

BEYOND LIOUVILLIAN TRANSCENDENCE

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ABSTRACT. To a codimension one foliation \mathcal{F} defined by a meromorphic 1-form ω , one may associate a *Godbillon-Vey sequence* $(\omega_j), j = 0, 1, \dots$, of meromorphic 1-forms ω_j with $\omega_0 = \omega$. The sequence is said to have *finite length* k if $\omega_k \neq 0$ and $\omega_j = 0$ for $j > k$. The case $k = 0, 1$ or 2 corresponds, respectively, to the case where the foliation \mathcal{F} has additive, affine or projective transverse structure and $k \leq 1$ is equivalent to the existence of a Liouvillian first integral. These are the only possible cases where the transverse structures come from an action of a Lie group on $\overline{\mathbb{C}}$ and a non-trivial model for these foliations is the Riccati differential equation. We propose to go beyond the Lie group transverse structure by studying the case of general k and, for this case, we determine a model differential equation, which generalizes the Riccati equation. We also discuss some other related topics.

1. Introduction

Let M be a complex manifold and $X \subset M$ an analytic subset of codimension ≥ 2 . A complex codimension one analytic foliation \mathcal{F} on M , with singular set $X = \text{sing}(\mathcal{F})$, is defined by a maximal atlas of submersions $y_i : U_i \rightarrow \mathbb{C}$, covering $M \setminus X$, defining \mathcal{F} outside X . This atlas induces a *transverse structure cocycle* (g_{ij}) , where the g_{ij} are local biholomorphisms of \mathbb{C} defined by $y_i = g_{ij}(y_j)$, provided that $U_i \cap U_j \neq \emptyset$. In a natural way this cocycle can be regarded as the holonomy pseudogroup of \mathcal{F} defined by the given atlas. The simplest pseudogroup structure of \mathcal{F} is obtained as restriction of a Lie group action on $\overline{\mathbb{C}}$. In this case we have only three possibilities, *additive*, *affine* and *projective* transverse structures (see [5],[17]). A nontrivial model for these foliations is the *Riccati differential equation* $\mathcal{R} : p(x)dy - (a_2(x)y^2 + a_1(x)y + a_0(x))dx = 0$, where the $a_j(x)$ and $p(x)$ are polynomials [17]. Indeed, if we take for instance M a simply-connected projective manifold such that the complementary of any analytic codimension one subset is a Stein manifold, then any foliation \mathcal{F} admitting a compact invariant divisor $\Lambda \subset M$ with solvable holonomy, will have a projective structure in a Zariski's open subset of M (containing $\Lambda \setminus \text{sing}(\mathcal{F})$) and will be, under generic hypothesis on $\text{sing}(\mathcal{F})$, either a Darboux (linear logarithmic)

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foliation, or the pull back, by a rational map, of a Bernoulli differential equation ([2],[17]). The linear case corresponds to the case we have a compact invariant divisor with abelian holonomy, [1]. On the other hand it can be proved [2],[17] that foliations with an affine transverse structure outside some compact invariant divisor $\Lambda \subset M$, and satisfying certain (generic) hypothesis on $\text{sing}(\mathcal{F}) \cap \Lambda$, correspond to the ones which have a Liouvillian first integral. More precisely it is shown in [2],[17] that a foliation on M , as above, admitting a Liouvillian first integral is in fact either a Darboux foliation or it is the rational pull-back of a suitable Riccati (Bernoulli) differential equation.

The Riccati differential equation was frequently encountered in the classical study of complex differential equations since last century. This is the case, for instance, of the theorem of Painlevé, [14],[15], asserting that a foliation on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ with a n to n holonomy map is a pull-back of a Riccati foliation, and also in several other works of P. Painlevé [15]. More recently, a well-known theorem of Malmquist, [10], states that if a complex differential equation on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ has a solution which is the graph of a nonalgebraic meromorphic function then \mathcal{F} is a Riccati foliation. This illustrates the fact that the highest level of transcendence of the solutions of a differential equation with a Lie group transverse structure is contained in the Riccati equation. The purpose of this paper is to go beyond the Lie group transverse structure of a foliation and to enter the unknown domain of the pseudogroup structures. We do that by introducing on this subject the concept of *Godbillon-Vey sequences*. Assuming that \mathcal{F} is defined by a meromorphic integrable 1-form ω_0 , we define a sequence of meromorphic 1-forms $\omega_0, \omega_1, \dots$, satisfying certain relations, $d\omega_0 = \omega_0 \wedge \omega_1, d\omega_1 = \omega_0 \wedge \omega_2, d\omega_2 = \omega_0 \wedge \omega_3 + \omega_1 \wedge \omega_2, \dots$. The sequences of “length” 0,1 and 2, correspond to the transversely additive, affine and projective cases respectively. The general case of finite length k is a sequence (ω_j) , with $\omega_k \neq 0$, and $\omega_j \equiv 0, \forall j > k$. The main result of this paper associates to each $k \in \mathbb{N}$ a differential equation which models any foliation with finite Godbillon-Vey sequence of length k (section 2). This differential equation reduces to a Riccati equation whenever $k = 0, 1, 2$. The proposal of this paper is thus to use these models in order to exhibit higher level of transcendence beyond the Liouvillian functions.

