau(h, C^{(\tau)}, 0)\|_{q,\Omega'} \right) . \tag{4.24}
\end{aligned}
$$

According to (4.6), we have

$$
\begin{aligned}
\|W_\tau\|_{\Omega,1} &\leq \| \left(\Phi_e^{(\tau)} \right)^+ \|_{\Omega,1} + \| \left(\Phi_s^{(\tau)} \right)^+ \|_{\Omega',1} \\
&\leq c \left(L^{(\tau)} \right)^{\frac{3d+3+2\alpha_0}{4}} \|H_\tau(h, C^{(\tau)}, 0)\|_{2,\Omega'}.
\end{aligned}
$$

This together with (4.24) implies

$$
\begin{aligned}
&\operatorname{ess\,sup}_\Omega \left(\Phi_e^{(\tau)} \right)^+ + \operatorname{ess\,sup}_{\Omega'} \left(\Phi_s^{(\tau)} \right)^+ \\
&\leq c \left(L^{(\tau)} \right)^{\frac{qN\alpha_0}{2q-N} + \frac{3d+3+2\alpha_0}{4}} \|H_\tau(h, C^{(\tau)}, 0)\|_{q,\Omega'}. \tag{4.25}
\end{aligned}
$$

In an entirely similar manner, we can also prove

$$
\operatorname{ess\,sup}_\Omega \left(\Phi_e^{(\tau)} \right)^- + \operatorname{ess\,sup}_{\Omega'} \left(\Phi_s^{(\tau)} \right)^- \leq c \left(L^{(\tau)} \right)^{\frac{qN\alpha_0}{2q-N} + \frac{3d+3+2\alpha_0}{4}} \|H_\tau(h, C^{(\tau)}, 0)\|_{q,\Omega'}.
$$

Combining this with(4.25) yields (4.16). The proof is complete. $\square$

To continue the proof of Theorem 3.1, we let

$$M^{(\tau)} = M^{(\tau)}(T) = \max\{\max_{\overline{\Omega}_T} C^{(\tau)}, 1\} \geq 1.$$

We can easily see from (3.11) that

$$\left|H_\tau(h, C^{(\tau)}, 0)\right| \leq c\left(\left(M^{(\tau)}\right)^{\frac{1}{2}} + \left(M^{(\tau)}\right)^{d+\frac{1}{2}}\right)\left(L^{(\tau)}\right)^d \leq c\left(M^{(\tau)}\right)^{d+\frac{1}{2}}\left(L^{(\tau)}\right)^d.$$

This together with (4.16) implies

$$\text{ess sup}_\Omega|\Phi_e^{(\tau)}| + \text{ess sup}_{\Omega'}|\Phi_s^{(\tau)}| \leq c\left(M^{(\tau)}\right)^{d+\frac{1}{2}}\left(L^{(\tau)}\right)^{\frac{qN\alpha_0}{2q-N} + \frac{7d+3+2\alpha_0}{4}}.$$

We conclude from Lemmas 3.3 and 4.1 that

$$M^{(\tau)} \leq c(1+T)^{\frac{2N}{(N+2)}}T^{\frac{2(2q-2-N)}{q(N+2)}}\left\|e^{\alpha_2(\Phi_s-\Phi_e)}\right\|_{q,\Omega'_T}^2 + 4\|C_0\|_{\infty,\Omega} + 2$$

$$\leq c(1+T)^{\frac{2N}{(N+2)}}T^{\frac{4}{N+2}}e^{c\left(M^{(\tau)}\right)^{d+\frac{1}{2}}\left(L^{(\tau)}\right)^{\frac{qN\alpha_0}{2q-N} + \frac{7d+3+2\alpha_0}{4}}} + 4\|C_0\|_{\infty,\Omega} + 2, \quad (4.26)$$

$$L^{(\tau)} \leq c(1+T)^{\frac{2N}{(N+2)(2d-1)}}\left\|e^{-\alpha_2(\Phi_s^{(\tau)}-\Phi_e^{(\tau)})}\right\|_{q,\Omega'_T}^{\frac{2}{2d-1}} + \frac{1}{\min_{\overline{\Omega}} C_0}$$

$$\leq c(1+T)^{\frac{2N}{(N+2)(2d-1)}}T^{\frac{2}{q(2d-1)}}e^{c\left(M^{(\tau)}\right)^{d+\frac{1}{2}}\left(L^{(\tau)}\right)^{\frac{qN\alpha_0}{2q-N} + \frac{7d+3+2\alpha_0}{4}}} + \frac{1}{\min_{\overline{\Omega}} C_0}. \quad (4.27)$$

