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A NOTE ON THE TOTAL CURVATURE OF A KÄHLER MANIFOLD

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Given a complete manifold with non-negative Ricci curvature, it is a very interesting geometric problem of how curvature decays at infinity. While it is not true that the curvature decays in a strong sense, it is possible that the average of the scalar curvature decays at least linearly. Such a statement is certainly consistent with the Cohn-Vossen inequality which holds for surfaces. The significance of such an inequality is also clear because of its relevance with the work of the first author [1] on the attempt to prove the conjecture of the second author that a complete noncompact Kähler manifold with positive bisectional curvature is biholomorphic to the complex euclidean space.

The purpose of this note is to prove a weaker version of the conjecture for Kähler manifolds.

Theorem 1. *Suppose M is a complex n-dimensional ($n \geq 3$) complete noncompact Kähler manifold with bounded and nonnegative holomorphic bisectional curvature such that*

$$R_{\alpha\overline{\alpha}\beta\overline{\beta}} \geq \epsilon R \qquad \text{on } M, \tag{1}$$

where $0 < \epsilon < +\infty$ is a constant and R is the scalar curvature. Then we have

$$\int_{B(x_0,\gamma)} R(x)dx \leq \frac{C(n,\epsilon)}{\gamma^2} \ \mathrm{Vol}\, B(x_0,\gamma) \tag{2}$$

for any $x_0 \in M$ and $0 < \gamma < +\infty$, where $B(x_0,\gamma)$ is the geodesic ball of radius γ centered at x_0 and the constant $C(n,\epsilon)$ depends only on n and ϵ.

Since both assumptions and conclusion are scaling invariant, we only need to prove (2) for $\gamma = 1$. That is, we only need to prove

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Theorem 2. *Suppose M is a complex n-dimensional ($n \geq 3$) complete noncompact Kähler manifold such that*

$$\epsilon R \leq R_{\alpha\overline{\alpha}\beta\overline{\beta}} \leq K_0 \qquad \text{on } M, \tag{3}$$

where $0 < \epsilon, K_0 < +\infty$ are constants. Then

$$\int_{B(x_0,1)} R(x)dx \leq c(n,\epsilon) \ \text{Vol}\ B(x_0,1) \tag{4}$$

for any $x_0 \in M$, where $c(n,\epsilon)$ depends only on n and ϵ, and is independent of K_0.

Fix a point $x_0 \in M$. Suppose $\varphi(x,t)$ is the solution of the heat equation

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \Delta\varphi & \text{on } M \times [0,\infty) \\ \varphi(x,0) = \left[\frac{1}{1+\gamma(x,x_0)}\right]^{2n} & x \in M, \end{cases} \tag{5}$$

where $\gamma(x,x_0)$ denotes the distance between x and x_0. We have $\varphi(x,t) \in C^\infty(M \times (0,\infty))$,

$$\begin{cases} \frac{C_1}{[1+\gamma(x,x_0)]^{2n}} \leq \varphi(x,t) \leq \frac{C_2}{[1+\gamma(x,x_0)]^{2n}} & \text{on } M \times [0,1] \\ |\nabla_i \varphi(x,t)| \leq \frac{C_3}{[1+\gamma(x,x_0)]^{2n+1}} & \text{on } M \times [0,1], \end{cases} \tag{6}$$

where $0 < C_1, C_2, C_3 < +\infty$ depend only on n.

Lemma 3.

$$|\nabla_\alpha \nabla_{\overline{\beta}} \varphi(x,t)| \leq \frac{C_4}{[1+\gamma(x,x_0)]^{2n+1}} \left(\frac{1}{t}\right) \qquad \text{on } M \times [0,1], \tag{7}$$

where $0 < C_4 < +\infty$ depends only on n.