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2. Transverse structures of holomorphic foliations

The maximal collection of the (g_{ij}) defined in the introduction satisfies the cocycle property $g_{ij} \circ g_{jk} = g_{ik}$ and defines the *transverse structure* of \mathcal{F} . The

foliation \mathcal{F} is called *transversely homogeneous* if there exists a Lie group G acting on $\overline{\mathbb{C}}$ such that there exists a collection in the transverse structure of \mathcal{F} , which consists of restrictions of the biholomorphisms induced by G [6]. For instance we can have $G = (\mathbb{C}, +)$, $\text{Aff}(\mathbb{C})$, and $\text{SL}(2, \mathbb{C})$, defining a transversely *additive*, *affine and projective* foliation respectively. Indeed these are the only possibilities for a holomorphic codimension-one transversely homogeneous foliation, [5],[17]. Let us fix a meromorphic integrable 1-form ω_o defining \mathcal{F} . In the affine case there exists a global holomorphic 1-form ω_1 on M which satisfies $d\omega_o = \omega_o \wedge \omega_1$ and $d\omega_1 = 0$. In the projective case if there exists ω_1 such that $d\omega_o = \omega_o \wedge \omega_1$, then there exists also a meromorphic 1-form ω_2 in M such that we have $d\omega_1 = \omega_o \wedge \omega_2$ and $d\omega_2 = \omega_1 \wedge \omega_2$, [6],[1]. These equations verified in the three cases above are particular cases of a set of relations which we proceed to introduce now.

Let X be a (meromorphic) vector field on M satisfying $\omega_o(X) = 1$. Recall that the *Lie derivative* is given by $L_X(\omega_o) = i_X d\omega_o + d(i_X \omega_o)$. We consider $\omega_j = L_X^j(\omega_o)$, where $L_X^{j+1}(\cdot) = L_X(L_X^j(\cdot))$. This construction is found in [7] for the nonsingular real case. The sequence $\omega_o, \omega_1, \dots$, of meromorphic 1-forms thus obtained satisfies the relations:

$$(E)_j : d\omega_j = \omega_o \wedge \omega_{j+1} + \sum_{k=1}^j \binom{j}{k} \omega_k \wedge \omega_{j+1-k}.$$

Any such a sequence is called a *Godbillon-Vey sequence*.

Denote by $\widehat{\mathcal{X}}(\mathbb{C}, 0)$ the Lie algebra of the formal vector fields in one complex variable with a singularity at the origin, endowed with the Lie bracket. According to [7] the only finite Godbillon-Vey sequences $\{\omega_o, \dots, \omega_k, 0, \dots\}$, defining Lie algebras, occur in the cases $k = 0, 1, 2$. Moreover the full Godbillon-Vey relations define the Lie algebra $\widehat{\mathcal{X}}(\mathbb{C}, 0)$. In particular there are *no* Lie algebras associated to finite Godbillon-Vey sequences, strictly between $SL(2, \mathbb{C})$ and $\mathcal{L}(\widehat{\mathcal{X}}(\mathbb{C}, 0))$.

3. Models of finite length

Suppose that \mathcal{F} is defined by a meromorphic integrable 1-form ω_o on a rational algebraic manifold M , and that there exists a meromorphic vector field X such that $\omega_o(X) = 1$. We say that the sequence $(\omega_j)_{j=1}^\infty$, where $\omega_j = L_X^j(\omega_o)$, has *finite length* k if $\omega_k \neq 0$ and $\omega_j \equiv 0, \forall j > k$.