Set

$$\gamma = \frac{qN\alpha_0}{2q-N} + \frac{7d+3+2\alpha_0}{4} > d + \frac{1}{2} > 1, \quad (4.28)$$

$$g(T) = \max\left\{(1+T)^{\frac{2N}{(N+2)}}T^{\frac{2(2q-2-N)}{q(N+2)}}, (1+T)^{\frac{2N}{(N+2)(2d-1)}}T^{\frac{2}{q(2d-1)}}\right\}. \quad (4.29)$$

We can conclude from (4.26) and (4.27) that there is a positive number c such that

$$M^{(\tau)}L^{(\tau)} < cg^2(T)e^{c\left(M^{(\tau)}L^{(\tau)}\right)^\gamma} + c.$$

We are in a position to invoke Lemma 2.3. For this purpose set

$$a(s) = M^{(\tau)}(s)L^{(\tau)}(s), \quad \varepsilon = cg^2(T).$$

Subsequently,

$$a(s) < cg^2(s)e^{ca^\gamma(s)} + c \leq \varepsilon e^{ca^\gamma(s)} + c \quad \text{for each } s \in [0, T].$$

Here we have used the fact that $g(T)$ strictly increases from 0 to infinity on the interval $[0, \infty)$. We are in a position to apply Lemma 2.3. Let ε_0, s_0 be determined by Lemma 2.3. Obviously, there is a unique $T_0 \in (0, \infty)$ such that

$$\varepsilon_0 = cg^2(T_0). \quad (4.30)$$

This implies

$$a(s) < s_0 \quad \text{for each } s \in [0, T_0] \text{ since we obviously have } a(0) \leq c.$$

Consequently,

$$\frac{1}{c} \leq C^{(\tau)} \leq c \quad \text{in } \Omega_{T_0} \text{ for some } c > 0. \tag{4.31}$$

LEMMA 4.4. *Let T_0 be given as in (4.30). For each $p > 1$ the sequence $\{\Phi_e^{(\tau)}\}$ is precompact in $L^p(\Omega_{T_0})$ and the sequence $\{\Phi_s^{(\tau)}\}$ is precompact in $L^p(\Omega'_{T_0})$.*

Proof. Let $t_1, t_2 \in [0, T_0]$. For simplicity, we employ the following notions

$$h_t(x) = h(x,t), \ C_t^{(\tau)}(x) = C^{(\tau)}(x,t), \ \left(\Phi_e^{(\tau)}\right)_t(x) = \Phi_e^{(\tau)}(x,t), \ \left(\Phi_s^{(\tau)}\right)_t(x) = \Phi_s^{(\tau)}(x,t).$$

We can derive from (3.35) and (3.36) that

$$-\text{div}\left[\kappa_\tau(C_{t_1}^{(\tau)})\nabla\left[\left(\Phi_e^{(\tau)}\right)_{t_1} - \left(\Phi_e^{(\tau)}\right)_{t_2}\right]\right] + \tau\left(\left(\Phi_e^{(\tau)}\right)_{t_1} - \left(\Phi_e^{(\tau)}\right)_{t_2}\right)$$

$$= \text{div}\left[\left(\kappa_\tau(C_{t_1}^{(\tau)}) - \kappa_\tau(C_{t_2}^{(\tau)})\right)\nabla\left(\Phi_e^{(\tau)}\right)_{t_2}\right]$$

$$+ \frac{1}{2}\alpha_4 H_\tau\left((h_{t_1}, C_{t_1}^{(\tau)}, \left(\Phi_s^{(\tau)}\right)_{t_1} - \left(\Phi_e^{(\tau)}\right)_{t_1}\right)\chi_{\Omega'}$$