Proof. From (5) we have

$$\begin{aligned} \frac{\partial}{\partial t}|\nabla_\alpha \nabla_{\overline{\beta}}\varphi|^2 =& \Delta|\nabla_\alpha\nabla_{\overline{\beta}}\varphi|^2 - |\nabla_\gamma\nabla_\alpha\nabla_{\overline{\beta}}\varphi|^2 - |\nabla_{\overline{\gamma}}\nabla_\alpha\nabla_{\overline{\beta}}\varphi|^2 \\ &+ 2R_{\alpha\overline{\beta}\gamma\overline{\delta}}\nabla_{\overline{\alpha}}\nabla_\beta\varphi \cdot \nabla_\delta\nabla_{\overline{\gamma}}\varphi - 2R_{\alpha\overline{\beta}}\nabla_\beta\nabla_{\overline{\gamma}}\varphi \cdot \nabla_\gamma\nabla_{\overline{\alpha}}\varphi. \end{aligned} \tag{8}$$

Choose a coordinate system such that at one point

$$\nabla_\alpha\nabla_{\overline{\beta}}\varphi = \begin{cases} 0 & \alpha \neq \beta \\ \ell\ell_\alpha & \alpha = \beta. \end{cases}$$

Then

$$
\begin{aligned}
&2R_{\alpha\overline{\beta}\gamma\overline{\delta}}\nabla_{\overline{\alpha}}\nabla_{\beta}\varphi\cdot\nabla_{\delta}\nabla_{\overline{\gamma}}\varphi - 2R_{\alpha\overline{\beta}}\nabla_{\beta}\nabla_{\overline{\gamma}}\varphi\cdot\nabla_{\gamma}\nabla_{\overline{\alpha}}\varphi \\
&= 2R_{\alpha\overline{\alpha}\gamma\overline{\gamma}}\ell\ell_{\alpha}\ell\ell_{\gamma} - 2R_{\alpha\overline{\alpha}}\ell\ell_{\alpha}^{2} \\
&= -\sum_{\alpha,\beta} R_{\alpha\overline{\alpha}\beta\overline{\beta}}\ell\ell_{\alpha}\ell\ell_{\gamma} - 2R_{\alpha\overline{\alpha}}\ell\ell_{\alpha}^{2} \\
&= -\sum_{\alpha,\beta} R_{\alpha\overline{\alpha}\beta\overline{\beta}}(\ell\ell_{\alpha}-\ell\ell_{\beta})^{2} \leq 0.
\end{aligned}
\tag{9}
$$

Thus

$$
\frac{\partial}{\partial t}|\nabla_{\alpha}\nabla_{\overline{\beta}}\varphi|^{2} \leq \Delta|\nabla_{\alpha}\nabla_{\overline{\beta}}\varphi|^{2} - |\nabla_{\gamma}\nabla_{\alpha}\nabla_{\overline{\beta}}\varphi|^{2} - |\nabla_{\overline{\gamma}}\nabla_{\alpha}\nabla_{\overline{\beta}}\varphi|^{2}. \tag{10}
$$

Combining (6) and (10) we can establish (7). For the details, one can see Shi-Yau [2]. □

Remark. If we do not assume the nonnegativity of the holomorphic bisectional curvature, estimate (7) is still true with the constant C_4 depends *not only* on n *but also* on K_0.

Let

$$
\psi(x) = \varphi(x,1) \qquad x \in M. \tag{11}
$$

Then

$$
\psi(x) \in C^{\infty}(M), \tag{12}
$$

$$
\frac{C_1}{(1+\gamma)^{2n}} \leq \psi(x) \leq \frac{C_2}{(1+\gamma)^{2n}}, \tag{13}
$$

$$
|\nabla_{\alpha}\psi(x)| \leq \frac{C_3}{(1+\gamma)^{2n+1}}, \tag{14}
$$

$$
|\nabla_{\alpha}\nabla_{\overline{\beta}}\psi(x)| \leq \frac{C_4}{(1+\gamma)^{2n+1}}. \tag{15}
$$

For the remaining part of this note, we *always* denote

$$
\gamma = \gamma(x) = \gamma(x, x_0), \qquad x \in M. \tag{16}
$$

Suppose $U(x)$ is the function defined by

$$U(x) = -\int_M G(x,y)\psi(y)dy, \qquad x \in M \tag{17}$$

where $G(x,y) > 0$ is the Green function on M. Then

$$\Delta U(x) = \psi(x) \qquad x \in M. \tag{18}$$

If $G(x,y)$ does not exist on M, we can use elliptic equation theory to solve (18). Thus (13), (14), and (15) give

$$\begin{cases} \frac{C_1}{(1+\gamma)^{2n}} \le \Delta U \le \frac{C_2}{(1+\gamma)^{2n}} \\ |\nabla_\alpha(\Delta U)| \le \frac{C_3}{(1+\gamma)^{2n+1}} \\ |\nabla_\alpha \nabla_{\overline{\beta}}(\Delta U)| \le \frac{C_4}{(1+\gamma)^{2n+1}}. \end{cases} \qquad \text{on } M \tag{19}$$