Theorem 1. *The 1-form ω_k is closed and there are two possibilities:*

- (i) ω_k admits a meromorphic first integral and then there exists a meromorphic map $\pi: M \rightarrow \overline{\mathbb{C}} \times \overline{\mathbb{C}}$ and a model equation of length k in $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$

$$\mathcal{S}_k : \eta_k = dy - (a_o(x)y^k + \dots + a_{k-1}(x)y + a_k(x))dx = 0,$$

where the $a_j(x)$ are meromorphic functions, such that $\omega_o = \pi^*(\eta_k)$.

(ii) ω_k does not admit a meromorphic first integral and then we have

$$\omega_0 = df + (f^k \lambda_0 + \cdots + \lambda_k) \omega_k,$$

for some meromorphic function f on M and for some constants $\lambda_0, \dots, \lambda_k \in \mathbb{C}, \lambda_k \neq 0$.

Proof. First we prove that ω_k is closed.

Choose an open set $U \subset M$, nonvoid, such that $X|_U$ is holomorphic nonsingular and has a flowbox, that is, there are coordinates $(x_1, \dots, x_n, y) \in U$ such that in these coordinates we have $X = \frac{\partial}{\partial y}$. For simplicity of our notation we assume that $n = 1$ and write $x_1 = x$. The condition $i_X \omega_0 = 1$ therefore, implies that $\omega_0 = dy - R(x, y)dx$, where R is meromorphic in U . Thus it is easy to see that we have, $\omega_j = L_X^j \omega_0 = (-1)^{j+1} \frac{\partial^j}{\partial y^j} R(x, y)dx$. Thus $\omega_k = 0$ implies that R is polynomial of degree k in the variable y , say,

$$R(x, y) = \sum_{j=1}^k a_j(x) y^j,$$

and in particular $\omega_k = k! a_k(x) dx$ depends only on the variable x and is therefore closed.

Now we prove (i): Let us first assume that $\omega_k = dx$ for a meromorphic nonconstant function $x: M \rightarrow \overline{\mathbb{C}}$.

Claim. $\omega_j \wedge dx \equiv 0, \forall 1 \leq j \leq k$.

Proof. The proof is by induction on j from $j = k$ to $j = 1$. We take the equations

$$(E)_j : d\omega_j = \omega_0 \wedge \omega_{j+1} + \sum_{k=1}^j \binom{j}{k} \omega_k \wedge \omega_{j+1-k},$$

for $j = 1, \dots, 2k-2$. For $j = 2k-2$ it gives $\omega_k \wedge \omega_{k-1} \equiv 0$ because $\omega_\ell \equiv 0, \forall \ell \geq k+1$. Thus we have that the claim holds for $j = k-1$. Assume now that we have proved that $\omega_j \wedge dx \equiv 0, \forall j = k-r+1, \dots, k$. Then using the equation $(E)_{2k-1-r}$ we conclude that $\omega_{k-r} \wedge 0$. Thus the claim follows by induction.

The claim above and the equations $(E)_j$ give:

$$d\omega_j = \omega_o \wedge \omega_{j+1}, \forall j = 1, \dots, k. \quad \square$$

Now we proceed the proof of (i). According to [17] we can assume $k \geq 3$. Using the claim above we write $\omega_{k-1} = ydx$ for some meromorphic function $y: M \rightarrow \overline{\mathbb{C}}$. Since $\omega_k \neq 0$, it follows that y is not constant on M . Moreover

since M is algebraic rational it follows from the claim that $\omega_j = h_j dx, \forall j = 1, \dots, k-1$. First we assume that x is *primitive*, i.e., it has connected fibers. We have $\omega_{k-2} = \omega_o \wedge y dx = \omega_o \wedge d(\frac{y^2}{2} dx)$. Now we write $\omega_{k-2} = h_2 dx$ for some meromorphic function h_2 on M . Therefore it follows that $d((h_2 - \frac{y^2}{2}) dx) \wedge \omega_o \equiv 0$. This implies that $d((h_2 - \frac{y^2}{2}) dx) = g_2 \omega_o$ for some meromorphic function g_2 , and moreover since the left hand 1-form is closed it follows that $g_2 \omega_o$ is also closed. By the hypothesis it implies that $g_2 \equiv 0$ and therefore $d(h_2 - \frac{y^2}{2}) \wedge dx \equiv 0$. Since we have chosen x to be primitive it follows that $h_2 = \frac{y^2}{2} + a_2(x)$ for some rational function $a_2(z)$ on $\overline{\mathbb{C}}$. Thus we have $\omega_{k-2} = (\frac{y^2}{2} + a_2(x)) dx$. Now using the induction and the same arguments above we can prove that