$$- \frac{1}{2}\alpha_4 H_\tau\left(h_{t_2}, C_{t_2}^{(\tau)}, \left(\Phi_s^{(\tau)}\right)_{t_2} - \left(\Phi_e^{(\tau)}\right)_{t_2}\right)\chi_{\Omega'} \quad \text{in } \Omega, \tag{4.32}$$

$$-\text{div}\left[\sigma\nabla\left[\left(\Phi_s^{(\tau)}\right)_{t_1} - \left(\Phi_s^{(\tau)}\right)_{t_2}\right]\right] + \tau\left(\left(\Phi_s^{(\tau)}\right)_{t_1} - \left(\Phi_s^{(\tau)}\right)_{t_2}\right)$$

$$= -\frac{1}{2}\alpha_4 H_\tau\left((h_{t_1}, C_{t_1}^{(\tau)}, \left(\Phi_s^{(\tau)}\right)_{t_1} - \left(\Phi_e^{(\tau)}\right)_{t_1}\right)$$

$$+ \frac{1}{2}\alpha_4 H_\tau\left(h_{t_2}, C_{t_2}^{(\tau)}, \left(\Phi_s^{(\tau)}\right)_{t_2} - \left(\Phi_e^{(\tau)}\right)_{t_2}\right) \quad \text{in } \Omega'. \tag{4.33}$$

Observe that

$$H_\tau\left((h_{t_1}, C_{t_1}^{(\tau)}, \left(\Phi_s^{(\tau)}\right)_{t_1} - \left(\Phi_e^{(\tau)}\right)_{t_1}\right) - H_\tau\left(h_{t_2}, C_{t_2}^{(\tau)}, \left(\Phi_s^{(\tau)}\right)_{t_2} - \left(\Phi_e^{(\tau)}\right)_{t_2}\right)$$

$$= H_\tau\left((h_{t_1}, C_{t_1}^{(\tau)}, \left(\Phi_s^{(\tau)}\right)_{t_1} - \left(\Phi_e^{(\tau)}\right)_{t_1}\right) - H_\tau\left(h_{t_2}, C_{t_1}^{(\tau)}, \left(\Phi_s^{(\tau)}\right)_{t_1} - \left(\Phi_e^{(\tau)}\right)_{t_1}\right)$$

$$+ H_\tau\left(h_{t_2}, C_{t_1}^{(\tau)}, \left(\Phi_s^{(\tau)}\right)_{t_1} - \left(\Phi_e^{(\tau)}\right)_{t_1}\right) - H_\tau\left(h_{t_2}, C_{t_2}^{(\tau)}, \left(\Phi_s^{(\tau)}\right)_{t_1} - \left(\Phi_e^{(\tau)}\right)_{t_1}\right)$$

$$+ H_\tau\left(h_{t_2}, C_{t_2}^{(\tau)}, \left(\Phi_s^{(\tau)}\right)_{t_1} - \left(\Phi_e^{(\tau)}\right)_{t_1}\right) - H_\tau\left(h_{t_2}, C_{t_2}^{(\tau)}, \left(\Phi_s^{(\tau)}\right)_{t_2} - \left(\Phi_e^{(\tau)}\right)_{t_2}\right)$$

$$\equiv J_1 + J_2 + J_3.$$

We can easily conclude from (4.31) and (4.16) that

$$|J_1| \leq c\|h_{t_1} - h_{t_2}\|_{\infty,\Omega'},$$

$$|J_2| \leq c\|C_{t_1}^{(\tau)} - C_{t_2}^{(\tau)}\|_{\infty,\Omega'}.$$

By an argument similar to (4.10), we can obtain

$$J_3\left[\left[\left(\Phi_s^{(\tau)}\right)_{t_1} - \left(\Phi_e^{(\tau)}\right)_{t_1}\right] - \left[\left(\Phi_s^{(\tau)}\right)_{t_2} - \left(\Phi_e^{(\tau)}\right)_{t_2}\right]\right]$$

$$\geq c\left[\left[\left(\Phi_s^{(\tau)}\right)_{t_1} - \left(\Phi_s^{(\tau)}\right)_{t_2}\right] - \left[\left(\Phi_e^{(\tau)}\right)_{t_1} - \left(\Phi_e^{(\tau)}\right)_{t_2}\right]\right]^2.$$