Since $R_{ij} \ge 0$ on M, we have

$$\frac{C(n)\gamma(x,y)^2}{\text{Vol } B(x,\gamma(x,y))} \le G(x,y) \le \frac{\widetilde{C(n)}\gamma(x,y)^2}{\text{Vol } B(x,\gamma(x,y))}, \quad \forall x,y \in M \tag{20}$$

where $0 < C(n), \widetilde{C(n)} < +\infty$ depend only on n.

Combining (13), (17), (18), (19) and (20) we can show that

$$-\frac{C_7(\gamma+1)^2 \text{Vol } B(x_0,1)}{\text{Vol } B(x_0,\gamma+1)} \le U(x) \le -\frac{C_8(\gamma+1)^2 \text{Vol } B(x_0,1)}{\text{Vol } B(x_0,\gamma+1)} \quad \forall x \in M \tag{21}$$

$$|\nabla_\alpha U(x)| \le \frac{C_9(\gamma+1)^2 \text{Vol } B(x_0,1)}{\text{Vol } B(x_0,\gamma+1)} \qquad x \in M \tag{22}$$

$$|\nabla_\alpha \nabla_{\overline{\beta}} U(x)| \le \frac{C_9(\gamma+1)^2 \text{Vol } B(x_0,1)}{\text{Vol } B(x_0,\gamma+1)} \qquad x \in M, \tag{23}$$

where $0 < C_7, C_8, C_9 < +\infty$ depend only on n. The proof of (23) is similar to the proof of (7). The only difference is to replace $\frac{\partial \varphi}{\partial t}$ in (5) by $\psi(x)$ in (18).

Using the interchange formula for covariant derivatives, we have (convention: $\Delta = \frac{1}{2}\nabla_\alpha\nabla_{\overline{\alpha}} + \frac{1}{2}\nabla_{\overline{\alpha}}\nabla_\alpha$)

$$\begin{aligned}\Delta(\nabla_{\overline{\beta}}\nabla_\gamma U) =& \nabla_{\overline{\beta}}\nabla_\gamma(\Delta U) + \frac{1}{2}R_{\theta\overline{\beta}}\nabla_{\overline{\theta}}\nabla_\gamma U \\ &+ \frac{1}{2}R_{\gamma\overline{\theta}}\nabla_{\overline{\beta}}\nabla_\theta U - R_{\alpha\overline{\beta}\gamma\overline{\theta}}\nabla_{\overline{\alpha}}\nabla_\theta U\end{aligned}$$

$$\begin{aligned}\Delta|\nabla_{\overline{\beta}}\nabla_\gamma U| = 2\mathrm{Re}\{&\nabla_\beta\nabla_{\overline{\gamma}}U\cdot\nabla_{\overline{\beta}}\nabla_\gamma(\Delta U) \\ &+ |\nabla_\alpha\nabla_{\overline{\beta}}\nabla_\gamma U|^2 + |\nabla_{\overline{\alpha}}\nabla_{\overline{\beta}}\nabla_\gamma U|^2 \\ &+ 2R_{\theta\overline{\beta}}\nabla_{\overline{\theta}}\nabla_\gamma U\cdot\nabla_\beta\nabla_{\overline{\gamma}}U - 2R_{\alpha\overline{\beta}\gamma\overline{\theta}}\nabla_{\overline{\alpha}}\nabla_\theta U\cdot\nabla_\beta\nabla_{\overline{\gamma}}U\}.\end{aligned}$$