$$\omega_{k-j} = (-1)^j k(k-1) \cdots (j+1) j \frac{y^j}{k!} \left(\sum_{\ell=1}^j a_\ell(x) \frac{y^\ell}{\ell!} \right) dx, \forall j = 1, \dots, k-1,$$

where $a_\ell(z)$ is a rational function on $\overline{\mathbb{C}}$. In particular it follows that $\omega_1 = (ky^{k-1}\alpha_o(x) + (k-1)y^{k-2}\alpha_1(x) + \cdots + \alpha_k(x)) dx$ for some meromorphic functions $\alpha_j(z)$ on $\overline{\mathbb{C}}$. The first equation $d\omega_o = \omega_1 \wedge \omega_o$ now gives that $\omega_o = dy - (\alpha_o(x)y^k + \alpha_1(x)y^{k-1} + \cdots + \alpha_{k-1}(x)y + \alpha_k(x)) dx + b(x) dx$, for some rational function $b(z)$ on $\overline{\mathbb{C}}$.

Now, for the general case we recall that according to the Stein Factorization Theorem [8] (since the leaves of a foliation are always connected) there exists a primitive rational map $x: M \rightarrow \overline{\mathbb{C}}$, and a rational function $a_k(z)$ on $\overline{\mathbb{C}}$, such that $\omega_k = (-1)^{k+1} k! a_k(x) dx$. Using now essentially the same arguments above one can prove this case. This proves the first part of the theorem.

Part (ii) in the theorem is proved the same way that the first part. The difference is that since by the hypothesis there exists no rational first integral for ω_k , the functions a_j we obtain in (i) are constant. \square

Remark 1. When $M = \overline{\mathbb{C}} \times \overline{\mathbb{C}}$, there exists an independent way of finding these models, using the approach of P. Painlevé [14]. In his memoir Painlevé used frequently the fact that a foliation on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$, which is transverse to a generic vertical line $\overline{\mathbb{C}}_{x_o} = \{x_o\} \times \overline{\mathbb{C}}$ is in fact a Riccati differential equation. The next case to this is the case of foliations which are transverse to generic vertical lines $\overline{\mathbb{C}}_{x_o}$, except at infinity, where there is a tangency with these lines of order $k-2$. These are precisely the models \mathcal{S}_k .

The following theorem, whose proof is easy from [2],[17], shows that for $k \geq 3$ a dense subset of differential equations in \mathcal{S}_k have no Liouvillian first integral.

Theorem 2. *The set of differential equations in \mathcal{S}_k which are not rational pull-backs of Riccati differential equations, form a second category subset of the space \mathcal{S}_k on the space of foliations on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$.*

Remark 2. Perhaps the simplest example where a model \mathcal{S}_k is present is the saddle-node situation:

$$\eta_k = x^{p+1}dy - [(1 + \lambda x^p)y + a_2(x)y^2 + \cdots + a_k(x)y^k]dx = 0, \quad k \geq 3.$$

It would be interesting to find the natural mathematical object generalizing a Liouvillian function through this example. A natural question is what are the *sectorial isotropies* of Martinet-Ramis [11] associated to $\eta_k=0$.

4. Holonomy and Godbillon-Vey sequences

We consider a germ of holomorphic singular foliation \mathcal{F} on $(\mathbb{C}^2, 0)$ given by a holomorphic 1-form ω , defined in a neighborhood U of the origin. Let S be a local smooth separatrix of \mathcal{F} through 0 and choose a local coordinate system $(x, y) \in U$, such that $S = (y = 0)$. In a small disk $\overline{\mathbb{D}} \subset S$ we consider the simple path $\gamma(t) = (e^{2\pi it}, 0)$, $t \in [0, 2\pi]$ so if $\Sigma = \{(1, y)\}$, then $\Sigma \cap S = \{(1, 0)\} = \{p_1\}$ and the product $\Pi = \gamma([0, 2\pi i]) \times \Sigma$ is a solid torus which is foliated by a transversely holomorphic real flow \mathcal{L} , obtained from \mathcal{F} , this flow \mathcal{L} has a closed orbit which corresponds to the curve $t \mapsto \gamma(t)$ and whose *Poincaré Map* is a holomorphic map $h : \Sigma \rightarrow \Sigma$ which fixes the point p_1 . This map is called *holonomy map* of the separatrix S . The identification $(\Sigma, p_1) = (\mathbb{C}, 0)$ allows us to consider h as an element of $\text{Diff}(\mathbb{C}, 0)$.