Use $\left(\Phi_e^{(\tau)}\right)_{t_1} - \left(\Phi_e^{(\tau)}\right)_{t_2}$ as a test function in (4.32), $\left(\Phi_s^{(\tau)}\right)_{t_1} - \left(\Phi_s^{(\tau)}\right)_{t_2}$ in (4.33), add up the two resulting equations to deduce

$$\int_\Omega \left| \nabla \left[\left(\Phi_e^{(\tau)}\right)_{t_1} - \left(\Phi_e^{(\tau)}\right)_{t_2} \right] \right|^2 dx + \int_{\Omega'} \left| \nabla \left[\left(\Phi_s^{(\tau)}\right)_{t_1} - \left(\Phi_s^{(\tau)}\right)_{t_2} \right] \right|^2 dx$$

$$c \int_{\Omega'} \left[\left[\left(\Phi_s^{(\tau)}\right)_{t_1} - \left(\Phi_s^{(\tau)}\right)_{t_2} \right] - \left[\left(\Phi_e^{(\tau)}\right)_{t_1} - \left(\Phi_e^{(\tau)}\right)_{t_2} \right] \right]^2 dx$$

$$\leq c\|\kappa(C_{t_1}^{(\tau)} + \tau) - \kappa(C_{t_2}^{(\tau)} + \tau)\|_{\infty,\Omega'} + c\|h_{t_1} - h_{t_2}\|_{\infty,\Omega'} + c\|C_{t_1}^{(\tau)} - C_{t_2}^{(\tau)}\|_{\infty,\Omega'}.$$

Integrate (4.32) over Ω and (4.33) over Ω' and add up the two resulting equations to get

$$\int_\Omega \left[\left(\Phi_e^{(\tau)}\right)_{t_1} - \left(\Phi_e^{(\tau)}\right)_{t_2} \right] dx + \int_{\Omega'} \left[\left(\Phi_s^{(\tau)}\right)_{t_1} - \left(\Phi_s^{(\tau)}\right)_{t_2} \right] dx = 0.$$

This puts us in a position to employ the proof of (4.6). Upon doing so, we arrive at

$$\int_\Omega \left[\left(\Phi_e^{(\tau)}\right)_{t_1} - \left(\Phi_e^{(\tau)}\right)_{t_2} \right]^2 dx + \int_{\Omega'} \left[\left(\Phi_s^{(\tau)}\right)_{t_1} - \left(\Phi_s^{(\tau)}\right)_{t_2} \right]^2 dx$$

$$\leq c\|\kappa(C_{t_1}^{(\tau)} + \tau) - \kappa(C_{t_2}^{(\tau)} + \tau)\|_{\infty,\Omega'} + c\|h_{t_1} - h_{t_2}\|_{\infty,\Omega'}$$

$$+ c\|C_{t_1}^{(\tau)} - C_{t_2}^{(\tau)}\|_{\infty,\Omega'}. \tag{4.34}$$

Classical regularity results for linear parabolic equations ([12], Chap. III) assert that $\{C^{(\tau)}\}$ is bounded in $C^{\frac{\delta}{2},\delta}(\overline{\Omega_{T_0}})$ for some $\delta \in (0,1)$. This combined with (4.34) implies that $\{\Phi_e^{(\tau)}\}$ is uniformly equicontinuous in $C([0,T_0];L^2(\Omega))$ and $\{\Phi_s^{(\tau)}\}$ in $C([0,T_0];L^2(\Omega'))$. Furthermore, $\{\Phi_e^{(\tau)}\}$ is bounded in $L^\infty(0,T_0;W^{1,2}(\Omega))$ and $\{\Phi_e^{(\tau)}\}$ in $L^\infty(0,T_0;W^{1,2}(\Omega'))$. By virtue of the Ascoli theorem ([16], Lemma 1), the two sequences are precompact in $C([0,T_0];L^2(\Omega))$ and $C([0,T_0];L^2(\Omega'))$, respectively. This implies the lemma. □

Now we have all the ingredients necessary to pass to the limit in (3.4)-(3.10). The proof of Theorem 1.1 is completed.

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