Thus

$$\begin{aligned}&2\int_M \mathrm{Re}\{\nabla_\beta\nabla_{\overline{\gamma}}U\cdot\nabla_{\overline{\beta}}\nabla_\gamma(\Delta U)\}dx \\ &+\int_M[|\nabla_\alpha\nabla_{\overline{\beta}}\nabla_\gamma U|^2 + |\nabla_{\overline{\alpha}}\nabla_{\overline{\beta}}\nabla_\gamma U|^2]dx \\ &+2\int_M[R_{\theta\overline{\beta}}\nabla_{\overline{\theta}}\nabla_\gamma U\cdot\nabla_\beta\nabla_{\overline{\gamma}}U - R_{\alpha\overline{\beta}\gamma\overline{\theta}}\nabla_{\overline{\alpha}}\nabla_\theta U\cdot\nabla_\beta\nabla_{\overline{\gamma}}U]dx \\ &= 0.\end{aligned} \tag{24}$$

Choose a coordinate system such that at one point

$$\nabla_\alpha\nabla_{\overline{\beta}}U = \begin{cases} 0 & \alpha\neq\beta \\ \lambda_\alpha & \alpha=\beta.\end{cases} \tag{25}$$

Then

$$\begin{aligned}&R_{\theta\overline{\beta}}\nabla_{\overline{\theta}}\nabla_\gamma U\cdot\nabla_\beta\nabla_{\overline{\gamma}}U - R_{\alpha\overline{\beta}\gamma\overline{\theta}}\nabla_{\overline{\alpha}}\nabla_\theta U\cdot\nabla_\beta\nabla_{\overline{\gamma}}U \\ &= \sum_{\alpha,\beta} R_{\alpha\overline{\alpha}\beta\overline{\beta}}(\lambda_\alpha-\lambda_\beta)^2 \geq 0.\end{aligned} \tag{26}$$

(24) can be written as

$$\begin{aligned}&2\int_M \mathrm{Re}\{\nabla_\beta\nabla_{\overline{\gamma}}U\cdot\nabla_{\overline{\beta}}\nabla_\gamma(\Delta U)\}dx \\ &+\int_M[|\nabla_\alpha\nabla_{\overline{\beta}}\nabla_\gamma U|^2 + |\nabla_{\overline{\alpha}}\nabla_{\overline{\beta}}\nabla_\gamma U|^2]dx \\ &+2\int_M\sum_{\alpha,\beta}R_{\alpha\overline{\alpha}\beta\overline{\beta}}(\lambda_\alpha-\lambda_\beta)^2dx = 0.\end{aligned} \tag{27}$$

But from (19) and (23) we have

$$
\begin{aligned}
&2\left|\int_M \mathrm{Re}\{\nabla_\beta \nabla_{\overline{\gamma}} U \cdot \nabla_{\overline{\beta}} \nabla_\gamma (\Delta U)\} dx\right| \\
&\leq 2 \int_M |\nabla_\beta \nabla_{\overline{\gamma}} U| \cdot |\nabla_{\overline{\beta}} \nabla_\gamma (\Delta U)| dx \\
&\leq \int_M \frac{2C_4 C_9 \mathrm{Vol} B(x_0,1)}{(1+\gamma)^{2n+1} \mathrm{Vol} B(x_0, \gamma+1)} dx \leq C_{10} \mathrm{Vol}\ B(x_0,1),
\end{aligned} \tag{28}
$$

where $0 < C_{10} < +\infty$ depends only on n.

Combining (27) and (28) we get

$$
\int_M \{|\nabla_\alpha \nabla_{\overline{\beta}} \nabla_\gamma U|^2 + |\nabla_{\overline{\alpha}} \nabla_{\overline{\beta}} \nabla_\gamma U|^2\} dx \leq C_{10} \mathrm{Vol}\ B(x_0,1) \tag{29}
$$

$$
2 \int_M \sum_{\alpha,\beta} R_{\alpha\overline{\alpha}\beta\overline{\beta}} (\lambda_\alpha - \lambda_\beta)^2 dx \leq C_{10} \mathrm{Vol}\ B(x_0,1). \tag{30}
$$