Problem. Describe the relation between h and the transverse structure of \mathcal{F} , by means of the Godbillon-Vey sequence associated to ω .

We consider the natural holomorphic coordinate induced by y on Σ and obtain a convergent series

$$h(y) = \sum_{j=1}^{+\infty} a_j y^j \in \mathbb{C}\{\{y\}\},$$

for h in this coordinate. On the other hand we define $X = \frac{\partial}{\partial y}$ a holomorphic vector field in U and write $\omega = -Pdx + Qdy$, $P, Q \in \mathcal{O}(U)$, so that $\omega_0 = dy - \frac{P}{Q}dx$ defines the same foliation, and $\omega_0(X) = 1$. Now proceeding as in the Godbillon-Vey algorithm we obtain a *meromorphic* sequence for \mathcal{F} in U which is in fact given by $\omega_j = (-1)^{j+1} \frac{\partial^j}{\partial y^j}(R)dx$, where $R := \frac{P}{Q}$ is meromorphic. The map h is therefore described as follows: Let $\phi(t, y)$ be the flow \mathcal{L} on Π , that is, it is the solution of the real differential equation

$$(0) \quad \frac{dy(t)}{dt} = R(\gamma(t), y(t))\gamma'(t), \quad \gamma(t) = e^{it},$$

satisfying the initial condition $y(0, y_0) = y_0$. Then the holonomy map is given by $h(y) = \phi(2\pi, y)$. We also remark that since $\gamma(t)$ is a solution we have $\phi(t, 0) = 0$,

for all t . Given any $y \in \Sigma$ we define the path $\gamma_y(t) = (e^{it}, \phi(t, y))$; $t \in [0, 2\pi]$. Thus $\gamma = \gamma_0$. We have :

$$(1) \quad \frac{\partial \phi}{\partial t}(t, y) = R(e^{it}, \phi(t, y))ie^{it},$$

so that

$$(2) \quad \frac{\partial}{\partial t} \left(\frac{\partial \phi}{\partial y}(t, y) \right) = \frac{\partial R}{\partial y}(\gamma_y(t)) \frac{\partial \phi}{\partial y}(t, y)ie^{it},$$

and then the solution is

$$(3) \quad \frac{\partial \phi}{\partial y}(t, y) = \exp F_1(t, y),$$

where we define,

$$(4) \quad F_1(t, y) = \int_{\gamma} \omega_1,$$

and we also define $f_1(x) = \int_1^x \omega_1(x, 0)$.

Thus, we have

$$(5) \quad h'(0) = \exp \int_{\gamma} \omega_1,$$

(recall that $\phi(0, y) = y$).

Derivating (2) we obtain:

$$(6) \quad \frac{\partial}{\partial t} \left(\frac{\partial^2 \phi}{\partial y^2} \right) = \frac{\partial^2 R}{\partial y^2}(\gamma_y(t)) \left(\frac{\partial \phi}{\partial y}(t, y) \right) ie^{it} + \frac{\partial R}{\partial y}(\gamma_y(t)) \frac{\partial^2 \phi}{\partial y^2}(t, y)ie^{it}.$$

Remark 3. The solution of the differential equation

$$(7) \quad f' = af + b,$$

is $f = A \int \frac{b}{A}$ where $A = \exp \int a$.

Thus, we obtain

$$(8) \quad \frac{\partial^2 \phi}{\partial y^2}(t, y) = -\exp F_1(t, y) \int_{\gamma_y} F_2(t, y)$$

where

$$(9) \quad F_2(t, y) = \int_{\gamma_y} \exp F_1(t, y) \omega_2,$$

we also define $f_2(x) = -expf_1(x) \int_1^x \omega_2(x, 0)$.

Thus, we obtain,

$$(10) \quad h^{(2)}(0) = exp \int_1^x \omega_1 \int_1^x expf_1 \omega_2.$$

Proceeding this same way we obtain for the k -th step:

$$(11) \quad \frac{\partial^k \phi}{\partial y^k}(t, y) = [expF_1(t, y)].F_k(t, y),$$

and $h^{(k)}(0) = [exp \int_\gamma] . \omega_1 \int_\gamma \Omega_k$, for some $F_k(t, y), f_k(x)$ and some differential 1-form Ω_k which we describe inductively below, but first we need the following well-known lemma:

Lemma 1. *Let f, g be C^∞ real functions such that the composition $f \circ g$ is defined, then:*

$$(12) \quad \frac{1}{k!} (f \circ g)^{(k)}(x) = \sum_{p=1}^k \frac{1}{p!} f^{(p)}(g(x)) \sum_{n \in \mathbb{N}(p, k)} \binom{k-1}{p-1} \left(\frac{g^{(1)}(x)}{1!} \right)^{n_1} \dots \left(\frac{g^{(k)}(x)}{k!} \right)^{n_k},$$

where $\mathbb{N}(p, k) := \{(n_1, \dots, n_k); \sum n_j = p, \sum j n_j = k\}$, for all $k \in \mathbb{N}$.

Using the formula above we obtain the following:

$$(13) \quad \frac{1}{n!} \cdot \frac{\partial^n}{\partial y^n} \left(\frac{\partial \phi}{\partial t}(t, y) \right) = \left[\sum_{p=1}^n \frac{1}{p!} \frac{\partial^p R}{\partial y^p} \left(e^{(it)}, \phi(t, y) \right) \sum_{k \in \mathbb{N}(n, p)} \binom{k-1}{p-1} \left(\frac{1}{1!} \frac{\partial \phi}{\partial y}(t, y) \right)^{(k_1)} \dots \left(\frac{1}{n!} \frac{\partial^n \phi}{\partial y^n}(t, y) \right)^{k_n} \right] i e^{it}.$$

Now we remark that if $(n_1, \dots, n_k) \in \mathbb{N}(p, k)$ is such that $\sum_{j=1}^k j n_j = n$, then,

if $n_k \neq 1$, we must have: $n_k = 1$ and $n_j = 0$ for the others. Moreover $\sum_{j=1}^k n_j = 1$ and then using also (11) we obtain :

$$(14) \quad \frac{\partial^n \phi}{\partial y^n}(t, y) = [expF_1(t, y)]. \int_{\gamma_y} \sum_{p=1}^n \frac{\partial^{(p)} R}{\partial y^{(p)}(\gamma_y(t))} \sum_{n \in \mathbb{N}(p, k)} \binom{n}{p} \frac{(n-1)!}{(p-1)!} \times (expF_1(t, y))^{p-1} \left(\frac{F_2(t, y)}{2!} \right)^{n_2} \dots \left(\frac{F_{n-1}(t, y)}{(n-1)!} \right)^{n_{k-1}} i e^{it} dt.$$

Using this we get the inductive definition for F_n and therefore for $f_n(x)$ and Ω_n , these expressions can be used to prove that:

Lemma 2. $\Omega_n(x)$ is a closed Liouvillian differential 1-form on $\{y = 0\}$ for every n .

Proof. As we have remarked above, this is a consequence of the recurrence relations stated above, but we can give an intuition of the geometric reason; in fact all one need to recall is that the periods $\int_\gamma \Omega$ are defined intrinsically by the equations above so that in particular they do not depend on the chosen closed path γ on $(y = 0)$ as above. This is enough to assure that the restriction $\Omega_n|_{(y=0)}$ is closed. \square

Definition 1. The *realization equations* for the diffeomorphism $h(y)$ are the equations (RE1),(RE2) and (RE3) below:

$$(RE1) \quad \begin{aligned} h^{(1)}(0) &= \exp \int_\gamma \omega_1(x, 0), \\ h^{(k)}(0) &= h^{(1)}(0) \int_\gamma \Omega_k, \end{aligned}$$

$$(RE2) \quad f_{k+1}(x) = \int_\gamma \Omega_k(x),$$

$$(RE3) \quad \Omega_k = \sum_{p=1}^n (-1)^{p+1} \sum_{n \in \mathbb{N}(p,k)} \binom{n}{p} \frac{(n-1)!}{(p-1)!} \times (\exp f_1(x))^{p-1} \left(\frac{f_2(x)}{2!} \right)^{n_2} \dots$$

$$\left(\frac{f_{n-1}(x)}{(n-1)!} \right)^{n_{k-1}} \omega_p(x, 0).$$

Remark 4. The realization equations introduced above may be applied in order to give a proof of the local synthese result [16], and the main idea is that the convergence of the series

$$h(y) = \sum_{j=0}^{+\infty} \frac{h^{(j)}(0)}{j!} y^j$$

may be used to prove the convergence of a suitable (formal) expression $R(x, y)$ solution for the realization equations.