On the other hand, we have

$$
\begin{aligned}
&\int_M R(\Delta U)^2 dx = \int_M R\Delta U \cdot \Delta U dx \\
&= \int_M R\Delta U \cdot \nabla_\alpha \nabla_{\overline{\alpha}} U dx \\
&= -\int_M R\nabla_\alpha (\Delta U) \cdot \nabla_{\overline{\alpha}} U dx - \int_M \nabla_\alpha R \cdot \Delta U \cdot \nabla_{\overline{\alpha}} U dx \\
&= -\int_M R\nabla_\alpha \nabla_\beta \nabla_{\overline{\beta}} U \cdot \nabla_{\overline{\alpha}} U dx - \int_M \nabla_\gamma R_{\alpha\overline{\gamma}} \cdot \Delta U \cdot \nabla_{\overline{\alpha}} U dx \\
&= -\int_M R\nabla_\beta \nabla_\alpha \nabla_{\overline{\beta}} U \cdot \nabla_{\overline{\alpha}} U dx + \int_M R_{\alpha\overline{\gamma}} \nabla_\gamma [\Delta U \cdot \nabla_{\overline{\alpha}} U] dx \\
&= \int_M \nabla_\alpha \nabla_{\overline{\beta}} U \cdot \nabla_\beta [R\nabla_{\overline{\alpha}} U] dx + \int_M R_{\alpha\overline{\gamma}} \nabla_\gamma [\Delta U \cdot \nabla_{\overline{\alpha}} U] dx \\
&= \int_M R\nabla_\alpha \nabla_{\overline{\beta}} U \cdot \nabla_\beta \nabla_{\overline{\alpha}} U dx + \int_M \nabla_\beta R \cdot \nabla_\alpha \nabla_{\overline{\beta}} U \cdot \nabla_{\overline{\alpha}} U dx \\
&+ \int_M R_{\alpha\overline{\gamma}} \nabla_\gamma \nabla_{\overline{\alpha}} U \cdot \Delta U dx + \int_M R_{\alpha\overline{\gamma}} \nabla_\gamma (\Delta U) \cdot \nabla_{\overline{\alpha}} U dx.
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
&\int_M \nabla_\beta R \cdot \nabla_\alpha \nabla_{\overline{\beta}} U \cdot \nabla_{\overline{\alpha}} U dx \\
&= \int_M \nabla_\gamma R_{\beta\overline{\gamma}} \cdot \nabla_\alpha \nabla_{\overline{\beta}} U \cdot \nabla_{\overline{\alpha}} U dx \\
&= -\int_M R_{\beta\overline{\gamma}} \nabla_\gamma \nabla_\alpha \nabla_{\overline{\beta}} U \cdot \nabla_{\underline{\alpha}} U dx - \int_M R_{\beta\overline{\gamma}} \nabla_\alpha \nabla_{\overline{\beta}} U \cdot \nabla_\gamma \nabla_{\overline{\alpha}} U dx \\
&= -\int_M R_{\beta\overline{\gamma}} \nabla_\alpha \nabla_\gamma \nabla_{\overline{\beta}} U \cdot \nabla_{\overline{\alpha}} U dx - \int_M R_{\beta\overline{\gamma}} \nabla_\alpha \nabla_{\overline{\beta}} U \cdot \nabla_\gamma \nabla_{\overline{\alpha}} U dx \\
&= \int_M \nabla_\gamma \nabla_{\overline{\beta}} U \cdot \nabla_\alpha [R_{\beta\overline{\gamma}} \nabla_{\overline{\alpha}} U] dx - \int_M R_{\beta\overline{\gamma}} \nabla_\alpha \nabla_{\overline{\beta}} U \cdot \nabla_\gamma \nabla_{\overline{\alpha}} U dx \\
&= \int_M \nabla_\alpha R_{\beta\overline{\gamma}} \cdot \nabla_\gamma \nabla_{\overline{\beta}} U \cdot \nabla_{\overline{\alpha}} U dx + \int_M R_{\beta\overline{\gamma}} \nabla_\gamma \nabla_{\overline{\beta}} U \cdot \Delta U dx \\
&- \int_M R_{\beta\overline{\gamma}} \nabla_\alpha \nabla_{\overline{\beta}} U \cdot \nabla_\gamma \nabla_{\overline{\alpha}} U dx \\
&= \int_M \nabla_\beta R_{\alpha\overline{\gamma}} \cdot \nabla_\gamma \nabla_{\overline{\beta}} U \cdot \nabla_{\overline{\alpha}} U dx + \int_M R_{\beta\overline{\gamma}} \nabla_\gamma \nabla_{\overline{\beta}} U \cdot \Delta U dx \\
&- \int_M R_{\beta\overline{\gamma}} \nabla_\alpha \nabla_{\overline{\beta}} U \cdot \nabla_\gamma \nabla_{\overline{\alpha}} U dx \\
&= -\int_M R_{\alpha\overline{\gamma}} \nabla_\gamma (\Delta U) \cdot \nabla_{\overline{\alpha}} U dx - \int_M R_{\alpha\overline{\gamma}} \nabla_\gamma \nabla_{\overline{\beta}} U \cdot \nabla_\beta \nabla_{\overline{\alpha}} U dx \\
&+ \int_M R_{\beta\overline{\gamma}} \nabla_\gamma \nabla_{\overline{\beta}} U \cdot \Delta U dx - \int_M R_{\beta\overline{\gamma}} \nabla_\alpha \nabla_{\overline{\beta}} U \cdot \nabla_\gamma \nabla_{\overline{\alpha}} U dx.
\end{aligned} \tag{32}
$$