Remark 5 (Finite length groups): We recall that a germ of foliation \mathcal{F} at $0 \in \mathbb{C}^2$, singular at the origin, is *not-much degenerate* (*n.m.d.* for short) if: (i) \mathcal{F} is nondicritical and can be desingularized with one sole blow-up, $\pi : (\widetilde{\mathbb{C}^2}, \mathbb{P}) \rightarrow (\mathbb{C}^2, 0)$, (ii) the singularities of the foliation $\widetilde{\mathcal{F}} = \pi^* \mathcal{F}$, appearing over

the invariant projective line $\mathbb{P} = \pi^{-1}(0)$ are of the *reduced* form $xdy - \lambda ydx + \dots, \lambda \notin \mathbb{Q}_+$, for some local chart with $\mathbb{P} = (y = 0)$. In this case we consider the holonomy group of the leaf $\mathbb{P} \setminus \text{sing } \tilde{\mathcal{F}}$; which is called the *projective holonomy* of \mathcal{F} (see [3]). In [3] we study the analytical invariants of such foliations. We recall that two germs of foliations $\mathcal{F}, \mathcal{F}_1$ at $0 \in \mathbb{C}^2$ are *strongly analytically equivalent* if there exists a germ of biholomorphism $\tilde{\Phi}: (\tilde{\mathbb{C}}^2, \mathbb{P}) \rightarrow (\tilde{\mathbb{C}}^2, \mathbb{P})$ which is an equivalence for $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}_1$. Such an equivalence induces a *strong analytical equivalence* $\Phi: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ between \mathcal{F} and \mathcal{F}_1 [3]. In [3] we prove that if two germs of n.m.d. foliations have their separatrices contained in complex lines, the same projective singular points and singularities, and conjugated projective holonomy groups, then they are strongly analytically equivalent. We also prove that any finitely generated subgroup of $G \subset \text{Diff}(\mathbb{C}, 0)$ is conjugated to the projective holonomy group of a n.m.d germ $\mathcal{F}(G)$ called a *realization of G* , which has separatrices contained in complex lines. Finally, from [1],[2],[9],[13],[17] we have that if $G \subset \text{Diff}(\mathbb{C}, 0)$ is a finitely generated subgroup and $\mathcal{F}(G)$ is its realization then:

- (i) G is finite $\Leftrightarrow \mathcal{F}(G)$ has a local holomorphic first integral $f \in \mathcal{O}_2$;
- (ii) G is abelian linearizable $\Leftrightarrow \mathcal{F}(G)$ is a germ of Darboux (logarithmic) foliation $\sum_j \lambda_j \frac{df_j}{f_j} = 0$;
- (iii) G is abelian analytically normalizable $\Leftrightarrow \mathcal{F}(G)$ is given by a closed meromorphic 1-form ω ;
- (iv) G is solvable analytically normalizable $\Leftrightarrow \mathcal{F}(G)$ is given by a closed meromorphic 1-form or a pull-back of a germ of *Riccati* foliation (i.e., of model \mathcal{S}_2).

Motivated by these facts we introduce the following definition:

Definition 2. A finitely generated subgroup of $\text{Diff}(\mathbb{C}, 0)$ has *finite length k* if its realization $\mathcal{F}(G)$ is of model \mathcal{S}_k .

Nakai's Theorem [12] and the above remark imply that if $\mathcal{F}(G)$ has closed leaves outside its set of separatrices, then $\mathcal{F}(G)$ is of model $\mathcal{S}_k, k \leq 2$ [4]. Also according to [2],[17] if $\mathcal{F}(G)$ admits a Liouvillian first integral, then it is of model $\mathcal{S}_k, k \leq 2$. The question now is to decide whether there exists some dynamical or some integration hypothesis on the foliation $\mathcal{F}(G)$, which assures that a subgroup $G \subset \text{Diff}(\mathbb{C}, 0)$, has finite length $k \geq 3$. This problem remains open.

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