Combining (31) and (32) we get

$$
\begin{aligned}
\int_M R(\Delta U)^2 dx = &\int_M R \nabla_\alpha \nabla_{\overline{\beta}} U \cdot \nabla_{\overline{\alpha}} \nabla_\beta U dx \\
&+ 2 \int_M R_{\alpha\overline{\gamma}} \nabla_\gamma \nabla_{\overline{\alpha}} U \cdot \Delta U dx \\
&- 2 \int_M R_{\alpha\overline{\gamma}} \nabla_\gamma \nabla_{\overline{\beta}} U \cdot \nabla_\beta \nabla_{\overline{\alpha}} U dx.
\end{aligned} \tag{33}
$$

If we define a function

$$
\begin{aligned}
F(x) = &R(\Delta U)^2 - R \nabla_\alpha \nabla_{\overline{\beta}} U \cdot \nabla_{\overline{\alpha}} \nabla_\beta U \\
&+ 2 R_{\alpha\overline{\gamma}} \nabla_\gamma \nabla_{\overline{\beta}} U \cdot \nabla_\beta \nabla_{\overline{\alpha}} U \\
&- 2 R_{\alpha\overline{\gamma}} \nabla_\gamma \nabla_{\overline{\alpha}} U \cdot \Delta U.
\end{aligned} \tag{34}
$$

Then

$$
\int_M F(x) dx = 0. \tag{35}
$$

With the coordinate (25) we know that

$$|\nabla_\alpha \nabla_{\overline{\beta}} U| = \sqrt{\sum_\alpha \lambda_\alpha^2}. \tag{36}$$

Define $\Omega \subseteq M$ such that

$$\Omega = \{x \in M \Big| |\lambda_\alpha - \frac{|\nabla_\beta \nabla_{\overline{\gamma}} U|}{\sqrt{n}}| \leq \frac{|\nabla_\beta \nabla_{\overline{\gamma}} U|}{(2n+10)^5} \qquad \text{for } \alpha = 1, 2, \cdots, n\}. \tag{37}$$

It is easy to see that for any $x \in M \setminus \Omega$, there exist α and β (say $\alpha = 1, \beta = 2$) such that

$$|\lambda_1 - \lambda_2| \geq \frac{|\nabla_\beta \nabla_{\overline{\gamma}} U|}{(2n+10)^5}.$$

Thus

$$\sum_{\alpha,\beta} R_{\alpha\overline{\alpha}\beta\overline{\beta}}(\lambda_\alpha - \lambda_\beta)^2 \geq R_{1\overline{1}2\overline{2}} \frac{|\nabla_\alpha \nabla_{\overline{\beta}} U|^2}{(2n+10)^{10}} \geq \frac{\epsilon}{(2n+10)^{10}} R |\nabla_\alpha \nabla_{\overline{\beta}} U|^2 \qquad \forall x \in M \setminus \Omega \tag{38}$$

$$\begin{aligned} \int_{M \setminus \Omega} R |\nabla_\alpha \nabla_{\overline{\beta}} U|^2 dx &\leq \frac{(2n+10)^{10}}{\epsilon} \int_{M \setminus \Omega} \sum_{\alpha,\beta} R_{\alpha\overline{\alpha}\beta\overline{\beta}} (\lambda_\alpha - \lambda_\beta)^2 dx \\ &\leq \frac{(2n+10)^{10}}{\epsilon} \int_M \sum_{\alpha,\beta} R_{\alpha\overline{\alpha}\beta\overline{\beta}} (\lambda_\alpha - \lambda_\beta)^2 dx. \end{aligned} \tag{39}$$

Combining (30) and (39) we know that

$$\int_{M \setminus \Omega} R |\nabla_\alpha \nabla_{\overline{\beta}} U|^2 dx \leq \frac{(2n+10)^{10}}{2\epsilon} C_{10} \text{Vol } B(x_0, 1). \tag{40}$$

It is easy to see that

$$|F(x)| \leq C_{11}(n) R |\nabla_\alpha \nabla_{\overline{\beta}} U|^2 \qquad \forall x \in M. \tag{41}$$

Thus

$$\begin{aligned} &|\int_{M \setminus \Omega} F(x) dx| \leq \int_{M \setminus \Omega} |F(x)| dx \leq \\ &\leq C_{11} \int_{M \setminus \Omega} R |\nabla_\alpha \nabla_{\overline{\beta}} U|^2 dx \leq C_{12}(n, \epsilon) \text{Vol } B(x_0, 1). \end{aligned} \tag{42}$$

From (35) we get

$$\int_\Omega F(x)dx = -\int_{M\setminus\Omega} F(x)dx.$$

Thus from (42)

$$|\int_\Omega F(x)dx| \le C_{12}(n,\epsilon) \text{ Vol } B(x_0,1). \tag{43}$$

Since on Ω we have

$$\lambda_\alpha \sim \frac{|\nabla_\beta\nabla_{\overline{\gamma}}U|}{\sqrt{n}} \qquad \alpha = 1,2,\cdots,n.$$

From (34) we have

$$\begin{aligned} F(x) &\sim R\cdot n|\nabla_\beta\nabla_{\overline{\gamma}}U|^2 - R|\nabla_\beta\nabla_{\overline{\gamma}}U|^2 + \\ &\quad + 2\sum_\alpha R_{\alpha\overline{\alpha}}\cdot\frac{1}{n}|\nabla_\beta\nabla_{\overline{\gamma}}U|^2 - 2\sum_\alpha R_{\alpha\overline{\alpha}}|\nabla_\beta\nabla_{\overline{\gamma}}U|^2, \\ F(x) &\sim \sum_\alpha R_{\alpha\overline{\alpha}}(n-3+\frac{2}{n})|\nabla_\beta\nabla_{\overline{\gamma}}U|^2, \qquad \forall x\in\Omega. \end{aligned} \tag{44}$$

Since $n \ge 3$, we have $n-3+\frac{2}{n} \ge \frac{2}{n}$. (44) is not precise. Precisely we have

$$F(x) \ge \frac{1}{n}R|\nabla_\beta\nabla_{\overline{\gamma}}U|^2 \qquad x\in\Omega. \tag{45}$$

(43), (45) $\Longrightarrow$

$$\int_\Omega R|\nabla_\beta\nabla_{\overline{\gamma}}U|^2dx \le C_{13}(n,\epsilon) \text{ Vol } B(x_0,1). \tag{46}$$

(40), (46) $\Longrightarrow$

$$\int_M R|\nabla_\beta\nabla_{\overline{\gamma}}U|^2dx \le C_{14}(n,\epsilon) \text{ Vol } B(x_0,1). \tag{47}$$

(19) $\Longrightarrow$

$$|\nabla_\beta\nabla_{\overline{\gamma}}U|^2 \ge \frac{C_1^2}{n(1+\gamma)^{4n}}, \qquad x\in M. \tag{48}$$

(47), (48) $\Longrightarrow$

$$\int_M \frac{R(x)dx}{[1+\gamma(x,x_0)]^{4n}} \le C_{15}(n,\epsilon) \text{ Vol } B(x_0,1). \tag{49}$$

Thus finally we have

$$\int_{B(x_0,1)} R(x)dx \le C_{16}(n,\epsilon) \text{ Vol } B(x_0,1). \tag{50}$$

Theorem 2 is proved.

References

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